Construction of the Real Numbers $\mathbb{R}$

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Abstract

This paper presents a constructive definition of the real numbers $\mathbb{R}$ directly from the non-negative integers. Every real number is defined as a definite, finite value expressed through finite or generative sums of rationals, where infinity is treated as a process rather than a completed entity. This framework explains why Weierstrass' epsilon–delta formulation works, resolves classical paradoxes as in set theory and topology, and establishes a foundation for analysis that is simple, intuitive, and contradiction-free.

Keywords: Real numbers, Constructive mathematics, Infinity, Continuity, One-to-one correspondence, Bijection, Epsilon–delta framework

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1 Preface — Values, Numbers, Processes

Kronecker once remarked:

"Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk." God created the integers; all else is the work of man.

The paradoxes of set theory and analysis arise from treating *infinite processes* as if they were completed entities. In reality, decimals such as $0.999\ldots$ or series like $\sum 9/10^n$ denote *definite, finite real numbers.* Multiple forms (e.g. $1.0 = 0.999\ldots$) are simply different representations of the same element in the set of real numbers, just as $2/4 = 1/2$ in the rationals.

The aim of this paper is to rebuild the real numbers constructively, avoiding such confusions and restoring clarity.

2 Construction of the Real Numbers $\mathbb{R}$

2.1 Radix Foundation

Fix a base $b \geq 2$. Define the digit set

$$K = \{0, 1, 2, \ldots, b - 1\}.$$

2.2 Integer Part

The integer part is constructed as a finite sum:

$$i = \sum_{p=0}^{P} k_p b^p, \quad k_p \in K,\ P \in \mathbb{Z}_{\geq 0}. \tag{2.1}$$

2.3 Fractional Part

The fractional part is defined by the generative process:

$$f = \sum_{q=1}^{\infty} \frac{k_q}{b^q}, \quad k_q \in K, \quad 0 \leq f < 1. \tag{2.2}$$

Here "∞" denotes either a generative process, or a definite $M \in \mathbb{N}$, but not both at the same time. The value f is definite and finite. The index q can (1) terminate at some finite integer $M \in \mathbb{N}$, or (2) continue **generatively, indefinitely**, (3) but not both at the same time. The value of f is always definite and finite.

2.4 Real Numbers

A non-negative real number is the sum of its integer and fractional parts:

$$x = i + f. \tag{2.3}$$

Thus,

$$\mathbb{R} = \left\{ \pm \left(\sum_{p=0}^{P} k_p b^p + \sum_{q=1}^{\infty} \frac{k_q}{b^q} \right) \middle| \, k_p, k_q \in K \right\}. \tag{2.4}$$

2.5 Remarks

This construction uses only $\mathbb{Z}_{\geq 0}$. Integers, fractions, and infinite processes are all derived from the generative set of non-negative integers. Negativity is introduced contextually as additive inverse.

Why Weierstrass' Epsilon–Delta Framework Works. Define the partial sums

$$s_M = \left(\sum_{q=1}^{M} \frac{k_q}{b^q} \right) \in \mathbb{Q}, \quad M = 1, 2, \ldots$$

and let

$$f = \sum_{q=1}^{\infty} \frac{k_q}{b^q}.$$

Then for all $f \geq 0$ and $f < 1$, there exist $s_M \in \mathbb{Q}$ and $M \in \mathbb{N}$ such that

$$0 \leq |f - s_M| < \frac{1}{b^{M+1}}.$$

Hence, for **every rational number**[1] $\varepsilon > 0$, there exists M with $\varepsilon = 1/b^{M+1}$, so that $|f - s_M| < \varepsilon$.

[1] Cauchy's equivalence-class definition of real numbers was flawed because it quantified "for every $\varepsilon > 0$" where ε was silently understood as a real number. This is circular: it

This is precisely why the epsilon–delta framework of Weierstrass works: the real f is definite and finite, approximated by rational sums, where $\varepsilon > 0$ must not be a real number. We cannot use reals to define reals. ε can be either (1) a definite, finite rational number or (2) an infinitesimal process that preserves the definiteness and finiteness of f, but never both at once.

2.6 Why Continuity, Not Limit

The classical definition of *limit* speaks only of approaching a value, while excluding the value itself (in this case the value f). In contrast, the definition of *continuity* includes the value:

$$\forall \varepsilon > 0,\, \exists M \in \mathbb{Z}_{\geq 0} \text{ such that } |f - s_M| < \varepsilon,$$

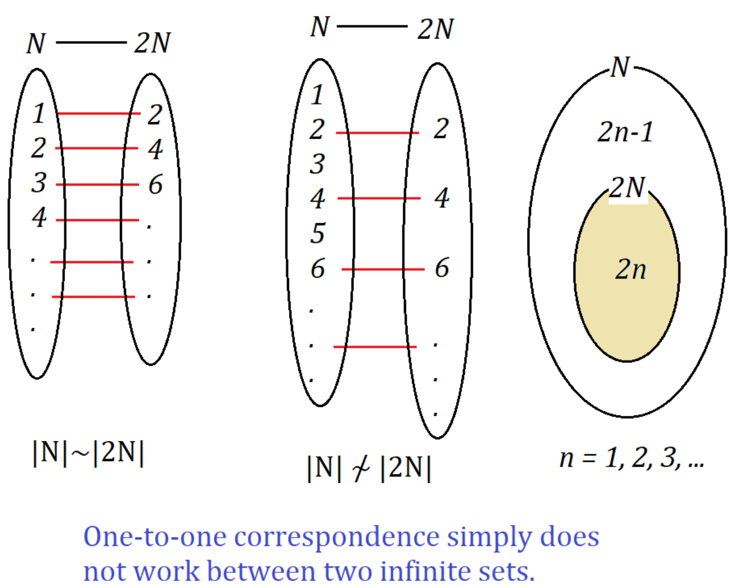
where s_M is the rational approximation and f is the real number being defined.

Continuity affirms the definite, finite existence of f, while limit excludes it. Therefore, continuity is the correct foundation for real numbers in this constructive setting.

presupposes the existence of arbitrary small positive reals in order to define what a real number is. In a constructive foundation, ε must be taken as a rational number ($\varepsilon \in \mathbb{Q}_{>0}$) or as a generative infinitesimal process, but never as an already-given real.

3 Infinity and One-to-One Correspondence

Cantor compared the sizes of infinite sets by one-to-one correspondence, for example asserting that $|\mathbb{N}| \sim |2\mathbb{N}|$. This creates unavoidable contradictions.



As the figure shows: Cantor's bijection declares $|\mathbb{N}| = |2\mathbb{N}|$, while finite intuition insists $|\mathbb{N}| \neq |2\mathbb{N}|$, since $2\mathbb{N}$ is a proper subset of $\mathbb{N}$. Thus one-to-one correspondence simply does not work between two infinite sets.

If $|\mathbb{N}| \sim |2\mathbb{N}|$, then $|\mathbb{N}| \nsim |2\mathbb{N}|$ must not also hold. If $|\mathbb{N}| \nsim |2\mathbb{N}|$, then $|\mathbb{N}| \sim |2\mathbb{N}|$ must not also hold. Yet both intuitions are forced when bijection is extended to infinity.

Infinity, however, is not a completed entity but a generative process. As in Peano's induction, numbers are generated step by step without end. Assigning a fixed property to infinity terminates its generativity.

Therefore, one-to-one correspondence applies only to finite sets, never to infinite processes. Removing bijection from the infinite case dissolves the paradoxes of Cantor's set theory: no $|\mathbb{N}| = |2\mathbb{N}|$, no diagonal paradox, and no uncountable hierarchies.

Dedekind's definition of infinite sets is fundamentally flawed: bijection or one-to-one correspondence cannot serve as a valid criterion between two infinite sets.

This reinforces the constructive foundation of $\mathbb{R}$: real numbers are built only from integers and generative processes, not from completed infinite sets.

Note to Readers and Referees

The section is included to show that the same principle — infinity as generative process, not a transfinite entity — strengthens both the construction of real numbers and the resolution of set-theoretic paradoxes.

4 Conclusion

With this constructive definition of the real numbers $\mathbb{R}$, it becomes clear why Weierstrass' epsilon–delta framework works: real numbers are approximated by finite or generative sums of rationals. The epsilon–delta framework is valid only **because real numbers are constructively defined as rational approximations**[2]. It functions as a method, but it does not itself define or construct the real numbers. Every real number is definite and finite, which the epsilon–delta framework makes explicit.

Infinity is never a static, definite entity. It is a generative process, as in Peano's principle of mathematical induction. The moment one assigns a fixed property (such as cardinality or a transfinite number) to infinity, its generativity ceases and it becomes definite and finite. The same is true with $\varepsilon \in \mathbb{Q}$ in the context of this paper: ε is either (1) a definite, finite rational number, or (2) an infinitesimal process — but not both at the same time.

Thus the real numbers are reconstructed as a set of simple, intuitive, contradiction-free objects, firmly grounded in the non-negative integers.

[2] My confession: before I wrote this paper, I did not know why Weierstrass' epsilon–delta framework works. I knew it worked since I was a freshman in college, but not why until I constructed the real numbers from rationals. It works because we are approximating reals with rationals.

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