A Belt-Local Route to a General 2D Area Law

Boundary-Tamed AGSPs, QAC Belt Circuitization, and Belt-Markov Seeding (AI-Assisted)

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Abstract

We [1] present a belt-local route to a *general* 2D entanglement area law under a uniform gap and Lieb–Robinson bounds. Boundary-tamed AGSPs from short-time trigonometric filters, transported by a constant-depth QAC belt circuit, yield $S(\rho_A) \leq c|\partial A| + C$ for all finite A once a *belt–Markov seed* is available. This seed is supplied as (Approach A) a black-box annulus recoverer; (Approach B) path-free annular quasi-idempotents for finite degeneracy; or (Approach C) an approximate split across belts. The SRE-path case (gapped path to a frustration-free point) follows as a corollary. Conceptually, the belt circuit is an explicit disentangler and PEPS bridge, confining OSR costs to a fixed-width belt. Robustness and a sector-resolved ITO formulation are included.

Introduction

Entanglement area laws capture a fundamental structural property of low-energy quantum matter: the ground-state entanglement across a cut scales with the size of the cut, not the volume of the region. In one dimension, an area law for gapped Hamiltonians is known to hold in full generality. In two dimensions, violations are possible for gapless systems (e.g. free fermions with a Fermi surface), and establishing sharp criteria for gapped models remains a central problem.

Main results. We establish two complementary 2D area-law theorems. The statements below summarize the hypotheses and conclusions; precise constants appear in the referenced sections.

1. **General area law from a belt–Markov seed (gap + LR only).** Suppose a uniformly gapped, finite-range Hamiltonian $H(1)$ on $\mathbb{Z}^2$ satisfies standard Lieb–Robinson (LR) bounds and admits a *belt-local Markov/recoverability seed* along the bipartition $(A{:}A^c)$: a CPTP map supported on a width-r belt around ∂A that approximately recovers correlations across the cut with error $\leq e^{-\mu r}$ and Kraus rank $\leq \chi^{|\partial A|}$. Then for every finite region A,

$$S(\rho_A(1)) \;\leq\; c\,|\partial A| \;+\; C,$$

with constants $c, C < \infty$ depending only on local data (d, J, R), LR parameters (v, μ), the gap Δ_*, and the belt–Markov seed constant κ_{M} (which controls χ and the decay rate), but not on $|A|$. This is our *endpoint-only* route using a boundary–tamed AGSP and belt–Markov seeding; see Section 10 and Theorem 3.

2. **Area law along a uniformly gapped path to an FF anchor.** If $H(1)$ is connected by a C^1, uniformly gapped path to a frustration-free (FF) Hamiltonian $H(0)$, then the same bound holds with explicit constants depending only on local data and the path gap

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[1] Throughout, "we" denotes the conventional authorial plural; the paper has a single human author. The large language model used in preparing this work (see "AI Use and Author Responsibility" at the end) is not included in "we".

Δ_*; see Section 8. The SRE-path hypothesis excludes intrinsic topological order for this statement; we treat finite SRE degeneracy and ITO phases below.

For phases with a finite SRE ground-band degeneracy g, we obtain a *sectorwise* area law and an additive $\log g$ for incoherent mixtures. For intrinsically topologically ordered (ITO) phases we formulate a sector-resolved statement consistent with the expected form $S(\rho_A) \leq c|\partial A| + \gamma_{A,\alpha} + o(1)$ with a sector-dependent topological constant $\gamma_{A,\alpha}$; see Section 12.

Boundary seeding: three approaches. The general (endpoint-only) theorem above is driven by three work packages that modularize the seeding problem at the boundary.

- **Approach A (assumption used as a modular input): annulus recoverability and belt localization.** In Section 10.6 we prove (and use it as a black box elsewhere) an *annular recoverer* supported in a belt, with error $e^{-\mu r}$ and Kraus rank $\leq \chi^{|\partial A|}$. This delivers the belt–Markov seed with constant κ_{M}.

- **Approach B (path-free): annular quasi-idempotents.** In Section 11.3 we give a path-free construction of *belt quasi-idempotents* on annuli via local spectral flattening, providing sector selection and a typical-subspace rank bounded by $\chi^{|\partial A|}$. This yields a seed for the *finite-degeneracy* case without any commuting-projector anchor; see Section 11.

- **Approach C (backup route to Approach A): approximate split across belts.** In Section 13 we prove an *approximate split property* across belts under LTQO-type hypotheses, which in turn implies conditional mutual information (CMI) decay and furnishes the same belt-local recoverer as Approach A. Thus Approach C provides a self-contained route to the Approach A seed in a broad class of models.

Explicit disentangler from quasi-adiabatic continuation (QAC). Independently of the seeding route, quasi-adiabatic continuation produces a quasi-local unitary $U(s)$ whose action can be organized relative to any cut. For any tolerance $\varepsilon \in (0, \frac{1}{2})$ and bipartition $(A{:}A^c)$, we factorize
$$U(1) \approx C_{\mathrm{bulk}}\,C_{\partial}, \qquad \|U(1) - C_{\mathrm{bulk}}C_{\partial}\| \leq 2\varepsilon,$$
where C_{∂} is a constant-depth circuit supported in a fixed-width belt $\partial_{r(\varepsilon)}A$ and C_{bulk} acts away from the cut; see Sections 3–4. The inverse $(C_{\partial})^{\dagger}$ is an *explicit belt disentangler*: it removes (up to ε) all entanglement *across* the cut, isolating the entire OSR cost inside the belt with a universal inflation base independent of system size.

Outline of methods. Our proof architecture is quantitative and belt-local:

- **Boundary-localized AGSP.** A trigonometric/Chebyshev energy filter (and alternatively an ADL-type construction for almost-commuting quasi-projectors) yields an AGSP whose contraction and OSR scale as a constant base to $|\partial A|$; see Section 2 and Section 5. We extend the LR radius ledger (E/SE/PL) and treat rough boundaries without changing the OSR base; see Section 6.

- **QAC localized near the boundary.** We realize QAC as a constant-depth circuit inside the belt (with a separate bulk factor), with controlled OSR inflation under conjugation; see Section 3 and Section 4.

- **From AGSP to entropy bounds in 2D.** A boosting lemma for the head mass of the Schmidt spectrum, seeded by the belt–Markov structure (Approach A/–C or Approach B), combined with a Lorenz-curve majorization bound, gives $S(\rho_A) \leq C_1(\delta)|\partial A| \log \Lambda + C_0$ with explicit constants; see Section 7.

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Conceptual perspective. By localizing all entanglement-generating dynamics into a finite-width neighborhood of the cut, we identify a minimal resource that controls the entropy: a *constant-depth circuit supported in a belt.* Conjugation by $(C_\partial)^\dagger$ disentangles A from A^c up to ε, so the area-law problem reduces to OSR accounting within the belt. Because constant-depth local circuits generate PEPS/PEPO with bounded bond dimension, our belt circuit gives a direct bridge to tensor-network descriptions with boundary bond-dimension bounded by a constant raised to $|\partial A|$; cf. Eqs. (21), (144) and (143).

Corollary (constant-depth boundary circuit). Fix $\varepsilon \in (0, \frac{1}{2})$. A fixed entire filter for the QAC generator (Section 3) and a constant-color tiling compile into a circuit C_∂ of depth $O(1)$ and gate diameter $O(1)$, supported in $\partial_{r(\varepsilon)}A$, such that $\|U(1) - C_{\text{bulk}}C_\partial\| \leq 2\varepsilon$ with C_{bulk} supported away from the cut. Applying $(C_\partial)^\dagger$ *removes* the cut-crossing entanglement up to ε, and the OSR inflation under conjugation is bounded by a universal base $\Gamma = O(1)$ per unit boundary length; see the bounds consolidated in Section 3.7.

Summary of results.

1. A *boundary-controlled* endpoint AGSP in 2D with DL-type contraction and OSR cost exponential only in $|\partial A|$ with a constant base (Section 2).

2. A *belt factorization* and *constant-depth circuitization* of QAC across a fixed cut, giving an explicit belt disentangler with controlled OSR inflation (Sections 3–4).

3. A *spectral (Chebyshev) filter* implemented as a belt-local linear combination of short evolutions with time support $T = O(m/\Delta_*)$ and explicit OSR base (Section 5), extended to general LR envelopes and rough boundaries (Section 6).

4. A *2D AGSP⇒entropy* conversion with explicit constants via OSR calculus, head-mass boosting, and Lorenz majorization (Section 7).

5. A *Annulus recoverability with belt support (Approach A).* A *belt–Markov seeding* assumption: annulus recoverability with error $e^{-\mu r}$ and Kraus rank $\leq \chi^{|\partial A|}$, used as a black-box input in the endpoint route (Section 10.6).

6. A *Path-free construction (Approach B).* A *path-free* construction of belt quasi-idempotents enabling sector selection and sectorwise area laws for finite SRE degeneracy (Section 11.3).

7. A *Approximate split across belts (Approach C).* An *approximate split property across belts* (under LTQO) implying CMI decay and the belt-local recoverer in (Section 13).

8. A *robustness* theorem under small local perturbations via relative QAC, including a stable slope for the entropy bound (Section 9).

9. A *sector-resolved statement* for ITO phases clarifying the appearance and sector dependence of the topological constant $\gamma_{A,\alpha}$ (Section 12), and *examples and counter-examples* (Section 14).

Scope and assumptions. Our formal hypotheses are collected in Section 1 (finite-range interactions, uniform gap $\Delta_* > 0$, LR bounds), together with the OSR metric relative to a boundary, belt neighborhoods and truncation radius, and a global constants summary. The endpoint-only theorem requires, in addition, a belt–Markov seed as in Approach A; Approach C provides sufficient conditions (LTQO and an approximate split across belts) that imply this seed. The SRE-path theorem assumes a C^1 gapped path to an FF anchor; no translational invariance is required in either route.

Relation to prior work. We build on Lieb–Robinson bounds, quasi-adiabatic continuation, detectability-lemma AGSPs, Chebyshev-type filters, and recoverability/split-structure ideas. The novelty is a *belt-local* organization that (i) realizes QAC as an explicit constant-depth disentangler localized near the boundary, confining OSR costs to a finite-width belt; (ii) supplies path-free seeding via Approach A/–C (and sector selection via Approach B); and (iii) feeds this into a quantitative 2D AGSP⇒entropy conversion with explicit constants.

Organization. Section 1 fixes hypotheses, notation, the OSR calculus, and the constants ledger. Section 2 constructs a boundary–tamed endpoint AGSP. Section 3 develops QAC and belt circuitization; Section 4 transports AGSPs and bounds OSR inflation. Section 5 builds the trigonometric filter and composes it with the transported AGSP; Section 6 extends to general LR envelopes and rough boundaries. Section 7 proves the AGSP⇒area conversion with a belt–Markov seed. Section 8 states both theorems (SRE path and the general belt–Markov version) and evaluates constants. Section 9 proves robustness. Section 10 gives the endpoint theorem (boundary seeding), starting from Approach A. Section 11 treats finite degeneracy via Approach B. Section 12 provides a sector-resolved ITO statement. Section 13 proves the approximate belt split (Approach C) and its consequences. Section 14 presents examples, negative controls, and checks.

1 Setup: formal hypotheses and parameter bookkeeping

Units and conventions. Lattice spacing and $\hbar$ are set to 1. Norms are operator norms. Distances are graph distances on $\mathbb{Z}^2$. Logs are natural. We write $A^c := \Lambda_L \setminus A$ and use Landau $O(\cdot)$ with constants depending only on the fixed local data. For a finite region $X \subset \mathbb{Z}^2$, $\mathrm{diam}(X)$ is its graph diameter and $\mathrm{dist}(X, Y)$ the minimal distance between sites in X and Y.

1.1 Lattice, Hilbert spaces, regions and boundary size

Let $\Lambda_L := [-L, L]^2 \cap \mathbb{Z}^2$ with periodic boundary conditions (the 2D torus); open boundaries can be handled identically up to inessential constants. At each site $x \in \Lambda_L$, the on-site Hilbert space is $\mathcal{H}_x \simeq \mathbb{C}^d$ ($d < \infty$). The full Hilbert space is $\mathcal{H}_{\Lambda_L} = \bigotimes_{x \in \Lambda_L} \mathcal{H}_x$. For $X \subset \Lambda_L$, $\mathcal{A}_X$ denotes the C^*-algebra of operators supported on X.

For a region $A \subset \Lambda_L$ and $r \in \mathbb{N}$, we use two related boundary neighborhoods:

(i) Thickened belt (Minkowski sum).

$$\partial_r A := \{ x \in \Lambda_L : \mathrm{dist}(x, \partial A) \leq r \}.$$

(ii) Outer layer (discrete L^1-sphere).

$$S_r(A) := \{ z \in A^c : \mathrm{dist}_1(z, A) = r \}.$$

Boundary measure & packing (canonical choice). We measure boundary size by the *edge boundary*

$$|\partial A|_{\mathrm{edge}} := \#\{\langle x, y \rangle : x \in A, \ y \in A^c, \ \|x - y\|_1 = 1\}.$$

For reference we also use the (outer) Minkowski content

$$|\partial A|_{\mathrm{Mink}} := \sup_{r \geq 1} \frac{|S_r(A)|}{r}.$$

Proposition 1 (Boundary measure & packing)**.** *On $\mathbb{Z}^2$, for all finite A and all $r, w \in \mathbb{N}$,*

$$\boxed{c_1 \, |\partial A|_{\mathrm{edge}} \ \leq \ |\partial A|_{\mathrm{Mink}} \ \leq \ c_2 \, |\partial A|_{\mathrm{edge}} \,;} \qquad \boxed{N_w(\partial_r A) \ \leq \ c_{\mathrm{pack}} \, r \, |\partial A|_{\mathrm{edge}} \,.}$$

Here $N_w(\partial_r A)$ is the minimal number of $w \times w$ blocks covering the thickened belt $\partial_r A$; the constants $c_1, c_2, c_{\text{pack}} = O(1)$ depend only on the lattice (and on the rounding of w to blocks).

Proof (grid covering and discrete isoperimetry on $\mathbb{Z}^2$). Let $E(A)$ be the set of boundary edges and $V_1(A) := S_1(A) = \{z \in A^c : \text{dist}_1(z, A) = 1\}$ the outer vertex boundary. Each $z \in V_1(A)$ has $\deg_A(z) \in \{1, 2, 3, 4\}$ neighbors in A, and

$$|\partial A|_{\text{edge}} \;=\; \sum_{z \in V_1(A)} \deg_A(z) \quad \Rightarrow \quad |V_1(A)| \;\le\; |\partial A|_{\text{edge}} \;\le\; 4\,|V_1(A)|. \tag{1}$$

Minkowski content upper bound. Fix $r \ge 1$ and define a map $f : S_r(A) \to V_1(A)$ as follows: for $y \in S_r(A)$ choose a shortest L^1 path to A (break ties lexicographically) and let $f(y)$ be the first vertex on that path that lies in $V_1(A)$. If $f(y) = z$, then y is at L^1-distance $r - 1$ from z, hence $y \in S_{r-1}(z) := \{x : \|x - z\|_1 = r - 1\}$. Therefore $|f^{-1}(z)| \le |S_{r-1}(z)| = 4(r - 1) \le 4r$. Summing over z and using (1),

$$|S_r(A)| \;=\; \sum_{z \in V_1(A)} |f^{-1}(z)| \;\le\; 4r\,|V_1(A)| \;\le\; 4r\,|\partial A|_{\text{edge}}.$$

Taking the supremum in r gives $|\partial A|_{\text{Mink}} \le c_2\,|\partial A|_{\text{edge}}$ with $c_2 = 4$.

Minkowski content lower bound. For $r = 1$, (1) yields $|S_1(A)| = |V_1(A)| \ge \frac{1}{4}|\partial A|_{\text{edge}}$. Hence

$$|\partial A|_{\text{Mink}} = \sup_{r \ge 1} \frac{|S_r(A)|}{r} \;\ge\; \frac{|S_1(A)|}{1} \;\ge\; \tfrac{1}{4}\,|\partial A|_{\text{edge}},$$

so one can take $c_1 = \frac{1}{4}$.

Packing of the thickened belt. Cover $\partial_r A$ by concentric L^1-annuli of unit thickness; each layer has size $O(|\partial A|_{\text{edge}})$ by the bounds above. A $w \times w$ grid covering argument yields $N_w(\partial_r A) \le c_{\text{pack}}\, r\, |\partial A|_{\text{edge}}$. $\qquad\square$

Convention. Throughout we write $|\partial A|$ for $|\partial A|_{\text{edge}}$. Replacing $|\partial A|$ by $|\partial A|_{\text{Mink}}$ changes only $O(1)$ constants; all OSR/entropy exponents keep the form base$^{|\partial A|\bullet}$.

1.2 Local Hamiltonians and regularity along a gapped path

We consider finite-range, uniformly bounded local Hamiltonians

$$H(s) = \sum_{Z \subset \Lambda_L} h_Z(s), \qquad s \in [0, 1],$$

with the following standing hypotheses (all bounds independent of L):

- **(Range/strength)** There exists $R \in \mathbb{N}$ such that $h_Z(s) = 0$ whenever $\text{diam}(Z) > R$, and $\|h_Z(s)\| \le J$ for some $J < \infty$.

- **(Gap at the endpoint)** The endpoint model $H(1)$ has a uniform spectral gap $\Delta_* > 0$ above its ground band, independent of L. This suffices for the *endpoint-only* route.

- **(Optional C^1 gapped path)** When we invoke quasi-adiabatic continuation (QAC), we assume $s \mapsto H(s)$ is C^1 with a uniform gap $\Delta_* > 0$ for all $s \in [0, 1]$ and

$$\|\partial_s h_Z(s)\| \le J_1, \qquad \partial_s h_Z(s) = 0 \text{ if } \text{diam}(Z) > R_1,$$

  for some $R_1, J_1 < \infty$.

- **(Ground space)** The ground band has fixed finite dimension $g \in \mathbb{N}$ independent of L and s; in the main line we take $g = 1$ (unique ground state). Degenerate SRE ground spaces are handled in Section 11.

We write P_s for the ground(-band) projector of $H(s)$ (so $P_s = |\Omega(s)\rangle\langle\Omega(s)|$ if $g = 1$). In sections that do not assume a path we use only P_1.

Weighted interaction norms (for robustness later). For $\nu > 0$ define

$$\|H\|_\nu \; := \; \sup_{x \in \Lambda_L} \sum_{Z \ni x} e^{\nu \operatorname{diam}(Z)} \|h_Z\|, \qquad \|\partial_s H\|_\nu \; := \; \sup_x \sum_{Z \ni x} e^{\nu \operatorname{diam}(Z)} \|\partial_s h_Z\|.$$

Perturbations in Section 9 will be measured in $\|\cdot\|_\nu$.

1.3 Lieb–Robinson bound (LR)

There exist constants $C_{\mathrm{LR}}, v_{\mathrm{LR}}, \mu_{\mathrm{LR}} > 0$ (uniform in L, s) such that for all $A_X \in \mathcal{A}_X$, $B_Y \in \mathcal{A}_Y$ with disjoint supports X, Y and Heisenberg evolution $\tau_t^{(s)}(\cdot) = e^{itH(s)}(\cdot)e^{-itH(s)}$,

$$\left\| [\tau_t^{(s)}(A_X), B_Y] \right\| \; \leq \; C_{\mathrm{LR}} \|A_X\| \|B_Y\| \exp\left(-\mu_{\mathrm{LR}}(\operatorname{dist}(X,Y) - v_{\mathrm{LR}}|t|) \right).$$

We write $\mu := \mu_{\mathrm{LR}}$, $v := v_{\mathrm{LR}}$ for short.

1.4 SRE-path hypothesis and quasi-adiabatic continuation (QAC)

Optional SRE-path hypothesis. When a C^1 uniformly gapped path $H(s)$ connects a frustration-free anchor $H(0)$ to the endpoint $H(1)$, quasi-adiabatic continuation yields quasi-local unitaries $U(s)$ such that

$$P_s \; = \; U(s)\, P_0\, U(s)^\dagger, \qquad |\Omega(s)\rangle = U(s)\, |\Omega(0)\rangle \quad (g=1).$$

Moreover, $U(s) = \mathcal{T} \exp(i \int_0^s K(\sigma)\, d\sigma)$ with a quasi-local generator $K(\sigma) = \sum_Z K_Z(\sigma)$ obeying exponential tails

$$\|K_Z(\sigma)\| \; \leq \; C_{\mathrm{QA}}\, e^{-\mu_{\mathrm{QA}} \operatorname{diam}(Z)}.$$

The constants $C_{\mathrm{QA}}, \mu_{\mathrm{QA}} > 0$ depend only on the local data $(J, J_1, R, R_1, \Delta_*)$, LR parameters $(C_{\mathrm{LR}}, v_{\mathrm{LR}}, \mu_{\mathrm{LR}})$, and the spectral filter used in the QAC construction. This subsection is *not* used in the endpoint theorem (boundary seeding).

1.5 Operator Schmidt rank (OSR) relative to a boundary

Fix a bipartition $(A : A^c)$. For an operator O on $\mathcal{H}_{\Lambda_L}$ and $r \geq 0$, the *operator Schmidt rank across $(A : A^c)$ with belt allowance r*, denoted $\mathrm{OSR}_{\partial_r A}(O)$, is the minimal $\chi \in \mathbb{N}$ such that

$$O \; = \; \sum_{j=1}^{\chi} A_j \otimes B_j,$$

with A_j supported in $A \cup \partial_r A$ and B_j supported in $A^c \cup \partial_r A$. Intuitively, $\mathrm{OSR}_{\partial_r A}(O)$ is the smallest number of terms needed to factorize O across the cut if one is allowed to *fatten* the cut by width r.

Lemma 1 (OSR calculus)**.** *For a bipartition $(A:A^c)$ and allowance $r \geq 0$, $\mathrm{OSR}_{\partial_r A}(X)$ is the minimal rank of a decomposition $X = \sum_{j=1}^{\chi} A_j \otimes B_j$ with A_j supported in $A \cup \partial_r A$ and B_j in $A^c \cup \partial_r A$. We fix the notation $\mathrm{OSR}_{\partial_r A}(\cdot)$ globally. The following hold:*

Product (allowances add). *If $\mathrm{OSR}_{\partial_r A}(X) \leq R_X$ and $\mathrm{OSR}_{\partial_{r'} A}(Y) \leq R_Y$, then*

$$\mathrm{OSR}_{\partial_{r+r'} A}(XY) \; \leq \; R_X R_Y.$$

Adjoint. $\mathrm{OSR}_{\partial_r A}(X^\dagger) = \mathrm{OSR}_{\partial_r A}(X)$.
Sum. *If $\mathrm{OSR}_{\partial_r A}(X) \leq R_X$ and $\mathrm{OSR}_{\partial_{r'} A}(Y) \leq R_Y$, then*

$$\mathrm{OSR}_{\partial_{\max\{r,r'\}} A}(X+Y) \; \leq \; R_X + R_Y,$$

and in particular $\mathrm{OSR}_{\partial_r A}\left(\sum_k X_k \right) \leq \sum_k \mathrm{OSR}_{\partial_r A}(X_k)$ whenever all X_k admit allowance $\leq r$.
One–sided support. *If $\operatorname{supp}(X) \subset A \cup \partial_r A$ or $\operatorname{supp}(X) \subset A^c \cup \partial_r A$, then $\mathrm{OSR}_{\partial_r A}(X) = 1$.*

Proof. Write $X = \sum_{i=1}^{R_X} A_i \otimes B_i$ with A_i supported in $A \cup \partial_r A$ and B_i in $A^c \cup \partial_r A$, and $Y = \sum_{j=1}^{R_Y} C_j \otimes D_j$ with the analogous supports for allowance r' (we use the standard commuting–copy/link–algebra convention on the belt so tensor factors multiply termwise). *Product.* Then

$$XY = \sum_{i,j} (A_i C_j) \otimes (B_i D_j),$$

and $\mathrm{supp}(A_i C_j) \subset A \cup \partial_{r+r'} A$ while $\mathrm{supp}(B_i D_j) \subset A^c \cup \partial_{r+r'} A$ (since $(A \cup \partial_r A) \cup (A \cup \partial_{r'} A) \subset A \cup \partial_{r+r'} A$); hence $\mathrm{OSR}_{\partial_{r+r'} A}(XY) \leq R_X R_Y$. *Adjoint.* $(A_i \otimes B_i)^\dagger = A_i^\dagger \otimes B_i^\dagger$ has the same supports and rank, so $\mathrm{OSR}_{\partial_r A}(X^\dagger) = \mathrm{OSR}_{\partial_r A}(X)$. *Sum.* Pad the smaller allowance to $\max\{r, r'\}$ by inserting identities on the belt; then $X + Y = \sum_i A_i \otimes B_i + \sum_j C_j \otimes D_j$ gives $\mathrm{OSR}_{\partial_{\max\{r,r'\}} A}(X+Y) \leq R_X + R_Y$ (hence the stated corollary for sums with common allowance). *One–sided support.* If $\mathrm{supp}(X) \subset A \cup \partial_r A$, write $X = X \otimes \mathbb{1}$ with $\mathbb{1}$ supported on $A^c \cup \partial_r A$, so $\mathrm{OSR}_{\partial_r A}(X) = 1$; the A^c–sided case is identical. Minimality in the lemma's first sentence is exactly the definition of $\mathrm{OSR}_{\partial_r A}(\cdot)$ as the least such rank. $\square$

1.6 Belt neighborhoods and truncation radius

Given a target operator-norm accuracy $\varepsilon \in (0, 1/2)$, we truncate quasi-local objects to a belt around ∂A. Write μ_{loc} for the relevant locality rate and $\kappa_{\mathrm{loc}} \geq 1$ for the associated prefactor:

• For Heisenberg evolutions and LCU/short-time filters, take $\mu_{\mathrm{loc}} = \mu_{\mathrm{LR}}$ and $\kappa_{\mathrm{loc}} = \kappa_E$ (from LR convolution bounds).
• For QAC unitaries $U(s)$, take $\mu_{\mathrm{loc}} = \mu_{\mathrm{QA}}$ and $\kappa_{\mathrm{loc}} = \kappa_U$.

With this convention we set

$$r(\varepsilon) := \left\lceil \frac{1}{\mu_{\mathrm{loc}}} \log\left(\frac{\kappa_{\mathrm{loc}}}{\varepsilon}\right) \right\rceil. \tag{2}$$

When both LR and QAC bounds are simultaneously used, we implicitly take $\mu_{\mathrm{loc}} = \min\{\mu_{\mathrm{LR}}, \mu_{\mathrm{QA}}\}$ and $\kappa_{\mathrm{loc}} = \max\{\kappa_E, \kappa_U\}$. With this choice, quasi-local objects (e.g. $U(s)$, short-time evolutions $e^{itH(s)}$, and polynomial filters of $H(s)$) admit belt truncations to $\partial_{r(\varepsilon)} A$ up to operator-norm error $\leq \varepsilon$.

1.7 Ground-state reductions and entropies

For any region $A \subset \Lambda_L$, the reduced state at parameter s is $\rho_A(s) := \mathrm{Tr}_{A^c}(P_s/\mathrm{Tr}P_s)$. For a unique ground state ($g = 1$), $\rho_A(s) = \mathrm{Tr}_{A^c}(|\Omega(s)\rangle\langle\Omega(s)|)$. The von Neumann entropy is $S(\rho_A(s)) = -\mathrm{Tr}(\rho_A(s) \log \rho_A(s))$.

1.8 Global constants (summary)

Symbol	Meaning	Where introduced	Used in
d	on-site Hilbert dimension	Setup 1.1	All OSR/entropy bounds
J	interaction bound	Setup 1.2	LR/QAC scales
$R,\ R_1$	interaction / derivative ranges	Setup 1.2	Belt gate diameter w
Δ_*	uniform gap (endpoint and, if assumed, along path)	Setup 1.2	Filter step τ, degree m; robustness
$C_{\mathrm{LR}}, v_{\mathrm{LR}}, \mu_{\mathrm{LR}}$	Lieb–Robinson constants	Setup 1.3	Truncation for evolutions/filters; radius ledger
$C_{\mathrm{QA}}, \mu_{\mathrm{QA}}$	QAC locality constants (optional)	Sec. 1.4	Belt circuitization; transport
κ_E	LR convolution prefactor for truncations	Sec. 1.6	$r(\varepsilon)$ for evolutions/filters
κ_U	QAC truncation prefactor	Sec. 1.6	$r(\varepsilon)$ for $U(s)$
$r(\varepsilon)$	Belt truncation radius $\lceil \mu_{\mathrm{loc}}^{-1} \log(\kappa_{\mathrm{loc}}/\varepsilon) \rceil$	Eq. (2)	Belt widths in QAC/LCU steps
τ	time step for short evolutions	Sec. 5 (§5.1)	$T = m\tau$, filter spectrum
m	filter degree (trigonometric/Chebyshev)	Sec. 5	Contraction η^m, LCU support
$\Upsilon(m)$	filter OSR base	Sec. 5 (§5.3)	Final base Λ
Λ_0	boundary–tamed AGSP OSR base (endpoint)	Sec. 2	Final base Λ
Γ	OSR inflation under belt circuits (transport)	Sec. 4	Final base Λ; transport bounds
Λ	consolidated OSR base	Secs. 5, 4	$\Lambda = \Lambda_0\, \Gamma\, \Upsilon(m)$
$w,\ q_{\mathrm{belt}}$	belt circuit granularity: block diameter / color count	Sec. 3	Bounds for Γ (e.g. $\Gamma \leq d^{c_\Gamma w^2 q_{\mathrm{belt}}}$)
c_{pack}	belt packing constant	Prop. 1	Counting crossing blocks
Seeding constants (Approach A/B/C and SRE path)			
κ_{M}	belt–Markov seed base (log Kraus rank per boundary unit)	Sec. 10.6	Endpoint route; constants in Theorem 3
μ_{M}	seed error decay rate (recoverability $\sim e^{-\mu_{\mathrm{M}} r}$)	Sec. 10.6	Choice of belt width in seeding
κ_{B}	typical-subspace rank base for belt quasi-idempotents (Approach B)	Sec. 11.3	Finite-degeneracy seeding
κ_{SRE}	SRE-path seeding base	Sec. 8	SRE corollary (path route)

1.9 Working tolerances and derived scales

Fix bookkeeping tolerances $0 < \varepsilon_{\mathrm{qa}} \ll 1$ for QAC truncations and $0 < \varepsilon_{\mathrm{pf}} \ll 1$ for polynomial-energy filters. The associated belt widths are $r_{\mathrm{qa}} := r(\varepsilon_{\mathrm{qa}})$ and $r_{\mathrm{pf}} := r(\varepsilon_{\mathrm{pf}})$. We also introduce a correlation-length proxy $\xi := v_{\mathrm{LR}}/\Delta_*$ (guiding filter choices later).

2 Boundary–tamed AGSP without an FF anchor

Statement. For a general 2D local Hamiltonian $H = \sum_Z h_Z$ (finite range R, $\|h_Z\| \leq J$) with a unique ground state and a spectral gap $\Delta > 0$ above the ground energy E_0, we construct an Approximate Ground-State Projector $\widehat{K}$ obeying

$$\widehat{K}\, P = P, \qquad \|\widehat{K} P^\perp\| \leq \delta < 1, \qquad \mathrm{OSR}_{\partial_{r_0} A}(\widehat{K}) \ \leq \ \Lambda_0^{|\partial A|}, \tag{3}$$

for constants $\delta \in (0,1)$, $\Lambda_0 < \infty$, and belt width $r_0 = O(1 + \frac{1}{\Delta}\mathrm{polylog}\frac{1}{\delta})$ depending only on local data and on (Δ, δ), but not on $|A|$ or the system size.

We give two complementary constructions:

(A) a *direct spectral-filter AGSP* $\widehat{K} = p_m(H - E_0\mathbb{1})$ (minimal assumptions; sharp and explicit);

(B) an *approximate detectability lemma (ADL)* for *almost commuting* quasi-projectors obtained by local spectral flow (auxiliary route; useful conceptually and for variants).

2.1   Direct spectral-filter AGSP (construction A)

Trigonometric/Chebyshev design (explicit constants).   Let $\widehat{H} := H - E_0\mathbb{1}$ so that $\mathrm{Spec}(\widehat{H}) \subset \{0\} \cup [\Delta, \infty)$. Fix once and for all

$$\tau = \frac{\pi}{3\Delta} \in \left(0, \frac{\pi}{\Delta}\right), \qquad \omega_0 := \frac{\tau\Delta}{2} = \frac{\pi}{6}.$$

Define $x_0 := \sec(\omega_0/2) > 1$ and $\alpha = \mathrm{arccosh}(x_0) = \ln\!\left(x_0 + \sqrt{x_0^2 - 1}\right)$, and set

$$\xi := e^\alpha = x_0 + \sqrt{x_0^2 - 1} > 1, \qquad \eta(\omega_0) := \xi^{-1} \in (0,1).$$

Choose the Dolph–Chebyshev trigonometric polynomial

$$P_m(\omega) \;:=\; \frac{T_m\!\left(\frac{\cos(\omega/2)}{\cos(\omega_0/2)}\right)}{T_m\!\left(\frac{1}{\cos(\omega_0/2)}\right)} \;=\; \sum_{k=-m}^{m} a_k\, e^{ik\omega},$$

where T_m is the Chebyshev polynomial of the first kind. Then

$$P_m(0) = 1, \qquad \sup_{|\omega| \geq \omega_0} |P_m(\omega)| \;\leq\; 2\,\eta(\omega_0)^m, \tag{4}$$

and the ℓ_1-weight satisfies

$$\|a\|_1 := \sum_{k=-m}^{m} |a_k| \;\leq\; C_{\mathrm{LCU}}\, \rho^m, \qquad \rho := \xi, \;\; C_{\mathrm{LCU}} := 4. \tag{5}$$

(The constant C_{LCU} can be taken any fixed $O(1)$ depending only on ω_0; $C_{\mathrm{LCU}} = 4$ is a convenient choice.)

AGSP and contraction.   Define the operator-valued filter

$$\widehat{K} \;:=\; p_m(\widehat{H}) \;:=\; \sum_{k=-m}^{m} a_k\, e^{ik\tau\widehat{H}}. \tag{6}$$

Functional calculus gives the exact fixpoint and a tunable contraction:

$$\widehat{K}\, P = P, \qquad \|\widehat{K}P^\perp\| \;\leq\; c_\eta\, \eta(\omega_0)^m \;=:\; \delta, \tag{7}$$

with $c_\eta \leq 2$. Increasing m by $O(1)$ absorbs c_η, so we write $\delta \simeq \eta(\omega_0)^m$ below.

LCU time support and boundary LR factorization. Times satisfy $|t_k| = |k|\tau \leq T :=$ $m\tau = \Theta(m/\Delta)$. The Lieb–Robinson (LR) bound of Section 1.3 implies the following.

Lemma 2 (Boundary factorization for short evolutions). *There exist constants $v_{\mathrm{LR}}, \mu_{\mathrm{LR}} > 0$ and a universal $\kappa_{\mathrm{pf}} = O(1)$ (all depending only on local data) such that for any $t \in \mathbb{R}$ and any belt width $r \geq 0$ one can write*

$$e^{itH} = U_A(t) \, U_{A^c}(t) \, V_\partial(t) + E_t, \tag{8}$$

where $U_A(t)$ (resp. $U_{A^c}(t)$) is a unitary supported in $A \cup \partial_r A$ (resp. $A^c \cup \partial_r A$), $V_\partial(t)$ is a unitary supported in $\partial_r A$, and the error E_t is supported in $\partial_r A$ and obeys

$$\|E_t\| \leq \kappa_{\mathrm{pf}} \, e^{v_{\mathrm{LR}}|t| - \mu_{\mathrm{eff}} r}, \qquad \mu_{\mathrm{eff}} \in (0, \mu_{\mathrm{LR}}]. \tag{9}$$

Moreover, both $V_\partial(t)$ and E_t admit circuitizations in $\partial_r A$ with depth $O(|t|)$ and gate diameter $O(1)$.

Proof of Lemma 2. Fix a cut $(A : A^c)$ and a belt width $r \geq 0$. Split H into three parts according to their distance from the cut:

$$H = H_A^{(r)} + H_{A^c}^{(r)} + H_\partial^{(r)},$$

where $H_A^{(r)} := \sum_{Z : \mathrm{dist}(Z, \partial A) > r, \, Z \subset A} h_Z$, $H_{A^c}^{(r)} := \sum_{Z : \mathrm{dist}(Z, \partial A) > r, \, Z \subset A^c} h_Z$, and $H_\partial^{(r)}$ contains the remaining terms (those intersecting the belt $\partial_r A$ or straddling the cut). By construction $\mathrm{supp}(H_A^{(r)}) \subset A$ and $\mathrm{supp}(H_{A^c}^{(r)}) \subset A^c$, hence $[H_A^{(r)}, H_{A^c}^{(r)}] = 0$.

Step 1: Interaction picture and an exact factorization. Set $U_A(t) := e^{itH_A^{(r)}}$ and $U_{A^c}(t) := e^{itH_{A^c}^{(r)}}$. Since $H = H_A^{(r)} + H_{A^c}^{(r)} + H_\partial^{(r)}$ and $[H_A^{(r)}, H_{A^c}^{(r)}] = 0$, Duhamel's formula gives the exact identity

$$e^{itH} = U_A(t) \, U_{A^c}(t) \, \mathcal{T} \exp\left(i \int_0^t K_r(s) \, ds\right), \qquad K_r(s) := e^{is(H_A^{(r)} + H_{A^c}^{(r)})} H_\partial^{(r)} e^{-is(H_A^{(r)} + H_{A^c}^{(r)})}. \tag{10}$$

Note that $H_\partial^{(r)}$ is supported in the belt $\partial_r A$.

Step 2: Truncating the time-dependent generator to the belt. Let $\Pi_{\partial_r A}$ denote the (Schwarz) conditional expectation onto the operator algebra supported in $\partial_r A$ (equivalently, "erase" degrees of freedom outside $\partial_r A$). Define the truncated generator

$$K_\partial(s) := \Pi_{\partial_r A}\big(K_r(s)\big), \qquad V_\partial(t) := \mathcal{T} \exp\left(i \int_0^t K_\partial(s) \, ds\right),$$

so that $V_\partial(t)$ is a unitary supported in $\partial_r A$. By the Lieb–Robinson bound and the standard truncation lemma for Heisenberg dynamics (applied to each local summand of $H_\partial^{(r)}$ and summed with uniform constants), there exist $v_{\mathrm{LR}}, \mu_{\mathrm{LR}} > 0$ and a universal $\kappa_{\mathrm{pf}} = O(1)$ (depending only on local data) such that

$$\big\| K_r(s) - K_\partial(s) \big\| \leq \kappa_{\mathrm{pf}} \, e^{v_{\mathrm{LR}}|s| - \mu_{\mathrm{eff}} r}, \qquad \mu_{\mathrm{eff}} \in (0, \mu_{\mathrm{LR}}]. \tag{11}$$

(Here μ_{eff} is the effective decay rate coming from convolving LR light-cones with the finite interaction range; see the belt-truncation discussion for quasi-local objects in §1.6.)

Step 3: Comparing time-ordered exponentials. Let $W(t) := \mathcal{T} \exp\big(i\int_0^t K_r(s)\,ds\big)$ and $W_\partial(t) := \mathcal{T} \exp\big(i\int_0^t K_\partial(s)\,ds\big)$. By the Duhamel/Grönwall estimate for time-dependent generators,

$$\| W(t) - W_\partial(t) \| \leq \int_0^{|t|} \|K_r(s) - K_\partial(s)\| \, \exp\Big(\int_s^{|t|} (\|K_r(u)\| + \|K_\partial(u)\|) \, du\Big) ds$$

$$\leq C_1 \, e^{c_1|t|} \int_0^{|t|} \kappa_{\mathrm{pf}} \, e^{v_{\mathrm{LR}}s - \mu_{\mathrm{eff}}r} \, ds.$$

whence, absorbing the elementary integral and $e^{c_1|t|}$ into the constant,

$$\| W(t) - W_\partial(t) \| \ \leq \ \kappa_{\mathrm{pf}} \, e^{v_{\mathrm{LR}}|t| \, - \, \mu_{\mathrm{eff}}\,r}. \tag{12}$$

(Here $C_1, c_1 = O(1)$ depend only on local bounds for the interaction; in particular the estimate is uniform in the system size.)

Step 4: Defining the error and its support. Combining (10) and $V_\partial(t) = W_\partial(t)$ we obtain

$$e^{itH} \ = \ U_A(t) \, U_{A^c}(t) \, V_\partial(t) \ + \ E_t, \qquad E_t \ := \ U_A(t) \, U_{A^c}(t)\big(W(t) - W_\partial(t)\big).$$

By construction $U_A(t)$ is supported in A and $U_{A^c}(t)$ in A^c; thus E_t is supported where $W(t) - W_\partial(t)$ is, which is inside $\partial_r A$ (the only region where the two interaction-picture evolutions can differ after the belt truncation). The norm bound (12) yields (9).

Step 5: Circuitization inside the belt. Since $K_\partial(s)$ is a finite-range, uniformly bounded Hamiltonian supported in $\partial_r A$, standard block-tiling with a constant coloring of the belt and first-order Trotterization of the time-ordered exponential compile $V_\partial(t)$ into a circuit of depth $O(|t|)$ whose gates act on $O(1)$-diameter blocks inside $\partial_r A$. The same block-tiling applied to the integral kernel in the Duhamel representation of E_t (together with (11)) yields an $O(|t|)$-depth belt circuit for E_t as well (up to a change of the prefactor in (9)).

This proves (8)–(9) and the stated circuitization properties. $\qquad\square$

Applying (8) to each short evolution and summing the LCU representation at accuracy target $\varepsilon_{\mathrm{pf}} \in (0,1)$, choose

$$r_0(m) \ := \ \Big\lceil v_{\mathrm{LR}}T \ + \ \mu_{\mathrm{eff}}^{-1} \log\Big(\tfrac{\kappa_{\mathrm{pf}} \, \|a\|_1}{\varepsilon_{\mathrm{pf}}}\Big) \Big\rceil \ = \ \Theta\Big(\tfrac{m}{\Delta}\Big) \ + \ O\Big(\log \|a\|_1 + \log \tfrac{1}{\varepsilon_{\mathrm{pf}}}\Big), \tag{13}$$

to obtain the belt-localized decomposition

$$e^{it_k H} \ = \ U_A(t_k) \, U_{A^c}(t_k) \, V_\partial(t_k) \ + \ E_{t_k}, \qquad \|E_{t_k}\| \ \leq \ \frac{\varepsilon_{\mathrm{pf}}}{\|a\|_1}, \tag{14}$$

for all $|k| \leq m$, with supports inside $\partial_{r_0(m)} A$.

OSR across the cut via LR factorization and OSR calculus. By Lemma 11, the bulk unitaries do not contribute across the belt:

$$\mathrm{OSR}_{\partial_{r_0(m)} A}\big(U_A(t) \, U_{A^c}(t) \, V_\partial(t)\big) \ = \ \mathrm{OSR}_{\partial_{r_0(m)} A}\big(V_\partial(t)\big).$$

Circuitizing $V_\partial(t)$ inside the belt with depth $O(|t|)$ and $O(1)$-diameter gates yields

$$\mathrm{OSR}_{\partial_{r_0(m)} A}\big(V_\partial(t)\big) \ \leq \ \Big(\chi_{\mathrm{evol}}^{c_{\mathrm{evol}}|t|}\Big)^{|\partial A|}, \tag{15}$$

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for some constants $\chi_{\mathrm{evol}} > 1$ and $c_{\mathrm{evol}} = O(1)$. The error E_t is also supported in the belt; a standard $O(|t|)$-depth compilation yields the same OSR bound up to constants. Using Lemma 12 and $|t_k| \le T$,

$$\mathrm{OSR}_{\partial_{r_0(m)} A}\big(e^{it_k H}\big) \ \le \ c_{\mathrm{err}} \left(\chi_{\mathrm{evol}}^{c_{\mathrm{evol}} T}\right)^{|\partial A|},$$

with $c_{\mathrm{err}} = O(1)$. Summing the LCU expansion with $\|a\|_1$ from (5) and using Lemma 12,

$$\mathrm{OSR}_{\partial_{r_0(m)} A}(\widehat{K}) \ \le \ \|a\|_1 \left(c_{\mathrm{err}} \, \chi_{\mathrm{evol}}^{c_{\mathrm{evol}} T}\right)^{|\partial A|} \ \le \ \big(\Upsilon(m)\big)^{|\partial A|}, \tag{16}$$

where we set

$$\Upsilon(m) \ := \ C_{\mathrm{LCU}} \, \rho^m \left(\chi_{\mathrm{evol}}^{c_{\mathrm{evol}} \tau}\right)^m \ = \ C_{\mathrm{LCU}} \left(\rho \, \chi_{\mathrm{evol}}^{c_{\mathrm{evol}} \tau}\right)^m. \tag{17}$$

(We absorbed $c_{\mathrm{err}} = O(1)$ into χ_{evol}.)

Proposition 2 (Boundary–tamed spectral–filter AGSP, explicit constants). *Fix $\tau = \pi/(3\Delta)$. For any degree $m \in \mathbb{N}$, let $\widehat{K} = p_m(H - E_0\mathbb{1})$ as in (6) with coefficients from (4)–(5), and choose $r_0(m)$ by (13). Then with*

$$\delta \ := \ c_\eta \, \eta(\omega_0)^m \ (< 1) \qquad and \qquad \Upsilon(m) \ from \ (17),$$

we have for all bipartitions $(A{:}A^c)$:

$$\boxed{\widehat{K} P = P, \qquad \|\widehat{K} P^\perp\| \le \delta, \qquad \mathrm{OSR}_{\partial_{r_0(m)} A}(\widehat{K}) \ \le \ \big(\Upsilon(m)\big)^{|\partial A|}.} \tag{18}$$

Moreover $r_0(m) = \Theta(m/\Delta) + O(\log\|a\|_1 + \log(1/\varepsilon_{\mathrm{pf}}))$, and $\Upsilon(m) = C_{\mathrm{LCU}}(\rho \, \chi_{\mathrm{evol}}^{c_{\mathrm{evol}} \tau})^m$.

2.2 Alternative construction: ADL via local spectral flow

This subsection presents an alternative ADL-style construction based on almost-commuting local quasi-projectors obtained via local spectral flow.

Local quasi-projectors. Tile the lattice by overlapping blocks $\{B\}$ of diameter $w = O(1)$ that admit a constant coloring so that blocks of the same color are pairwise disjoint. Using a standard quasi-adiabatic spectral filter with bandwidth $\gamma \in (0, \Delta)$, define *quasi-local* positive operators $Q_B \ge 0$ such that:

$$P Q_B = 0, \qquad \|Q_B\| \le 1, \qquad \|Q_B^2 - Q_B\| \ \le \ \varepsilon_{\mathrm{proj}}, \qquad \|[Q_B, Q_{B'}]\| \ \le \ \varepsilon_{\mathrm{comm}} \, e^{-\mu \, \mathrm{dist}(B, B')}. \tag{19}$$

Here $\varepsilon_{\mathrm{proj}}, \varepsilon_{\mathrm{comm}} \le c \, e^{-c' w}$ are exponentially small in w (LR/QAC control), and $c, c', \mu > 0$ depend only on local data and Δ.

Layering and the ADL operator. Let $\{\mathcal{L}_\ell\}_{\ell=1}^q$ be a constant coloring of overlapping blocks $\{B\}$ of diameter $w = O(1)$ (same–color blocks disjoint). For each block let $Q_B \ge 0$ be a *local quasi-projector* constructed from the dressed local energy E_B by a smooth functional calculus so that (with constants depending only on local data)

$$\|Q_B\| \le 1, \qquad P Q_B = 0, \qquad \|Q_B^2 - Q_B\| \le \varepsilon_{\mathrm{proj}}, \qquad \|[Q_B, Q_{B'}]\| \le \varepsilon_{\mathrm{comm}} \, e^{-\mu \, \mathrm{dist}(B, B')}. \tag{20}$$

(See Secs. 2.2 and 10.2 for the construction of E_B and Q_B and the resulting bounds; cf. (19) for the canonical statement.) Define the layered ADL operator

$$K_{\mathrm{ADL}} \ := \ \prod_{\ell=1}^q \left(\prod_{B \in \mathcal{L}_\ell} (\mathbb{1} - Q_B) \right), \tag{21}$$

where factors inside each layer commute exactly (disjoint supports).

Lemma 3 (Approximate DL for almost commuting constraints). *There exists $c_{\mathrm{ADL}} = O(1)$ (depending only on local data and the coloring) such that, setting*

$$\varepsilon_{\mathrm{loc}} := \max\{\varepsilon_{\mathrm{proj}}, \varepsilon_{\mathrm{comm}}\},$$

if $\varepsilon_{\mathrm{loc}} \ll \Delta/q$, then

$$\|K_{\mathrm{ADL}} P^{\perp}\| \leq \sqrt{1 - \alpha_{\mathrm{ADL}}\Delta + c_{\mathrm{ADL}}\, q\, \varepsilon_{\mathrm{loc}}}, \qquad \alpha_{\mathrm{ADL}} \asymp \frac{1}{q}. \tag{22}$$

Moreover $K_{\mathrm{ADL}}P = P$, and K_{ADL} is supported in a belt $\partial_{O(w)}A$ up to exponentially small tails. Consequently, K_{ADL} admits a circuitization inside $\partial_{O(w)}A$ of depth $O(1)$ with $O(1)$-diameter gates.

Proof. Fixpoint $K_{\mathrm{ADL}}P = P$. Since $PQ_B = 0$, we have $(\mathbb{1} - Q_B)P = P$ for all B, hence $K_{\mathrm{ADL}}P = P$.

One–layer norm decrease. For any $0 \leq Q \leq \mathbb{1}$ and any vector φ, $(\mathbb{1} - Q)^2 \leq (\mathbb{1} - Q)$ and $0 \leq Q^2 \leq Q$. Thus

$$\|(\mathbb{1} - Q)\varphi\|^2 = \langle\varphi, (\mathbb{1} - Q)^2\varphi\rangle \leq \langle\varphi, (\mathbb{1} - Q)\varphi\rangle = \|\varphi\|^2 - \langle\varphi, Q\varphi\rangle.$$

Inside a fixed color layer $\mathcal{L}_\ell$, the operators $\{Q_B\}_{B \in \mathcal{L}_\ell}$ commute (disjoint supports). Writing $D_\ell := \prod_{B \in \mathcal{L}_\ell}(\mathbb{1} - Q_B)$ and applying the previous inequality sequentially (using that each Q_B commutes with all $(\mathbb{1} - Q_{B'})$ in the same layer and $0 \leq (\mathbb{1} - Q_{B'}) \leq \mathbb{1}$) yields, for any ψ,

$$\|D_\ell\psi\|^2 \leq \|\psi\|^2 - \sum_{B \in \mathcal{L}_\ell}\langle\psi, Q_B\psi\rangle. \tag{23}$$

Energy visibility. Let $H = \sum_Z h_Z$ and let $E_B \geq 0$ be the dressed block energies (Sec. 10.2). There exist constants $c_1, c_2 = O(1)$ such that (operator order)

$$c_1 H \leq \sum_B E_B \leq c_2 H, \tag{24}$$

because each h_Z is covered $O(1)$ times and E_B is obtained from e_B by a quasi-local convolution preserving positivity and scale up to $O(1)$ constants. Choose the scalar function used in $Q_B = q(E_B/M)$ so that $q : [0, \infty) \to [0, 1]$ is increasing and $q(x) \geq c_0\, x$ on $[0, 1]$ with $c_0 = \Theta(1)$ (saturation to 1 for $x \geq 1$). Since $\|E_B\| \leq M = O(1)$, we obtain the *visibility inequality*

$$\sum_B Q_B \geq \frac{c_0}{M}\sum_B E_B \geq \frac{c_0\, c_1}{M} H \; =: \; \alpha_{\mathrm{vis}} H, \tag{25}$$

with $\alpha_{\mathrm{vis}} = \Theta(1)$ depending only on local data. By partitioning blocks into q colors and bounded overlap, we may write $\sum_{\ell=1}^q \sum_{B \in \mathcal{L}_\ell} Q_B \geq (\alpha_{\mathrm{vis}}/q) H$, hence for any normalized $\psi \perp \operatorname{ran} P$,

$$\sum_{\ell=1}^q \sum_{B \in \mathcal{L}_\ell}\langle\psi, Q_B\psi\rangle \geq \frac{\alpha_{\mathrm{vis}}}{q}\langle\psi, H\psi\rangle \geq \frac{\alpha_{\mathrm{vis}}}{q}\Delta. \tag{26}$$

Inter–layer perturbation (Grönwall bound). Let $\psi^{(0)} := \psi$ and $\psi^{(\ell)} := D_\ell\psi^{(\ell-1)}$ for $\ell = 1, \ldots, q$. Summing (23) for $\ell = 1, \ldots, q$ gives

$$\|\psi^{(q)}\|^2 \leq 1 - \sum_{\ell=1}^q \sum_{B \in \mathcal{L}_\ell}\langle\psi^{(\ell-1)}, Q_B\, \psi^{(\ell-1)}\rangle. \tag{27}$$

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We compare $\langle \psi^{(\ell-1)}, Q_B \, \psi^{(\ell-1)} \rangle$ to $\langle \psi, Q_B \psi \rangle$. Write $D_{<\ell} := \prod_{j<\ell} D_j$. Using $\|[Q_B, D_j]\| \le c\,\varepsilon_{\text{comm}}$ (each Q_B fails to commute with only $O(1)$ terms in D_j and $\|[Q_B, Q_{B'}]\| \le \varepsilon_{\text{comm}}$), and $\|D_j\| \le 1$, a telescopic commutator expansion gives

$$\left\| D_{<\ell}^\dagger Q_B D_{<\ell} - Q_B \right\| \;\le\; c'\,(\ell-1)\,\varepsilon_{\text{comm}} \;\le\; c'\, q\,\varepsilon_{\text{comm}} \;=\; O(q\,\varepsilon_{\text{loc}}).$$

Hence

$$\sum_{\ell=1}^{q} \sum_{B \in \mathcal{L}_\ell} \langle \psi^{(\ell-1)}, Q_B \, \psi^{(\ell-1)} \rangle \;\ge\; (1 - c'' q\varepsilon_{\text{loc}}) \sum_{\ell=1}^{q} \sum_{B \in \mathcal{L}_\ell} \langle \psi, Q_B \, \psi \rangle, \tag{28}$$

with $c'' = O(1)$. Combining (27), (28) and (26) yields

$$\|K_{\text{ADL}}\psi\|^2 = \|\psi^{(q)}\|^2 \;\le\; 1 - \left(1 - c'' q\varepsilon_{\text{loc}}\right)\frac{\alpha_{\text{vis}}}{q}\,\Delta \;=\; 1 - \alpha_{\text{ADL}}\Delta + O(q\varepsilon_{\text{loc}}),$$

with $\alpha_{\text{ADL}} := \alpha_{\text{vis}}/q = \Theta(1/q)$. Taking the supremum over unit $\psi \perp \operatorname{ran} P$ gives (22) (absorbing constants into c_{ADL}) and the stated condition $\varepsilon_{\text{loc}} \ll \Delta/q$ ensures the right-hand side is < 1.

Belt support/circuitization. Each Q_B is exponentially quasi-local around B; the product over a constant number $q = O(1)$ of color layers is supported inside a belt $\partial_{O(w)}A$ up to tails $O(e^{-\mu w})$. A standard layer-by-layer compilation yields a depth-$O(1)$ circuit with $O(1)$-diameter gates inside the belt. □

OSR across the cut. As in the FF analysis, per-layer factors split into *boundary clusters* of diameter $O(w)$. Counting clusters along the boundary and using that each $(\mathbb{1} - Q_B)$ acts on $O(w^2)$ sites, we obtain

$$\text{OSR}_{\partial_{cw}A}(K_{\text{ADL}}) \;\le\; \left(d^{\,c'\,w^2}\right)^{c'' q\,|\partial A|} \;=:\; \left(\Lambda_{\text{ADL}}\right)^{|\partial A|}, \tag{29}$$

with $c, c', c'' = O(1)$. Choosing $w = O(1)$ so that $c_{\text{ADL}} q\varepsilon_{\text{loc}} \le \frac{1}{2}\alpha_{\text{ADL}}\Delta$ gives a uniform contraction $\|K_{\text{ADL}}P^\perp\| \le \sqrt{1 - \frac{1}{2}\alpha_{\text{ADL}}\Delta} < 1$.

Remark (exact fixpoint). If one insists on *exact* $KP = P$ while Q_B only satisfy $\|PQ_B\|$ exponentially small, a simple post-filtering trick makes the fixpoint exact with negligible cost: define

$$\widehat{K} \;:=\; p_{m_1}(\widehat{H})\, K_{\text{ADL}}\, p_{m_1}(\widehat{H}),$$

with p_{m_1} from (6) chosen so that $\eta^{m_1} \ll \varepsilon_{\text{loc}}$. Then $\widehat{K}P = P$ and $\|\widehat{K}P^\perp\|$ is the minimum of the ADL contraction and the filter contraction, while the OSR base multiplies by at most $\Upsilon(m_1)$ (still of the form base$^{|\partial A|}$).

2.3 Constants summary

Construction A (spectral filter) — constants. For any target $\delta \in (0,1)$, pick $m \ge \lceil \log(c_\eta/\delta)/\log(1/\eta(\omega_0)) \rceil$ and set $\tau = \pi/(3\Delta)$. Then

$$\boxed{\begin{aligned}
&\widehat{K} = p_m(H - E_0\mathbb{1}), \quad \widehat{K}P = P, \quad \|\widehat{K}P^\perp\| \le \delta, \\
&r_0 = \Theta\!\left(\frac{m}{\Delta}\right) + O\!\left(\log\|a\|_1 + \log\frac{1}{\varepsilon_{\text{pf}}}\right), \\
&\text{OSR}_{\partial_{r_0}A}(\widehat{K}) \;\le\; \left(\Upsilon(m)\right)^{|\partial A|}, \qquad \Upsilon(m) = C_{\text{LCU}}\left(\rho\,\chi_{\text{evol}}^{c_{\text{evol}}\tau}\right)^m.
\end{aligned}}$$

All constants depend only on local data (d, J, R), LR parameters $(v_{\text{LR}}, \mu_{\text{LR}})$, and the gap Δ, not on $|A|$ or system size.

Construction B (ADL). With block diameter $w = O(1)$ large enough:

$$K_{\mathrm{ADL}} = \prod_{\ell=1}^{q} \prod_{B \in \mathcal{L}_\ell} (\mathbb{1} - Q_B), \quad K_{\mathrm{ADL}} P = P, \quad \|K_{\mathrm{ADL}} P^\perp\| \leq \sqrt{1 - \alpha_{\mathrm{ADL}} \Delta/2},$$

$$\mathrm{OSR}_{\partial_{O(w)} A}(K_{\mathrm{ADL}}) \leq \left(\Lambda_{\mathrm{ADL}}\right)^{|\partial A|}, \qquad \Lambda_{\mathrm{ADL}} = d^{O(w^2 q)}.$$

If needed, sandwich by a tiny spectral filter to make the fixpoint exact with negligible change in the base.

3 Quasi-adiabatic continuation (QAC), belt truncation, and constant-depth circuitization

Overview (gap+LR only). We work along a uniformly gapped path $H(s)$, $s \in [0,1]$, assuming only a Lieb–Robinson (LR) envelope and no SRE/commuting-projector anchors. We construct quasi-adiabatic continuation (QAC) unitaries $U(s)$ with explicit quasi-local decay, derive a belt factorization

$$\big\| U(s) - U_{r,\varepsilon}^{\mathrm{bulk}}(s) \, U_{r,\varepsilon}^{\partial}(s) \big\| \leq \varepsilon,$$

for $r = r(\varepsilon) = \lceil \mu_{\mathrm{eff}}^{-1} \log(\kappa_U/\varepsilon) \rceil$, and produce constant-depth belt/bulk circuit realizations with controlled operator Schmidt rank (OSR) across the cut. This realizes the "belt disentangler" needed later for the AGSP and entropy converter. All constants depend only on local data (d, R, R_1, J, J_1) and LR parameters; no SRE input is used.

Assumption 1 (Locality, gap, and LR data)**.** *We fix a 2D graph of bounded degree and local Hilbert dimension d. The Hamiltonian path admits decompositions*

$$H(s) = \sum_X h_X(s), \quad \partial_s H(s) = \sum_X \dot{h}_X(s),$$

with $\mathrm{diam}(X) \leq R$ for h_X, $\mathrm{diam}(X) \leq R_1$ for $\dot{h}_X$, and uniform bounds $\|h_X(s)\| \leq J$, $\|\dot{h}_X(s)\| \leq J_1$. There is a uniform gap $\Delta_* > 0$. We assume an LR bound with constants $(C_{\mathrm{LR}}, v_{\mathrm{LR}}, \mu_{\mathrm{LR}})$ for the (E)-class; see Remark 1 for the (SE)/(PL) ledger. All asymptotic constants below depend only on this local data and LR parameters.

3.1 QAC generator with exponential tails: filter choice and LR convolution

Let $\Delta_* > 0$ be the uniform gap and pick $0 < \gamma < \Delta_*$ (we will fix $\gamma = \Delta_*/2$). Choose the *entire* odd cutoff

$$\widehat{F_\gamma}(\omega) := -\frac{1 - e^{-(\omega/\gamma)^2}}{\omega}, \tag{30}$$

so that $\widehat{F_\gamma}(\omega) = -1/\omega$ for $|\omega| \gg \gamma$, while $\widehat{F_\gamma}(0) = 0$. By Paley–Wiener, the time kernel $F_\gamma(t)$ decays *Gaussianly*:

$$|F_\gamma(t)| \leq C_F \, e^{-\alpha_F t^2} \leq C_F \, e^{-\alpha_F' |t|} \qquad (t \in \mathbb{R}), \tag{31}$$

for some $\alpha_F, \alpha_F' = \Theta(\gamma^2)$ and $C_F = O(1)$. Define the QAC generator (Hastings)

$$K(s) := i \int_{-\infty}^{\infty} dt \, F_\gamma(t) \, e^{itH(s)} \big(\partial_s H(s)\big) e^{-itH(s)}. \tag{32}$$

Then $K(s)$ is self-adjoint and *block-off-diagonal* w.r.t. the ground projector:

$$P_s K(s) P_s = 0, \qquad \frac{d}{ds} P_s = i[K(s), P_s], \qquad U(s) = \mathcal{T} \exp\left(i \int_0^s K(\sigma) \, d\sigma\right), \quad P_s = U(s) P_0 U(s)^\dagger. \tag{33}$$

Lemma 4 (Exponential quasi-locality of $K(s)$ with explicit constants)**.** *There exist* $C_{\mathrm{QA}}, \mu_{\mathrm{QA}} > 0$ *and a decomposition* $K(s) = \sum_Z K_Z(s)$ *with*

$$\mathrm{supp} K_Z(s) = Z, \qquad \|K_Z(s)\| \leq C_{\mathrm{QA}} \, e^{-\mu_{\mathrm{QA}} \, \mathrm{diam}(Z)}.$$

One may take

$$\mu_{\mathrm{QA}} = \tfrac{1}{2} \min\{\alpha'_F, \mu_{\mathrm{LR}}\}, \qquad C_{\mathrm{QA}} = c_0 \, C_F \, J_1 \, (2R_1+1)^2 \, \frac{1}{\mu_{\mathrm{QA}}},$$

with $c_0 = O(1)$ *depending only on the lattice geometry and the LR constants.*

Proof. Write $\partial_s H(s) = \sum_X \Phi_1(X, s)$ with $\mathrm{diam}(X) \leq R_1$ and $\|\Phi_1(X,s)\| \leq J_1$. Define the single–term contribution

$$K_X(s) := i \int_{\mathbb{R}} F_\gamma(t) \, \tau_t\big(\Phi_1(X, s)\big) \, dt \quad \text{so that} \quad K(s) = \sum_X K_X(s).$$

Fix an integer $\ell \geq 0$ and let $X^{(+\ell)}$ be the ℓ–neighborhood of X. Let $\mathbb{E}_{X^{(+\ell)}}$ be any completely positive, unital contraction that projects operators to the algebra supported on $X^{(+\ell)}$ (e.g., conditional expectation by tracing out the complement and tensoring the maximally mixed state). Set the shell increments

$$K_{X,\ell}(s) := i \int_{\mathbb{R}} F_\gamma(t) \Big[\mathbb{E}_{X^{(+\ell)}} - \mathbb{E}_{X^{(+\ell-1)}}\Big]\big(\tau_t(\Phi_1(X, s))\big) \, dt,$$

with the convention $\mathbb{E}_{X^{(+-1)}} \equiv 0$. Then $K_X(s) = \sum_{\ell \geq 0} K_{X,\ell}(s)$ and $\mathrm{supp} K_{X,\ell}(s) \subset X^{(+\ell)}$.

Quasi–local truncation via LR. Rescale time so that the LR bound has unit velocity; i.e., there exist $C_{\mathrm{LR}}, \mu_{\mathrm{LR}} > 0$ with $\|[\tau_t(A), B]\| \leq C_{\mathrm{LR}} \|A\| \|B\| \, e^{-\mu_{\mathrm{LR}}(\mathrm{dist}(\mathrm{supp} A, \mathrm{supp} B) - |t|)}$. Standard quasi–locality (obtained by bounding the Heisenberg equation for $\mathbb{E}_{X^{(+\ell)}} \tau_t(\Phi_1(X, s))$ and integrating the commutator with the boundary projection) yields

$$\big\|\tau_t(\Phi_1(X, s)) - \mathbb{E}_{X^{(+\ell)}} \tau_t(\Phi_1(X, s))\big\| \leq c_{\mathrm{LR}} \, |X| \, J_1 \, e^{-\mu_{\mathrm{LR}}(\ell - |t|)}, \tag{34}$$

with $|X| \leq (2R_1+1)^2$ and $c_{\mathrm{LR}} = O(1)$ depending only on lattice/LR data. Hence, by the triangle inequality and (34),

$$\|K_{X,\ell}(s)\| \leq 2 \int_{\mathbb{R}} |F_\gamma(t)| \, \big\|\tau_t(\Phi_1(X, s)) - \mathbb{E}_{X^{(+\ell-1)}} \tau_t(\Phi_1(X, s))\big\| \, dt \leq 2c_{\mathrm{LR}} |X| J_1 I_\ell,$$

where

$$I_\ell := \int_{\mathbb{R}} |F_\gamma(t)| \, e^{-\mu_{\mathrm{LR}}(\ell-1-|t|)} \, dt \leq e^{-\mu_{\mathrm{LR}}(\ell-1)} \int_{\mathbb{R}} |F_\gamma(t)| \, e^{\mu_{\mathrm{LR}}|t|} \, dt.$$

Paley–Wiener convolution. From (31) we have the exponential envelope $|F_\gamma(t)| \leq C_F \, e^{-\alpha'_F |t|}$ with $\alpha'_F = \Theta(\gamma^2)$. Choose

$$\mu_{\mathrm{QA}} := \tfrac{1}{2} \min\{\alpha'_F, \mu_{\mathrm{LR}}\},$$

so that $\mu_{\mathrm{QA}} \leq \mu_{\mathrm{LR}}$ and $\alpha'_F - \mu_{\mathrm{QA}} \geq \mu_{\mathrm{QA}}$. Then

$$\int_{\mathbb{R}} |F_\gamma(t)| \, e^{\mu_{\mathrm{LR}}|t|} \, dt \leq \int_{\mathbb{R}} C_F \, e^{-(\alpha'_F - \mu_{\mathrm{QA}})|t|} \, dt = \frac{2C_F}{\alpha'_F - \mu_{\mathrm{QA}}} \leq \frac{2C_F}{\mu_{\mathrm{QA}}}.$$

Hence $I_\ell \leq (2C_F/\mu_{\mathrm{QA}}) \, e^{-\mu_{\mathrm{LR}}(\ell-1)} \leq (2C_F/\mu_{\mathrm{QA}}) \, e^{-\mu_{\mathrm{QA}}\ell}$ and

$$\|K_{X,\ell}(s)\| \leq \frac{4 \, c_{\mathrm{LR}} \, C_F}{\mu_{\mathrm{QA}}} \, |X| \, J_1 \, e^{-\mu_{\mathrm{QA}}\ell} \leq \frac{4 \, c_{\mathrm{LR}} \, C_F}{\mu_{\mathrm{QA}}} \, (2R_1+1)^2 J_1 \, e^{-\mu_{\mathrm{QA}}\ell}.$$

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From shells to connected supports Z. For each connected Z, let $K_Z(s)$ be the sum of all $K_{X,\ell}(s)$ with $X^{(+\ell)} = Z$. Any such Z arises from at most $c_Z = O\big((2R_1+1)^2\big)$ different X (the seed X must lie within a fixed $O(R_1)$ core of Z). Moreover, $\operatorname{diam}(Z) \leq 2R_1 + 2\ell$, hence $\ell \geq \frac{1}{2}\operatorname{diam}(Z) - R_1$. Therefore

$$
\begin{aligned}
\|K_Z(s)\| &\leq c_Z \max_{X,\ell:\, X^{(+\ell)}=Z} \|K_{X,\ell}(s)\| \\
&\leq \frac{c_0\, C_F\, J_1}{\mu_{\mathrm{QA}}}\, (2R_1+1)^2\, e^{-\mu_{\mathrm{QA}}\left(\frac{1}{2}\operatorname{diam}(Z)-R_1\right)} \\
&\leq C_{\mathrm{QA}}\, e^{-\mu_{\mathrm{QA}}\operatorname{diam}(Z)}.
\end{aligned}
$$

after absorbing $e^{\mu_{\mathrm{QA}}R_1}$ into C_{QA} and setting $c_0 := 8c_{\mathrm{LR}}$. This yields the claimed decomposition $K(s) = \sum_Z K_Z(s)$ with

$$
C_{\mathrm{QA}} \;=\; c_0\, C_F\, J_1\, (2R_1+1)^2\, \frac{1}{\mu_{\mathrm{QA}}}, \qquad \mu_{\mathrm{QA}} \;=\; \tfrac{1}{2}\min\{\alpha'_F, \mu_{\mathrm{LR}}\},
$$

uniform in s. $\qquad\square$

3.2 Belt factorization and near-factorization with constants

Fix a bipartition $(A{:}A^c)$ and $r \geq 0$. Let

$$
K_r^\partial(s) \;:=\; \sum_{Z:\, \operatorname{dist}(Z,\partial A)\leq r} K_Z(s), \qquad K_r^{\mathrm{bulk}}(s) \;:=\; \sum_{Z:\, \operatorname{dist}(Z,\partial A)>r} K_Z(s),
$$

and define

$$
U_r^\partial(s) \;:=\; \mathcal{T}\exp\!\Big(i\int_0^s K_r^\partial(\sigma)\,d\sigma\Big), \qquad U_r^{\mathrm{bulk}}(s) \;:=\; \mathcal{T}\exp\!\Big(i\int_0^s K_r^{\mathrm{bulk}}(\sigma)\,d\sigma\Big).
$$

Then $U_r^\partial(s)$ is supported in $\partial_r A$ and $U_r^{\mathrm{bulk}}(s)$ in $(\partial_r A)^c$.

Proposition 3 (Belt factorization with exponential accuracy). *Let $\mu_{\mathrm{eff}} := \min\{\mu_{\mathrm{QA}},\mu_{\mathrm{LR}}\}$. There is a constant*

$$
\kappa_U \;:=\; c_1\, C_{\mathrm{QA}}\, C_{\mathrm{LR}}\, (1 + v_{\mathrm{LR}}/\mu_{\mathrm{eff}}),
$$

with $c_1 = O(1)$ (lattice dependent), such that for all $s \in [0,1]$ and all $r \geq 0$,

$$
\big\| U(s) \;-\; U_r^{\mathrm{bulk}}(s)\, U_r^\partial(s) \big\| \;\leq\; \kappa_U\, e^{-\mu_{\mathrm{eff}}\, r}. \tag{35}
$$

Consequently, for any $\varepsilon \in (0,1/2)$, taking

$$
r(\varepsilon) \;:=\; \left\lceil \frac{1}{\mu_{\mathrm{eff}}} \log\!\Big(\frac{\kappa_U}{\varepsilon}\Big) \right\rceil, \tag{36}
$$

we have $\|U(s) - U_{r(\varepsilon)}^{\mathrm{bulk}}(s)U_{r(\varepsilon)}^\partial(s)\| \leq \varepsilon$ for all $s \in [0,1]$.

Proof (interaction picture + Grönwall with belt truncation). Fix $r \geq 0$ and $s \in [0,1]$. Write the full evolution in the interaction picture w.r.t. the bulk:

$$
U(s) \;=\; U_r^{\mathrm{bulk}}(s)\, U_I(s), \qquad \partial_s U_I(s) \;=\; i\, K_I(s)\, U_I(s), \quad K_I(s) \;:=\; \operatorname{Ad}_{U_r^{\mathrm{bulk}}(s)^\dagger}\!\big(K_r^\partial(s)\big),
$$

where $\operatorname{Ad}_V(X) := VXV^\dagger$ and $U_I(0) = \mathbf{1}$. Then

$$
U(s) - U_r^{\mathrm{bulk}}(s)\, U_r^\partial(s) \;=\; U_r^{\mathrm{bulk}}(s)\,\big(U_I(s) - U_r^\partial(s)\big),
$$

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so it suffices to bound $\|U_I(s) - U_r^\partial(s)\|$.

Set $W(s) := U_I(s)\, U_r^\partial(s)^\dagger$; then $W(0) = \mathbf{1}$ and

$$\partial_s W(s) = i\big(K_I(s) - W(s) K_r^\partial(s) W(s)^\dagger\big)\, W(s).$$

Applying the same Grönwall step as in Lemma 2 (the "short evolutions" bound applied to $\|W(s) - \mathbf{1}\|$) gives

$$\|U_I(s) - U_r^\partial(s)\| = \|W(s) - \mathbf{1}\| \leq \int_0^s \big\|K_I(\sigma) - K_r^\partial(\sigma)\big\|\, d\sigma. \tag{37}$$

It remains to control the generator mismatch. By Lemma 4 (belt truncation under Heisenberg flow) and the QAC locality of $K_r^\partial(\sigma)$, the bulk evolution—whose generator is supported at distance $\geq r$ from ∂A—cannot significantly dress a belt operator before its LR light cone reaches the belt. Quantitatively, for all $\sigma \in [0,1]$,

$$\big\|\mathrm{Ad}_{U_r^{\mathrm{bulk}}(\sigma)^\dagger}\big(K_r^\partial(\sigma)\big) - K_r^\partial(\sigma)\big\| \leq c_1\, C_{\mathrm{QA}}\, C_{\mathrm{LR}}\Big(1 + \frac{v_{\mathrm{LR}}}{\mu_{\mathrm{eff}}}\Big) e^{-\mu_{\mathrm{eff}}\, r}, \tag{38}$$

with $\mu_{\mathrm{eff}} = \min\{\mu_{\mathrm{QA}}, \mu_{\mathrm{LR}}\}$ and a lattice constant $c_1 = O(1)$. Inequality (38) follows by differentiating the Heisenberg orbit $X_\sigma(t) := \mathrm{Ad}_{U_r^{\mathrm{bulk}}(t,\sigma)^\dagger}\big(K_r^\partial(\sigma)\big)$, using

$$\partial_t X_\sigma(t) = i\, \mathrm{Ad}_{U_r^{\mathrm{bulk}}(t,\sigma)^\dagger}\ \big[\, K_r^{\mathrm{bulk}}(t),\, X_\sigma(t)\,\big],$$

and bounding the commutator by the LR estimate with separation

$$\mathrm{dist}\big(\mathrm{supp}\, K_r^{\mathrm{bulk}}(t),\ \mathrm{supp}\, K_r^\partial(\sigma)\big) \geq r.$$

together with the QAC tail; integrating the envelope $e^{-\mu_{\mathrm{eff}}(r - v_{\mathrm{LR}}|t-\sigma|)}$ yields the factor $\big(1 + v_{\mathrm{LR}}/\mu_{\mathrm{eff}}\big)$.

Combining (37) and (38), and using $s \leq 1$, we obtain

$$\|U_I(s) - U_r^\partial(s)\| \leq \kappa_U\, e^{-\mu_{\mathrm{eff}}\, r}, \qquad \kappa_U := c_1\, C_{\mathrm{QA}}\, C_{\mathrm{LR}}\Big(1 + \frac{v_{\mathrm{LR}}}{\mu_{\mathrm{eff}}}\Big).$$

Since left multiplication by the unitary $U_r^{\mathrm{bulk}}(s)$ preserves operator norm,

$$\big\|U(s) - U_r^{\mathrm{bulk}}(s)\, U_r^\partial(s)\big\| \leq \kappa_U\, e^{-\mu_{\mathrm{eff}}\, r},$$

which is (35). The choice of $r(\varepsilon)$ in (36) is immediate. For (SE)/(PL) LR envelopes, replace $e^{-\mu_{\mathrm{eff}} r}$ by the corresponding ledger; the proof is unchanged with the substituted envelope under the time integral. $\qquad\square$

Remark 1 (LR-envelope summary for this section). All occurrences of $e^{-\mu r}$ in this section use the (E)-class LR envelope. For (SE)/(PL) envelopes, replace $e^{-\mu r}$ by the corresponding LR ledger $\mathrm{env}_{\mathrm{LR}}(r)$ (stretched exponential or polynomial), and define μ_{eff} so that $\mathrm{env}_{\mathrm{QA}} \star \mathrm{env}_{\mathrm{LR}} \preceq \mathrm{env}_{\mathrm{LR}}$ under the standard LR–QAC convolution. The constructions, circuit depths, and OSR bookkeeping remain unchanged; only the radius ledger (error vs. belt width) changes.

3.3 Direct OSR bound for belt-supported operators

We use a coarse but very general OSR bound that will be used repeatedly.

Lemma 5 (OSR of belt-supported operators)**.** *Let V be any operator supported on $\partial_r A$. Then*

$$\mathrm{OSR}_{\partial_r A}(V) \leq d^{\, c_{\mathrm{belt}}\, r\, |\partial A|}, \qquad c_{\mathrm{belt}} = 8,$$

where c_{belt} depends only on the lattice geometry. In particular, for any $r = O(1)$ the OSR base grows like $\exp(O(|\partial A|))$.

Lemma 6 (General OSR inflation under belt circuits)**.** *Let X be any operator with $\mathrm{OSR}_{\partial_{r_0} A}(X) \leq B^{|\partial A|}$. With $\mathcal{C}_\partial$ and $\mathcal{C}_{\mathrm{bulk}}$ as in §3.4 and §3.5, one has*

$$\mathrm{OSR}_{\partial_{r_0 + r(\varepsilon) + w} A}\Big(\mathcal{C}_{\mathrm{bulk}}\, \mathcal{C}_\partial\, X\, \mathcal{C}_\partial^\dagger\, \mathcal{C}_{\mathrm{bulk}}^\dagger\Big) \;\leq\; \Big(\Gamma\, B\Big)^{|\partial A|}, \qquad \Gamma \;:=\; \chi_{\mathrm{gate}}^{2 c_{\mathrm{cnt}} q_{\mathrm{belt}}} \;=\; d^{\, 2 c_{\mathrm{cnt}} q_{\mathrm{belt}} c_{\mathrm{gate}} w^2}.$$
(39)

Moreover, conjugation by $\mathcal{C}_{\mathrm{bulk}}$ does not increase OSR across $(A{:}A^c)$.

Proof. Conjugation by $\mathcal{C}_{\mathrm{bulk}}$ preserves the cut since all bulk gates are supported in $(\partial_{r(\varepsilon)} A)^c$. For $\mathcal{C}_\partial$, write it as a product of q_{belt} color layers, each a product of gates supported on $O(w^2)$-site blocks. In each layer, at most $c_{\mathrm{cnt}}|\partial A|$ blocks cross the cut (boundary packing). Every crossing gate has OSR $\leq \chi_{\mathrm{gate}} = d^{c_{\mathrm{gate}} w^2}$, hence left multiplication by the layer multiplies the base by $\chi_{\mathrm{gate}}^{c_{\mathrm{cnt}}}$; across q_{belt} layers and including right multiplication by $\mathcal{C}_\partial^\dagger$ the inflation is $\chi_{\mathrm{gate}}^{2 c_{\mathrm{cnt}} q_{\mathrm{belt}}}$. Allowances add by at most w, giving the displayed radius. $\square$

3.4 Constant-depth circuitization inside the belt

While Lemma 5 suffices for coarse OSR control, our quantitative bounds benefit from an explicit constant-depth belt circuit.

Block tiling and coloring. Tile $\partial_{r(\varepsilon)} A$ by overlapping *blocks* $\{B_j\}$ of diameter

$$w \;:=\; 8R + 8R_1 + 8,$$
(40)

such that each site of the belt belongs to at most $m_0 = O(1)$ blocks and blocks can be colored with $q_{\mathrm{belt}} \leq 9$ (a 3×3 checkerboard) so that same-color blocks are pairwise disjoint and at distance $\geq w/2$. Define the block generators

$$K_{B_j}(s) \;:=\; \sum_{Z \subset B_j} K_Z(s), \qquad K_{r(\varepsilon)}^\partial(s) \;=\; \sum_j K_{B_j}(s) \;+\; \Delta_{\mathrm{blk}}(s), \quad \|\Delta_{\mathrm{blk}}(s)\| \leq C_{\mathrm{blk}} e^{-\mu_{\mathrm{QA}} w/4}.$$

(The small tail Δ_{blk} accounts for K_Z straddling block boundaries; its norm is exponentially small in w.)

Lemma 7 (Exact commuting within colors)**.** *For each color class $\mathcal{B}_\alpha$, the generators $\{K_B(s) : B \in \mathcal{B}_\alpha\}$ have disjoint supports at all s and hence commute at equal s. Consequently, the time-ordered exponential over a color factorizes exactly:*

$$\mathcal{U}_\alpha \;:=\; \mathcal{T}\exp\Big(i \int_0^1 \sum_{B \in \mathcal{B}_\alpha} K_B(s)\, ds\Big) \;=\; \prod_{B \in \mathcal{B}_\alpha} \mathcal{T}\exp\Big(i \int_0^1 K_B(s)\, ds\Big).$$

Proposition 4 (Belt circuit with exponentially small error)**.** *Let*

$$\mathcal{C}_\partial \;:=\; \prod_{\alpha=1}^{q_{\mathrm{belt}}} \mathcal{U}_\alpha \qquad (\textit{colors ordered arbitrarily}).$$

There exist constants C_∂, $\mu_\partial > 0$ (depending only on the local data and LR/QAC constants) such that

$$\big\| U_{r(\varepsilon)}^\partial(1) \;-\; \mathcal{C}_\partial \big\| \;\leq\; C_\partial\, e^{-\mu_\partial w}.$$
(41)

Moreover, $\mathcal{C}_\partial$ has depth $D_\partial = q_{\mathrm{belt}} = O(1)$ with gates $\mathcal{T}\exp(i\int_0^1 K_B)$ supported on blocks of diameter $w = O(1)$.

Proof of Proposition 4. Fix the belt radius $r(\varepsilon)$ and tile $\partial_{r(\varepsilon)}A$ by overlapping blocks $\{B\}$ of diameter

$$w := 8R + 8R_1 + 8,$$

with a constant coloring $\{\mathcal{B}_\alpha\}_{\alpha=1}^{q_{\text{belt}}}$ such that blocks of the same color are pairwise disjoint and at distance $\geq w/2$ (a 3×3 checkerboard suffices, hence $q_{\text{belt}} = O(1)$). Let $K_{\partial,r(\varepsilon)}(s)$ denote the belt generator (supported in $\partial_{r(\varepsilon)}A$) of the quasi-adiabatic continuation, and set

$$K_B(s) := \sum_{Z \subset B} K_Z(s), \qquad \Delta_{\text{blk}}(s) := K_{\partial,r(\varepsilon)}(s) - \sum_B K_B(s).$$

By exponential quasi-locality of $K_Z(s)$ (Lemma 4) and finite overlap of the tiling, there exist constants $C_{\text{blk}}, \mu_{\text{blk}} > 0$ (depending only on local data and LR/QAC constants) such that

$$\|\Delta_{\text{blk}}(s)\| \leq C_{\text{blk}} \, e^{-\mu_{\text{blk}} w} \qquad \text{uniformly in } s \in [0,1].$$

Commutativity within colors. For each color α, the supports of $\{K_B(s) : B \in \mathcal{B}_\alpha\}$ are disjoint at every fixed s, hence these generators commute at equal time. Consequently,

$$\mathcal{U}_\alpha := \prod_{B \in \mathcal{B}_\alpha} \mathcal{T} \exp\Big(i \int_0^1 K_B(s) \, ds\Big)$$

is exactly the time-ordered exponential of the (time-dependent) sum $\sum_{B \in \mathcal{B}_\alpha} K_B(s)$; moreover each factor is supported on its block B (diameter w).

Dyson expansion and error control. Write

$$U_\partial^{r(\varepsilon)}(1) = \mathcal{T} \exp\Big(i \int_0^1 K_{\partial,r(\varepsilon)}(s) \, ds\Big) = \mathcal{T} \exp\Big(i \int_0^1 \sum_{\alpha=1}^{q_{\text{belt}}} \sum_{B \in \mathcal{B}_\alpha} K_B(s) \, ds\Big) \cdot \mathcal{E}_{\text{blk}}.$$

By Duhamel,

$$\|\mathcal{E}_{\text{blk}} - I\| \leq \int_0^1 \|\Delta_{\text{blk}}(s)\| \, e^{\int_s^1 \|K_{\partial,r(\varepsilon)}(u)\| \, du} \, ds \leq C_1' \, e^{-\mu_{\text{blk}} w}.$$

Next, compare the time-ordered exponential of the sum with the product across layers:

$$\Big\| \mathcal{T} e^{i \int_0^1 \sum_\alpha H_\alpha} - \prod_{\alpha=1}^{q_{\text{belt}}} \mathcal{T} e^{i \int_0^1 H_\alpha} \Big\| \leq \sum_{\alpha < \beta} \int_0^1 \int_0^1 \big\| [H_\alpha(s), H_\beta(s')] \big\| \, ds \, ds',$$

where $H_\alpha(s) := \sum_{B \in \mathcal{B}_\alpha} K_B(s)$. For $\alpha \neq \beta$, blocks are separated by $\geq w/2$. Exponential quasi-locality gives $\big\| [H_\alpha(s), H_\beta(s')] \big\| \leq C_2' e^{-\mu' w}$, uniformly in s, s'. Integrating and summing over $O(1)$ pairs (α, β) yields the claim with constants $C_\partial, \mu_\partial > 0$. $\square$

OSR of the belt circuit. We package the OSR base for the belt circuit as a lemma.

Lemma 8 (OSR base for belt circuits). *Each gate $\mathcal{T} \exp\big(i \int_0^1 K_B\big)$ acts on $O(w^2)$ sites. If a block B intersects both sides of the cut, its OSR across $(A{:}A^c)$ with allowance $r(\varepsilon) + w$ is bounded by*

$$\chi_{\text{gate}} := d^{c_{\text{gate}} w^2} \qquad (c_{\text{gate}} = O(1)).$$

In every color there are at most $c_{\text{cnt}} |\partial A|$ crossing blocks, with $c_{\text{cnt}} = O(1)$ (boundary packing). Since the depth is $D_\partial = q_{\text{belt}}$, we have

$$\text{OSR}_{\partial_{r(\varepsilon)+w} A}(\mathcal{C}_\partial) \leq \Big(\chi_{\text{gate}}^{c_{\text{cnt}} q_{\text{belt}}}\Big)^{|\partial A|}. \tag{42}$$

Proof. Each layer is a product over disjoint blocks, hence OSR multiplies over crossing gates within the layer by Lemma 5 and the OSR product rule; the number of crossing blocks per layer is $\leq c_{\text{cnt}} |\partial A|$. Across q_{belt} layers the base exponent adds. Allowance radii add by at most w (block diameter), giving $r(\varepsilon) + w$. $\square$

3.5 Constant-depth circuitization of the bulk

Tile $(\partial_{r(\varepsilon)}A)^c$ by $O(1)$-diameter blocks with a constant coloring so that same-color blocks are disjoint. Repeating the color product-formula argument yields:

Proposition 5 (Bulk circuit). *There exists a circuit $\mathcal{C}_{\text{bulk}}$ of depth $D_{\text{bulk}} = O(1)$, with $O(1)$-diameter gates supported in $(\partial_{r(\varepsilon)}A)^c$, such that*

$$\big\| U_{r(\varepsilon)}^{\text{bulk}}(1) - \mathcal{C}_{\text{bulk}} \big\| \leq C_{\text{bulk}}\, e^{-\mu_{\text{bulk}} w},$$

for constants $C_{\text{bulk}}, \mu_{\text{bulk}} > 0$. In particular, $\text{OSR}_{\partial_{r(\varepsilon)}A}(\mathcal{C}_{\text{bulk}}) = 1$.

Proof. Let $S := (\partial_{r(\varepsilon)}A)^c$ be the bulk. By Lemma 4, the QAC generator admits a decomposition $K(s) = \sum_Z K_Z(s)$ with $\|K_Z(s)\| \leq C_{\text{QA}} e^{-\mu_{\text{QA}} \text{diam}(Z)}$. Let $K_S(s) := \sum_{Z \subset S} K_Z(s)$ (drop all terms intersecting the belt), so $U_{r(\varepsilon)}^{\text{bulk}}(1) = \mathcal{T} \exp\big(i \int_0^1 K_S(s)\,ds\big)$. Fix a block diameter $w \geq 1$ and truncate range by setting $K_S^{(w)}(s) := \sum_{Z \subset S,\ \text{diam}(Z) \leq w} K_Z(s)$.

Step 1 (range truncation error). Tile S by $O(1)$–diameter blocks and color them with a constant number $q = O(1)$ of colors so that same–color blocks are disjoint. Group the tail $\sum_{\text{diam}(Z) > w} K_Z(s)$ by color: within a color, supports are disjoint and the operator norm of the sum equals the maximum block contribution. Using the exponential bound for $K_Z(s)$ and that each block Y is touched only by $O(1)$ sets Z with $\text{diam}(Z) \geq w$, we get

$$\|K_S(s) - K_S^{(w)}(s)\| \leq C_1\, e^{-\mu_{\text{QA}} w},$$

with $C_1 = O(C_{\text{QA}})$. By the Dyson (Duhamel) estimate,

$$\Big\| \mathcal{T} e^{i \int_0^1 K_S} - \mathcal{T} e^{i \int_0^1 K_S^{(w)}} \Big\| \leq e^{\int_0^1 (\|K_S\| + \|K_S^{(w)}\|)ds} \int_0^1 \|K_S - K_S^{(w)}\|\, ds \leq C_2\, e^{-\mu_{\text{QA}} w}.$$

Step 2 (color tiling $\Rightarrow$ constant depth). Decompose $K_S^{(w)}(s) = \sum_{c=1}^q H_c^{(w)}(s)$, where $H_c^{(w)}(s)$ is a sum of terms supported on the blocks of color c; within a fixed color the supports are disjoint, so the local exponentials commute. Discretize $s \in [0, 1]$ into n steps of size $\delta = 1/n$ and apply a first–order product formula per step:

$$\prod_{\ell=1}^n \prod_{c=1}^q \exp\big(i\delta\, H_c^{(w)}(s_\ell)\big), \qquad s_\ell = \ell\delta.$$

Each $\exp(i\delta\, H_c^{(w)}(s_\ell))$ factors into parallel on–block gates of diameter $O(w)$; since w is fixed (chosen $O(1)$), these are $O(1)$–diameter gates. Hence the depth per step is $q = O(1)$ and the total depth is $D_{\text{bulk}} = q\, n = O(1)$ if $n = n_0$ is a fixed constant.

By locality, the operator norm of $\sum_{c \neq c'} [H_c^{(w)}(s), H_{c'}^{(w)}(s')]$ is $O(1)$ (each block overlaps only $O(1)$ neighbors across colors). Therefore the standard product–formula error in operator norm scales as $O(n\delta^2) = O(1/n)$. Choose n_0 (independent of system size) so that this error is $\leq C_3 e^{-\mu_{\text{QA}} w}$. Let $\mathcal{C}_{\text{bulk}}$ be the resulting circuit. Combining with Step 1 and setting $\mu_{\text{bulk}} := \mu_{\text{QA}}/2$ (absorbing constants into C_{bulk}) gives

$$\big\| U_{r(\varepsilon)}^{\text{bulk}}(1) - \mathcal{C}_{\text{bulk}} \big\| \leq C_{\text{bulk}}\, e^{-\mu_{\text{bulk}} w}.$$

OSR across the belt. All gates of $\mathcal{C}_{\text{bulk}}$ act inside $S = (\partial_{r(\varepsilon)}A)^c$, so it has no support on $A \cup \partial_{r(\varepsilon)}A$; hence $\text{OSR}_{\partial_{r(\varepsilon)}A}(\mathcal{C}_{\text{bulk}}) = 1$. $\qquad \square$

3.6 Consolidated belt factorization and circuit form

We collect the outputs of Proposition 3, Proposition 4, Lemma 8, and Proposition 5.

Theorem 1 (QAC = bulk × constant-depth belt circuit; OSR base)**.** *Fix $\varepsilon \in (0, 1/2)$ and set $r(\varepsilon) = \left\lceil \mu_{\text{eff}}^{-1} \log(\kappa_U/\varepsilon) \right\rceil$. There exist constant-depth circuits*

$$\mathcal{C}_{\text{bulk}} \ (depth \ D_{\text{bulk}} = O(1), \ \text{supp} \subset (\partial_{r(\varepsilon)}A)^c),$$
$$and \quad \mathcal{C}_{\partial} \ (depth \ D_{\partial} = q_{\text{belt}} = O(1), \ \text{supp} \subset \partial_{r(\varepsilon)}A \ with \ block \ diameter \ w).$$

such that

$$\left\| U(1) \ - \ \mathcal{C}_{\text{bulk}} \, \mathcal{C}_{\partial} \right\| \ \leq \ 2\varepsilon, \tag{43}$$

the belt width is $r(\varepsilon) + w$, and the OSR across $(A{:}A^c)$ obeys

$$\text{OSR}_{\partial_{r(\varepsilon)+w}A}(\mathcal{C}_{\partial}) \ \leq \ \left(\chi_{\text{gate}}^{c_{\text{cnt}}q_{\text{belt}}} \right)^{|\partial A|}, \qquad \text{OSR}_{\partial_{r(\varepsilon)}A}(\mathcal{C}_{\text{bulk}}) = 1. \tag{44}$$

Consequently, for any operator X with $\text{OSR}_{\partial_{r_0}A}(X) \leq B^{|\partial A|}$,

$$\text{OSR}_{\partial_{r_0+r(\varepsilon)+w}A}\left(\mathcal{C}_{\text{bulk}}\mathcal{C}_{\partial} X \mathcal{C}_{\partial}^{\dagger}\mathcal{C}_{\text{bulk}}^{\dagger} \right) \ \leq \ \left(\Gamma B \right)^{|\partial A|}, \qquad \Gamma \ := \ \chi_{\text{gate}}^{2c_{\text{cnt}}q_{\text{belt}}} \ = \ d^{2c_{\text{cnt}}q_{\text{belt}}c_{\text{gate}}w^2}. \tag{45}$$

3.7 Constants summary

Constants produced here.

filter:	$\gamma = \Delta_*/2$, $\quad F_\gamma$ as in (30), $\quad (\alpha_F', C_F)$ from (31),
QAC locality:	C_{QA}, $\mu_{\text{QA}} = \frac{1}{2}\min\{\alpha_F', \mu_{\text{LR}}\}$ (Lemma 4),
belt factorization:	$\mu_{\text{eff}} = \min\{\mu_{\text{QA}}, \mu_{\text{LR}}\}$, $\kappa_U = c_1 C_{\text{QA}} C_{\text{LR}}(1 + v_{\text{LR}}/\mu_{\text{eff}})$,
radius:	$r(\varepsilon) = \left\lceil \mu_{\text{eff}}^{-1} \log(\kappa_U/\varepsilon) \right\rceil$,
belt circuit:	$w = 8R + 8R_1 + 8$, $\quad q_{\text{belt}} \leq 9$, $\quad \chi_{\text{gate}} = d^{c_{\text{gate}}w^2}$, $\quad c_{\text{cnt}} = O(1)$,
OSR inflation base:	$\Gamma = \chi_{\text{gate}}^{2c_{\text{cnt}}q_{\text{belt}}} = d^{2c_{\text{cnt}}q_{\text{belt}}c_{\text{gate}}w^2}$ (Lemma 10).

4 AGSP transport and OSR inflation control

Objective   Let K_0 be the frustration-free AGSP from Section 2 with

$$K_0 P_0 = P_0, \qquad \|K_0 P_0^{\perp}\| = \delta_0 < \tfrac{1}{2}, \qquad \text{OSR}_{\partial_{r_0}A}(K_0) \leq \Lambda_0^{|\partial A|}, \ \ r_0 = R.$$

Let $U(s)$ be the QAC unitary along the gapped path (Assumption 1) constructed in Section 3. Define

$$K_s := U(s)K_0 U(s)^{\dagger}, \qquad P_s := U(s)P_0 U(s)^{\dagger}.$$

We establish:

(i) *Projection faithfulness and contraction invariance* for the exact conjugate:

$$K_s P_s = P_s, \qquad \|K_s P_s^{\perp}\| = \|K_0 P_0^{\perp}\| =: \delta_0 < \tfrac{1}{2}. \tag{46}$$

(ii) *Area-controlled OSR* for a circuitized transported AGSP at the endpoint $s = 1$. Writing $U := U(1)$, we build a constant-depth belt/bulk circuit approximation $U \approx \mathcal{C}_{\text{bulk}}\mathcal{C}_\partial$ (Theorem 1, Remark 1), and set

$$K_1^e := \mathcal{C}_{\text{bulk}}\mathcal{C}_\partial K_0 \mathcal{C}_\partial^\dagger \mathcal{C}_{\text{bulk}}^\dagger.$$

Then for a constant belt width $r_\star$ (defined below) one has

$$\text{OSR}_{\partial_{r_\star} A}(K_1^e) \ \leq\ \big(\Lambda_0 \Gamma\big)^{|\partial A|}, \qquad \text{with } \Gamma = O(1). \tag{47}$$

Moreover, $\|K_1 - K_1^e\|$ is exponentially small in the circuit block diameter.

4.1 Projection faithfulness and contraction invariance

Lemma 9. *For $K_s = U(s)K_0 U(s)^\dagger$ and $P_s = U(s)P_0 U(s)^\dagger$ one has* (46).

Proof. $K_0 P_0 = P_0$ by Section 2, and $U(s)$ is unitary. Thus $K_s P_s = U(s)K_0 P_0 U(s)^\dagger = P_s$. For the norm, $\|K_s P_s^\perp\| = \|U(s)K_0 U(s)^\dagger (I - P_s)\| = \|K_0 P_0^\perp\|$. $\qquad\square$

4.2 Circuit-level transport: belt/bulk circuitization

Fix an accuracy $\varepsilon_\star \in (0, 10^{-2}]$ and let

$$r(\varepsilon_\star) = \left\lceil \mu_{\text{eff}}^{-1} \log(\kappa_U/\varepsilon_\star) \right\rceil$$

as in (36). By Theorem 1 (alias of Proposition 1) and the LR radius ledger (Remark 1), there exist constant–depth circuits $\mathcal{C}_{\text{bulk}}$ and $\mathcal{C}_\partial$ such that

$$\left\| U - U_{r(\varepsilon_\star)}^{\text{bulk}} U_{r(\varepsilon_\star)}^\partial \right\| \leq \varepsilon_\star, \qquad \left\| U_{r(\varepsilon_\star)}^{\text{bulk}} - \mathcal{C}_{\text{bulk}} \right\| \leq C_{\text{bulk}} e^{-\mu_{\text{bulk}} w}, \qquad \left\| U_{r(\varepsilon_\star)}^\partial - \mathcal{C}_\partial \right\| \leq C_\partial e^{-\mu_\partial w},$$

with $\text{depth}(\mathcal{C}_{\text{bulk}}) = O(1)$, $\text{depth}(\mathcal{C}_\partial) = q_{\text{belt}} \leq 9$, and $\mathcal{C}_\partial$ supported in $\partial_{r(\varepsilon_\star)} A$ on disjoint blocks of diameter

$$w = 8R + 8R_1 + 8 \quad \text{(see Remark 1)}.$$

4.3 OSR inflation under belt circuits

Lemma 10 (OSR inflation under belt circuits)**.** *Let K_0 obey $\text{OSR}_{\partial_{r_0} A}(K_0) \leq \Lambda_0^{|\partial A|}$ with $r_0 = R$. Let U be the endpoint QAC unitary of Section 3. For fixed $\varepsilon_\star \in (0, 10^{-2}]$, let $r(\varepsilon_\star)$ be as above. There are constant–depth circuits $\mathcal{C}_{\text{bulk}}$ (supported in $(\partial_{r(\varepsilon_\star)} A)^c$) and $\mathcal{C}_\partial$ (supported in $\partial_{r(\varepsilon_\star)} A$, with block diameter w and $q_{\text{belt}} \leq 9$ layers) such that*

$$\left\| U - \mathcal{C}_{\text{bulk}}\mathcal{C}_\partial \right\| \ \leq\ \varepsilon_\star \ +\ C_{\text{bulk}} e^{-\mu_{\text{bulk}} w} \ +\ C_\partial e^{-\mu_\partial w}.$$

Define the circuit–transported AGSP

$$K_1^e \ :=\ \mathcal{C}_{\text{bulk}} \, \mathcal{C}_\partial \, K_0 \, \mathcal{C}_\partial^\dagger \, \mathcal{C}_{\text{bulk}}^\dagger, \tag{48}$$

and set the final allowance

$$r_\star \ :=\ r_0 \ +\ r(\varepsilon_\star) \ +\ w. \tag{49}$$

Then

$$\text{OSR}_{\partial_{r_\star} A}\big(K_1^e\big) \ \leq\ \big(\Lambda_0 \, \Gamma\big)^{|\partial A|}. \tag{50}$$

Here Γ is the belt OSR inflation base

$$\Gamma \ :=\ \chi_{\text{gate}}^{2 \, c_{\text{cnt}} \, q_{\text{belt}}} \ =\ d^{\, 2 \, c_{\text{cnt}} \, q_{\text{belt}} \, c_{\text{gate}} \, w^2}, \tag{51}$$

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where d is the on-site dimension, $\chi_{\text{gate}} = d^{c_{\text{gate}}w^2}$ bounds the OSR of any belt gate crossing the cut, $c_{\text{gate}} = O(1)$ is a geometric constant, and $c_{\text{cnt}} = O(1)$ is the crossing-block packing constant per color layer. Moreover, for the exact conjugate $K_1 := U K_0 U^\dagger$ one has

$$\|K_1 - K_1^e\| \leq 2\varepsilon_\star + 2C_{\text{bulk}}e^{-\mu_{\text{bulk}}w} + 2C_\partial e^{-\mu_\partial w} =: \varepsilon_{\text{circ}}. \tag{52}$$

Proof. Use the belt/bulk factorization and circuitization of Theorem 1 to write $U = \big(U_{r(\varepsilon_\star)}^{\text{bulk}} U_{r(\varepsilon_\star)}^\partial\big) + E$ with $\|E\| \leq \varepsilon_\star$, and approximate $U_{r(\varepsilon_\star)}^{\text{bulk}}$ and $U_{r(\varepsilon_\star)}^\partial$ by $\mathcal{C}_{\text{bulk}}$ and $\mathcal{C}_\partial$ with the stated errors, yielding (52) by triangle inequality.

For OSR, apply the OSR calculus: (1) $\text{OSR}_{\partial_{r(\varepsilon_\star)}A}(\mathcal{C}_{\text{bulk}}) = 1$ since it acts in the bulk allowance. (2) In the belt $\partial_{r(\varepsilon_\star)}A$, each color layer of $\mathcal{C}_\partial$ contributes at most $c_{\text{cnt}}|\partial A|$ disjoint crossing blocks, each gate having $\text{OSR} \leq \chi_{\text{gate}}$ across $(A{:}A^c)$ once the allowance is increased by w. Hence

$$\text{OSR}_{\partial_{r(\varepsilon_\star)+w}A}(\mathcal{C}_\partial) \leq \big(\chi_{\text{gate}}^{c_{\text{cnt}}q_{\text{belt}}}\big)^{|\partial A|}, \qquad \text{OSR}_{\partial_{r(\varepsilon_\star)+w}A}(\mathcal{C}_\partial^\dagger) \leq \big(\chi_{\text{gate}}^{c_{\text{cnt}}q_{\text{belt}}}\big)^{|\partial A|}.$$

(3) Product submultiplicativity and allowance additivity for $\mathcal{C}_{\text{bulk}}\mathcal{C}_\partial K_0 \mathcal{C}_\partial^\dagger \mathcal{C}_{\text{bulk}}^\dagger$ give (49) and

$$\text{OSR}_{\partial_{r_\star}A}(K_1^e) \leq \text{OSR}_{\partial_{r_0}A}(K_0)\,\text{OSR}_{\partial_{r(\varepsilon_\star)+w}A}(\mathcal{C}_\partial)\,\text{OSR}_{\partial_{r(\varepsilon_\star)+w}A}(\mathcal{C}_\partial^\dagger) \leq \big(\Lambda_0\,\Gamma\big)^{|\partial A|}.$$

$\square$

Constants (collected).

$$r_\star = r_0 + r(\varepsilon_\star) + w, \quad r_0 = R, \quad w = 8R + 8R_1 + 8, \quad q_{\text{belt}} \leq 9,$$
$$\chi_{\text{gate}} = d^{c_{\text{gate}}w^2}, \quad c_{\text{gate}} = O(1), \quad c_{\text{cnt}} = O(1), \qquad \Gamma = \chi_{\text{gate}}^{2c_{\text{cnt}}q_{\text{belt}}} = d^{2c_{\text{cnt}}q_{\text{belt}}c_{\text{gate}}w^2},$$
$$\varepsilon_{\text{circ}} = 2\varepsilon_\star + 2C_{\text{bulk}}e^{-\mu_{\text{bulk}}w} + 2C_\partial e^{-\mu_\partial w}.$$

5 Polynomial (trigonometric) energy filter via short evolutions

Objective Design a degree–m trigonometric filter p_m that is *flat* on the ground sector and *suppressed* on the excited spectrum, realize $p_m(H(s) - E_0(s)\mathbb{1})$ as a linear combination of short evolutions $e^{itH(s)}$ (LCU), belt–factorize each short evolution using Lieb–Robinson (LR) bounds, and compose with the transported AGSP K_s from Section 4 to obtain

$$\tilde{K}_s := p_m\big(H(s) - E_0(s)\mathbb{1}\big)K_s,$$
$$\tilde{K}_s P_s = P_s, \qquad \|\tilde{K}_s P_s^\perp\| \leq \delta, \tag{53}$$
$$\text{OSR}_{\partial_{r''}A}(\tilde{K}_s) \leq \big(\Lambda_0\,\Gamma\,\Upsilon(m)\big)^{|\partial A|_{\text{edge}}}.$$

with $\delta \ll 1$ tunable by m and belt width $r'' = r_\star + r'(m)$.

Proposition 6 (Filtered AGSP with tunable contraction and controlled belt width). *Assume the spectral gap $\Delta_* > 0$ and a Lieb–Robinson (LR) envelope in 2D as in Proposition 7. Let P_s be the ground projector of $H(s)$ and let K_s be the transported AGSP from Section 4, satisfying*

$$K_s P_s = P_s, \qquad \|K_s P_s^\perp\| \leq \delta_0 < \tfrac{1}{2}, \qquad \text{OSR}_{\partial_{r_\star}A}(K_s) \leq \big(\Lambda_0\,\Gamma\big)^{|\partial A|_{\text{edge}}},$$

with $r_\star$ as in (49) and the belt OSR base Γ as in (51).

Fix $\tau = \pi/(3\Delta_*)$ and define p_m by (59) using the Dolph–Chebyshev design of Section 5.1. Then for any target $\delta \in (0, \tfrac{1}{2})$ there exists a degree $m = O\big(\log(\delta_0/\delta)\big)$, an LCU tolerance ε_{pf}, and a belt allowance $r'(m)$ given by (61) (with regimes (E/SE/PL) from Proposition 7 and Remark 1) such that

$$\tilde{K}_s := p_m\big(H(s) - E_0(s)\mathbb{1}\big)K_s$$

obeys

$$\tilde{K}_s P_s = P_s, \qquad \|\tilde{K}_s P_s^\perp\| \leq \delta, \tag{54}$$

$$\mathrm{OSR}_{\partial_{r''} A}(\tilde{K}_s) \leq \left(\Lambda_0 \, \Gamma \, \Upsilon(m)\right)^{|\partial A|_{\mathrm{edge}}}, \qquad r'' = r_\star + r'(m), \tag{55}$$

where $\Upsilon(m)$ is defined in (66). The three-term error budget can be chosen as in (69).

Proof. The stopband leakage (60) with $\eta = \eta(\pi/6)$ and the projection/contraction identities (46) give (67); choosing the budget as in (69) yields (54). For the belt allowance, apply the LR factorization uniformly for $|t| \leq T = m\tau$ to obtain (61) and the LCU error (63). The belt OSR bound follows from (64) and (65); submultiplicativity (Lemma 1) combined with the hypothesis on K_s gives (55) with $r'' = r_\star + r'(m)$. $\qquad\square$

5.1 Trigonometric/Chebyshev design on the unit circle

Let $\widehat{H}(s) := H(s) - E_0(s)\mathbb{1}$ so that $\mathrm{Spec}(\widehat{H}(s)) \subset \{0\} \cup [\Delta_*, \infty)$. Fix a time step $\tau \in (0, \pi/\Delta_*)$ and define the normalized stopband edge

$$\omega_0 := \frac{\tau \Delta_*}{2} \in (0, \pi/2).$$

Consider even, real *cosine* polynomials

$$P_m(\omega) := a_0 + 2\sum_{k=1}^m a_k \cos(k\omega), \qquad a_k \in \mathbb{R}, \quad a_k = a_{-k}. \tag{56}$$

Classical Dolph–Chebyshev/equiripple design guarantees (for any $m \geq 1$) the existence of coefficients $\{a_k\}$ with

$$P_m(0) = 1, \qquad \sup_{|\omega| \geq \omega_0} |P_m(\omega)| \leq \eta(\omega_0)^m, \tag{57}$$

for some $\eta(\omega_0) \in (0, 1)$ depending only on ω_0 (hence on $\tau\Delta_*$). Moreover the ℓ^1–weight obeys

$$\|a\|_1 := \sum_{k=-m}^m |a_k| \leq C_{\mathrm{LCU}}(\omega_0) \, \rho(\omega_0)^m, \tag{58}$$

with $C_{\mathrm{LCU}}(\omega_0) = O(1)$ and $\rho(\omega_0) > 1$ depending only on ω_0. Define the operator-valued filter by *frequency sampling*:

$$p_m(\widehat{H}) := \sum_{k=-m}^m a_k \, e^{ik\tau\widehat{H}}. \tag{59}$$

Then $p_m(0) = 1$ and, by functional calculus,

$$\|p_m(\widehat{H}) P_s^\perp\| \leq \eta^m, \qquad \eta := \eta(\omega_0) \in (0, 1). \tag{60}$$

From now on we fix the canonical sampling step $\tau = \pi/(3\Delta_*)$ (so $\omega_0 = \pi/6$), and write $\eta = \eta(\pi/6) \in (0, 1)$ for the stopband factor.

5.2 Time support and LR belt factorization for e^{itH}

All times satisfy $|t_k| = |k|\tau \leq T := m\tau = \Theta(m/\Delta_*)$. Let $\mathsf{R}(t)$ denote the LR *envelope* (light-cone radius at time t), as in Proposition 7. For a target per–LCU error $\varepsilon_{\mathrm{pf}} \in (0, 1)$, define the belt allowance $r'(m)$ by

$$r'(m) := \left\lceil \mathsf{R}(T) + \Delta r_{\mathrm{tail}}(T, \varepsilon_{\mathrm{pf}}) \right\rceil, \tag{61}$$

where the tail term Δr_{tail} is chosen according to the LR regime (E/SE/PL) in Proposition 7:

$$\Delta r_{\text{tail}}(T, \varepsilon_{\text{pf}}) = \begin{cases} \mu^{-1} \log\big(\kappa_{\text{pf}}^{(\text{E})} \|a\|_1 / \varepsilon_{\text{pf}}\big), & \text{(E) exponential LR,} \\ \mu_\zeta^{-1} \log\big(\kappa_{\text{pf}}^{(\text{SE})} \|a\|_1 / \varepsilon_{\text{pf}}\big), & \text{(SE) stretched–exp. LR,} \\ \big(\kappa_{\text{pf}}^{(\text{PL})} \|a\|_1 (1+T)^\theta / \varepsilon_{\text{pf}}\big)^{1/(\sigma-2)}, & \text{(PL) polynomial LR in 2D.} \end{cases}$$

Then the LR belt factorization yields, *uniformly in* $|t| \leq T$,

$$\left\| e^{itH(s)} - V_t^{\text{bulk}}(s) V_t^\partial(s) \right\| \leq \frac{\varepsilon_{\text{pf}}}{\|a\|_1}, \qquad \text{supp} V_t^{\text{bulk}} \subset (\partial_{r'} A)^c, \quad \text{supp} V_t^\partial \subset \partial_{r'} A. \tag{62}$$

Hence the total LCU truncation error is bounded by

$$\left\| p_m(\widehat{H}) - \sum_{k=-m}^m a_k V_{t_k}^{\text{bulk}} V_{t_k}^\partial \right\| \leq \varepsilon_{\text{pf}}. \tag{63}$$

Remark 2 (LR ledger consistency). The choice (61) uses the LR envelope $\mathsf{R}(t)$ and tail calibration in the (E/SE/PL) regimes of Proposition 7; see also Remark 1. All constants can be taken uniform in s under Assumption 1.

5.3 OSR bound for $p_m(\widehat{H})$ (belt form)

Each belt unitary V_t^∂ can be realized by a circuit of block diameter $O(1)$ and depth $O(|t|)$. Hence there exist constants $\chi_{\text{evol}} > 1$ and $c_{\text{evol}} = O(1)$ (depending only on the local data in Assumption 1 and the LR envelope) such that

$$\text{OSR}_{\partial_{r'} A}\big(V_t^\partial\big) \leq \left(\chi_{\text{evol}}^{c_{\text{evol}}|t|}\right)^{|\partial A|_{\text{edge}}} \leq \left(\chi_{\text{evol}}^{c_{\text{evol}} T}\right)^{|\partial A|_{\text{edge}}}. \tag{64}$$

Using (59), (63), and the *subadditivity* of OSR under sums,

$$\text{OSR}_{\partial_{r'} A}\big(p_m(\widehat{H})\big) \leq \sum_{k=-m}^m \text{OSR}_{\partial_{r'} A}\big(V_{t_k}^\partial\big) \leq (2m+1) \left(\chi_{\text{evol}}^{c_{\text{evol}} T}\right)^{|\partial A|_{\text{edge}}}. \tag{65}$$

It is convenient to package this as

$$\text{OSR}_{\partial_{r'} A}\big(p_m(\widehat{H})\big) \leq \big(\Upsilon(m)\big)^{|\partial A|_{\text{edge}}}, \qquad \Upsilon(m) := \max\left\{ \chi_{\text{evol}}^{c_{\text{evol}} T}, 2m+1 \right\}. \tag{66}$$

(One may equivalently use the outer Minkowski content $|\partial A|_{\text{Mink}}$; in all cases OSR bases appear as base$^{|\partial A|_\bullet}$.)

5.4 Composition with K_s: boosted contraction and controlled OSR

By (60) and the projection/contraction identities (46),

$$\tilde{K}_s P_s = P_s, \qquad \|\tilde{K}_s P_s^\perp\| \leq \|p_m(\widehat{H}) P_s^\perp\| \|K_s P_s^\perp\| \leq \eta^m \delta_0. \tag{67}$$

For the OSR across $(A{:}A^c)$, combine (66) with (47) and submultiplicativity (Lemma 1) to get, with $r'' = r_\star + r'(m)$,

$$\text{OSR}_{\partial_{r''} A}(\tilde{K}_s) \leq \text{OSR}_{\partial_{r'} A}\big(p_m(\widehat{H})\big) \text{OSR}_{\partial_{r_\star} A}(K_s) \leq \big(\Lambda_0 \Gamma \Upsilon(m)\big)^{|\partial A|_{\text{edge}}}. \tag{68}$$

5.5 Parameter choice (locked constants) and error budget

Locked time step. We henceforth *fix* the sampling step

$$\tau = \frac{\pi}{3\Delta_*} \quad \implies \quad \omega_0 = \frac{\tau\Delta_*}{2} = \frac{\pi}{6},$$

so the Dolph–Chebyshev design yields a stopband factor $\eta = \eta(\omega_0) \in (0,1)$ and LCU weight $\|a\|_1 \leq C_{\mathrm{LCU}}\rho^m$ (Section 5.1).

Three-term error budget. There are three small contributions to control:

$$\underbrace{\varepsilon_{\mathrm{circ}}}_{\text{transport via QAC circuits}}, \qquad \underbrace{\varepsilon_{\mathrm{pf}}}_{\text{LCU belt truncation in (63)}}, \qquad \underbrace{\eta^m}_{\text{filter stopband leakage in (60)}} .$$

Given a target $\delta \in (0, \frac{1}{2})$ and the transported prefilter contraction $\delta_0 = \|K_s P_s^\perp\| < \frac{1}{2}$ (Section 4), choose

$$\varepsilon_{\mathrm{circ}} \leq \tfrac{1}{8}\delta, \qquad \varepsilon_{\mathrm{pf}} \leq \tfrac{1}{8}\delta, \qquad \eta^m \delta_0 \leq \tfrac{3}{4}\delta. \tag{69}$$

Then, by (67) and absorbing the two norm errors, the filtered operator obeys

$$\|\tilde{K}_s P_s^\perp\| \leq \eta^m \delta_0 + \varepsilon_{\mathrm{circ}} + \varepsilon_{\mathrm{pf}} \leq \delta.$$

Degree, radius, and OSR base. With τ fixed, set $T = m\tau$ and

$$m \geq \left\lceil \frac{\ln\left(\frac{3\delta}{4\delta_0}\right)}{\ln\eta} \right\rceil. \tag{70}$$

For the LCU belt factorization radius under the (exponential) LR bound (regime (E)),

$$r'(m) := \left\lceil vT + \mu^{-1}\log\left(\frac{\kappa_{\mathrm{pf}}\,\|a\|_1}{\varepsilon_{\mathrm{pf}}}\right) \right\rceil = \left\lceil \frac{v\pi}{3\Delta_*}m + \frac{1}{\mu}\log\left(\frac{\kappa_{\mathrm{pf}}\,C_{\mathrm{LCU}}\,\rho^m}{\varepsilon_{\mathrm{pf}}}\right) \right\rceil, \tag{71}$$

which is the (E) line of Proposition 7 and coincides with (61) specialized to $R(T) = vT$. (For stretched–exponential or polynomial LR, use the corresponding lines of Proposition 7.) The OSR base of the filter remains

$$\mathrm{OSR}_{\partial_{r'}A}\big(p_m(\widehat{H})\big) \leq \big(\Upsilon(m)\big)^{|\partial A|_{\mathrm{edge}}}, \qquad \Upsilon(m) := \max\Big\{\chi_{\mathrm{evol}}^{c_{\mathrm{evol}}T}, \, 2m+1\Big\}, \tag{72}$$

as in (66), so that the composed OSR bound (68) reads

$$\mathrm{OSR}_{\partial_{r''}A}(\tilde{K}_s) \leq \big(\Lambda_0\,\Gamma\,\Upsilon(m)\big)^{|\partial A|_{\mathrm{edge}}}, \qquad r'' = r_\star + r'(m).$$

Parameter mapping (exponential LR). With τ locked and the budget split (69), the following table summarizes the choices and dependencies:

Input (LR/gap data)	Locked choice / outcome
Δ_*	$\tau = \dfrac{\pi}{3\Delta_*}$
v, μ	enter $r'(m)$ via (71)
(target) $\delta \in (0, \frac{1}{2})$, given $\delta_0 < \frac{1}{2}$	degree $m = \left\lceil \dfrac{\ln(3\delta/4\delta_0)}{\ln\eta} \right\rceil$ from (70)
LCU tolerance $\varepsilon_{\mathrm{pf}} = \delta/8$	$r'(m) = \left\lceil vm\tau + \mu^{-1}\log\big(\kappa_{\mathrm{pf}}C_{\mathrm{LCU}}\rho^m/\varepsilon_{\mathrm{pf}}\big) \right\rceil$
transport tolerance $\varepsilon_{\mathrm{circ}} = \delta/8$	absorbed in the norm bound; no OSR cost (bulk)
OSR base	$\Upsilon(m) = \max\{\chi_{\mathrm{evol}}^{c_{\mathrm{evol}}m\tau}, \, 2m+1\}$
Final belt allowance	$r'' = r_\star + r'(m)$
Final contraction	$\|\tilde{K}_s P_s^\perp\| \leq \delta$

5.6 Remarks

- *Filter length and evolution time.* The choice $m = O(\log(1/\delta))$ with the τ-lock implies $T = \Theta(m/\Delta_*)$.

- *LR envelope swap.* By Proposition 7 one may change the LR envelope without affecting $\Upsilon(m)$.

- *Size independence.* All bases $(\Lambda_0, \Gamma, \Upsilon(m))$ and all radii are independent of $|A|$.

- *Even filter & real coefficients.* Taking $a_k = a_{-k} \in \mathbb{R}$ makes $p_m(\widehat{H})$ Hermitian and time–reversal symmetric; the LCU then uses only $\cos(k\tau\widehat{H})$.

- *Alternative windows.* Gaussian/Jackson kernels also give $\sup_{[\Delta_*/2,\infty)} |p_m| \leq e^{-\Omega(m)}$ with the same $T = \Theta(m/\Delta_*)$; constants in $\Upsilon(m)$ change only by $O(1)$ factors.

- *Boundary dependence.* Wherever OSR bases appear, replace $|\partial A|$ by $|\partial A|_{\text{edge}}$ (or by $|\partial A|_{\text{Mink}}$); the form base$^{|\partial A|\bullet}$ is preserved.

5.7 Outcome

With τ locked, m chosen as in (70), and $r'(m)$ from (61), Proposition 6 yields (53) with belt width

$$r'' = r_\star + r'(m), \qquad r_\star \text{ as in (49)}.$$

The OSR base factors as $\Lambda_0 \, \Gamma \, \Upsilon(m)$, with the belt OSR component Γ given in (51). This completes the filtered-AGSP ingredient for the AGSP$\Rightarrow$entropy converter.

6 Filter AGSP beyond spectral gaps: mobility gaps, LR tails, and rough boundaries

Overview. We extend the filter analysis to settings beyond uniform exponential Lieb–Robinson bounds:

6.1 General LR envelope: one proposition for all regimes

We use a single Lieb–Robinson *envelope* and tail:

$$\big\| [\tau_t(A_X), B_Y] \big\| \leq C_{\text{LR}} \|A_X\| \|B_Y\| \, \Xi\big(\text{dist}(X, Y) - \mathsf{R}(|t|)\big), \tag{73}$$

with nondecreasing radius R and nonincreasing tail Ξ, covering three cases:

$$
\begin{array}{lll}
\text{(E)} & \mathsf{R}(t) = v\,t, & \Xi(u) = e^{-\mu u}; \\
\text{(SE)} & \mathsf{R}(t) = v_\zeta\, t^\zeta \ (0 < \zeta < 1), & \Xi(u) = e^{-\mu_\zeta u}; \\
\text{(PL)} & \mathsf{R}(t) = 0, & \Xi(u) \asymp \dfrac{(1 + |t|)^\theta}{(1 + u)^\sigma}, \ \sigma > 2 \ \text{in 2D}.
\end{array}
$$

Proposition 7 (General LR envelope $\Rightarrow$ per–LCU belt radius; OSR base unchanged)**.** *Let $p_m(\widehat{H}) = \sum_{k=-m}^{m} a_k e^{ik\tau\widehat{H}}$ be the degree–m trigonometric filter with time support $T = m\tau$ and ℓ_1 weight $\|a\|_1$. Fix a per–LCU truncation target $\varepsilon_{\text{pf}} \in (0, 1)$. Under (73), for every $|t| \leq T$ there exist unitaries V_t^{bulk} and V_t^∂ supported on $(\partial_{r'} A)^c$ and $\partial_{r'} A$, respectively, such that*

$$\big\| e^{itH} - V_t^{\text{bulk}} V_t^\partial \big\| \leq \frac{\varepsilon_{\text{pf}}}{\|a\|_1}.$$

It suffices to choose the belt radius $r'(m)$ from the single ledger

$$\boxed{r'(m) \;\geq\; \mathsf{R}(T) \;+\; \Delta r_{\mathrm{tail}}(T, \varepsilon_{\mathrm{pf}})} \tag{74}$$

with tail term (constants $\kappa_{\mathrm{pf}}^{(\bullet)} = O(1)$ depend only on local/LR data)

$$\Delta r_{\mathrm{tail}}(T, \varepsilon_{\mathrm{pf}}) = \begin{cases} \mu^{-1} \log\!\big(\kappa_{\mathrm{pf}}^{(\mathrm{E})} \|a\|_1 / \varepsilon_{\mathrm{pf}}\big), & (\mathrm{E}), \\ \mu_\zeta^{-1} \log\!\big(\kappa_{\mathrm{pf}}^{(\mathrm{SE})} \|a\|_1 / \varepsilon_{\mathrm{pf}}\big), & (\mathrm{SE}), \\ \big(\kappa_{\mathrm{pf}}^{(\mathrm{PL})} \|a\|_1 (1+T)^\theta / \varepsilon_{\mathrm{pf}}\big)^{1/(\sigma-2)}, & (\mathrm{PL}) \text{ in 2D}. \end{cases}$$

Consequently, for the filter $p_m(\widehat{H})$ the packaged OSR base

$$\Upsilon(m) \;:=\; \max\Big\{ \chi_{\mathrm{evol}}^{c_{\mathrm{evol}} T}, \; 2m+1 \Big\}$$

is unchanged *in all three regimes; only the belt allowance grows to $r'' = r_\star + r'(m)$ when composing with transport, and*

$$\mathrm{OSR}_{\partial_{r''} A}\big(\tilde{K}_s\big) \;\leq\; \big(\Lambda_0 \, \Gamma \, \Upsilon(m)\big)^{|\partial A|_{\mathrm{edge}}}. \tag{75}$$

Here $r_\star$ is the transport allowance from (49) and Γ is the belt circuit inflation from (51).

Proof. Fix a region A and an $r' > 0$ (to be chosen). Let $B := \partial_{r'} A$ be the belt and $S := (\partial_{r'} A)^c$ the bulk. Split the Hamiltonian into the bulk part H_S (sum of all interaction terms fully supported in S) and the belt part $H_B := H - H_S$ (all terms intersecting B). For $t \in \mathbb{R}$ define

$$U(t) := e^{itH}, \qquad U_S(t) := e^{itH_S}, \qquad W(t) := U_S(t)^\dagger U(t).$$

Then $U(t) = U_S(t) \, W(t)$ and $W(t)$ solves the interaction–picture equation

$$\frac{d}{dt} W(t) = iK(t) \, W(t), \qquad K(t) := \tau_t^{H_S}(H_B) := U_S(t)^\dagger H_B \, U_S(t). \tag{76}$$

We shall approximate $W(t)$ by a unitary $\widetilde{W}(t)$ supported *inside the belt* and set

$$V_t^{\mathrm{bulk}} := U_S(t), \qquad V_t^{\partial} := \widetilde{W}(t).$$

The factorization error satisfies (Dyson/Duhamel bound for time–ordered exponentials)

$$\|W(t) - \widetilde{W}(t)\| \;\leq\; \exp\!\Big(\int_0^{|t|} \|K(s)\| + \|\widetilde{K}(s)\| \, ds\Big) \int_0^{|t|} \|K(s) - \widetilde{K}(s)\| \, ds, \tag{77}$$

where $\widetilde{K}$ is any (belt–supported) generator defining $\widetilde{W}$ via $\dot{\widetilde{W}} = i\widetilde{K} \, \widetilde{W}$, $\widetilde{W}(0) = \mathbb{1}$.

Belt tiling and color layers. Tile the belt B by a family $\mathcal{Y}$ of *crossing blocks* Y of diameter $O(r')$ whose c-neighborhoods are disjoint within each of $q = O(1)$ *colors* (standard 2D coloring). Let $\{J_Y\}_{Y \in \mathcal{Y}}$ be a smooth partition of unity subordinate to this tiling (bounded overlap $O(1)$). Decompose

$$H_B = \sum_{Y \in \mathcal{Y}} H_Y, \qquad H_Y := J_Y H_B,$$

so $K(t) = \sum_Y K_Y(t)$ with $K_Y(t) := \tau_t^{H_S}(H_Y)$. For a thickness parameter $\rho \geq 0$ write $Y^{(+\rho)}$ for the ρ–neighborhood of Y. Using the LR envelope (73) (with $A_X := H_Y$ and B_Y any local observable outside $Y^{(+\rho)}$), the standard quasi–locality estimate gives

$$\big\|\big(K_Y(t)\big)_{(Y^{(+\rho)})^c}\big\| \;\leq\; C_1 \sum_{u \geq \rho} N_Y(u) \, \Xi\big(u - \mathsf{R}(|t|)\big), \tag{78}$$

where $N_Y(u)$ counts site pairs at graph distance u with one site in Y and the other in $Y^{(+u)^c}$ and $C_1 = O(1)$ depends only on local/LR data. In 2D, for each block Y there is a geometric constant $c_{\mathrm{pc}} = O(1)$ such that

$$N_Y(u) \;\leq\; c_{\mathrm{pc}}\, u \qquad \text{for all } u \geq 1, \tag{79}$$

since a spherical shell of radius u intersects $O(u)$ sites along the normal to the boundary within a block of $O(1)$ longitudinal extent.

Choice of the belt generator and colorwise disjointness. Fix $T \geq 0$ and choose $\rho := r' - \mathsf{R}(T) \geq 0$. Define the truncated (belt–supported) generators

$$\widetilde{K}_Y(t) := \big(K_Y(t)\big)_{Y^{(+\mathsf{R}(T))}}, \qquad \widetilde{K}(t) := \sum_{Y \in \mathcal{Y}} \widetilde{K}_Y(t),$$

which are all supported inside B provided $r' \geq \mathsf{R}(T)$. By construction, within any fixed color class the supports of $\{\widetilde{K}_Y(t)\}_Y$ are pairwise disjoint for all $|t| \leq T$, hence commute; in particular

$$\Big\| \sum_{Y \in \text{one color}} \big(K_Y(t) - \widetilde{K}_Y(t)\big) \Big\| = \max_Y \|K_Y(t) - \widetilde{K}_Y(t)\|. \tag{80}$$

Summing over the constant number $q = O(1)$ of colors and using (78)–(79) gives

$$\|K(t) - \widetilde{K}(t)\| \;\leq\; C_2 \sum_{u \geq \rho} u\, \Xi\big(u - \mathsf{R}(|t|)\big), \qquad |t| \leq T, \tag{81}$$

for a constant $C_2 = O(1)$ (depends only on LR/local data and the fixed tiling). Likewise $\|K(t)\|, \|\widetilde{K}(t)\| \leq C_3$ uniformly in A, r', t (the colorwise disjointness bounds the norm by the *max* block contribution).

Tail summations in the three LR regimes. We now bound the spatial tail in (81) and integrate in time. Write $\rho = r' - \mathsf{R}(T)$; note that $u - \mathsf{R}(|t|) \geq u - \mathsf{R}(T)$ since R is nondecreasing.
(E) $\mathsf{R}(t) = vt$, $\Xi(u) = e^{-\mu u}$. Using $u - \mathsf{R}(|t|) \geq u - v|t|$ and summing a geometric series,

$$\sum_{u \geq \rho} u\, e^{-\mu(u - \mathsf{R}(|t|))} \;\leq\; e^{\mu v |t|} \sum_{u \geq \rho} u e^{-\mu u} \;\leq\; \frac{c_E}{\mu^2}\, e^{-\mu(\rho - \mathsf{R}(|t|))},$$

for a universal $c_E = O(1)$. Integrating over $s \in [0, |t|]$ and using $\int_0^{|t|} e^{\mu v s} ds \leq (\mu v)^{-1} e^{\mu v |t|}$ yields

$$\int_0^{|t|} \|K(s) - \widetilde{K}(s)\|\, ds \;\leq\; \frac{C_E}{\mu^2\, \mu v}\, e^{-\mu(\rho - \mathsf{R}(|t|))} \;\leq\; \kappa_{\mathrm{pf}}^{(E)}\, e^{-\mu(\rho - \mathsf{R}(T))}, \tag{82}$$

with $\kappa_{\mathrm{pf}}^{(E)} = O(1)$ (absorbing C_2 and constants).
(SE) $\mathsf{R}(t) = v_\zeta t^\zeta$ *(0 < \zeta < 1)*, $\Xi(u) = e^{-\mu_\zeta u}$. As above,

$$\sum_{u \geq \rho} u\, e^{-\mu_\zeta(u - \mathsf{R}(|t|))} \;\leq\; \frac{c_E}{\mu_\zeta^2}\, e^{-\mu_\zeta(\rho - \mathsf{R}(|t|))}.$$

A crude time integration gives $\int_0^{|t|} e^{\mu_\zeta \mathsf{R}(s)} ds \leq c_\zeta e^{\mu_\zeta \mathsf{R}(|t|)}$ for a constant $c_\zeta = O(1)$ depending only on (ζ, v_ζ).[2] Thus

$$\int_0^{|t|} \|K(s) - \widetilde{K}(s)\|\, ds \;\leq\; \kappa_{\mathrm{pf}}^{(SE)}\, e^{-\mu_\zeta(\rho - \mathsf{R}(|t|))} \;\leq\; \kappa_{\mathrm{pf}}^{(SE)}\, e^{-\mu_\zeta(\rho - \mathsf{R}(T))}, \tag{83}$$

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[2]One may take $c_\zeta := 1 + \frac{1}{\mu_\zeta v_\zeta} \sup_{x>0} x^{1-\zeta} e^{-\mu_\zeta v_\zeta x^\zeta}$; the supremum is finite for fixed (ζ, v_ζ).

with $\kappa_{\mathrm{pf}}^{(\mathrm{SE})} = O(1)$.

(PL) $\mathsf{R}(t) = 0$, $\Xi(u) \asymp (1+|t|)^\theta/(1+u)^\sigma$, $\sigma > 2$ *(2D)*. Using the pair counting and the integral test,

$$\sum_{u \geq \rho} \frac{u}{(1+u)^\sigma} \ \leq \ \int_{\rho-1}^\infty \frac{x}{(1+x)^\sigma}\, dx \ \leq \ \frac{c_\sigma}{(\sigma-2)}\, \rho^{2-\sigma},$$

for $c_\sigma = O(1)$. Hence, for $|t| \leq T$,

$$\|K(t) - \widetilde{K}(t)\| \ \leq \ C_2\, c_\sigma\, \frac{(1+|t|)^\theta}{(\sigma-2)}\, \rho^{2-\sigma} \ \leq \ \frac{\tilde{c}_\sigma}{\sigma-2}\, (1+T)^\theta\, \rho^{2-\sigma}.$$

Integrating in time simply multiplies by another $O(1)$ factor (absorbed), giving

$$\int_0^{|t|} \|K(s) - \widetilde{K}(s)\|\, ds \ \leq \ \kappa_{\mathrm{pf}}^{(\mathrm{PL})}\, (1+T)^\theta\, \rho^{2-\sigma}, \tag{84}$$

with $\kappa_{\mathrm{pf}}^{(\mathrm{PL})} = O(1)$.

From generator tail to unitary error (summary). By the uniform bounds $\|K\|, \|\widetilde{K}\| \leq C_3 = O(1)$, the exponential prefactor in (77) is a harmless $O(1)$ and can be absorbed into the $\kappa_{\mathrm{pf}}^{(\bullet)}$. Combining (77) with (82)–(84), taking $|t| \leq T$ and writing $\rho = r' - \mathsf{R}(T)$, we obtain

$$\| U(t) - V_t^{\mathrm{bulk}} V_t^\partial \| = \| W(t) - \widetilde{W}(t) \| \ \leq \ \begin{cases} \kappa_{\mathrm{pf}}^{(\mathrm{E})}\, e^{-\mu(r'-\mathsf{R}(T))}, \\ \kappa_{\mathrm{pf}}^{(\mathrm{SE})}\, e^{-\mu_\zeta(r'-\mathsf{R}(T))}, \\ \kappa_{\mathrm{pf}}^{(\mathrm{PL})}\, (1+T)^\theta\, (r'-\mathsf{R}(T))^{2-\sigma}. \end{cases}$$

For the LCU filter $p_m(\widehat{H}) = \sum_{k=-m}^m a_k e^{ik\tau\widehat{H}}$ with $T = m\tau$, choose r' so that the *per–time* error is $\leq \varepsilon_{\mathrm{pf}}/\|a\|_1$. This is achieved by the single summary bound

$$r'(m) \ \geq \ \mathsf{R}(T) \ + \ \Delta r_{\mathrm{tail}}(T, \varepsilon_{\mathrm{pf}}),$$

with

$$\Delta r_{\mathrm{tail}}(T, \varepsilon_{\mathrm{pf}}) = \begin{cases} \mu^{-1} \log\big(\kappa_{\mathrm{pf}}^{(\mathrm{E})}\, \|a\|_1/\varepsilon_{\mathrm{pf}}\big), & (\mathrm{E}), \\ \mu_\zeta^{-1} \log\big(\kappa_{\mathrm{pf}}^{(\mathrm{SE})}\, \|a\|_1/\varepsilon_{\mathrm{pf}}\big), & (\mathrm{SE}), \\ \big(\kappa_{\mathrm{pf}}^{(\mathrm{PL})}\, \|a\|_1(1+T)^\theta/\varepsilon_{\mathrm{pf}}\big)^{1/(\sigma-2)}, & (\mathrm{PL, 2D}), \end{cases}$$

exactly as stated in (74). Then

$$\sum_{k=-m}^m |a_k|\, \big\| e^{ik\tau H} - V_{k\tau}^{\mathrm{bulk}} V_{k\tau}^\partial \big\| \ \leq \ \|a\|_1 \cdot \frac{\varepsilon_{\mathrm{pf}}}{\|a\|_1} \ = \ \varepsilon_{\mathrm{pf}}.$$

OSR base unchanged. The belt unitary V_t^∂ is generated inside B and is obtained by a color–layered product of commuting block evolutions; its circuitization within B has the same evolution base $\chi_{\mathrm{evol}}^{c_{\mathrm{evol}}T}$ as in the exponential LR case. The LCU superposition contributes the $(2m+1)$ base. Hence

$$\Upsilon(m) = \max\{\chi_{\mathrm{evol}}^{c_{\mathrm{evol}}T},\, 2m+1\}$$

is unchanged across (E)/(SE)/(PL). Composing with transport inflates only the belt allowance to $r'' = r_\star + r'(m)$ and multiplies the base by Γ, yielding (75). □

Summary tie-in. The radius choice (61) is the only place where the LR regime enters; all OSR bases remain Λ_0 (filter algebra), Γ (belt circuit inflation; cf. Assumption 1), and $\Upsilon(m)$ (LCU/evolution), as encoded in (96) and summarized in Remark 1.

Corollary 1 (Rough boundaries). *With $|\partial A|_{\mathrm{edge}}$ and $N_w(\partial_r A)$ as in Proposition 1,*

$$N_w(\partial_r A) \ \leq \ c_{\mathrm{pack}}\, r\, |\partial A|_{\mathrm{edge}}.$$

Hence every belt circuit layer uses at most $O(r\,|\partial A|_{\mathrm{edge}})$ crossing blocks, and all OSR bases stated as $\mathrm{base}^{|\partial A|}$ remain valid with $|\partial A|_{\mathrm{edge}}$ (or $|\partial A|_{\mathrm{Mink}}$) up to $O(1)$ constant changes.

Compatibility with the filtered AGSP. Combining Proposition 7 with Proposition 6 keeps the same OSR base $\Lambda = \Lambda_0\,\Gamma\,\Upsilon(m)$ and only updates the belt width to $r'' = r_\star + r'(m)$, with $r_\star$ from (49) and $r'(m)$ from (61). No other constants change across (E/SE/PL).

7 AGSP to area law: conversion with belt–Markov seeding

Scope & status. This section is a *conditional* conversion: it proves an area law from an AGSP under the *belt approximate–Markov* assumption stated in (88). It is *not* required for the unconditional main result in Main theorem for gapped SRE phases (gapped path to FF), which uses SRE seeding. We retain this section for the endpoint-only route and variants where Markov seeding is available.

Objective. Assume that for a fixed Hamiltonian H (the endpoint model) we have constructed an AGSP

$$\tilde{K} \quad \text{with} \quad \tilde{K}P = P, \qquad \|\tilde{K}P^\perp\| \leq \delta < \tfrac{1}{2}, \qquad \mathrm{OSR}_{\partial_{r''}A}(\tilde{K}) \ \leq \ \Lambda^{|\partial A|},$$

where $\Lambda < \infty$ is independent of $|A|$ and the system size, and $r'' = O(1)$ (the belt allowance coming from the AGSP construction and the short-time evolutions used in its realization). *Assuming* belt approximate–Markov behavior at some width $r_\star = O(\xi)$ (see (88) below), we prove a quantitative area law,

$$S(\rho_A) \ \leq \ C_1(\delta)\,|\partial A|\log\Lambda \ + \ \frac{\log\kappa_{\mathrm{M}}}{1 - \delta^2}\,|\partial A| \ + \ C_0, \qquad C_1(\delta) = \frac{1}{(1 - \delta^2)^2}, \quad C_0 = O(1), \quad (85)$$

where $\kappa_{\mathrm{M}} < \infty$ is a *Markov seeding constant* obtained from approximate recoverability across a width-$O(1)$ belt. This removes any SRE-circuit hypothesis from the seeding step but relies on (88).

7.1 OSR calculus: product and sum bounds

Recall that $\mathrm{OSR}_{\partial_r A}(X)$ is the minimal R for which X admits a decomposition across $(A:A^c)$

$$X \ = \ \sum_{j=1}^{R} L_j \otimes R_j, \qquad \mathrm{supp}(L_j) \subseteq A \cup \partial_r A, \ \ \mathrm{supp}(R_j) \subseteq A^c \cup \partial_r A.$$

We use only the following two elementary properties.

Lemma 11 (Product bound). *If $\mathrm{OSR}_{\partial_r A}(X) \leq R_X$ and $\mathrm{OSR}_{\partial_{r'} A}(Y) \leq R_Y$, then*

$$\mathrm{OSR}_{\partial_{r+r'} A}(XY) \ \leq \ R_X\, R_Y.$$

Proof. Write $X = \sum_{i=1}^{R_X} L_i \otimes R_i$ and $Y = \sum_{j=1}^{R_Y} L_j' \otimes R_j'$ with the stated support allowances. Then $XY = \sum_{i,j}(L_i L_j') \otimes (R_i R_j')$ has allowance $r + r'$ and at most $R_X R_Y$ terms. □

Lemma 12 (Sum bound). *If* $\mathrm{OSR}_{\partial_r A}(X) \le R_X$ *and* $\mathrm{OSR}_{\partial_{r'} A}(Y) \le R_Y$, *then*

$$\mathrm{OSR}_{\partial_{\max\{r,r'\}} A}(X + Y) \ \le \ R_X + R_Y.$$

More generally, $\mathrm{OSR}_{\partial_r A}\big(\sum_k X_k\big) \ \le \ \sum_k \mathrm{OSR}_{\partial_r A}(X_k)$ *whenever each* X_k *has allowance* $\le r$.

Proof. Concatenate the two decompositions under the common allowance $\max\{r, r'\}$. $\qquad\square$

Rank growth under OSR-limited operators. If X acts across $(A{:}A^c)$ with $\mathrm{OSR}_{\partial_r A}(X) \le R$ and $|\phi\rangle$ has Schmidt rank $\mathrm{rank}_A(|\phi\rangle) \le s$, then

$$\mathrm{rank}_A\big(X\,|\phi\rangle\big) \ \le \ R\,s.$$

Indeed, for $X = \sum_{j=1}^{R} L_j \otimes R_j$ each summand preserves the Schmidt rank bound s, and the sum adds ranks.

7.2   Schmidt data and head mass

Let $|\Omega\rangle$ be the (normalized) unique ground state and

$$|\Omega\rangle \ = \ \sum_{i \ge 1} \sqrt{\lambda_i}\,|i_A\rangle\,|i_{A^c}\rangle, \qquad \lambda_1 \ge \lambda_2 \ge \cdots, \qquad \sum_i \lambda_i = 1,$$

its Schmidt decomposition across $(A{:}A^c)$. For $R \in \mathbb{N}$ define the head mass $p^{(R)} := \sum_{i=1}^{R} \lambda_i$ and tail $\varepsilon^{(R)} := 1 - p^{(R)}$. Let Q_R be the projector onto the span of the R leading Schmidt vectors on A.

7.3   One-step boosting lemma

Lemma 13 (Boosting by an AGSP). *Let* Q_R *be the projector onto the span of the* R *leading Schmidt vectors on* A *for* $|\Omega\rangle$, *and set* $p := \langle\Omega|\,Q_R\,|\Omega\rangle = p^{(R)}$. *Let* $|\psi'\rangle := \tilde{K}\,Q_R\,|\Omega\rangle$. *Then*

$$\mathrm{rank}_A(|\psi'\rangle) \ \le \ R\,\Lambda^{|\partial A|}, \qquad \frac{|\,\langle\Omega|\psi'\rangle\,|^2}{\|\psi'\|^2} \ \ge \ \frac{p}{p + (1-p)\delta^2}.$$

Proof. Rank bound. The state $Q_R\,|\Omega\rangle$ has Schmidt rank $\le R$. By the paragraph "Rank growth under OSR-limited operators" (with $X = \tilde{K}$ and $s = R$) and $\mathrm{OSR}_{\partial_{r''} A}(\tilde{K}) \le \Lambda^{|\partial A|}$,

$$\mathrm{rank}_A(|\psi'\rangle) \ \le \ \mathrm{OSR}_{\partial_{r''} A}(\tilde{K}) \cdot R \ \le \ R\,\Lambda^{|\partial A|}.$$

Overlap bound. Decompose $Q_R\,|\Omega\rangle = P\,Q_R\,|\Omega\rangle + P^{\perp}Q_R\,|\Omega\rangle$, where P projects onto the ground space of H. Using $\tilde{K}P = P$ and $\|\tilde{K}P^{\perp}\| \le \delta$,

$$\langle\Omega|\psi'\rangle = \langle\Omega|\,\tilde{K}PQ_R\,|\Omega\rangle = \langle\Omega|\,PQ_R\,|\Omega\rangle = \langle\Omega|\,Q_R\,|\Omega\rangle = \sqrt{p}.$$

Moreover,

$$\|\psi'\|^2 = \|\tilde{K}(PQ_R\,|\Omega\rangle + P^{\perp}Q_R\,|\Omega\rangle)\|^2 \ \le \ \|PQ_R\,|\Omega\rangle\|^2 + \delta^2\|P^{\perp}Q_R\,|\Omega\rangle\|^2 \ = \ p + \delta^2(1-p).$$

Therefore the claimed fidelity bound follows. $\qquad\square$

7.4   Closed-form recursion for the head mass

Let $R_{t+1} := R_t\,\Lambda^{|\partial A|}$ and $p_t := p^{(R_t)}$. Lemma 13 implies

$$p_{t+1} \ \ge \ \frac{p_t}{p_t + (1 - p_t)\delta^2}. \tag{86}$$

Hence (by induction)

$$p_t \ \ge \ \frac{p_0}{p_0 + (1 - p_0)\delta^{2t}}, \qquad \varepsilon_t := 1 - p_t \ \le \ \frac{1 - p_0}{p_0}\,\delta^{2t}. \tag{87}$$

7.5 Belt–Markov seeding

Fix $A \subset \mathbb{Z}^2$, a belt $B := \partial_r A$ and $C := (A \cup B)^c$. Assume the *belt approximate–Markov* behavior

$$I(A : C \,|\, B)_\rho \;\leq\; c_{\mathrm{MI}}\, e^{-\mu_{\mathrm{MI}} r/\xi}, \tag{88}$$

for universal constants $c_{\mathrm{MI}}, \mu_{\mathrm{MI}} > 0$ (independent of A), where $\xi = O(1)$ denotes the correlation length.

Localized Fawzi–Renner and belt channel. By Fawzi–Renner recoverability, there exists a CPTP map $\mathcal{R}_{B \to AB}$ such that

$$\|\rho_{AB} - \mathcal{R}_{B \to AB}(\rho_B)\|_1 \;\leq\; \varepsilon_{\mathrm{rec}}(r) \;:=\; C_{\mathrm{rec}}\, e^{-\frac{1}{2}\mu_{\mathrm{MI}} r/\xi}, \qquad C_{\mathrm{rec}} = O(1).$$

To act only within the belt, approximate $\mathcal{R}_{B \to AB}$ by a *belt–supported* channel $\mathcal{R}^{(r)}_{B \to AB}$ using Lieb–Robinson (LR) bounds: the Stinespring unitary of $\mathcal{R}_{B \to AB}$ can be written as a time–ordered exponential of local generators on B; truncating its light cone to $\partial_{r+O(1)} A$ and compiling it into a constant–depth circuit yields

$$\left\| \rho_{AB} \;-\; \mathcal{R}^{(r)}_{B \to AB}(\rho_B) \right\|_1 \;\leq\; \varepsilon_{\mathrm{rec}}(r) \;=\; C_{\mathrm{rec}}\, e^{-\frac{1}{2}\mu_{\mathrm{MI}} r/\xi}. \tag{89}$$

Let $\mathcal{N}^{(r)}_{B \to A} := \mathrm{Tr}_B \circ \mathcal{R}^{(r)}_{B \to AB}$. The circuit form implies a *Kraus–rank* bound

$$\mathrm{KrausRank}\!\left(\mathcal{N}^{(r)}_{B \to A}\right) \;\leq\; \chi^{|B|}_{\mathrm{rec}}, \qquad \chi_{\mathrm{rec}} = d^{c_{\mathrm{rec}} w^2 q_{\mathrm{belt}}} = O(1), \tag{90}$$

where $w = O(1)$ is the gate diameter, $q_{\mathrm{belt}} = O(1)$ the circuit depth, and $c_{\mathrm{rec}} = O(1)$ a packing constant. The belt size satisfies $|B| \leq c_{\mathrm{belt}}\, r\, |\partial A|$ with $c_{\mathrm{belt}} = O(1)$.

Lemma 14 (Markov seeding). *Pick $r_\star = O(\xi)$ so that $\varepsilon_{\mathrm{rec}}(r_\star) \leq 1/10$. Let Π_B be any typical projector on B with $\mathrm{Tr}[\Pi_B \rho_B] \geq 1 - \varepsilon_{\mathrm{typ}}$ (e.g. $\varepsilon_{\mathrm{typ}} = 2/5$), and define*

$$Q_A \;:=\; \mathrm{Proj}\Big(\mathrm{supp}\big(\mathcal{N}^{(r_\star)}_{B \to A}(\Pi_B\, \rho_B\, \Pi_B)\big)\Big).$$

Then $\mathrm{Tr}[Q_A\, \rho_A] \geq \frac{1}{2}$ and

$$\mathrm{rank}(Q_A) \;\leq\; \mathrm{KrausRank}\!\left(\mathcal{N}^{(r_\star)}_{B \to A}\right) \mathrm{rank}(\Pi_B) \;\leq\; \big(\chi_{\mathrm{rec}}\, d\big)^{c_{\mathrm{belt}}\, r_\star\, |\partial A|} \;=:\; \kappa_{\mathrm{M}}^{|\partial A|},$$

i.e.

$$\kappa_{\mathrm{M}} \;:=\; \big(\chi_{\mathrm{rec}}\, d\big)^{c_{\mathrm{belt}}\, r_\star}. \tag{91}$$

Proof. From (89) and data processing, $\|\rho_A - \mathcal{N}^{(r_\star)}(\rho_B)\|_1 \leq \varepsilon_{\mathrm{rec}}(r_\star)$. Since $\mathrm{Tr}[\Pi_B \rho_B] \geq 1 - \varepsilon_{\mathrm{typ}}$ and $\varepsilon_{\mathrm{typ}} + \varepsilon_{\mathrm{rec}} \leq \frac{1}{2}$, we get $\mathrm{Tr}[Q_A \rho_A] \geq 1 - \varepsilon_{\mathrm{typ}} - \varepsilon_{\mathrm{rec}} \geq \frac{1}{2}$. The rank bound uses (90) and $\mathrm{rank}(\Pi_B) \leq d^{|B|}$. $\square$

Thus we obtain a seed (R_0, p_0) with $R_0 \leq \kappa_{\mathrm{M}}^{|\partial A|}$ and $p_0 \geq \frac{1}{2}$, *without* any SRE circuit.

7.6 A Lorenz-curve majorization bound

Lemma 15 (Entropy from a multi-scale tail). *If $\sum_{i > R_t} \lambda_i \leq \varepsilon_t$ for nondecreasing R_t and nonincreasing $\varepsilon_t \downarrow 0$, then*

$$S(\{\lambda_i\}) \;\leq\; \sum_{t \geq 1} \big(\varepsilon_{t-1} - \varepsilon_t\big) \log R_t \;+\; H_2(\varepsilon_0). \tag{92}$$

Proof of Lemma 15. Write $\lambda_1 \geq \lambda_2 \geq \cdots$ and put $\varepsilon_{-1} := 1$, $R_0 := 0$, and $\delta_t := \varepsilon_{t-1} - \varepsilon_t \geq 0$ for $t \geq 0$. Then $\sum_{t \geq 0} \delta_t = 1$ and, by hypothesis, the cumulative mass inside the top R_t positions is at least $1 - \varepsilon_t = \sum_{j=0}^{t} \delta_j$.

A block–uniform majorant. Consider the explicit nonincreasing distribution $\mu = \{\mu_i\}_{i \geq 1}$ that saturates these prefix constraints with minimal possible prefix sums: place mass $\delta_0 = 1 - \varepsilon_0$ uniformly on indices $1, \ldots, R_1$, then place mass δ_1 uniformly on indices $R_1 + 1, \ldots, R_2$, and so on. Concretely,

$$\mu_i = \sum_{t \geq 1} \frac{\delta_t}{R_t - R_{t-1}} \, \mathbf{1}\{R_{t-1} < i \leq R_t\}, \qquad \text{and} \qquad \sum_{i \leq R_t} \mu_i = \sum_{j=0}^{t} \delta_j = 1 - \varepsilon_t.$$

By construction μ is feasible (it obeys the same tail inequalities) and has the smallest possible prefix sums among all feasible nonincreasing distributions; therefore it majorizes every other feasible $\{\lambda_i\}$ in the sense of Lorenz curves. Since entropy is Schur–concave, $S(\{\lambda_i\}) \leq S(\{\mu_i\})$.

Computing $S(\mu)$. The distribution μ is block–uniform: on block $t \geq 1$ of length $L_t := R_t - R_{t-1}$ each entry equals δ_t/L_t, while the "head block" $t = 0$ contributes a binary split between total head mass δ_0 and its complement ε_0. Hence

$$S(\mu) = H_2(\varepsilon_0) \; + \; \sum_{t \geq 1} \delta_t \log L_t \; \leq \; H_2(\varepsilon_0) \; + \; \sum_{t \geq 1} \delta_t \log R_t,$$

using $L_t \leq R_t$.

Conclusion. Because $S(\{\lambda_i\}) \leq S(\mu)$ and $\delta_t = \varepsilon_{t-1} - \varepsilon_t$, we obtain

$$S(\{\lambda_i\}) \; \leq \; \sum_{t \geq 1} \big(\varepsilon_{t-1} - \varepsilon_t\big) \log R_t \; + \; H_2(\varepsilon_0),$$

which is precisely (92). $\qquad\square$

7.7 Summation, telescoping, and constants

Apply Lemma 15 with

$$R_t \; = \; R_0 \, \Lambda^{t|\partial A|}, \qquad \varepsilon_t \; \leq \; \theta \, \delta^{2t}, \quad \theta := \frac{1 - p_0}{p_0} \; \leq \; 1 \quad (\text{since } p_0 \geq \tfrac{1}{2}).$$

Using the ratio form (restated here for convenience),

$$S(\{\lambda_i\}) \; \leq \; \sum_{t \geq 0} \varepsilon_t \log\!\Big(\frac{R_{t+1}}{R_t}\Big) \; + \; (1 - \varepsilon_0) \log R_0 \; + \; H_2(\varepsilon_0), \tag{93}$$

we obtain

$$S(\rho_A) \; \leq \; (1 - \varepsilon_0) \log R_0 \; + \; \sum_{t \geq 0} \varepsilon_t \log\!\Big(\frac{R_{t+1}}{R_t}\Big) \; + \; H_2(\varepsilon_0).$$

Now $\log(R_{t+1}/R_t) = |\partial A| \log \Lambda$ is constant, while

$$\sum_{t \geq 0} \varepsilon_t \; \leq \; \sum_{t \geq 0} \theta \, \delta^{2t} \; = \; \frac{\theta}{1 - \delta^2} \; \leq \; \frac{1}{1 - \delta^2},$$

and

$$\sum_{t \geq 0} (t+1)\varepsilon_t \; \leq \; \sum_{t \geq 0} (t+1) \, \theta \, \delta^{2t} \; = \; \frac{\theta}{(1 - \delta^2)^2} \; \leq \; \frac{1}{(1 - \delta^2)^2}.$$

Writing $\log R_0 \le |\partial A| \log \kappa_{\mathrm{M}}$ from Lemma 14 and absorbing $(1 - \varepsilon_0) \le 1$ and $H_2(\varepsilon_0) \le \log 2$ into a universal $C_0 = O(1)$ gives

$$S(\rho_A) \;\le\; \frac{\log \kappa_{\mathrm{M}}}{1 - \delta^2} \, |\partial A| \;+\; \frac{1}{(1 - \delta^2)^2} \, |\partial A| \, \log \Lambda \;+\; C_0. \tag{94}$$

Equivalently, combining the two slope terms,

$$S(\rho_A) \;\le\; \frac{1}{(1 - \delta^2)^2} \, |\partial A| \, \log\!\Big(\kappa_{\mathrm{M}}^{1 - \delta^2} \, \Lambda \Big) \;+\; C_0. \tag{95}$$

Remark (sharpening). One can keep the slightly tighter coefficient $\sum_{t \ge 0} \varepsilon_t \le \theta/(1 - \delta^2)$ in the seed term and the exact $\sum_{t \ge 0} (t + 1)\varepsilon_t = \theta/(1 - \delta^2)^2$ in the AGSP term; we state the clean uniform bounds shown above.

7.8 Main conversion lemma

Lemma 16 (AGSP $\Rightarrow$ area law under belt–Markov seeding (summary compatibility)). *Let $\tilde{K}$ be the filtered AGSP from Proposition 6 built with the LR envelope of (73). Then*

$$\tilde{K}P = P, \qquad \|\tilde{K}P^\perp\| \le \delta < \tfrac{1}{2}, \qquad \mathrm{OSR}_{\partial_{r''} A}(\tilde{K}) \;\le\; \Lambda^{|\partial A|_{\mathrm{edge}}},$$

with

$$\Lambda \;=\; \Lambda_0 \, \Gamma \, \Upsilon(m), \qquad r'' \;=\; r_\star \;+\; r'(m),$$

where $r_\star$ is the transport allowance from (49), $r'(m)$ is the per–LCU belt width from (61), and Γ is the belt circuit inflation from (51). Assume the belt approximate–Markov behavior (88) at some $r_\star = O(\xi)$ so that Lemma 14 yields a seed (R_0, p_0) with $R_0 \le \kappa_{\mathrm{M}}^{|\partial A|_{\mathrm{edge}}}$ and $p_0 \ge \tfrac{1}{2}$. Then

$$S(\rho_A) \;\le\; \frac{1}{(1 - \delta^2)^2} \, |\partial A|_{\mathrm{edge}} \, \log\!\Big(\kappa_{\mathrm{M}}^{1 - \delta^2} \, \Lambda \Big) \;+\; C_0,$$

with $C_0 = O(1)$ independent of A and system size.

Proof. By Lemma 13, with $R_{t+1} := R_t \Lambda^{|\partial A|}$ the head masses satisfy $p_{t+1} \ge \frac{p_t}{p_t + (1 - p_t)\delta^2}$, hence $\varepsilon_t := 1 - p_t \le \theta\,\delta^{2t}$ with $\theta \le 1$. Apply Lemma 15 in the ratio form (92) and sum as in Section 7.7, using $\log R_0 \le |\partial A| \log \kappa_{\mathrm{M}}$. This yields (95). $\qquad\square$

7.9 Constants at a glance

Symbol	Meaning	Where fixed / used
δ	AGSP contraction $\|\tilde{K}P^\perp\| \le \delta$	built via filter (Sec. 5)
Λ_0	OSR base from filter algebra	Sec. 5; multiplies into Λ
Γ	OSR inflation of belt circuit	(51) (QAC belt circuitization)
$\Upsilon(m)$	LCU/evolution OSR base	Prop. 7
Λ	Total AGSP OSR base	$\Lambda = \Lambda_0 \, \Gamma \, \Upsilon(m)$; used in Lemma 13, §7.7
$r_\star$	Transport allowance	(49) (AGSP transport)
$r'(m)$	Per–LCU belt width (LR ledger)	(61) (E/SE/PL via (73))
r''	Total belt allowance of $\tilde{K}$	$r'' = r_\star + r'(m)$; appears in $\mathrm{OSR}_{\partial_{r''} A}$
ε_t	Tail after t boosts	§7.3 (closed recursion)
R_t	Head rank after t boosts	$R_t = R_0 \Lambda^{t|\partial A|_{\mathrm{edge}}}$
κ_{M}	Markov seeding base	(91); from Lemma 14
$c_{\mathrm{MI}}, \mu_{\mathrm{MI}}, \xi$	belt–Markov/length scales	assumption (88)
χ_{rec}	per–site Kraus inflation	(90)
C_0	universal additive constant	collects $H_2(\varepsilon_0)$ and $O(1)$ terms

8 Main theorem for gapped SRE phases (gapped path to FF)

Scope & status. This section now states two parallel area–law theorems that share the same AGSP summary of constants and belt accounting. *(i) General gap+LR route (conditional until Sec. 10.6):* assuming the belt–Markov seed provided by the Annulus Recoverability Theorem in Approach A, we obtain an area law with an explicit coefficient c_{gen} (Theorem 3). *(ii) SRE–path route (unconditional under a gapped SRE path to an FF anchor):* we recover the SRE theorem (Theorem 2) as a special case where the seed is supplied by a constant–depth belt circuit preparing the ground state from a product. Both routes use the same filtered, boundary–tamed AGSP with OSR base $\Lambda = \Lambda_0 \, \Gamma \, \Upsilon(m)$ and belt width $r'' = r_\star + r'(m)$; in the endpoint–only path one may set $\Gamma = 1$ (no transport), while in the SRE–path route $\Gamma > 1$ accounts for QAC belt circuitization.

8.1 Theorem and short proof

Theorem 2 (Area law for gapped SRE phases: explicit coefficient). *Under Setup: formal hypotheses and parameter bookkeeping with $g = 1$, assume $H(0)$ is frustration-free and there is a C^1 uniformly gapped path $H(s)$, $s \in [0,1]$, to $H(1)$ with uniform gap $\Delta_\star > 0$ and Lieb–Robinson bounds. Then for every finite region $A \subset \mathbb{Z}^2$,*

$$\boxed{S\big(\rho_A(1)\big) \;\leq\; c\,|\partial A| + C \;,}$$

with

$$\boxed{c \;=\; \frac{\log \kappa_{\text{SRE}}}{1 - \delta^2} \;+\; \frac{\log \Lambda_0 + \log \Gamma + \log \Upsilon(m)}{(1 - \delta^2)^2} \;, \qquad C = O(1) \;.}$$

Here κ_{SRE} is the SRE seeding base from the belt circuit, Λ_0 is the endpoint AGSP base, Γ is the belt-transport OSR inflation base, $\Upsilon(m)$ is the filter base for degree m, and $\delta \in (0, \frac{1}{2})$ is the AGSP contraction after filtering. All are $O(1)$ functions of local/LR data and the fixed design choices.

Proof (belt ledger). (1) *Boundary–tamed AGSP at the FF anchor.* For $H(0)$ (frustration-free), build K_0 with $K_0 P_0 = P_0$, $\|K_0 P_0^\perp\| \leq \delta_0 < \frac{1}{2}$, and $\text{OSR}_{\partial_{r_0} A}(K_0) \leq \Lambda_0^{|\partial A|}$.

(2) *QAC belt factorization and circuitization.* By Theorem 1, $U(s)$ admits bulk×belt factorization and belt circuitization with allowance $r_\star$ (see (49)) and depth $O(1)$.

(3) *Transport and circuitization.* Conjugate K_0 and circuitize to obtain K_1^e approximating $U(1)K_0 U(1)^\dagger$ with $\|K_1^e - U(1)K_0 U(1)^\dagger\| \leq \varepsilon_{\text{circ}}$ (see (52)), and $\text{OSR}_{\partial_{r_0 + r_\star} A}(K_1^e) \leq (\Lambda_0 \Gamma)^{|\partial A|}$ with Γ given by (51).

(4) *Polynomial (trigonometric) filtering.* Apply the degree-m filter $p_m(H(1) - E_0(1)\mathbb{1})$ to K_1^e to get $\widetilde{K}_1^e$ with $\widetilde{K}_1^e P_1 = P_1$, $\|\widetilde{K}_1^e P_1^\perp\| \leq \delta < \frac{1}{2}$, and $\text{OSR}_{\partial_{r''} A}(\widetilde{K}_1^e) \leq (\Lambda_0 \Gamma \Upsilon(m))^{|\partial A|}$ with $r'' := r_\star + r'(m)$ (ledger (61)). This is the display (96).

(5) *SRE seeding.* By the SRE–path hypothesis and belt circuitization, the ground state has Schmidt rank at most $\kappa_{\text{SRE}}^{|\partial A|}$ across $(A{:}A^c)$ with success probability $p_0 \geq \frac{1}{2}$ (seeding constant from the belt circuit).

(6) *Conversion.* Apply the standard AGSP⇒entropy converter (Sec. AGSP to area law: conversion with belt–Markov seeding) with seed (R_0, p_0) and AGSP (δ, Λ) to get

$$S\big(\rho_A(1)\big) \;\leq\; \frac{\log \kappa_{\text{SRE}}}{1 - \delta^2}\,|\partial A| \;+\; \frac{\log(\Lambda_0 \Gamma \Upsilon(m))}{(1 - \delta^2)^2}\,|\partial A| \;+\; C.$$

(7) Identify the coefficient c as in (99).

(8) The additive $C = O(1)$ collects universal constants (Lorenz bound, fixed truncation tolerances) and is independent of $|A|$.

(9) All constants depend only on local/LR data and fixed design choices and are uniform in system size.

(10) This completes the proof. □

Where the constants live. See the ledgers in *Setup: formal hypotheses and parameter book-keeping* (uniformity), *Boundary–tamed AGSP without an FF anchor* (AGSP Λ_0, δ_0), *Quasi-adiabatic continuation (QAC), belt truncation, and constant-depth circuitization* (belt radius, circuit depth), *AGSP transport and OSR inflation control* (Γ), and *Polynomial (trigonometric) energy filter via short evolutions* ($\Upsilon(m), \tau, m$).

8.2 Theorem: General 2D area law via belt–Markov seeding

Theorem 3 (Area law from belt–Markov seed; explicit coefficient). *Under Setup: formal hypotheses and parameter bookkeeping with $g = 1$, assume $H(1)$ is uniformly gapped with gap $\Delta_* > 0$ and admits Lieb–Robinson bounds. Let κ_{M} and $p_0 \geq \frac{1}{2}$ be the seed constants supplied by the belt–Markov (annulus recoverability) theorem in (Sec. 10.6), i.e., for every finite region A there exists a projector Q_A of rank at most $\kappa_{\mathrm{M}}^{|\partial A|}$ with $\mathrm{Tr}(Q_A \rho_A(1)) \geq p_0$. Then, for every finite $A \subset \mathbb{Z}^2$,*

$$\boxed{S(\rho_A(1)) \ \leq \ c_{\mathrm{gen}} |\partial A| \ + \ C \, ,}$$

with explicit

$$\boxed{c_{\mathrm{gen}} \ = \ \frac{\log \kappa_{\mathrm{M}}}{1 - \delta^2} \ + \ \frac{\log \Lambda_0 + \log \Gamma + \log \Upsilon(m)}{(1 - \delta^2)^2} \ , \qquad C = O(1).}$$

Here $\delta \in (0, \frac{1}{2})$ is the contraction of the filtered, boundary–tamed AGSP for $H(1)$ (Prop. 6); the OSR base factorizes as $\Lambda = \Lambda_0 \, \Gamma \, \Upsilon(m)$ and the belt width is $r'' = r_\star + r'(m)$ (ledger (49) and (61)).

Proof. By Proposition 6 (boundary–tamed filtered AGSP at the endpoint) there exists an operator K that is an (δ, Λ)–AGSP for $H(1)$ across the bipartition $(A{:}A^c)$ with belt allowance $r'' = r_\star + r'(m)$ (cf. the ledger displays (49) and (61)), contraction $\delta \in (0, \frac{1}{2})$ on the excited subspace, and OSR base

$$\Lambda \ = \ \Lambda_0 \, \Gamma \, \Upsilon(m),$$

as stated in display (96).

By Approach A (Sec. 10.6) there exists, for every finite region A, a *belt–Markov seed* projector Q_A of rank $R_0 \leq \kappa_{\mathrm{M}}^{|\partial A|}$ such that $\mathrm{Tr}(Q_A \, \rho_A(1)) \geq p_0$ with $p_0 \geq \frac{1}{2}$.

Apply the *AGSP⇒entropy conversion* (Sec. 7), concretely the Main Conversion Lemma (Lemma 16), with seed (R_0, p_0) and AGSP parameters (δ, Λ). The lemma yields, for a universal $C = O(1)$ independent of $|A|$,

$$S(\rho_A(1)) \ \leq \ \frac{\log R_0}{1 - \delta^2} \ + \ \frac{\log \Lambda}{(1 - \delta^2)^2} \ + \ C.$$

Substituting $R_0 \leq \kappa_{\mathrm{M}}^{|\partial A|}$ and $\Lambda = \Lambda_0 \Gamma \Upsilon(m)$ gives

$$S(\rho_A(1)) \ \leq \ \left[\frac{\log \kappa_{\mathrm{M}}}{1 - \delta^2} \ + \ \frac{\log \Lambda_0 + \log \Gamma + \log \Upsilon(m)}{(1 - \delta^2)^2} \right] |\partial A| \ + \ C,$$

which is exactly the asserted coefficient c_{gen} and completes the proof. In the endpoint–only route without transport one may set $\Gamma = 1$, leaving the factorized form to match the ledger. □

Belt width bookkeeping. Throughout this section we use the unified radius ledger

$$\mathrm{OSR}_{\partial_{r''}A}\big(\widetilde{K}_1^e\big) \;\leq\; \big(\Lambda_0\,\Gamma\,\Upsilon(m)\big)^{|\partial A|}, \qquad r'' \;:=\; r_\star \,+\, r'(m), \tag{96}$$

with $r_\star$ from (49) (belt circuitization allowance) and $r'(m)$ from (61) (filter radius).

8.3 Closed–form coefficient (general seed dictionary)

Let $\kappa_\bullet \in \{\kappa_{\mathrm{SRE}}, \kappa_{\mathrm{M}}\}$ denote the seed constant (SRE circuit seed or belt–Markov seed). With the same AGSP parameters $(\delta, \Lambda_0, \Gamma, \Upsilon(m))$ and belt width $r'' = r_\star + r'(m)$, the coefficient reads

$$\boxed{\;c(\kappa_\bullet) \;=\; \frac{\log \kappa_\bullet}{1-\delta^2} \;+\; \frac{1}{(1-\delta^2)^2}\Big[\log\Lambda_0 + \log\Gamma + \log\Upsilon(m)\Big], \qquad C = O(1).\;} \tag{97}$$

Choosing $\kappa_\bullet = \kappa_{\mathrm{SRE}}$ reproduces (99); choosing $\kappa_\bullet = \kappa_{\mathrm{M}}$ yields Theorem 3.

8.4 SRE seeding and the belt constant κ_{SRE}

By the SRE–path hypothesis and Quasi-adiabatic continuation (QAC), belt truncation, and constant-depth circuitization, for any fixed accuracy there is a constant–depth circuit supported in a belt $\partial_{r_\circ}A$ such that

$$|\Omega(1)\rangle \;\approx\; \big(\mathcal{C}_{\mathrm{bulk}}\,\mathcal{C}_\partial\big)\,|\Phi_{\mathrm{prod}}\rangle, \qquad \mathrm{depth}(\mathcal{C}_\partial) =: D_{\mathrm{seed}} = O(1), \quad \mathrm{diam(gates)} = w = O(1).$$

Each crossing gate on a $w \times w$ block has OSR across $(A{:}A^c)$ (with allowance $r_\circ + w$) bounded by

$$\chi_{\mathrm{gate}} \;:=\; d^{\,c_{\mathrm{gate}}\,w^2} \qquad (c_{\mathrm{gate}} = O(1)).$$

Since each layer uses at most $c_{\mathrm{cnt}}|\partial A|$ crossing blocks (boundary packing, $c_{\mathrm{cnt}} = O(1)$), the Schmidt rank of $|\Omega(1)\rangle$ satisfies

$$\mathrm{rank}_A\big(|\Omega(1)\rangle\big) \;\leq\; \Big(\chi_{\mathrm{gate}}^{c_{\mathrm{cnt}} D_{\mathrm{seed}}}\Big)^{|\partial A|} \;=:\; \kappa_{\mathrm{SRE}}^{|\partial A|}, \qquad \log\kappa_{\mathrm{SRE}} = O\big(w^2 \log d\big).$$

Let Q_A be the projector onto the span of the leading $R_0 := \kappa_{\mathrm{SRE}}^{|\partial A|}$ Schmidt vectors; then

$$\mathrm{Tr}(Q_A \rho_A(1)) \;\geq\; \tfrac{1}{2}, \qquad R_0 \;\leq\; \kappa_{\mathrm{SRE}}^{|\partial A|}.$$

(With exact circuits one has $p_0 = 1$; with norm–$o(1)$ truncation we keep $p_0 \geq \tfrac{1}{2}$ for fixed accuracy.)

Dictionary. In the general (gap+LR) route, the role of κ_{SRE} is played by the belt–Markov seed κ_{M} from Approach A; all subsequent formulas remain unchanged after the substitution $\kappa_{\mathrm{SRE}} \mapsto \kappa_{\mathrm{M}}$.

Closed–form coefficient (SRE seed). Applying the AGSP to area law: conversion with belt–Markov seeding conversion with the SRE seed κ_{SRE} (in place of κ_{M}) and $\Lambda = \Lambda_0\,\Gamma\,\Upsilon(m)$ from Boundary–tamed AGSP without an FF anchor–Polynomial (trigonometric) energy filter via short evolutions gives

$$S(\rho_A(1)) \;\leq\; \underbrace{\frac{\log\kappa_{\mathrm{SRE}}}{1-\delta^2}}_{\text{seeding (SRE belt circuit)}} |\partial A| \;+\; \underbrace{\frac{1}{(1-\delta^2)^2}\log\big(\Lambda_0\,\Gamma\,\Upsilon(m)\big)}_{\text{AGSP base (FF DL + transport + filter)}} |\partial A| \;+\; C_0. \tag{98}$$

Equivalently,

$$c = \frac{\log \kappa_{\mathrm{SRE}}}{1 - \delta^2} + \frac{1}{(1 - \delta^2)^2} \Big[\log \Lambda_0 + \log \Gamma + \log \Upsilon(m) \Big], \qquad C = O(1). \qquad (99)$$

Here Λ_0 and r_0 come from the FF–AGSP of Boundary–tamed AGSP without an FF anchor; Γ is the constant OSR inflation base from AGSP transport and OSR inflation control (belt circuitization of QAC),

$$\Gamma = d^{\,2c_{\mathrm{cnt}} q_{\mathrm{belt}} c_{\mathrm{gate}} w^2}, \qquad q_{\mathrm{belt}} \le 9, \quad w = 8R + 8R_1 + 8,$$

and $\Upsilon(m)$ is the filter OSR base of Polynomial (trigonometric) energy filter via short evolutions, with

$$\log \Upsilon(m) \le \alpha_* m, \qquad \alpha_* = \frac{\pi c_{\mathrm{evol}}}{2 \Delta_*} \log \chi_{\mathrm{evol}}(d, J, R, v, \mu) + \log \rho(\tau \Delta_*),$$

for $\tau \in (0, \pi/\Delta_*)$, $c_{\mathrm{evol}} = O(1)$.

8.5 Canonical parameter set and scaling

Pick

$$\delta = \tfrac{1}{4}, \qquad \tau = \frac{\pi}{3 \Delta_*}, \qquad m = \left\lceil \frac{\ln(\delta_0/\delta)}{\ln(1/\eta(\tau \Delta_*))} \right\rceil.$$

Then $C_1(\delta) = (1 - \delta^2)^{-2} \le (4/3)^2$ and

$$\log \kappa_{\mathrm{SRE}} = O(w^2 \log d), \qquad \log \Gamma = O(w^2 \log d), \qquad \log \Upsilon(m) \le \alpha_* m,$$

so c in (99) has an $O(1)$ contribution from seeding/transport and a tunable $O(m)$ contribution from the filter. All constants depend only on $(\Delta_*, v, \mu, d, J, R, R_1)$ and fixed circuit/filter choices, and are independent of $|A|$.

Summary of constants (SRE path). The area-law coefficient c in (99) depends on the FF–AGSP constants $(\delta_0, \Lambda_0, r_0)$, the transport inflation Γ (with belt width $w = 8R + 8R_1 + 8$ and $q_{\mathrm{belt}} \le 9$), the filter parameters (τ, m) determining $\Upsilon(m)$, and the SRE seeding constant κ_{SRE}.

9 Robustness under local perturbations

Statement (both seeding routes). Let $H'(s) := H(s) + V(s)$ with a local perturbation $V(s) = \sum_Z v_Z(s)$ of finite range and bounded strength. If $\|V\|_\nu$ is sufficiently small (precisely below a constant multiple of the uniform path gap Δ_*), then $H'(s)$ remains uniformly gapped, and the AGSP construction of Sections 3–5 transports through a *relative* quasi-adiabatic circuit supported in a fixed-width belt. Concretely, the filtered, circuitized AGSP keeps the same shrinkage δ and base $\Lambda = \Lambda_0 \Gamma \Upsilon(m)$ from the unperturbed arguments, with only two robust effects: *(i)* the belt width increases by r_{stab}, and *(ii)* the OSR bases multiply by Γ_{stab} for the seed and Γ_{stab}^2 for the AGSP. Consequently, along either seeding route (SRE or belt–Markov, cf. Theorems 2 and 3), the area-law slope is stable in the linear-response window

$$S(\rho'_A(1)) \le c_{\mathrm{stab}} |\partial A| + C_{\mathrm{stab}}, \qquad c_{\mathrm{stab}} = c_\bullet + O\Big(\frac{\|V\|_\nu}{\Delta_*}\Big), \qquad (100)$$

where $c_\bullet$ is the coefficient obtained from the chosen seed (use κ_{SRE} or κ_{M} in the unified formula (97)). The robustness constants are uniform in A and in the system size.

9.1 Standing perturbative hypotheses and norm

Assume the same locality data as in Section 1. Quantify smallness in the weighted interaction norm

$$\|V\|_\nu := \sup_{x\in\mathbb{Z}^2} \sum_{Z\ni x} e^{\nu\,\mathrm{diam}(Z)} \sup_{s\in[0,1]} \|v_Z(s)\| \qquad (\nu > 0). \tag{101}$$

9.2 Gap stability and perturbed LR/QAC constants

Uniform stability of the spectral gap under small quasi–local perturbations and the continuity of Lieb–Robinson (LR) and quasi–adiabatic (QAC) locality scales with respect to the interaction norm are standard. Concretely, for the interpolating family $H'(s) = H(s) + V(s)$, if the weighted interaction norm $\|V\|_\nu$ is sufficiently small (quantified below), then the path remains uniformly gapped and the locality parameters change at most linearly in $\|V\|_\nu$. One convenient formulation is: there exists $c_{\mathrm{gap}} = O(1)$ such that if $\|V\|_\nu \le c_{\mathrm{gap}}\Delta_*$, then the deformed Hamiltonians are uniformly gapped with $\Delta'_* \ge \Delta_*/2$ (spectral–flow/automorphic–equivalence methods Bachmann et al. (2012); Michalakis and Zwolak (2013)). LR constants depend Lipschitz–continuously on the interaction strength (velocity linear in the interaction norm, decay rate stable up to an $O(\|V\|_\nu)$ loss), giving

$$\begin{aligned}
v'_{\mathrm{LR}} &\le v_{\mathrm{LR}} + c_v\|V\|_\nu, \\
\mu'_{\mathrm{LR}} &\ge \mu_{\mathrm{LR}} - c_\mu\|V\|_\nu, \\
\mu'_{\mathrm{QA}} &\ge \mu_{\mathrm{QA}} - c_{\mathrm{qa}}\|V\|_\nu,
\end{aligned} \tag{102}$$

for constants $c_v, c_\mu, c_{\mathrm{qa}} = O(1)$ determined by the LR/QAC constructions Nachtergaele and Sims (2010); Hastings and Wen (2005); Bachmann et al. (2012). Set $\mu'_{\mathrm{eff}} := \min\{\mu'_{\mathrm{LR}}, \mu'_{\mathrm{QA}}\}$. The locality of the QAC generator is obtained by convolving the LR light–cone with a fixed entire filter; hence its decay rate μ_{QA} depends only on the LR envelope and the gap. These references provide the precise spectral–flow construction, gap stability, and the dependence of LR/QAC locality constants on the interaction norm; our use here is purely as a black box.

9.3 Relative QAC (R-QAC), belt circuitization, and OSR inflation

Interpolate $H_\lambda(s) := H(s) + \lambda V(s)$, $\lambda \in [0,1]$. Fix $\gamma \in (0, \Delta'_*)$ and the odd filter F_γ from (30)–(31). Define the *relative* generator

$$G(\lambda, s) := i \int_\mathbb{R} dt \, F_\gamma(t) \, e^{itH_\lambda(s)} V(s) e^{-itH_\lambda(s)}. \tag{103}$$

Then $G(\lambda, s)$ is block-off-diagonal w.r.t. $P_\lambda(s)$ and admits an exponentially localized decomposition $G = \sum_Z G_Z$ with

$$\|G_Z(\lambda, s)\| \le C_{\mathrm{rel}} \frac{\|V\|_\nu}{\Delta'_*} e^{-\mu_{\mathrm{rel}}\mathrm{diam}(Z)}. \tag{104}$$

Let

$$W(s) := \mathcal{T}_\lambda \exp\left(i \int_0^1 G(\lambda, s) \, d\lambda\right), \qquad P'_s = W(s) P_s W(s)^\dagger. \tag{105}$$

Belt form and constants. Exactly as in Proposition 3, for any $\varepsilon_{\mathrm{rel}} \in (0, 1/2)$ there is a radius

$$r_{\mathrm{stab}} := \left\lceil \frac{1}{\mu'_{\mathrm{eff}}} \log\left(\frac{\kappa''}{\varepsilon_{\mathrm{rel}}}\right) \right\rceil \tag{106}$$

such that $W(s)$ admits a belt/bulk factorization and, by the color-tiling argument of Section 3.4, a constant-depth belt circuit supported in $\partial_{r_{\mathrm{stab}}} A$. Each crossing gate has OSR $\le \chi_{\mathrm{rel}}$ with

$\log \chi_{\text{rel}} = O(r_{\text{stab}}^2 \log d)$; per layer there are $\leq c_{\text{cnt}}|\partial A|$ crossing blocks and a constant number of color layers. Hence

$$\text{OSR}_{\partial_{r_{\text{stab}}}A}\big(W(s)\big) \;\leq\; \Gamma_{\text{stab}}^{|\partial A|}, \qquad \log \Gamma_{\text{stab}} \;=\; O\big(r_{\text{stab}}^2 \log d\big) \;=\; O\Big(\tfrac{1}{(\mu'_{\text{eff}})^2} \log d\Big). \qquad (107)$$

For fixed ε_{rel} this gives $\log \Gamma_{\text{stab}} = O(1/\mu'^2_{\text{eff}}) + O(\|V\|_\nu/\Delta_*)$ via (102) and $\Delta'_* \geq \Delta_*/2$.

Remark 3 (Ledger compatibility). In the radius ledger of Remark 1, robustness simply *adds* the allowance r_{stab} to the belt width and *multiplies* the OSR bases by Γ_{stab} (seed) and Γ_{stab}^2 (AGSP). In particular, the filtered, circuitized AGSP of Section 5 across $\partial_{r''}A$ with OSR $\leq (\Lambda_0\Gamma\Upsilon(m))^{|\partial A|}$ (cf. (96)) becomes, after robustness, an AGSP across $\partial_{r''+r_{\text{stab}}}A$ with OSR base $\Lambda\Gamma_{\text{stab}}^2$. The seed base $\kappa_\bullet$ (SRE or Markov, cf. (97)) acquires at most a factor Γ_{stab}.

9.4 Transported AGSP and stable slope

Let $\widetilde{K}_s$ be the filtered transported AGSP from Section 5 with

$$\widetilde{K}_s P_s = P_s, \qquad \|\widetilde{K}_s P_s^\perp\| \leq \delta < \tfrac{1}{2}, \qquad \text{OSR}_{\partial_{r''}A}(\widetilde{K}_s) \;\leq\; \Lambda^{|\partial A|}, \quad \Lambda = \Lambda_0\,\Gamma\,\Upsilon(m), \;\; r'' = r_\star + r'(m).$$

Define the *robustly transported* operator $\widetilde{K}'_s := W(s)\,\widetilde{K}_s\,W(s)^\dagger$, where $W(s)$ is the relative QAC unitary from (105). Then

$$r_{\text{final}} \;:=\; r'' + r_{\text{stab}} \qquad (108)$$

and

$$\widetilde{K}'_s P'_s = P'_s, \quad \|\widetilde{K}'_s (P'_s)^\perp\| = \|\widetilde{K}_s P_s^\perp\| \leq \delta, \quad \text{OSR}_{\partial_{r_{\text{final}}}A}(\widetilde{K}'_s) \;\leq\; \big(\Lambda\,\Gamma_{\text{stab}}^2\big)^{|\partial A|}. \qquad (109)$$

In particular, the shrinkage δ and the filter/composition parameters (τ, m) can be kept unchanged, while the belt width and base ledger update deterministically via (108)–(109).

9.5 Corollary (stable area law; both seeding routes)

Corollary 2 (Stable area law under small local perturbations). *If $\|V\|_\nu \leq c_{\text{gap}}\Delta_*$, then there exist $c_{\text{stab}}, C_{\text{stab}} < \infty$, uniform in A and system size, such that*

$$S(\rho'_A(1)) \;\leq\; c_{\text{stab}}\,|\partial A| \;+\; C_{\text{stab}}, \qquad c_{\text{stab}} = c_\bullet + O\Big(\tfrac{\|V\|_\nu}{\Delta_*}\Big),$$

where $c_\bullet$ is the coefficient of the unperturbed route chosen for seeding: use $c_\bullet = c$ from Theorem 2 (SRE path) or $c_\bullet$ from Theorem 3 (belt–Markov seed), with the unified base dictionary in (97).

Proof. Apply the converter of Section 7 to $\widetilde{K}'_1$. Relative QAC multiplies the *seed* base by $\kappa_{\text{stab}} := \Gamma_{\text{stab}}$ and the *AGSP* base by Γ_{stab}^2, cf. (109). The slope contributions scale respectively like $\frac{\log \kappa_{\text{stab}}}{1-\delta^2}$ and $\frac{2}{(1-\delta^2)^2} \log \Gamma_{\text{stab}}$. Using (107) together with $\Delta'_* \geq \Delta_*/2$ and the perturbed LR/QAC constants in (102) yields the stated $O(\|V\|_\nu/\Delta_*)$ correction, added to the baseline $c_\bullet$. $\qquad \square$

Constants at a glance. Gap: $\Delta'_* \geq \Delta_*/2$. Envelope: $\mu'_{\text{eff}} = \min\{\mu'_{\text{LR}}, \mu'_{\text{QA}}\}$ with (102). Robust belt addition: $r_{\text{stab}} = \lceil \mu'^{-1}_{\text{eff}} \log(\kappa''/\varepsilon_{\text{rel}}) \rceil$ and $r_{\text{final}} = r'' + r_{\text{stab}}$ by (108). Inflation: $\Gamma_{\text{stab}} = \exp\{O(r_{\text{stab}}^2 \log d)\}$ with the bound (107). Bases: $\Lambda \mapsto \Lambda\Gamma_{\text{stab}}^2$ and $\kappa_\bullet \mapsto \kappa_\bullet \Gamma_{\text{stab}}$. Final slope: $c_{\text{stab}} = c_\bullet + O(\|V\|_\nu/\Delta_*)$ as in (100).

Summary pointer. Robustness modifies the belt ledger by $r'' \rightsquigarrow r_{\text{final}} = r'' + r_{\text{stab}}$ and $\Lambda \rightsquigarrow \Lambda\Gamma_{\text{stab}}^2$; seeds acquire Γ_{stab}. No other entries change.

10 Endpoint route (general): boundary–tamed AGSP and belt–Markov seeding

Objective (general route). Remove any *uniform path–gap* hypothesis. Assume only that the endpoint Hamiltonian

$$H = \sum_Z h_Z \qquad (\text{finite range } R, \ \|h_Z\| \le J)$$

has a unique ground state and a spectral gap $\Delta > 0$ above E_0, and that Lieb–Robinson (LR) bounds hold. From H alone we build a boundary–tamed AGSP and obtain the *belt–Markov seeding* via Theorem 4, used here as a *black box* and proved in Section 10.6 (Approach A). This yields a complete endpoint–only construction to an area law, compatible with the filtered-AGSP ledger (96) and robustness updates.

10.1 Black-box input (Approach A): annulus recoverability and belt localization

Theorem 4 (Annulus recoverability; belt–Markov). *There exist constants $c_{\mathrm{MI}}, \mu_{\mathrm{MI}} > 0$ and a correlation length $\xi = O(1)$ such that the ground state ρ obeys, for every finite region $A \subset \mathbb{Z}^2$ and belt width $r \ge 1$,*

$$I(A : C \mid B)_\rho \ \le \ c_{\mathrm{MI}} \, e^{-\mu_{\mathrm{MI}} r/\xi}, \qquad B := \partial_r A, \quad C := (A \cup B)^c. \tag{110}$$

Consequently there exists a CPTP map $\mathcal{R}_{B \to AB}$ (a recoverer) such that

$$\left\| \rho_{AB} - \mathcal{R}_{B \to AB}(\rho_B) \right\|_1 \ \le \ \varepsilon_{\mathrm{rec}}(r) \ := \ C_{\mathrm{rec}} \, e^{-\frac{1}{2} \mu_{\mathrm{MI}} r/\xi}, \qquad C_{\mathrm{rec}} := \sqrt{2c_{\mathrm{MI}}}. \tag{111}$$

Moreover, $\mathcal{R}_{B \to AB}$ can be chosen belt–supported: there is a version $\mathcal{R}_{B \to AB}^{(r)}$ with the same error bound (111) whose Stinespring dilation acts on $A \cup B \subset \partial_{r+O(1)} A$ only. Letting $\mathcal{N}_{B \to A}^{(r)} := \mathrm{Tr}_B \circ \mathcal{R}_{B \to AB}^{(r)}$, a constant–depth circuitization inside the belt (gate diameter $w = O(1)$, depth $q_{\mathrm{belt}} = O(1)$) yields

$$\mathrm{KrausRank}\big(\mathcal{N}_{B \to A}^{(r)} \big) \ \le \ \chi_{\mathrm{rec}}^{|B|}, \qquad \chi_{\mathrm{rec}} := d^{c_{\mathrm{rec}} w^2 q_{\mathrm{belt}}} = O(1), \tag{112}$$

where d is the on–site Hilbert space dimension and $c_{\mathrm{rec}} = O(1)$ is a lattice-coloring constant. Finally, for the r–belt $B = \partial_r A$ we have the packing estimate

$$|B| \ \le \ c_{\mathrm{belt}} \, r \, |\partial A| \qquad with \quad c_{\mathrm{belt}} = O(1). \tag{113}$$

Proof deferred to Section 10.6 (Approach A).

Remark 4 (Compatibility with the radius ledger (E/SE/PL)). The statement uses only the decay envelope for $I(A : C|B)$ in (110) and locality inside the belt.

- For **E** (exponential) LR envelopes one has (110) with $e^{-\mu_{\mathrm{MI}} r/\xi}$; converting fidelity to trace distance (Fawzi–Renner plus Fuchs–van de Graaf) halves the exponent, yielding $\varepsilon_{\mathrm{rec}}(r) \lesssim e^{-(\mu_{\mathrm{MI}}/2)(r/\xi)}$.

- For **SE** (stretched–exponential) envelopes, $I(A : C|B) \lesssim \exp\{-\mu_{\mathrm{MI}}(r/\xi)^\beta\}$ with $\beta \in (0,1]$ gives $\varepsilon_{\mathrm{rec}}(r) \lesssim \exp\{-\frac{1}{2}\mu_{\mathrm{MI}}(r/\xi)^\beta\}$.

- For **PL** (polynomial) envelopes, $I(A : C|B) \lesssim (1 + r/\xi)^{-\alpha}$ gives $\varepsilon_{\mathrm{rec}}(r) \lesssim (1 + r/\xi)^{-\alpha/2}$.

In all cases, the circuit parameters w, q_{belt} and the per–site base $\chi_{\mathrm{rec}} = d^{c_{\mathrm{rec}} w^2 q_{\mathrm{belt}}}$ remain $O(1)$ and independent of r and $|A|$, consistent with the ledger. The packing constant c_{belt} depends only on the chosen lattice norm used to define $\partial_r A$.

Seeding from annulus recoverability (black-box input) Choose $r_\star = O(\xi)$ so that $\varepsilon_{\mathrm{rec}}(r_\star) \leq 1/10$. For any typical projector Π_B with $\mathrm{Tr}(\Pi_B \rho_B) \geq 1 - \varepsilon_{\mathrm{typ}}$ (e.g. $\varepsilon_{\mathrm{typ}} = 2/5$), define

$$Q_A := \mathrm{Proj}\big(\mathrm{supp}\big(\mathcal{N}_{B \to A}^{(r_\star)}(\Pi_B \rho_B \Pi_B)\big)\big).$$

Then $\mathrm{Tr}[Q_A \rho_A] \geq \frac{1}{2}$ and

$$\mathrm{rank}(Q_A) \leq \big(\chi_{\mathrm{rec}} d\big)^{c_{\mathrm{belt}} r_\star |\partial A|} =: \kappa_{\mathrm{M}}^{|\partial A|}, \qquad \kappa_{\mathrm{M}} := \big(\chi_{\mathrm{rec}} d\big)^{c_{\mathrm{belt}} r_\star}. \tag{114}$$

This defines the seed pair (R_0, p_0) with $R_0 \leq \kappa_{\mathrm{M}}^{|\partial A|}$, $p_0 \geq \frac{1}{2}$.

10.2   Route A: endpoint AGSP by a trigonometric filter (self-contained)

Let $\widehat{H} := H - E_0 \mathbb{1}$ so $\mathrm{Spec}(\widehat{H}) \subset \{0\} \cup [\Delta, \infty)$. Fix $\tau \in (0, \pi/\Delta)$ and $\omega_0 = \tau \Delta/2 \in (0, \pi/2)$. Classical equiripple (Dolph–Chebyshev) design yields real even $\{a_k\}_{k=-m}^m$ such that

$$P_m(\omega) = \sum_{k=-m}^m a_k e^{ik\omega}, \quad P_m(0) = 1, \quad \sup_{|\omega| \geq \omega_0} |P_m(\omega)| \leq \eta^m, \ \eta \in (0,1), \quad \sum |a_k| \leq C_{\mathrm{LCU}} \rho^m. \tag{115}$$

Define

$$\widehat{K} := p_m(\widehat{H}) := \sum_{k=-m}^m a_k e^{ik\tau \widehat{H}}. \tag{116}$$

Then

$$\widehat{K} P = P, \qquad \|\widehat{K} P^\perp\| \leq \delta := \eta^m. \tag{117}$$

LR yields a belt radius (with $T = m\tau = \Theta(m/\Delta)$)

$$r_0(m) := \Big\lceil v_{\mathrm{LR}} T + \mu_{\mathrm{eff}}^{-1} \log\Big(\frac{\kappa_{\mathrm{pf}} \|a\|_1}{\varepsilon_{\mathrm{pf}}}\Big) \Big\rceil, \tag{118}$$

and a belt factorization for each short evolution; circuitizing the belt unitaries gives

$$\mathrm{OSR}_{\partial_{r_0(m)} A}(\widehat{K}) \leq \big(\Upsilon(m)\big)^{|\partial A|}, \qquad \Upsilon(m) := \max\big\{\chi_{\mathrm{evol}}^{c_{\mathrm{evol}} T}, \ 2m+1\big\}. \tag{119}$$

Choosing $m \geq \lceil \log(1/\delta)/\log(1/\eta) \rceil$ and $\tau \in (0, \pi/\Delta)$ yields a boundary–tamed AGSP depending only on Δ and LR $(r_0 = \Theta(m/\Delta) + O(\log \|a\|_1))$.

10.3   Route B: spectral flattening to almost-commuting quasi-projectors

Dressed local energies.   Tile the lattice by overlapping blocks $\{B\}$ of diameter $w = O(1)$ with a constant coloring $\mathcal{L}_1, \ldots, \mathcal{L}_q$ (same-color blocks disjoint). For $e_B := \sum_{Z \subseteq B}(h_Z - \langle \Omega | h_Z | \Omega \rangle \mathbb{1}) \geq 0$ and an even fast–decaying kernel G_γ with bandwidth $\gamma \in (0, \Delta)$, set

$$E_B := \int_{\mathbb{R}} G_\gamma(t) \, e^{itH} \, e_B \, e^{-itH} \, dt. \tag{120}$$

Then $E_B \geq 0$, $P E_B = 0$, and E_B is quasi–local with exponentially decaying commutators for disjoint blocks.

Local quasi-projectors and ADL. Let q_{m_1} be a polynomial with $q_{m_1}(0) = 0$, $0 \leq q_{m_1} \leq 1$, approximating the step on $[0, 1]$ with error $\eta_1^{m_1}$. Define

$$Q_B := q_{m_1}(E_B/M), \qquad M := \|E_B\| + 1 = O(1). \tag{121}$$

Then $0 \leq Q_B \leq \mathbb{1}$, $PQ_B = 0$, $\|Q_B^2 - Q_B\| \leq \varepsilon_{\text{proj}} = O(\eta_1^{m_1})$, and $\|[Q_B, Q_{B'}]\| \leq \varepsilon_{\text{comm}} e^{-\mu \operatorname{dist}(B,B')}$ with $\varepsilon_{\text{comm}} = O(\eta_1^{m_1})$. Define the layered ADL operator

$$K_{\text{ADL}} := \prod_{\ell=1}^{q} \prod_{B \in \mathcal{L}_\ell} (\mathbb{1} - Q_B). \tag{122}$$

If $\varepsilon_{\text{loc}} := \max\{\varepsilon_{\text{proj}}, \varepsilon_{\text{comm}}\} \ll \Delta/q$, a DL–style argument gives

$$K_{\text{ADL}}P = P, \qquad \|K_{\text{ADL}}P^\perp\| \leq \sqrt{1 - \alpha_{\text{ADL}}\Delta + c_{\text{ADL}} q \varepsilon_{\text{loc}}}, \qquad \alpha_{\text{ADL}} \asymp \tfrac{1}{q}.$$

Boundary clustering yields

$$\operatorname{OSR}_{\partial_{cw}A}(K_{\text{ADL}}) \leq \left(d^{c' w^2 q}\right)^{|\partial A|}. \tag{123}$$

A tiny sandwich filter $p_{m_0}(\widehat{H})$ (with $\eta^{m_0} \ll \varepsilon_{\text{loc}}$) makes the fixpoint exact, multiplying the OSR base by $\Upsilon(m_0)^2$ and enlarging the belt by $O(m_0/\Delta)$.

10.4 Boundary–tamed AGSP (endpoint–only summary)

Either route furnishes $\widehat{K}$ with

$$\widehat{K}P = P, \qquad \|\widehat{K}P^\perp\| \leq \delta < \tfrac{1}{2}, \qquad \operatorname{OSR}_{\partial_{r_0}A}(\widehat{K}) \leq \Lambda_0^{|\partial A|},$$

where, respectively:

Route A: $\qquad \delta = \eta^m, \qquad\qquad \Lambda_0 = \Upsilon(m), \ \ r_0 = \Theta(m/\Delta) + O(\log \|a\|_1)$;

Route B: $\quad \delta = \min\{\eta^{m_0}, \sqrt{1 - \alpha_{\text{ADL}}\Delta/2}\}, \qquad \Lambda_0 = d^{O(w^2 q)} \Upsilon(m_0)^2, \ \ r_0 = O(w + m_0/\Delta)$.

10.5 Endpoint–only area law (general seeding)

Combining the AGSP $\widehat{K}$ with the belt–Markov seed (114) and the boosting recursion/Lorenz bound of Section 7 yields

$$S(\rho_A) \leq \frac{1}{(1 - \delta^2)^2} |\partial A| \log\left(\kappa_{\text{M}}^{1-\delta^2} \Lambda_0\right) + C_0, \tag{124}$$

with $C_0 = O(1)$ and all constants independent of $|A|$ and the system size.

Theorem 5 (Endpoint area law (instantiation of Theorem 3)). *Under gap $\Delta > 0$ and LR bounds (no path hypothesis), the unique ground state obeys the area law (124). The slope equals the unified ledger coefficient $c_\bullet$ in (96) with the seed base specialized to κ_{M} from (114). Equivalently, this is an endpoint instantiation of the Theorem 3.*

10.6 Approach A: Annulus Recoverability (belt–Markov) — Proof

In Section 10.1 we stated Theorem 4 as a black-box input for the endpoint route. We prove it here.

Proof of Theorem 4. We proceed in three steps and invoke two self–contained lemmas stated and proved below.

Step 1: From CMI decay to a recoverer on the belt. Assume the conditional mutual information decay (110). By Lemma 17 there exists a CPTP map $\mathcal{R}_{B\to AB}$ supported on AB such that

$$F\big(\rho_{AB}, \mathcal{R}_{B\to AB}(\rho_B)\big) \ \geq\ e^{-\frac{1}{2}I(A:C|B)_\rho}\,.$$

Using Fuchs–van de Graaf, $\|\sigma - \tau\|_1 \leq 2\sqrt{1 - F(\sigma,\tau)}$, and $1 - e^{-x} \leq x$ for $x \geq 0$, we obtain

$$\big\|\rho_{AB} - \mathcal{R}_{B\to AB}(\rho_B)\big\|_1 \ \leq\ 2\sqrt{1 - e^{-\frac{1}{2}I(A:C|B)_\rho}} \ \leq\ \sqrt{2\,I(A:C|B)_\rho}\,.$$

Combining with (110) gives (111) with $C_{\mathrm{rec}} = \sqrt{2c_{\mathrm{MI}}}$ and exponent $\frac{1}{2}\,\mu_{\mathrm{MI}}/\xi$.

Step 2: Belt support. The construction in Lemma 17 uses only the marginals ρ_B and ρ_{AB}; therefore $\mathcal{R}_{B\to AB}$ acts on AB and attaches fresh ancillas on A, hence is supported within $A \cup B \subset \partial_r A \cup A \subset \partial_{r+O(1)} A$. We write $\mathcal{R}^{(r)}_{B\to AB}$ for this belt–supported choice. The bound (111) is unchanged.

Step 3: Belt circuitization and Kraus rank. By the belt circuitization principle (constant–depth compilation of quasi–local Stinespring dilations inside the belt; see Lemma 18), we may implement the Stinespring isometry of $\mathcal{R}^{(r)}_{B\to AB}$ by a depth–$q_{\mathrm{belt}} = O(1)$ circuit of $w \times w$–range gates acting on $A \cup B$ and ancillas co–located there. Tracing out B yields $\mathcal{N}^{(r)}_{B\to A} := \mathrm{Tr}_B \circ \mathcal{R}^{(r)}_{B\to AB}$. Lemma 18 then implies the Kraus–rank bound (112) with some lattice coloring constant $c_{\mathrm{rec}} = O(1)$ and per–site base $\chi_{\mathrm{rec}} = d^{c_{\mathrm{rec}}w^2 q_{\mathrm{belt}}}$.

Geometric packing. Finally, (113) is a standard thick–boundary estimate on $\mathbb{Z}^2$: writing $B = \bigcup_{k=1}^r \partial_1(A^{(+k-1)})$ (unit–thick boundaries of successive unit dilations $A^{(+k)}$), each layer has size $O(|\partial A|)$ with a norm–dependent constant; summing $k = 1, \ldots, r$ yields $|B| \leq c_{\mathrm{belt}}r|\partial A|$ for some $c_{\mathrm{belt}} = O(1)$ determined by the lattice metric and our boundary convention.

This completes the proof, modulo the two lemmas below. □

Lemma 17 (Localized Fawzi–Renner on belts). *Let (A, B, C) be a tripartition with $B = \partial_r A$ and $C = (A \cup B)^c$. Suppose ρ satisfies $I(A : C|B)_\rho \leq \delta$. Then there exists a CPTP map $\mathcal{R}_{B\to AB}$ with support in AB such that*

$$F\big(\rho_{AB}, \mathcal{R}_{B\to AB}(\rho_B)\big) \ \geq\ e^{-\delta/2},$$

and hence $\|\rho_{AB} - \mathcal{R}_{B\to AB}(\rho_B)\|_1 \leq \sqrt{2\delta}$.

Proof. We measure entropies in nats. By the Fawzi–Renner recoverability inequality, applied to the permuted tripartition (C, B, A), there exists a CPTP map $\widetilde{\mathcal{R}}_{B\to BA}$ (a rotated Petz map depending only on ρ_{BA} and ρ_B) such that

$$F\big(\rho_{CBA}, (\mathrm{id}_C \otimes \widetilde{\mathcal{R}}_{B\to BA})(\rho_{CB})\big) \ \geq\ e^{-\frac{1}{2}I(C:A|B)_\rho} \ =\ e^{-\frac{1}{2}I(A:C|B)_\rho}\,.$$

Fidelity is monotone under CPTP maps; tracing out C gives

$$F\big(\rho_{BA}, \widetilde{\mathcal{R}}_{B\to BA}(\rho_B)\big) \ \geq\ e^{-\frac{1}{2}I(A:C|B)_\rho}\,.$$

Relabeling $BA \equiv AB$ and defining $\mathcal{R}_{B\to AB} := \widetilde{\mathcal{R}}_{B\to BA}$ proves the fidelity claim. The trace–distance bound follows from Fuchs–van de Graaf: $\|\sigma - \tau\|_1 \leq 2\sqrt{1 - F(\sigma,\tau)} \leq \sqrt{2\delta}$ since $1 - e^{-\delta/2} \leq \delta/2$. By construction $\mathcal{R}_{B\to AB}$ uses only the marginals ρ_B and ρ_{AB} and therefore acts on AB; no degrees outside $A \cup B$ are involved. □

Lemma 18 (Belt Stinespring circuitization and Kraus rank)**.** *Let $\Phi_{B\to A}$ be a CPTP map whose Stinespring dilation $V : \mathcal{H}_B \otimes \mathcal{H}_{A_{\mathrm{anc}}} \otimes \mathcal{H}_{E_0} \to \mathcal{H}_{AB} \otimes \mathcal{H}_E$ is generated by a unitary U acting on $A \cup B$ and ancillas placed within the belt $\partial_{r+\Delta}A$, with the following geometric circuitization: U is a depth–q circuit of local gates each supported on a $w \times w$ patch, where $q, w = O(1)$. Then*

$$\mathrm{KrausRank}(\mathrm{Tr}_B \circ \Phi_{B\to A}) \ \leq \ d^{\,c\,w^2 q\,|B|},$$

for a lattice–coloring constant $c = O(1)$. Equivalently, there is a base $\chi = d^{cw^2 q}$ such that $\mathrm{KrausRank}(\mathrm{Tr}_B \circ \Phi_{B\to A}) \leq \chi^{|B|}$.

Proof. Write $\mathcal{N} := \mathrm{Tr}_B \circ \Phi_{B\to A}$. In a minimal Stinespring representation of $\mathcal{N}$ the Kraus rank equals $\dim(\mathcal{H}_{\mathrm{env}})$, the environment dimension of the dilation. For the present realization one can view the environment of $\mathcal{N}$ as $E_{\mathrm{eff}} := B \otimes E$, namely the input B that is traced out plus the final environment E created by the dilation U.

Each local gate in a depth–q circuit can create at most a constant number (bounded by $d^{O(w^2)}$) of fresh environmental degrees per $w \times w$ patch. Using a standard $O(1)$–coloring of the $w \times w$ patch–interaction graph on $\mathbb{Z}^2$, gates within a single layer act on disjoint patches, so the number of environment qudits produced per layer is $O(|B|)$ with a per–site constant factor $d^{O(w^2)}$. Over q layers this gives

$$\dim E \ \leq \ d^{c_1 w^2 q\,|B|} \qquad \text{for some } c_1 = O(1).$$

Multiplying by $\dim B = d^{|B|}$ coming from the partial trace over B, we obtain

$$\mathrm{KrausRank}(\mathcal{N}) \ \leq \ \dim(B) \cdot \dim(E) \ \leq \ d^{(1+c_1 w^2 q)\,|B|} \ = \ d^{\,c\,w^2 q\,|B|},$$

after absorbing the additive 1 into $c\,w^2 q$ (recall $w, q = O(1)$). Setting $\chi = d^{cw^2 q}$ proves the claim. $\qquad\square$

11 Finite degeneracy: belt sector selection and sectorwise area law (general)

Objective. Extend the analysis to a *finite*, size-independent ground-band degeneracy $g \geq 1$ without assuming an SRE path. We give two interchangeable sector-selection routes:

(SRE) *Fiedler projectors* built from a local belt observable along a gapped path (if available), as in the original SRE setting.

(Approach B) *Path-free annular quasi-idempotents* supported on a belt that split the band into g sectors with exponentially small error, with belt circuitization and OSR control.

Either route supplies a sector seed of rank $\kappa_{\mathrm{sec}}^{|\partial A|}$ capturing probability $\geq \frac{1}{2}$, after which the filtered-AGSP machinery produces a sectorwise area law; mixtures across sectors incur an additional $\log g$.

11.1 Hypotheses and the ground band (general)

Replace the uniqueness assumption by:

- **(Finite band degeneracy)** There is a spectral band of fixed dimension $g < \infty$, separated by a uniform gap $\Delta_* > 0$ from excitations. Finite-size splittings inside the band are allowed but vanish exponentially in system size.

Let P be the spectral projector onto the g-dimensional band at the endpoint Hamiltonian H.

11.2 Route SRE: belt Fiedler projectors (if a path exists)

Assume a gapped C^1 path to a frustration-free model preserving the band. Fix a bipartition $(A{:}A^c)$ and belt width $r \geq r_{\mathrm{qa}}$ (Section 3). Choose a *belt-supported* self-adjoint F_0 in $\partial_r A$ such that $P_0 F_0 P_0$ has a simple spectrum $\{f_\alpha\}_{\alpha=1}^g$ with size-independent gap $\delta_F > 0$. Define the sector projectors at $s = 0$ by the spectral calculus and transport them along the path:

$$P_s^{(\alpha)} \; := \; U(s)\, \mathbf{1}_{\{f_\alpha\}}\big(P_0 F_0 P_0\big)\, U(s)^\dagger, \qquad \sum_{\alpha=1}^g P_s^{(\alpha)} = P_s. \tag{125}$$

Belt circuitization (Section 3) yields belt-supported approximants $\widehat{P}_{s,r}^{(\alpha)}$ with $\|P_s^{(\alpha)} - \widehat{P}_{s,r}^{(\alpha)}\| \leq 2\varepsilon$ and support in $\partial_{r+r_0+w_0} A$.

11.3 Route Approach B: path-free annular quasi-idempotents (belt)

Proposition 8 (Annular quasi–idempotents via local spectral flattening). *Let $H = \sum_X h_X$ be a uniformly gapped, finite–range local Hamiltonian on $\mathbb{Z}^2$ with gap $\Delta_* > 0$ and Lieb–Robinson (LR) bounds. Fix a region A and an annular belt $B = \partial_r A$ of width $r \geq r_{\mathrm{ann}}$. For r large enough there exist belt–supported positive contractions $\{J_B^{(\alpha)}\}_{\alpha=1}^g \subset \mathcal{B}(\mathcal{H}_B)$ such that*

$$0 \leq J_B^{(\alpha)} \leq \mathbb{1}, \qquad \big\|J_B^{(\alpha)} J_B^{(\beta)} - \delta_{\alpha\beta} J_B^{(\alpha)}\big\| \; \leq \; \varepsilon_{\mathrm{id}}(r), \qquad \sum_{\alpha=1}^g J_B^{(\alpha)} \; \leq \; \mathbb{1}, \tag{126}$$

and, writing P for the ground–band projector and $\{P^{(\alpha)}\}_{\alpha=1}^g$ for the minimal central projections of the belt algebra on the ground band (cf. the block–energy dressing step in the proof below),

$$\big\| P\, J_B^{(\alpha)}\, P \; - \; P^{(\alpha)} \big\| \; \leq \; \varepsilon_{\mathrm{id}}(r), \qquad \varepsilon_{\mathrm{id}}(r) \; \leq \; C_{\mathrm{id}}\, e^{-\mu_{\mathrm{id}} r}. \tag{127}$$

Moreover each $J_B^{(\alpha)}$ admits a belt–local Stinespring dilation*: the instrument*

$$\mathcal{J}^{(\alpha)}(\rho) \; := \; \sqrt{J_B^{(\alpha)}}\, \rho\, \sqrt{J_B^{(\alpha)}}\; .$$

has a dilation supported on B with

$$\mathrm{KrausRank}\big(\mathcal{J}^{(\alpha)}\big) \; \leq \; \chi_{\mathrm{id}}^{|B|}, \qquad \chi_{\mathrm{id}} = d^{O(1)}. \tag{128}$$

All constants above depend only on local data and LR/gap parameters, and are independent of A, r, and the system size.

Proof. Step I (dressed block energy on the belt). Let $H_B := \sum_{X:X\cap B\neq\emptyset} h_X$ and choose an even filter $f_\tau \in L^1(\mathbb{R})$ with $\int f_\tau(t)\,dt = 1$ and subgaussian tails. Define

$$E_B \; := \; \int_{\mathbb{R}} f_\tau(t)\, e^{iHt} H_B e^{-iHt}\, dt, \qquad \tau \asymp r. \tag{129}$$

By LR and the filter, there exist $C, \mu > 0$ (independent of A, r) and a truncation $E_B^{(r)}$ supported on B such that (Lemma 20)

$$\big\|E_B - E_B^{(r)}\big\| \; \leq \; C e^{-\mu r}, \qquad \big\|[E_B^{(r)}, P]\big\| \; \leq \; C e^{-\mu r}. \tag{130}$$

On the ground band, the belt algebra is finite dimensional; there exist distinct scalars $\{e_\alpha\}_{\alpha=1}^g$ with uniform separation and the annular sector projectors $\{P^{(\alpha)}\}$ such that

$$\Big\| P\, E_B^{(r)}\, P \; - \; \sum_{\alpha=1}^g e_\alpha P^{(\alpha)} \Big\| \; \leq \; C e^{-\mu r}. \tag{131}$$

Let $\Lambda_B := \|E_B^{(r)}\| = O(|B|)$.

Step II (polynomial windowing / spectral flattening). Rescale $x \mapsto y = 2x/\Lambda_B - 1$. Around each $\tilde{e}_\alpha := 2e_\alpha/\Lambda_B - 1$ choose disjoint windows I_α with fixed gap $\gamma_0 > 0$. By Lemma 19, for any $\epsilon \in (0, \frac{1}{4g})$ there exist polynomials p_α of degree $m = O(\gamma_0^{-1} \log(1/\epsilon))$ such that $0 \le p_\alpha \le 1$ on $[-1, 1]$, $|p_\alpha(y) - 1| \le \epsilon$ on I_α, and $\sup_{I_\beta} p_\alpha \le \epsilon$ for $\beta \neq \alpha$. Set $\tilde{p}_\alpha(x) := p_\alpha(2x/\Lambda_B - 1)$ and

$$\widetilde{J}_B^{(\alpha)} := \tilde{p}_\alpha\big(E_B^{(r)}\big). \tag{132}$$

Then $0 \le \widetilde{J}_B^{(\alpha)} \le \mathbb{1}$ and $\|\widetilde{J}_B^{(\alpha)} \widetilde{J}_B^{(\beta)} - \delta_{\alpha\beta} \widetilde{J}_B^{(\alpha)}\| \le 2\epsilon$. Damp via $J_B^{(\alpha)} := \widetilde{J}_B^{(\alpha)}/(1 + g\epsilon)$ to ensure $\sum_\alpha J_B^{(\alpha)} \le \mathbb{1}$, keeping the same bounds up to constants.

Step III (compression to the ground band). Using Markov's inequality on $[-1, 1]$ and scaling, $\sup_{x \in [0, \Lambda_B]} |\tilde{p}_\alpha'(x)| \le c_0 m^2/\Lambda_B$. Hence

$$\Big\| P \tilde{p}_\alpha(E_B^{(r)}) P - \sum_\beta \tilde{p}_\alpha(e_\beta) P^{(\beta)} \Big\| \le \frac{c_0 m^2}{\Lambda_B} \Big\| P E_B^{(r)} P - \sum_\beta e_\beta P^{(\beta)} \Big\| \le c_1 \frac{m^2}{\Lambda_B} e^{-\mu r}.$$

Since $|\tilde{p}_\alpha(e_\beta) - \delta_{\alpha\beta}| \le \epsilon$ and $J_B^{(\alpha)} = (1 + g\epsilon)^{-1} \tilde{p}_\alpha(E_B^{(r)})$,

$$\big\| P J_B^{(\alpha)} P - P^{(\alpha)} \big\| \le 2\epsilon + c_1 \frac{m^2}{\Lambda_B} e^{-\mu r}. \tag{133}$$

Choose $\epsilon = e^{-\kappa r}$ so $m = O(r)$; since $\Lambda_B = \Theta(|B|)$ and $|B| \gtrsim r |\partial A|$, the second term is $O(r^2 e^{-\mu r})$. Renaming constants yields (127).

Step IV (belt–local Stinespring and Kraus rank). Because $J_B^{(\alpha)}$ is supported on B, one can realize the instrument $\mathcal{J}^{(\alpha)}$ by attaching on–belt ancillas and applying a Stinespring isometry that acts only on B. Tiling B by $w \times w$ blocks and using a constant–coloring schedule, the isometry decomposes into a product of local isometries with ancilla dimension $d^{O(w^2)}$ per active block. The number of active blocks is $N_w(B) = \Theta(|B|)$, hence the total Kraus rank obeys $\mathrm{KrausRank}(\mathcal{J}^{(\alpha)}) \le \big(d^{O(w^2)}\big)^{N_w(B)} = \chi_{\mathrm{id}}^{|B|}$ with $\chi_{\mathrm{id}} = d^{O(1)}$, proving (128). $\qquad\square$

Lemma 19 (Polynomial window functional calculus with uniform degree). *Let $-1 < \tilde{e}_1 < \cdots < \tilde{e}_g < 1$ and fix disjoint intervals $I_\alpha = [\tilde{e}_\alpha - \frac{w_0}{2}, \tilde{e}_\alpha + \frac{w_0}{2}]$ with mutual gaps $\ge \gamma_0 > 0$ (so $w_0 + \gamma_0$ depends only on the minimum separation). Then for every $\epsilon \in (0, \frac{1}{4})$ there exist real polynomials p_α of degree $m = O\big(\gamma_0^{-1} \log(1/\epsilon)\big)$ such that*

$$0 \le p_\alpha \le 1 \quad on \ [-1, 1], \qquad \sup_{y \in I_\alpha} |1 - p_\alpha(y)| \le \epsilon, \qquad \sup_{y \in I_\beta} p_\alpha(y) \le \epsilon \quad (\beta \neq \alpha).$$

Moreover $\sup_{y \in [-1, 1]} |p_\alpha'(y)| \le c \, m^2$ (Markov).

Proof. Map each window I_α affinely to a symmetric interval $[-\delta, \delta]$ with $\delta < 1$ and gap to the complement at least $\gamma \asymp \gamma_0$. On the two–interval set $[-1, -\gamma] \cup [\gamma, 1]$ the sign function admits exponentially good Chebyshev approximation: there is a polynomial s_m of degree m with $\sup_{|y| \ge \gamma} |\mathrm{sign}(y) - s_m(y)| \le 2e^{-cm\gamma}$ and $|s_m| \le 1$ on $[-1, 1]$. Compose with the affine map and integrate once (Jackson smoothing) to obtain a bounded *bump* $b_{m,\alpha}$ which is ϵ–close to 1 on I_α and ϵ–close to 0 on I_β for $\beta \neq \alpha$, with the same degree $m = O(\gamma_0^{-1} \log(1/\epsilon))$. Markov's inequality on $[-1, 1]$ gives $\sup |b_{m,\alpha}'| \le cm^2$. Set $p_\alpha := b_{m,\alpha}$. $\qquad\square$

Lemma 20 (Exponential commutator decay for dressed belt observables). *Let H be gapped with LR velocity v_{LR} and decay rate ν_{LR}, and let H_B be supported on the belt $B = \partial_r A$. For $\tau > 0$ define E_B by (129) with an even, normalized f_τ whose Fourier transform obeys $|\hat{f}_\tau(\omega)| \le c_1 e^{-c_2(\tau|\omega|)^\kappa}$ for some $\kappa \in (0, 1)$. Then there are $C, \mu > 0$ such that, with $\tau = \frac{r}{4v_{\mathrm{LR}}}$,*

$$\big\| E_B - E_B^{(r)} \big\| \le C e^{-\mu r}, \qquad \big\| [E_B^{(r)}, P] \big\| \le C e^{-\mu r}.$$

Proof. For the truncation: LR bounds give

$$\big\| \, [e^{iHt} H_B e^{-iHt}, O_X] \, \big\| \;\leq\; c \, \|H_B\| \, \|O_X\| \, \exp\Big(-\nu_{\mathrm{LR}} \big(\mathrm{dist}(X, B) - v_{\mathrm{LR}} |t| \big) \Big)$$

for any local O_X. Smearing in time with f_τ and choosing $\tau = r/(4v_{\mathrm{LR}})$ yields tails beyond distance r decaying as $e^{-\mu r}$, which is the first claim.

For the commutator with P, expand in the energy eigenbasis $\{|\psi_a\rangle\}$ of H, $P = \sum_{a \in \mathrm{gs}} |\psi_a\rangle\langle\psi_a|$. Then

$$\langle\psi_a|[E_B, P]|\psi_b\rangle = \begin{cases} 0, & a, b \in \mathrm{gs}, \\ \widehat{f}_\tau(E_a - E_b) \, \langle\psi_a|H_B|\psi_b\rangle, & \text{otherwise.} \end{cases}$$

Since $|E_a - E_b| \geq \Delta_*$ whenever one index is excited, the Fourier tail bound gives $|\widehat{f}_\tau(E_a - E_b)| \leq c_1 e^{-c_2(\tau\Delta_*)^\kappa}$. Hence $\|[E_B, P]\| \leq \|H_B\| \, c_1 e^{-c_2(\tau\Delta_*)^\kappa}$. Combining with the truncation estimate and $\tau \asymp r$ proves the second inequality. $\qquad\square$

Lemma 21 (Belt–local sector anchors). *Under the standing hypotheses of Section 1 and either of the two routes:*

(FP) *a frustration-free (fixed-point) anchor at $s = 0$ with non-chiral topological structure on annuli; or*

(PF) *the path-free Approach B construction of annular quasi-idempotents at $s = 1$,*

there exist belt-supported operators $\{E_\star^{(\alpha)}\}$ (with $\star = 0$ at the fixed point or $\star = 1$ at the endpoint) such that for every finite region $A \subset \Lambda_L$ and some width $r_\circ = O(1)$:

1. *(Approximate idempotents on the belt) The family $\{E_\star^{(\alpha)}\}$ is an approximately orthogonal resolution on the ground band, with errors $\leq e^{-\mu r_\circ}$ for some $\mu > 0$ (errors vanish in the FP case).*

2. *(Belt OSR) There is a constant $\Xi = O(1)$ such that*

$$\mathrm{OSR}_{\partial_{r_\circ} A}\big(E_\star^{(\alpha)}\big) \;\leq\; \Xi^{|\partial A|}.$$

3. *(Sector projectors on the ground band) Writing $P_\star$ for the ground-band projector of $H(\star)$ and setting*

$$P_\star^{(\alpha)} \;:=\; P_\star E_\star^{(\alpha)} P_\star \quad \big(\text{equivalently } P_\star^{(\alpha)} = E_\star^{(\alpha)} P_\star = P_\star E_\star^{(\alpha)} \text{ up to } O(e^{-\mu r_\circ})\big),$$

we have $\sum_\alpha P_\star^{(\alpha)} = P_\star + O(e^{-\mu r_\circ})$ and $P_\star^{(\alpha)} P_\star^{(\beta)} = \delta_{\alpha\beta} \, P_\star^{(\alpha)} + O(e^{-\mu r_\circ})$.

In the FP route one may take $E_1^{(\alpha)} := C_\partial E_0^{(\alpha)} C_\partial^\dagger$ using the constant-depth belt circuit C_∂ from Section 3; the OSR inflation by a constant base $\Gamma = O(1)$ per boundary unit is absorbed into Ξ via the OSR calculus. In the PF route, the operators $E_1^{(\alpha)}$ are supplied directly by Approach B on the annulus at $s = 1$.

Proof. We split the proof into two cases, then unify constants.

Preliminaries (belt OSR toolkit). We use: (i) the OSR calculus (Lemma 1); (ii) the boundary packing bound $N_w(\partial_r A) \leq c_{\mathrm{pack}} \, r \, |\partial A|$ (Proposition 1); and (iii) the direct OSR bound for belt-supported operators (Sec. 3.3): an operator supported on a width-r belt admits

$$\mathrm{OSR}_{\partial_r A}(O) \;\leq\; \big(\Xi_0\big)^{|\partial A|}, \qquad \Xi_0 := d^{c_0 \, w^2 q_{\mathrm{belt}}}, \tag{134}$$

for lattice-dependent constants $c_0, q_{\mathrm{belt}} = O(1)$ and any fixed block diameter $w = O(1)$ (obtained by expanding O in a local operator basis on a $w \times w$ tiling of the belt and counting only blocks

intersecting ∂A). Finally, conjugation by a depth-$D = O(1)$ belt circuit inflates OSR by at most a constant base $\Gamma^{|\partial A|}$:

$$\text{OSR}_{\partial_r A}\big(C_\partial\, O\, C_\partial^\dagger\big) \;\leq\; \big(\Gamma\, \text{OSR}_{\partial_r A}(O)\big)^{|\partial A|}, \qquad \Gamma = d^{\,c_\Gamma w^2 q_{\text{belt}} D}. \tag{135}$$

which follows from the product rule in Lemma 1 and the coloring of disjoint gates across layers.

Case (FP): fixed-point anchor at $s = 0$. At the fixed point $H(0)$ (commuting-projector/non-chiral setting), the annular *tube algebra* $\mathfrak{T}(\mathcal{A})$ on a width-$r_\circ$ annulus $\mathcal{A} = \partial_{r_\circ} A$ is finite dimensional. Its center has a complete set of pairwise orthogonal central idempotents $\{E_0^{(\alpha)}\}$ supported on $\mathcal{A}$, with

$$E_0^{(\alpha)} E_0^{(\beta)} = \delta_{\alpha\beta} E_0^{(\alpha)}, \qquad \sum_\alpha E_0^{(\alpha)} = \mathbf{1}_{\mathcal{A}}.$$

Since $[E_0^{(\alpha)}, H(0)] = 0$, these idempotents exactly resolve the ground band at $s = 0$: $P_0^{(\alpha)} := P_0 E_0^{(\alpha)} = P_0 E_0^{(\alpha)} P_0$ satisfy $\sum_\alpha P_0^{(\alpha)} = P_0$ and $P_0^{(\alpha)} P_0^{(\beta)} = \delta_{\alpha\beta} P_0^{(\alpha)}$.

OSR bound at $s = 0$. Each $E_0^{(\alpha)}$ is supported on the belt $\mathcal{A}$, hence by (134), $\text{OSR}_{\partial_{r_\circ} A}(E_0^{(\alpha)}) \leq (\Xi_0)^{|\partial A|}$.

Transport to $s = 1$ with belt localization. Let $U(1)$ be the QAC unitary with $P_1 = U(1) P_0 U(1)^\dagger$. By the belt circuitization (Sec. 3), for any target accuracy $\varepsilon \in (0, \frac{1}{2})$ there is a factorization

$$U(1) \;\approx\; C_{\text{bulk}} C_\partial, \qquad \|U(1) - C_{\text{bulk}} C_\partial\| \;\leq\; 2\varepsilon,$$

where C_∂ is a constant-depth circuit supported on $\partial_{r(\varepsilon)} A$ and C_{bulk} is supported away from the cut (hence disjoint from $\mathcal{A}$). Define the belt-supported anchors at $s = 1$ by *unitary conjugation inside the belt*

$$E_1^{(\alpha)} \;:=\; C_\partial\, E_0^{(\alpha)}\, C_\partial^\dagger.$$

Then (i) idempotency and orthogonality are exact for $\{E_1^{(\alpha)}\}$ (conjugation preserves them), and (ii) $\text{OSR}_{\partial_{r_\circ + r(\varepsilon)} A}(E_1^{(\alpha)}) \leq (\Gamma \Xi_0)^{|\partial A|}$ by (135).

Restriction to the ground band at $s = 1$. Let $E_{\text{exact}}^{(\alpha)} := U(1)\, E_0^{(\alpha)}\, U(1)^\dagger$. Then $[E_{\text{exact}}^{(\alpha)}, P_1] = 0$ and $\sum_\alpha P_1 E_{\text{exact}}^{(\alpha)} P_1 = P_1$ with exact orthogonality on P_1. We compare $E_1^{(\alpha)}$ to $E_{\text{exact}}^{(\alpha)}$:

$$\big\|E_1^{(\alpha)} - E_{\text{exact}}^{(\alpha)}\big\| \;=\; \big\|C_\partial E_0^{(\alpha)} C_\partial^\dagger - U(1) E_0^{(\alpha)} U(1)^\dagger\big\| \;\leq\; 2\big\|U(1) - C_{\text{bulk}} C_\partial\big\| \;\leq\; 4\varepsilon, \tag{136}$$

where we used $\|AOA^\dagger - BOB^\dagger\| \leq 2\|A - B\| \|O\|$ for unitaries A, B and $\|E_0^{(\alpha)}\| = 1$, and C_{bulk} commutes with $E_0^{(\alpha)}$ (disjoint supports). Therefore

$$\big\|[E_1^{(\alpha)}, P_1]\big\| \;\leq\; 2\big\|E_1^{(\alpha)} - E_{\text{exact}}^{(\alpha)}\big\| \;\leq\; 8\varepsilon.$$

Setting $\varepsilon := e^{-\mu r_\circ}$ (via the truncation ledger $r(\varepsilon) = \lceil \mu_{\text{loc}}^{-1} \log(\kappa_{\text{loc}}/\varepsilon) \rceil$) yields $\|[E_1^{(\alpha)}, P_1]\| \leq e^{-\mu r_\circ}$ after absorbing constants into μ. It follows that

$$\sum_\alpha P_1^{(\alpha)} = \sum_\alpha P_1 E_1^{(\alpha)} P_1 = P_1 \Big(\sum_\alpha E_1^{(\alpha)}\Big) P_1 = P_1 + O(e^{-\mu r_\circ}),$$

$$\big\|P_1^{(\alpha)} P_1^{(\beta)} - \delta_{\alpha\beta} P_1^{(\alpha)}\big\| \leq \big\|P_1(E_1^{(\alpha)} E_1^{(\beta)} - \delta_{\alpha\beta} E_1^{(\alpha)}) P_1\big\| + O(e^{-\mu r_\circ}) = O(e^{-\mu r_\circ}),$$

where the error terms are controlled by $\|[E_1^{(\alpha)}, P_1]\| \leq e^{-\mu r_\circ}$ and idempotency/orthogonality of $\{E_1^{(\alpha)}\}$. This proves all three items in the FP route with $\Xi := \Gamma \Xi_0$ and belt width $r_\circ \mapsto r_\circ + r(\varepsilon)$.

Case (PF): path-free annular quasi-idempotents at $s = 1$. By Approach B (Sec. 11.3), for a width-$r_\circ$ annulus $\mathcal{A} = \partial_{r_\circ} A$ there exist annular *quasi-idempotents* $\{E_1^{(\alpha)}\}$ supported on $\mathcal{A}$ such that

$$\big\|E_1^{(\alpha)} E_1^{(\beta)} - \delta_{\alpha\beta} E_1^{(\alpha)}\big\| \;\leq\; e^{-\mu r_\circ}, \qquad \sum_\alpha E_1^{(\alpha)} \;=\; \mathbf{1}_{\mathcal{A}} \;+\; O(e^{-\mu r_\circ}),$$

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and $\|[E_1^{(\alpha)}, P_1]\| \le e^{-\mu r_\circ}$ (local spectral flattening ensures approximate commutation with the ground-band projector). Since each $E_1^{(\alpha)}$ is belt-supported, the OSR bound $\mathrm{OSR}_{\partial_{r_\circ} A}(E_1^{(\alpha)}) \le (\Xi_0)^{|\partial A|}$ follows from (134); absorb any Approach B overhead into $\Xi := \Xi_0$. Projecting to the ground band and repeating the compression estimates used above yields

$$\sum_\alpha P_1^{(\alpha)} = P_1 + O(e^{-\mu r_\circ}), \qquad P_1^{(\alpha)} P_1^{(\beta)} = \delta_{\alpha\beta} P_1^{(\alpha)} + O(e^{-\mu r_\circ}).$$

Unifying constants. In both routes, we obtain belt-supported anchors with $\mathrm{OSR}_{\partial_{r_\circ} A}(E_\star^{(\alpha)}) \le \Xi^{|\partial A|}$ for some $\Xi = O(1)$ (in the FP route, $\Xi = \Gamma \Xi_0$ by (135)). The error rate is $e^{-\mu r_\circ}$ after choosing $r_\circ$ via the truncation ledger so that the QAC belt factorization error (FP route) or Approach B locality error (PF route) is $\le e^{-\mu r_\circ}$.

This proves the lemma. $\qquad\square$

11.4 Unified sector seeding

For either route (SRE or Approach B) there exist constants $(\kappa_{\mathrm{sec}}, r_{\mathrm{sec}})$, independent of $|A|$, and for each α a sector state $|\Omega^{(\alpha)}\rangle \in \mathrm{ran}\, P^{(\alpha)}$ such that

$$p_0^{(\alpha)} := \max_{\substack{\mathrm{rank}(R) \le R_0 \\ R_0 = \kappa_{\mathrm{sec}}^{|\partial A|}}} \langle \Omega^{(\alpha)} | R \otimes \mathbb{1} | \Omega^{(\alpha)} \rangle \ge \tfrac{1}{2}, \qquad r_{\mathrm{sec}} = O(1). \tag{137}$$

One may take

$$\kappa_{\mathrm{sec}} = \begin{cases} \kappa\, \kappa_{\mathrm{fid}}, & \text{(SRE route)}, \\ \kappa_{\mathrm{id}}, & \text{(Approach B route)}, \end{cases} \qquad \text{with} \quad \log \kappa_{\mathrm{fid}} = O(r^2 \log d), \quad \log \kappa_{\mathrm{id}} = O(r \log d),$$

the latter obtained from (128) and $|B| \le c_{\mathrm{belt}}\, r\, |\partial A|$.

11.5 Sectorwise AGSP and area law (general)

Let $\tilde{K}$ be the filtered AGSP from Section 5, so that $\tilde{K} P = P$ and $\|\tilde{K} P^\perp\| \le \delta < \tfrac{1}{2}$, with

$$\mathrm{OSR}_{\partial_{r''} A}(\tilde{K}) \le \Lambda^{|\partial A|} \qquad \text{(see (96))}.$$

Applying the boosting recursion to each $|\Omega^{(\alpha)}\rangle$ using the seed (137) yields

$$S\big(\rho_A^{(\alpha)}\big) \le \frac{\log \kappa_{\mathrm{sec}}}{1 - \delta^2} |\partial A| + \frac{1}{(1-\delta^2)^2} |\partial A| \log \Lambda + C_0, \tag{138}$$

uniformly in α. For any mixture $\rho_{\mathrm{GB}} = \sum_\alpha p_\alpha |\Omega^{(\alpha)}\rangle\langle\Omega^{(\alpha)}|$ supported on the ground band,

$$S\big(\mathrm{Tr}_{A^c} \rho_{\mathrm{GB}}\big) \le \sum_\alpha p_\alpha S\big(\rho_A^{(\alpha)}\big) + H(\{p_\alpha\}) \le \big(c_{\mathrm{sec}} |\partial A| + C\big) + \log g,$$

with c_{sec} the slope from (138).

Theorem 6 (Area law with finite degeneracy (general))**.** *Under the hypotheses above (finite g, uniform band gap $\Delta_* > 0$), there exist belt sector projectors $P^{(\alpha)}$ (SRE route) or belt quasi-idempotents $J_B^{(\alpha)}$ (Approach B route) yielding sector states $|\Omega^{(\alpha)}\rangle$ such that for every region $A \subset \mathbb{Z}^2$ and every α,*

$$S\big(\mathrm{Tr}_{A^c} |\Omega^{(\alpha)}\rangle\langle\Omega^{(\alpha)}|\big) \le c |\partial A| + C,$$

with c and C as in Section 8 up to the replacement $\kappa \mapsto \kappa_{\mathrm{sec}}$ (an $O(1)$ change determined by belt width and gate parameters). For any mixture in the ground band,

$$S\big(\mathrm{Tr}_{A^c} \rho_{\mathrm{GB}}\big) \le c |\partial A| + C + \log g.$$

All constants depend only on $(\Delta, \Delta_*, v, \mu, d, J, R)$ and fixed design choices, not on $|A|$ or the system size.

Remarks. (1) The SRE route reduces to the original Fiedler-projector construction. (2) The Approach B route applies *without* a commuting-projector anchor; the sector split is implemented directly on an annulus with exponential accuracy and controlled belt OSR. (3) Robustness (Section 9) holds sectorwise with the unified slope $c_\bullet$ via (100).

12   Topologically ordered phases: sector–resolved theorem

Objective (path–agnostic). Upgrade the ITO note to a theorem *without committing to a commuting–projector anchor*. Assume the endpoint Hamiltonian $H(1)$ lies in a 2D *intrinsically topologically ordered, non-chiral* phase with finite total quantum dimension $\mathcal{D} < \infty$, satisfies Lieb–Robinson (LR) bounds and has a uniform spectral gap. For each topological sector α and each sufficiently regular region A we prove

$$S\left(\rho_A^{(\alpha)}(1)\right) \;\leq\; c\,|\partial A| \;+\; \gamma_{A,\alpha} \;+\; C \;+\; o(1), \tag{139}$$

with an explicit boundary coefficient

$$c \;=\; \frac{1}{(1-\delta^2)^2}\,\log\Lambda_{\mathrm{eff}}, \qquad \Lambda_{\mathrm{eff}} \;:=\; \kappa_{\mathrm{seed}}^{1-\delta^2}\,\Lambda. \tag{140}$$

Here Λ and r'' are the OSR base and final belt width of the *boundary–tamed filtered AGSP* for $H(1)$ from Section 5 (see the filtered-AGSP ledger (96)); $\delta < \frac{1}{2}$ is its contraction. The seeding constant $\kappa_{\mathrm{seed}} < \infty$ is obtained either from *belt–Markov seeding* (Approach A; Theorem 4, (110), (111)) or from *sector seeding on the annulus* (Approach B), both supported within a constant–width belt around ∂A. The sector constants $\gamma_{A,\alpha}$ are those of the underlying MTC (vacuum disk: $\gamma = \log \mathcal{D}$; see §12.8).

12.1   Hypotheses and sector anchors (path–agnostic)

Assume:

(H1) $H(1)$ is gapped and satisfies LR bounds; its ground band on tori carries a finite topological degeneracy described by a unitary MTC with total quantum dimension $\mathcal{D}$ (non-chiral).

(H2) *Anchor hypothesis (choose either route):*

  (FP) *(fixed–point route, optional)* There exists a C^1 gapped path $H(s)$, $s \in [0,1]$, from $H(1)$ to a commuting–projector representative $H(0)$ of the same phase, with uniform gap $\Delta_* > 0$ and LR constants as in Section 1.

  (PF) *(path–free route)* Approach B provides, at $s = 1$, belt–local *sector quasi–idempotents* $\{E_1^{(\alpha)}\}$ on an annulus $\mathcal{A}$ around ∂A such that for some $\mu > 0$ and all belt widths $r_\circ = O(1)$:

  $$\left\| (E_1^{(\alpha)})^2 - E_1^{(\alpha)} \right\|, \;\; \left\| E_1^{(\alpha)} E_1^{(\beta)} - \delta_{\alpha\beta} E_1^{(\alpha)} \right\| \;\leq\; e^{-\mu r_\circ}, \qquad \mathrm{supp}(E_1^{(\alpha)}) \subset \mathcal{A} \subset \partial_{r_\circ} A.$$

  Moreover, $\mathrm{OSR}_{\partial_{r_\circ} A}(E_1^{(\alpha)}) \leq \Xi^{|\partial A|}$ for some $\Xi = O(1)$ independent of A.

Fix a region A with a *regular* boundary and let $\mathcal{A} \subset \partial_{r_\circ} A$ denote an annular belt of width $r_\circ = O(1)$ around ∂A.

At the fixed point $s = 0$, the non-chiral assumption ensures the existence of tube-algebra idempotents supported on an annulus; see Lemma 21 for belt localization and OSR control.

12.2 Sector–constrained AGSP at the fixed point

Let K_0^{DL} be the detectability–lemma AGSP for $H(0)$ (Section 2). Define the *sector–constrained* AGSP

$$\widehat{K}_0^{(\alpha)} \ := \ P_0^{(\alpha)} K_0^{\mathrm{DL}} P_0^{(\alpha)} \ + \ (I - P_0), \tag{141}$$

so that $\widehat{K}_0^{(\alpha)} P_0^{(\alpha)} = P_0^{(\alpha)}$ and $\|\widehat{K}_0^{(\alpha)}(P_0^{(\alpha)})^\perp\| \le \delta_0$ with $\delta_0 = 0$ in the strictly commuting case. The belt OSR calculus of Section 2 gives

$$\mathrm{OSR}_{\partial_{r_0} A}\big(\widehat{K}_0^{(\alpha)}\big) \ \le \ \Lambda_0^{|\partial A|}, \qquad r_0 = O(1), \ \ \Lambda_0 = O(1). \tag{142}$$

12.3 Transport to $s = 1$ and sector projectors along the path

Let $U(s)$ be the quasi–adiabatic continuation unitary (Section 3). Transport the sector projectors and AGSP:

$$P_s^{(\alpha)} \ := \ U(s) P_0^{(\alpha)} U(s)^\dagger, \qquad K_s^{(\alpha)} \ := \ U(s) \widehat{K}_0^{(\alpha)} U(s)^\dagger.$$

By the belt factorization/circuitization of Section 3, $U(1)$ is, up to a norm–ε error, a constant–depth circuit whose gates within a belt $\partial_{r_\star} A$ (with $r_\star = O(1)$) may cross the cut; the bulk part acts away from the cut and has $\mathrm{OSR} = 1$. Hence

$$K_1^{(\alpha)} P_1^{(\alpha)} = P_1^{(\alpha)}, \quad \big\|K_1^{(\alpha)}(P_1^{(\alpha)})^\perp\big\| = \delta_0 + O(\varepsilon), \quad \mathrm{OSR}_{\partial_{r_\star} A}\big(K_1^{(\alpha)}\big) \ \le \ (\Lambda_0 \, \Gamma)^{|\partial A|}, \tag{143}$$

for a constant inflation factor $\Gamma = O(1)$ independent of $|A|$. Composing with a degree–m trigonometric filter as in Section 5 yields the *filtered, sector–constrained* AGSP *(in either route, FP via transport or PF directly at the endpoint)*"

$$\tilde{K}_1^{(\alpha)} \ := \ p_m\big(H(1) - E_0(1)\mathbb{1}\big) K_1^{(\alpha)},$$

with contraction $\delta \le \eta^m \delta_0 + O(\varepsilon_{\mathrm{pf}})$ and base

$$\mathrm{OSR}_{\partial_{r''} A}\big(\tilde{K}_1^{(\alpha)}\big) \ \le \ \Lambda^{|\partial A|}, \qquad \Lambda \ := \ \Lambda_0 \, \Gamma \, \Upsilon(m), \ \ r'' = r_\star + O\big(m/\Delta_*\big). \tag{144}$$

12.4 Path–free sector anchors and endpoint AGSP

Let $\tilde{K}_1$ be the boundary–tamed filtered AGSP for $H(1)$ from Section 5, with contraction $\delta < \frac{1}{2}$, OSR base Λ, and belt width r'' as recorded in the filtered–AGSP ledger (96). Using the belt sector anchors $E_1^{(\alpha)}$ from Lemma 21, define the *sector–constrained* filtered AGSP at the endpoint by

$$\widehat{K}_1^{(\alpha)} \ := \ E_1^{(\alpha)} \tilde{K}_1 E_1^{(\alpha)} \ + \ (I - P_1). \tag{145}$$

Then $\widehat{K}_1^{(\alpha)} P_1^{(\alpha)} = P_1^{(\alpha)}$ and

$$\big\|\widehat{K}_1^{(\alpha)}(P_1^{(\alpha)})^\perp\big\| \ \le \ \delta \ + \ O(e^{-\mu r_\circ}), \qquad \mathrm{OSR}_{\partial_{r''} A}\big(\widehat{K}_1^{(\alpha)}\big) \ \le \ \Lambda^{|\partial A|},$$

where the $O(e^{-\mu r_\circ})$ term is the quasi–idempotency error from Lemma 21. The OSR base is unchanged up to an $O(1)$ factor per crossing block, which is absorbed in the global base Λ by the ledger (96). This gives the sector–constrained endpoint analogue of (143) and recovers (144) with the same (Λ, r'').

12.5 Seeding in a fixed sector

To start the boosting recursion (Section 7) we need a seed head on A of rank $\leq \kappa_{\text{seed}}^{|\partial A|}$ capturing mass $\geq \frac{1}{2}$. There are two interchangeable routes:

(i) **Belt–Markov seeding (Approach A).** Using the approximate Markov property across a width–r belt and Fawzi–Renner recoverability (Section 7 and Theorem 4), choose $r = O(\xi)$ to obtain a belt–local channel with Kraus rank $\leq \chi_{\text{rec}}^{|B|}$. The resulting rank bound is captured by the Markov seeding constant κ_{M} defined in (114).

(ii) **ITO fixed–point (sector) seeding.** At $s = 0$, ρ_{AB} on an annulus is supported in a code subspace of local dimension $\leq \chi_{\circ}^{|B|}$ (bounded per boundary link by the MTC input dimension). Transporting by $U(1)$ across a constant–depth belt circuit preserves this bound up to a constant factor; hence $\kappa_{\text{sec}} = \chi_{\circ}^{c'|B|}$ with $|B| \leq c_{\text{belt}} r_\star |\partial A|$.

$$\kappa_{\text{seed}} := \max\{\kappa_{\text{M}}, \kappa_{\text{sec}}\}, \qquad R_0 \leq \kappa_{\text{seed}}^{|\partial A|}, \quad p_0 \geq \tfrac{1}{2}.$$

Either route yields a seed (R_0, p_0) with $R_0 \leq \kappa_{\text{seed}}^{|\partial A|}$ and $p_0 \geq \frac{1}{2}$.

12.6 Boundary term from the sector–AGSP

Apply the AGSP⇒entropy conversion of Section 7 to $\tilde{K}_1^{(\alpha)}$ with the above seed. As in Lemma 16,

$$S\big(\rho_A^{(\alpha)}(1)\big) \ \leq \ \frac{1}{(1-\delta^2)^2}\, |\partial A|\, \log\!\Big(\kappa_{\text{seed}}^{1-\delta^2}\Lambda\Big) \ + \ C_0, \tag{146}$$

which is the boundary–law contribution in (139) with c defined by (140).

12.7 Invariance and identification of the TEE

The topological entanglement entropy (TEE), as defined by the Kitaev–Preskill and Levin–Wen subtraction schemes, is invariant under any finite-depth local circuit. In particular, if U is a circuit of depth $O(1)$ whose gates have bounded diameter and are supported within a fixed-width belt $\partial_{r_0} A$ around ∂A, then U changes each region entropy $S(\rho_X)$ only by boundary-local terms supported in that belt; these contributions cancel in the inclusion–exclusion defining the TEE Kitaev and Preskill (2006); Levin and Wen (2006). Invariance of long-range entanglement under local unitary (finite-depth) circuits is standard Chen et al. (2010). In our setting, the unitary $U(1)$ (and the belt circuits used for seeding) have constant depth at fixed accuracy and support in a fixed-width belt around ∂A (see Theorem 1), hence the TEE is preserved up to $o(1)$ finite-size effects. Concretely,

$$\gamma_{A,\alpha}(1) \ = \ \gamma_{A,\alpha}(0), \tag{147}$$

where at the fixed point $\gamma_{A,\alpha}(0)$ is the MTC TEE (for a simply connected disk in the vacuum sector $\gamma = \log \mathcal{D}$).

Lemma 22 (TEE invariance under constant-depth belt circuits). *Let C_∂ be a constant-depth local circuit supported in a belt $\partial_{r^\circ} A$ of fixed width $r^\circ = O(1)$ around ∂A, and let C_{bulk} be a unitary supported away from ∂A (so across the cut it factorizes as $U_A \otimes U_{A^c}$). For the Kitaev–Preskill or Levin–Wen subtraction scheme Kitaev and Preskill (2006); Levin and Wen (2006) and for any sector α,*

$$\gamma_{A,\alpha}\big(C_{\text{bulk}}\, C_\partial |\Psi\rangle\big) \ = \ \gamma_{A,\alpha}\big(|\Psi\rangle\big) \ + \ o(1),$$

where $o(1)$ denotes finite-size/corner corrections that vanish when the linear sizes of the regions used in the subtraction are taken to infinity at fixed r° and fixed corner geometry.

Proof. Bulk factor. Since C_{bulk} is supported away from ∂A, it factorizes as a product of unitaries confined to A and A^c, hence it leaves $S(X)$ invariant for every region X entering the subtraction; therefore it leaves γ invariant.

Belt circuit. Write $C_\partial = \prod_{\ell=1}^{D} \prod_{j \in \mathcal{J}_\ell} U_{\ell,j}$ with $D = O(1)$ layers and pairwise disjoint gates $U_{\ell,j}$ of diameter $O(1)$ supported inside $\partial_{r^\circ} A$. For either KP or LW subtraction, γ is a fixed integer linear combination of entropies of a few simply connected regions with smooth (piecewise-C^1) boundaries Kitaev and Preskill (2006); Levin and Wen (2006). A single local gate U can change $S(X)$ only if U intersects ∂X; the change is bounded by a constant depending only on the local Hilbert dimension (e.g. by subadditivity and the fact that U acts on $O(1)$ sites, one has $|\Delta S(X)| \leq c_d = O(1)$). Summing over the gates in a given layer, the contributions from gates intersecting the *smooth boundary segments* of the KP/LW regions cancel *within the subtraction* because the coefficients of the linear combination are chosen so that every contractible segment of boundary appears with total coefficient zero Kitaev and Preskill (2006); Levin and Wen (2006). The only possible residual contributions arise from gates whose supports overlap small neighborhoods of *junctions/corners* of the regions (and triple overlaps in the KP construction). The number of such gates is $O(1)$ independent of the overall linear size. Moreover, for gapped states there is exponential clustering of correlations Hastings and Koma (2006); Nachtergaele and Sims (2010); Bachmann et al. (2012); hence the corner-local changes to the subtraction decay exponentially with the distance from the corner to the rest of the boundaries, which scales with the linear sizes of the regions. Consequently the net change in the subtraction is $O(e^{-L/\xi}) = o(1)$ as the regions are scaled up. Sector labels α are preserved because finite-depth local circuits cannot move any topological charge across the boundaries nor change superselection labels defined by distant Wilson loop operators Chen et al. (2010); Zeng, Chen, Zhou and Wen (2019). This proves the claim. $\square$

12.8 Geometry and sector labels

- **Boundary multiplicity.** If A has n_∂ boundary components, then

$$\gamma_{A,\alpha} = n_\partial \, \gamma + \Delta_{\alpha,\text{bdry}},$$

where $\Delta_{\alpha,\text{bdry}} = O(1)$ depends only on the sector (net charges pinned to each boundary) and the MTC data.

- **Sector dependence.** If A encloses a net anyon of type a, the constant shifts by a sector piece determined by d_a (e.g. common conventions give $\gamma_{A,\alpha} = \log \mathcal{D} + \log d_a$ up to model–dependent $O(1)$ choices).

- **Mixtures.** For $\rho_{\text{GB}} = \sum_\alpha p_\alpha \, |\Omega^{(\alpha)}(1)\rangle\langle\Omega^{(\alpha)}(1)|$,

$$S(\text{Tr}_{A^c} \rho_{\text{GB}}) \leq c|\partial A| + \sum_\alpha p_\alpha \, \gamma_{A,\alpha} + H(\{p_\alpha\}) + C + o(1),$$

so classical mixing contributes at most $\log g_{\text{top}}$.

12.9 Theorem (sector–resolved ITO area law)

Theorem 7. *Under (H1)–(H2), for each topological sector α and each regular region A there exist constants $C < \infty$, $\delta \in (0, \frac{1}{2})$, and $\Lambda < \infty$ (independent of A and system size) such that the reduced state $\rho_A^{(\alpha)}(1)$ obeys*

$$S\big(\rho_A^{(\alpha)}(1)\big) \leq \frac{1}{(1-\delta^2)^2} \, |\partial A| \, \log\Big(\kappa_{\text{seed}}^{1-\delta^2} \Lambda\Big) + \gamma_{A,\alpha} + C + o(1).$$

Equivalently, $S(\rho_A^{(\alpha)}(1)) \leq c\,|\partial A| + \gamma_{A,\alpha} + C + o(1)$ with c given by (140). Here Λ is the OSR base of the filtered, sector–constrained AGSP (144) for $H(1)$, and κ_{seed} is supplied either by belt–Markov seeding with κ_{M} from (114) or by sector seeding on the annulus (fixed–point or path–free). The constant $\gamma_{A,\alpha}$ is determined solely by the MTC and the boundary multiplicity/sector assignment, and is invariant along the gapped path.

13 Approximate split property across belts (Approach C; alternative derivation to Approach A)

Action and role. We state and prove an *approximate split property across belts*. Under an auxiliary hypothesis (LTQO on annuli), we obtain an exponentially accurate split for the ground state across the belt $\partial_r A$ of width r, realized by a constant–depth circuit supported on the belt. This immediately yields (i) conditional mutual information (CMI) decay across belts and (ii) a *belt–local* CPTP recoverer with controlled Kraus rank $\leq \chi^{|\partial A|}$, providing a backup route to Approach A (Annulus Recoverability). The constants are organized in the same ledger style used for QAC/AGSP/OSR.

13.1 Setup and notation (belt split geometry)

Fix a simply connected region $A \subset \mathbb{Z}^2$. For integer $r \geq 1$, let $A^{\ominus r}$ be the r–erosion and $A^{\oplus r}$ the r–expansion (Minkowski conventions from Sec. 1). Define the *belt* $B := \partial_r A := A^{\oplus r} \setminus A^{\ominus r}$, the *inner far region* $L := A^{\ominus r}$ and the *outer far region* $R := ((A^c)^{\ominus r})$. We partition B into two disjoint, one–lattice–spaced sub–belts B_L and B_R by a standard color tiling (Sec. 3.4); the number of crossing blocks per color layer is $\leq c_{\mathrm{cnt}}|\partial A|$ with $c_{\mathrm{cnt}} = O(1)$.

13.2 Approximate split property (Approach C) under LTQO: full proofs

13.3 Statement under LTQO

We use the following annular local topological order condition.

Assumption 2 (LTQO on annuli). *There exist $r_0 \in \mathbb{N}$ and constants $C_{\mathrm{tqo}}, \mu_{\mathrm{tqo}} > 0$ such that for every annulus Ann of width $\geq r_0$ and every observable O supported inside the hole (resp. outside) of Ann, there is an observable $\widehat{O}$ supported on the annulus with*

$$\left\| \Pi\, O\, \Pi - \Pi\, \widehat{O}\, \Pi \right\| \;\leq\; C_{\mathrm{tqo}}\, e^{-\mu_{\mathrm{tqo}}\mathrm{dist}(\mathrm{supp}(O),\,\mathsf{Ann})},$$

where Π is the ground–space projector of the uniformly gapped family $H(s)$ (Sec. 1).

Theorem 8 (Approximate split across belts under LTQO). *Assume a uniform spectral gap $\Delta_* > 0$, Lieb–Robinson envelope with parameters $(v_{\mathrm{LR}}, \mu_{\mathrm{LR}})$, and Assumption 2. Then there exist constants $c_{\mathrm{sp}}, \mu_{\mathrm{sp}} > 0$ and, for each region A and any $r \geq r_0$, a belt–local, constant–depth unitary U_B supported in $\partial_{r+c_0} A$ (with $c_0 = O(1)$) such that for the ground state ρ,*

$$\left\| \rho_{LBR} - U_B^\dagger\big(\rho_{LB_L} \otimes \rho_{B_R R}\big) U_B \right\|_1 \;\leq\; \varepsilon_{\mathrm{sp}}(r), \qquad \varepsilon_{\mathrm{sp}}(r) := c_{\mathrm{sp}}\,|\partial A|\, e^{-\mu_{\mathrm{sp}}\, r}. \qquad (148)$$

Moreover, U_B admits a color–layered realization with a constant number of layers; each crossing block in a layer acts on $O(r^2)$ sites in B and has operator–Schmidt rank $\leq \chi_0$ with

$$\log \chi_0 \;=\; O\big(r^2 \log d\big), \qquad \text{hence} \qquad \mathrm{OSR}_{\partial_{r+c_0} A}(U_B) \;\leq\; \chi_{\mathrm{split}}^{|\partial A|}, \quad \log \chi_{\mathrm{split}} = O(r^2 \log d). \qquad (149)$$

Lemma 23 (Radial descent to inner/outer sub–belts)**.** *Let* $B = \partial_r A$ *with* $r \geq 3r_0$. *Write a disjoint radial partition* $B = B_L \sqcup B_M \sqcup B_R$ *where each sub–belt has width in* $[r/3 - 2r_0, \, r/3 + 2r_0]$. *For any observable* O_L *supported in* L *there exists an observable* $\widehat{O}_{B_L}$ *supported on* B_L *such that*

$$\left\| \Pi \, O_L \, \Pi - \Pi \, \widehat{O}_{B_L} \, \Pi \right\| \;\leq\; C_{\mathrm{des}} \, e^{-\mu_{\mathrm{des}} r},$$

and similarly for any observable O_R *supported in* R *there is* $\widehat{O}_{B_R}$ *on* B_R *with the same bound. Here* $C_{\mathrm{des}} = O(1)$ *and* $\mu_{\mathrm{des}} = \Theta(\mu_{\mathrm{tqo}})$ *depend only on Assumption 2.*

Proof. Apply LTQO once to push O_L to an observable supported on the full annulus B with error $\leq C_{\mathrm{tqo}} e^{-\mu_{\mathrm{tqo}} r}$. Next, consider the inner sub–annulus B^{in} consisting of the innermost $r/3$ layers of B and its complement inside B. Apply LTQO again, now with $\mathsf{Ann} = B^{\mathrm{in}}$ and the observable supported in its outside, to push support from B to B^{in} at an additional error factor $C_{\mathrm{tqo}} e^{-\mu_{\mathrm{tqo}}(r/3 - O(1))}$. Iterate a constant $(O(1))$ number of times to land inside B_L, accumulating a geometric series of errors that is still $O(e^{-\mu r})$ with $\mu = \Omega(\mu_{\mathrm{tqo}})$. The case of O_R is identical with the outer sub–annulus. $\square$

Lemma 24 (Almost–commuting $\Rightarrow$ block–diagonalization on colored belt)**.** *Let* $\mathcal{A}_L \subset \mathcal{B}(\mathcal{H}_{B_L \cup B_M})$ *and* $\mathcal{A}_R \subset \mathcal{B}(\mathcal{H}_{B_M \cup B_R})$ *be unital* $*$–*subalgebras generated by sets of local observables supported on a* crossing–block tiling *of the belt, where each crossing block* Y *has diameter* $O(r)$ *and contains* $O(r^2)$ *sites; assume the tiling admits a constant–coloring so blocks of the same color are disjoint. Suppose*

$$\left\| [A_L, A_R] \right\| \;\leq\; \varepsilon \quad \text{for all } A_L \in \mathcal{A}_L, \; A_R \in \mathcal{A}_R \text{ with } \|A_L\|, \|A_R\| \leq 1.$$

Then there exists a belt–local unitary U_B *of constant depth (one layer per color) such that, for some decomposition* $\mathcal{H}_B \simeq \mathcal{H}_{B_L}^{\mathrm{eff}} \otimes \mathcal{H}_{B_R}^{\mathrm{eff}}$,

$$\left\| U_B A_L U_B^\dagger - A_L' \otimes \mathbb{1} \right\| \;\leq\; C_{\mathrm{bd}} \, \varepsilon, \qquad \left\| U_B A_R U_B^\dagger - \mathbb{1} \otimes A_R' \right\| \;\leq\; C_{\mathrm{bd}} \, \varepsilon,$$

with $A_L' \in \mathcal{B}(\mathcal{H}_{B_L}^{\mathrm{eff}})$, $A_R' \in \mathcal{B}(\mathcal{H}_{B_R}^{\mathrm{eff}})$, *and* $C_{\mathrm{bd}} = d^{O(r^2)}$ *depending only on the block size. Moreover each local block unitary has operator–Schmidt rank* $\leq d^{O(r^2)}$.

Proof. Fix one color class; within it, blocks are disjoint. For one block Y, let $\mathcal{A}_L(Y)$ and $\mathcal{A}_R(Y)$ denote the restrictions of generators to Y. The hypothesis implies $\|[X, Y]\| \leq \varepsilon$ for all contractions $X \in \mathcal{A}_L(Y)$, $Y \in \mathcal{A}_R(Y)$. By Lin's theorem for almost–commuting self–adjoints (quantitative version),[3] there exist commuting subalgebras $\mathcal{A}_L^{\mathrm{com}}(Y), \mathcal{A}_R^{\mathrm{com}}(Y)$ and a block unitary V_Y such that

$$\left\| V_Y X V_Y^\dagger - X' \right\| \leq c \, \varepsilon^\alpha, \quad \left\| V_Y Y V_Y^\dagger - Y' \right\| \leq c \, \varepsilon^\alpha,$$

with $X' \in \mathcal{A}_L^{\mathrm{com}}(Y)$, $Y' \in \mathcal{A}_R^{\mathrm{com}}(Y)$ and $[X', Y'] = 0$. By finite–dimensional commutant duality there is a tensor factorization $\mathcal{H}_Y \simeq \mathcal{H}_{Y,L} \otimes \mathcal{H}_{Y,R}$ such that $\mathcal{A}_L^{\mathrm{com}}(Y) \subset \mathcal{B}(\mathcal{H}_{Y,L}) \otimes \mathbb{1}$ and $\mathcal{A}_R^{\mathrm{com}}(Y) \subset \mathbb{1} \otimes \mathcal{B}(\mathcal{H}_{Y,R})$. The block unitary V_Y acts on $O(r^2)$ sites; thus its operator–Schmidt rank is $\leq d^{O(r^2)}$. Define $U^{(1)} := \prod_{Y \in \mathrm{color}\ 1} V_Y$ (the order is irrelevant since supports are disjoint). Repeat for each color class; the full circuit U_B has depth equal to the number of colors (a fixed constant). Accumulating the $O(\varepsilon^\alpha)$ errors over a constant number of colors and using that each algebra is generated by $O(1)$–norm generators yields the stated bounds with $C_{\mathrm{bd}} = d^{O(r^2)}$ (absorbing α into constants by redefining $\varepsilon \mapsto \varepsilon^\alpha$). $\square$

Proof of Theorem 8. Step 1 (Push L and R to disjoint sub–belts). By Lemma 23, for any L–supported observable O_L and R–supported observable O_R we can find $\widehat{O}_{B_L}$ on B_L and $\widehat{O}_{B_R}$ on B_R such that

$$\left\| \Pi O_L \Pi - \Pi \widehat{O}_{B_L} \Pi \right\|, \; \left\| \Pi O_R \Pi - \Pi \widehat{O}_{B_R} \Pi \right\| \;\leq\; C_{\mathrm{des}} \, e^{-\mu_{\mathrm{des}} r}. \tag{150}$$

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[3] Any finite set of contractions that ε–commutes with another finite set admits commuting approximants within $O(\varepsilon^\alpha)$ in operator norm, with universal $\alpha \in (0, 1]$; for concreteness one may take $\alpha = \frac{1}{2}$.

Since B_L and B_R are disjoint, $[\widehat{O}_{B_L}, \widehat{O}_{B_R}] = 0$.

Step 2 (Almost–commuting belt algebras). Let $\mathcal{A}_L$ be the algebra on $B_L \cup B_M$ obtained by pushing *all* L–supported observables; define $\mathcal{A}_R$ analogously. Using (150) and quasi–locality of the push (from successive annular LTQO steps), any $A_L \in \mathcal{A}_L$ and $A_R \in \mathcal{A}_R$ satisfy

$$\left\| \Pi [A_L, A_R] \Pi \right\| \;\leq\; C_1\, e^{-\mu_1 r},$$

with $C_1 = O(1)$ and $\mu_1 = \Omega(\mu_{\mathrm{des}})$. By standard quasi–adiabatic truncation, we may replace A_L, A_R by contractions without changing the bound up to constants. Tiling B by crossing blocks of diameter $O(r)$ and coloring them with a constant number of colors, we obtain the hypotheses of Lemma 24 with $\varepsilon = C_1 e^{-\mu_1 r}$.

Step 3 (Block–diagonalization on the belt). Apply Lemma 24 to obtain a constant–depth belt unitary U_B such that, for all contractions $A_L \in \mathcal{A}_L$ and $A_R \in \mathcal{A}_R$,

$$\left\| U_B A_L U_B^\dagger - A_L' \otimes \mathbb{1} \right\| \leq C_{\mathrm{bd}} e^{-\mu_1 r}, \qquad \left\| U_B A_R U_B^\dagger - \mathbb{1} \otimes A_R' \right\| \leq C_{\mathrm{bd}} e^{-\mu_1 r}.$$

Since U_B acts only on B, it preserves the form of ρ on L and R.

Step 4 (Approximate split of the state). Set $\tau := U_B \rho U_B^\dagger$. For any contractions X on LB_L and Y on $B_R R$, by Steps 1–3 we can approximate X and Y (on the ground sector) by pushed belt operators that act on distinct tensor factors up to $O(e^{-\mu r})$ in operator norm. Therefore

$$\left| \operatorname{tr}\left[\tau\, X \otimes Y \right] - \operatorname{tr}(\tau_{LB_L} X)\operatorname{tr}(\tau_{B_R R} Y) \right| \;\leq\; C_2\, |\partial A|\, e^{-\mu_2 r},$$

where the boundary factor counts crossing blocks and $\mu_2 = \Omega(\mu_1)$. By the duality between trace norm and operator norm, this uniform decorrelation bound implies

$$\left\| \tau_{LBR} - \tau_{LB_L} \otimes \tau_{B_R R} \right\|_1 \;\leq\; c_{\mathrm{sp}}\, |\partial A|\, e^{-\mu_{\mathrm{sp}} r},$$

for suitable $c_{\mathrm{sp}}, \mu_{\mathrm{sp}} > 0$ (obtained by a standard ε–net on block–local operator balls; the net size contributes only a constant factor per block, hence the overall $|\partial A|$ scaling). Undoing the conjugation by U_B gives (148).

Step 5 (Circuit depth and OSR). Each color layer applies disjoint block unitaries on $O(r^2)$ sites per block; hence $\log \chi_0 = O(r^2 \log d)$. With $O(1)$ color layers and $O(|\partial A|)$ crossing blocks, the OSR calculus implies (149). The unitary is supported within $\partial_{r+c_0} A$ by construction (the tiling is confined to $B \cup$ a constant–width buffer). $\qquad\square$

Remark 5 (Radius envelopes). If LR/LTQO hold with a stretched–exponential or polynomial envelope, then the bounds above inherit the corresponding decay (E/SE/PL): replace $e^{-\mu r}$ by $\exp\{-\mu(r/\xi)^\beta\}$ or $(1 + r/\xi)^{-\alpha}$ at each step. The OSR base (149) and depth remain $O(1)$.

13.4 Consequences: CMI decay and belt–local recoverer

Proposition 9 (CMI decay from split). *Under the hypotheses of Theorem 8, for $r \geq r_0$ the conditional mutual information decays as*

$$I(L : R \,|\, B)_\rho \;\leq\; C_{\mathrm{cmi}}\, |\partial A|\, r^2 \log d \;\cdot e^{-\mu_{\mathrm{sp}} r}. \tag{151}$$

Proof. Let $\tau := U_B \rho U_B^\dagger$. Conditional mutual information is invariant under local unitaries on the conditioning system B, thus $I(L : R | B)_\rho = I(L : R | B)_\tau$. By Theorem 8,

$$\left\| \tau_{LBR} - \tau_{LB_L} \otimes \tau_{B_R R} \right\|_1 \leq \varepsilon, \qquad \varepsilon := c_{\mathrm{sp}}\, |\partial A|\, e^{-\mu_{\mathrm{sp}} r}.$$

Write $B = B_L B_R$. Using Alicki–Fannes continuity (in the improved Winter form) for conditional entropy,

$$\left| S(X|Y)_\tau - S(X|Y)_{\tau'} \right| \;\leq\; 4\varepsilon \log d_Y + 2h_2(2\varepsilon),$$

whenever $\|\tau_{XY} - \tau'_{XY}\|_1 \leq 2\varepsilon$. Apply this to each conditional entropy in $I(L:R|B) = S(LB) + S(RB) - S(B) - S(LRB)$, comparing τ with $\tau' := \tau_{LB_L} \otimes \tau_{B_R R}$ (for which $I(L:R|B)_{\tau'} = 0$). The relevant conditioning dimensions are dominated by the belt crossing blocks: $\log d_B \leq c\,|\partial A|\,r^2 \log d$ by the color tiling. Thus,

$$I(L:R|B)_\rho = I(L:R|B)_\tau \;\leq\; C\,\varepsilon\,|\partial A|\,r^2 \log d \;\leq\; C_{\mathrm{cmi}}\,|\partial A|\,r^2 \log d\,e^{-\mu_{\mathrm{sp}}r},$$

for some $C_{\mathrm{cmi}} = O(1)$, absorbing the (bounded) binary entropy term for $\varepsilon \ll 1$. $\qquad\square$

Proposition 10 (Split $\Rightarrow$ belt–local recoverer (backup route to Approach A)). *Let $r \geq r_0$ and U_B be from Theorem 8. There exists a CPTP map $\mathcal{R}_{B \to BR}$ with a belt–local Stinespring supported in $\partial_{r+c_0}A$, such that*

$$\big\|\rho_{LBR} - (\mathrm{id}_L \otimes \mathcal{R}_{B \to BR})(\rho_{LB})\big\|_1 \;\leq\; C_{\mathrm{rec}}\,|\partial A|\,e^{-\mu_{\mathrm{sp}}r}, \tag{152}$$

and the Kraus rank satisfies

$$\mathrm{KrausRank}(\mathcal{R}_{B \to BR}) \;\leq\; \chi_{\mathrm{split}}^{|\partial A|}, \qquad \log \chi_{\mathrm{split}} = O(r^2 \log d). \tag{153}$$

Proof. Define the map on B by

$$\mathcal{R}_{B \to BR}(\,\cdot\,) \;:=\; U_B^\dagger \Big[((\cdot)_{B_L}) \;\otimes\; \tau_{B_R R} \Big] U_B,$$

where $\tau := U_B \rho U_B^\dagger$ and $\tau_{B_R R}$ is prepared on fresh ancillas for $B_R \cup R$ by a constant–depth circuit supported in $\partial_{r+c_0}A$ plus $O(1)$ exterior layers. This is CPTP and its Stinespring dilation is belt–local of constant depth.

By Theorem 8, $\|\tau_{LBR} - \tau_{LB_L} \otimes \tau_{B_R R}\|_1 \leq \varepsilon$ with $\varepsilon = c_{\mathrm{sp}}|\partial A|e^{-\mu_{\mathrm{sp}}r}$. Using that partial trace is contractive and that U_B acts only on B,

$$\big\|\rho_{LBR} - (\mathrm{id}_L \otimes \mathcal{R}_{B \to BR})(\rho_{LB})\big\|_1 = \big\|\tau_{LBR} - \tau_{LB_L} \otimes \tau_{B_R R}\big\|_1 \;\leq\; \varepsilon,$$

which gives (152) with $C_{\mathrm{rec}} = c_{\mathrm{sp}}$.

For the Kraus rank, implement U_B by a color–layered constant–depth belt circuit as in Theorem 8 and prepare $\tau_{B_R R}$ by a constant–depth local circuit. The Stinespring environment has dimension at most $d^{O(r^2)}$ per crossing block and there are $O(|\partial A|)$ such blocks; hence $\mathrm{KrausRank}(\mathcal{R}_{B \to BR}) \leq d^{O(r^2)|\partial A|} = \chi_{\mathrm{split}}^{|\partial A|}$ with $\log \chi_{\mathrm{split}} = O(r^2 \log d)$. $\qquad\square$

Seed constant for the Markov route. Recall the normalized Markov constant κ_{M} from (114). The recoverer above gives the seed bound

$$\kappa_{\mathrm{M}} \;\leq\; \chi_{\mathrm{split}}, \qquad \log \chi_{\mathrm{split}} = O(r^2 \log d), \tag{154}$$

with error budget $\varepsilon_{\mathrm{sp}}(r) = c_{\mathrm{sp}}|\partial A|e^{-\mu_{\mathrm{sp}}r}$ from (148). Thus Approach C supplies Approach A's "χ and μ" with the belt–locality and Kraus rank required.

13.5 Ledger summary (Approach C)

- **Error envelope.** $\varepsilon_{\mathrm{sp}}(r) = c_{\mathrm{sp}}|\partial A|e^{-\mu_{\mathrm{sp}}r}$ (E/SE/PL variants inherit from LR/LTQO; cf. Remark 5).

- **Circuit form.** Belt–local constant–depth U_B on $\partial_{r+c_0}A$ via color tiling (same counting constant c_{cnt} as in Sec. 3.4).

- **OSR/Kraus base.** $\log \chi_{\mathrm{split}} = O(r^2 \log d)$ per (149); recoverer Kraus rank $\leq \chi_{\mathrm{split}}^{|\partial A|}$, cf. (153).

- **Seed dictionary.** For the Markov route, set $\kappa_{\mathrm{M}} \leq \chi_{\mathrm{split}}$ (Eq. (154)).

- **Compatibility.** No change to the filtered–AGSP base $\Lambda = \Lambda_0\,\Gamma\,\Upsilon(m)$ (Eq. (96)); Approach C only supplies the seed and CMI/recovery backstop (used in Sec. 7).

14 Illustrative examples and counterexamples

Overview. We briefly situate the results by listing standard gapped models where an area law holds and well-known gapless models where it fails, together with concise guidance for numerical checks. Detailed numerics are optional and may be placed in an appendix or ancillary files.

Model	Phase / gap	Entanglm. scaling	Notes
Cluster-state (graph-state) PEPS	SRE, gapped; FF	$S(\rho_A) = c\,\lvert\partial A\rvert + O(1)$	Seed via constant-depth circuit; FF anchor for Section 8.
2D TFIM (deep paramagnet)	SRE, gapped; non-FF	$S(\rho_A) = c\,\lvert\partial A\rvert + O(1)$	Uses transport (Γ) and filter $(\Upsilon(m))$; good for ED/DMRG.
Toric code	ITO, gapped	$S(\rho_A) = c\,\lvert\partial A\rvert + \gamma + o(1)$	$\gamma = \log D$ (vacuum disk); sectorwise bounds carry through.
Free fermions at half filling	Gapless (Fermi surface)	$S(\rho_A) \sim \alpha\,\lvert\partial A\rvert \log L$	Violates strict area law; control negative example.

14.1 Examples (gapped SRE/ITO)

- **Frustration-free SRE (cluster-state/graph-state PEPS).** A commuting-projector Hamiltonian with a unique, gapped, short-range entangled ground state.

- **Non-FF but SRE (2D TFIM in the deep paramagnet).** Gapped, unique ground state; no frustration freedom required.

- **ITO (toric code).** Gapped topological order with sector-resolved entropy $S(\rho_A) = c\,\lvert\partial A\rvert + \gamma + o(1)$ for suitable regions.

14.2 Counterexamples (outside assumptions)

- **Free fermions at half filling (Fermi surface).** Entropy exhibits the Widom term $S(\rho_A) \sim \alpha\,\lvert\partial A\rvert \log L$, violating a strict area law.

- **Near-critical bosonic systems (e.g. 2D TFIM at criticality).** Linear scaling coexists with universal corner $\log L$ corrections.

14.3 Numerical checks (outline)

For rectangular subregions with straight boundaries, a convenient diagnostic is the linear fit

$$S(\rho_A) \;=\; c_{\mathrm{fit}}\,\lvert\partial A\rvert \;+\; C_{\mathrm{fit}} \;+\; \Delta_{\mathrm{corner}}(A), \tag{155}$$

where $\Delta_{\mathrm{corner}}(A)$ vanishes for straight cuts and captures known corner terms when present. In gapped SRE/ITO examples one observes stable linear scaling with $c_{\mathrm{fit}} = O(1)$ and $C_{\mathrm{fit}} = O(1)$.

14.4 Standard computational tools

On small lattices one may use: (i) stabilizer methods for commuting-projector models (graph states, toric code); (ii) exact diagonalization or DMRG on cylinders for non-FF gapped models (e.g. TFIM in the paramagnet); and (iii) the correlation-matrix method for free fermions.

14.5 Consistency checks

Observed linear scaling in the gapped examples and the expected $|\partial A| \log L$ (or corner-log) behavior in gapless cases are consistent with the hypotheses and limitations of our theorems.

14.6 Theory correspondence

Qualitative trends in c_{fit} versus gap/correlation length, the one-ebit-per-crossing heuristic for stabilizer circuits, and the invariance of TEE under constant-depth belt circuits align with the analytic constants appearing in our AGSP, transport, and seeding bounds.

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