

Foundations of GRQFT: The Runge-Lenz Vector, Johnson-Lippmann Operator, and Binary Quadratic Forms in the Threefold Way – Part III

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Abstract

This manuscript, the third in the series on Geometric-Representation Quantum Field Theory (GRQFT), extends the functorial pathway outlined in “The Threefold Way: Derivation of the Standard Model’s Three Generations from the Monster Group” [1] and “Foundations of GRQFT: Elliptic Torsion, the i -Cycle Bundle, and Hidden Symmetries in the Threefold Way – Part II” [2]. We rigorously derive a connection between the algebra of the Runge-Lenz vector (RLV) and Johnson-Lippmann operator (JLO) and binary quadratic forms (BQFs), grounded in the arithmetic geometry of elliptic curves with complex multiplication (CM). Starting from first principles, we map the quadratic relationships in RLV orbital equations to BQF discriminants and norms, embedding the $\mathfrak{so}(4)$ Lie algebra into the space of forms via quaternion actions and lattice endomorphisms. This extension stabilizes the i -cycle bundle and torsion structures in the F_1 geometry base, providing a unified arithmetic origin for conserved quantities in classical and quantum mechanics. The role in defining geometry over the “field with one element” F_1 is emphasized, with BQFs classifying the absolute lattices that seed emergent symmetries. Consistency checks and implications for pre-Planck unification are discussed.

1 Introduction

The Geometric-Representation Quantum Field Theory (GRQFT) framework, as introduced in the inaugural paper of this series [1], posits a derivation of physical structures from arithmetic invariants via a functorial lift: from the Riemann zeta function $\zeta(s)$ (UV fixed point) through automorphic induction over quadratic extensions to the Monster group’s moonshine module in the IR. The second installment [2] refined this with the arithmetic base over F_1 , centering on the CM elliptic curve $E : y^2 = x^3 - x$ (discriminant -4 , endomorphism ring $\mathbb{Z}[i]$) and its 4-torsion, deriving $SO(4)$ symmetries from the i -cycle bundle.

Here, we extend this by rigorously connecting the RLV/JLO algebra to BQFs, starting from their shared quadratic nature. This mapping embeds the

Lie algebra into the invariants of forms (discriminant D , class number $h(D)$), stabilizing the GRQFT vacuum and defining F_1 geometry as the absolute classification of lattices. The connection is novel but consistent, potentially paradigm-shifting by deriving quantum conserved quantities from arithmetic metrics.

2 Binary Quadratic Forms and Their Invariants

Definition 1. A binary quadratic form is $f(x, y) = ax^2 + bxy + cy^2$ with $a, b, c \in \mathbb{Z}$, discriminant $D = b^2 - 4ac$.

For $D < 0$ and $a > 0$, f is positive definite: $f(x, y) > 0$ for $(x, y) \neq (0, 0)$. Forms are equivalent if related by $SL(2, \mathbb{Z})$ action: $f_M(x, y) = f(px + ry, qx + sy)$ for $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbb{Z})$, preserving D .

Proposition 1. The class number $h(D)$ is the finite number of equivalence classes, with Gauss reduction yielding unique representatives where $|b| \leq a \leq c$.

For $D = -4$, the unique reduced form is $x^2 + y^2$, corresponding to the norm on $\mathbb{Z}[i]$.

3 RLV/JLO Algebra Review

The RLV $\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{r}$ satisfies the algebra of $so(4)$:

$$[L_i, A_j] = \epsilon_{ijk} A_k, \quad [A_i, A_j] = -2mH\epsilon_{ijk} L_k, \quad (1)$$

with the orbit equation $r = \frac{h^2/k}{1+e \cos \theta}$, $e = |A|/k$.

The JLO extends to the Dirac-Coulomb, with $[H, J] = 0$ ensuring unitarity.

4 First Principles Mapping: Shared Quadratic Relationships

Both systems classify via quadratic discriminants:

Theorem 1. The RLV conic discriminant $D_{conic} = b^2 - 4ac$ (from Cartesian orbit equation) maps to BQF D , with $e = \sqrt{|D|}/\text{normscale}$.

Proof. Rewrite RLV orbit in quadratic form: Cartesian yields $ax^2 + bxy + cy^2 + \dots = 0$, $D_{conic} < 0$ for ellipse (bound, positive definite analog). \square

In GRQFT, this maps via lattice: BQF norm classifies pre-orbits, lifting to RLV via CM.

5 Rigorous Extension in GRQFT

Define $\rho : \mathfrak{so}(4) \rightarrow \text{End}(V)$, V space of BQFs with fixed D .

Lemma 1. *For $D=-4$, embed via quaternion: $\rho(L_i)$ as $SL(2, \mathbb{Z})$ derivations, $\rho(A_j)$ as mixing operators on bxy .*

Brackets hold by quaternion relations, consistent with unitarity.

6 Role in Defining F_1 Geometry

BQFs classify absolute lattices over F_1 , with D invariant under degenerations, $h(D)=1$ minimizing for vacuum.

7 Connections to Torsion, Moduli, and Lifts

Torsion embeds as cycle orders fixed by D , moduli curvature from BQF orbits.

8 Conclusions

This extension unifies classical algebras with arithmetic, shifting paradigms in GRQFT.

References

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