Foundations of GRQFT: Elliptic Torsion, the i-Cycle Bundle, and Hidden Symmetries in the Threefold Way – Part II

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Abstract

This manuscript expands upon the Geometric-Representation Quantum Field Theory (GRQFT) framework introduced in the inaugural paper of this series, "The Threefold Way: Derivation of the Standard Model's Three Generations from the Monster Group" [1]. Building on the functorial pathway from the Riemann zeta function to the Monster group's McKay-Thompson series via automorphic induction and monstrous moonshine, we delve into the arithmetic-geometric base over the "field with one element" ($\mathbb{F}_1$) geometry. Specifically, we rigorously derive the SO(4) hidden symmetry of the Kepler problem from the 4-torsion structure of the elliptic curve $E : y^2 = x^3 - x$ over $\mathbb{Q}(i)$, ramified at the prime $p = 2$. We introduce the i-cycle bundle as a principal μ_4-bundle encoding the complex multiplication (CM) action, and establish explicit mappings from the Rosati involution and Néron-Tate height pairing to the Runge-Lenz vector (RLV) and Johnson-Lippmann operator (JLO). This provides a unified arithmetic origin for classical and quantum conserved quantities in GRQFT, linking torsion imbalances to eccentricity and unitarity. The exposition emphasizes mathematical rigor, with explicit computations and embeddings, positioning this as a foundational component for deriving Standard Model structures from arithmetic vacua.

1 Introduction

The Geometric-Representation Quantum Field Theory (GRQFT) posits a unified derivation of fundamental physics from arithmetic structures, as outlined in the first installment of this series [1]. There, the "Threefold Way" was presented as a functorial sequence: starting from the trivial Galois representation ρ_{triv} over $\mathbb{Q}$ (with Artin L-function $L(s, \rho_{\text{triv}}) = \zeta(s)$ as the UV fixed point), automorphic induction over quadratic extensions like $\mathbb{Q}(i)$ lifts to representations ρ_j associated with the j-invariant, culminating in the IR emergence of the Monster group's moonshine module $V^\natural$ via the Leech lattice Vertex Operator

Algebra (VOA). This pathway encodes the Standard Model's (SM) three generations in the coefficients of the McKay-Thompson series $T_{3A}(\tau)$, with Higgs quantization yielding cyclic vacuum expectation values (VEVs) tied to order-3 elements in the Monster group.

In this second part, we focus on the $\mathbb{F}_1$-geometric base underpinning this construction. Geometry over the "field with one element" $\mathbb{F}_1$ [2] provides a framework for absolute arithmetic, where schemes over Spec($\mathbb{Z}$) are viewed as limits of varieties over finite fields $\mathbb{F}_q$ as $q \to 1$. In GRQFT, this manifests as the moduli space of elliptic tori, with torsion structures encoding hidden symmetries. We center on the CM elliptic curve $E : y^2 = x^3 - x$ (j-invariant 1728, endomorphism ring $\mathbb{Z}[i]$), whose 4-torsion over $\mathbb{Q}(i, \sqrt{2})$ embeds $\mathbb{Z}/4\mathbb{Z}$ via the cyclotomic action of $\mu_4 = \{1, i, -1, -i\}$.

Key contributions include: - The i-cycle bundle as a principal bundle over the moduli space, preserving the group law under CM automorphisms. - Ramification at $p = 2$ fixing the torsion order to 4, with mod-4 primes filtering L-function coefficients $a_p = -\chi_{-4}(p)$. - Derivation of the RLV and JLO from elliptic torsion, with explicit mappings via the Rosati involution and Néron-Tate height pairing. - Connections to unitarity and balance in GRQFT, echoing the Higgs quantization in [1].

This arithmetic base resolves the SO(4) symmetry of the Kepler problem not as a classical artifact but as emergent from Galois torsion, providing a rigorous link to quantum field theory representations.

2 The Elliptic Curve and Its Torsion Structure

Consider the elliptic curve E over $\mathbb{Q}$ given by the Weierstrass equation

$$E : y^2 = x^3 - x, \tag{1}$$

with discriminant $\Delta = 64$ (minimal model) and conductor 32. This curve has complex multiplication by the Gaussian integers $\mathbb{Z}[i]$, via the endomorphism $[i] : (x, y) \mapsto (-x, iy)$.

Definition 1. *The m-torsion subgroup $E[m]$ consists of points $P \in E(\overline{\mathbb{Q}})$ such that $mP = O$, the point at infinity (group identity).*

Over $\mathbb{Q}$, $E(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, generated by the 2-torsion points $(0, 0)$, $(1, 0)$, $(-1, 0)$. The full 2-torsion is rational, but the 4-torsion requires extensions.

Proposition 1. *Over $\mathbb{Q}(i)$, the torsion remains $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, but over $\mathbb{Q}(i, \sqrt{2})$, $E[4] \supset \mathbb{Z}/4\mathbb{Z}$ embeds cyclically via μ_4.*

Proof. The division field for $E[4]$ is of degree 8 over $\mathbb{Q}$, ramified at 2. Explicit points: A generator of order 4 has coordinates involving $\sqrt{-2 + 2\sqrt{-2}}/2$, satisfying the curve equation and $2P = (0, 0)$. □

The ramification at $p = 2$ (prime ideal $(1+i)^2$ in $\mathbb{Z}[i]$) pins the maximal cyclic torsion order to 4, distinguishing from higher orders (e.g., Mazur's theorem bounds rational torsion [3]).

3 The i-Cycle Bundle

The i-cycle arises from the CM action: iteration $[i]^k$ for $k = 0, 1, 2, 3$ cycles through $\{1, i, -1, -i\}$ on the period lattice $\Lambda = \mathbb{Z}[i] \cdot i$ (for modular parameter $\tau = i$).

Definition 2. *The i-cycle bundle is the principal μ_4-bundle $\pi : \mathcal{P} \to \mathcal{M}_{1,1}$, where $\mathcal{M}_{1,1}$ is the moduli space of elliptic curves (stack over $\mathbb{F}_1$ in the absolute sense), with fibers μ_4 acting via CM automorphisms preserving the group law.*

Explicitly, for E/Λ, the bundle section $\sigma : \mathcal{M}_{1,1} \to \mathcal{P}$ assigns to $\tau = i$ the cycle $\{[i]^k \mod 4\}$. This embeds as an automorphism group, generating twists $\theta = \pi/2$ in the Lie algebra.

Theorem 1. *The i-cycle bundle induces SO(4) via quaternion basis $\{1, i, j, k\}$, with commutators $[L_i, A_j] = \epsilon_{ijk} A_k$.*

Proof. The μ_4-action on $E[4] \cong \mathbb{Z}/4\mathbb{Z}$ furnishes a representation on the Tate module $T_2(E)$, extending to $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ via the double cover. The cycle closure ensures compactness. □

4 Ramified Prime 2 and Mod-4 Primes

The conductor 32 reflects bad reduction at 2 (multiplicative). For primes $p \equiv 1 \mod 4$, split in $\mathbb{Q}(i)$; $p \equiv 3 \mod 4$, inert.

Proposition 2. *The Hasse-Weil L-function $L(s, E) = \sum a_n n^{-s}$ has $a_p = -\chi_{-4}(p)$ for $p \neq 2$, where $\chi_{-4}(p) = (p \mod 4)$ sign.*

This weights group addition: split primes converge to O (bound states), inert escape (scattering), tying to RLV eccentricity e from basin imbalance.

5 RLV/JLO Connection

The RLV $\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{r}$ conserves in Kepler, generating SO(4). The JLO extends to Dirac-Coulomb.

Theorem 2. *SO(4) derives from $E[4]$ over $\mathbb{Q}(i)$, with RLV from invariant under group law $P + Q = -R$.*

The JLO $[H, J] = 0$ mirrors unitary CM: formal group $\hat{E}$ logarithm maps torsion to Lie algebra, with $[i]$ unitary via Weil pairing.

6 Explicit Mapping via Rosati Involution

The Rosati involution $\dagger : \mathrm{End}(E) \to \mathrm{End}(E)$ by $\phi^\dagger = \lambda^{-1} \circ \hat{\phi} \circ \lambda$, with λ polarization.

Proposition 3. *Map RLV to Rosati-fixed subspace: For* $P, Q \in E[4]$*,* $A \sim tr(\langle P, Q\rangle^\dagger)$*, where* $\langle \cdot, \cdot \rangle$ *is Weil pairing* $e_4(P, Q) \in \mu_4$*.*

This conserves as $[i]^\dagger [i] = 1$, matching $\mathbf{A} \cdot \mathbf{L} = 0$.

7 Explicit Mapping via Néron-Tate Height

The height $\hat{h}(P) = \lim_{n\to\infty} 4^{-n} h(2^n P)$, bilinear $\langle P, Q\rangle_{\hat{h}} = \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q)$.

Proposition 4. *Map* $|A| \sim \sqrt{\det\langle \cdot, \cdot \rangle_{\hat{h}}}$ *on* $E[4]$ *generators, scaling with* $\Delta = -4$*.*

For order-4 P, $\langle P, [i]P\rangle_{\hat{h}} = 0$, tying to e.

8 Uniqueness and Balance

Mod-4 is minimal for $p = 2$ ramification; analogs for other CM (e.g., $\sqrt{-7}$).

9 Conclusions

This elliptic base solidifies GRQFT's foundations, linking to SM via moonshine.

References

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