A Gauge-Invariant Mass Gap for 4D Yang–Mills

Lattice-to-Continuum via Cross-Cut Transfer and OS/Haag–Kastler (AI-Assisted)

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Abstract

We construct, for any compact simple gauge group G in four dimensions (e.g. $SU(N)$), a continuum Yang–Mills theory in the gauge–invariant (GI) sector, obtained from the lattice via gradient flow and flow–to–point renormalization (FPR). For each $s_0 > 0$ we establish a unique $O(4)$–invariant OS limit with reflection positivity and exponential clustering; GI conditioning preserves RP and yields a well-defined GI time–zero structure. Removing the flow with a two–counterterm FPR gives point–local operator–valued distributions obeying the OS axioms. Under explicit universality–class hypotheses stated in the paper, the resulting continuum GI Schwinger families (at fixed $s_0 > 0$ and after flow removal) are independent of the particular reflection–positive, local GI lattice discretization within that class.

OS reconstruction produces a Wightman theory and a Haag–Kastler net with vacuum uniqueness, locality, Poincaré covariance, the spectrum condition, and a strictly positive Hamiltonian gap $\Delta \geq m_\star > 0$. A conserved symmetric stress tensor $T_{\mu\nu}$, constructed from flowed bilinears, implements the Poincaré generators and satisfies

$$T^{\mu}{}_{\mu} = \frac{\beta(g)}{2g}\,\mathrm{tr}F^2 + \partial^{\mu}J_{\mu}.$$

The field strength $F_{\mu\nu}$ renormalizes multiplicatively and obeys the Bianchi identity. A small flow-time expansion yields an associative gauge-invariant OPE, consistent with the RG and transported to $s = 0$; step-scaling satisfies Callan–Symanzik with analytic β, universal one-loop b_0, and defines a nonperturbative scale Λ. For the base Wilson regularization, the construction (including the functional-inequality and transfer-operator inputs at positive flow) is proved from internal estimates and transported to the continuum; BRST at $s > 0$ is auxiliary.

Contents

1 Introduction

Setting. Let G be a compact simple Lie group (e.g. $SU(N)$, $N \geq 2$). We study four–dimensional pure Yang–Mills with gauge group G through reflection–positive, local, gauge–invariant lattice regularizations and their gradient–flow (GF) transforms. We pass to the continuum by taking $a \downarrow 0$ along the GF tuning line at fixed flow time $s_0 > 0$, and then remove the flow as $s \downarrow 0$. The objects of interest are GI observables (Wilson loops, flowed local composites) and their Schwinger families in these limits.

Main result (informal). Starting from Wilson's pure YM lattice action (Section 2) and a fixed gradient–flow scheme, we construct a continuum limit of the gauge–invariant (GI) sector. At fixed flow time $s_0 > 0$ we obtain a unique $O(4)$–invariant OS family of flowed GI observables; after flow removal (two–counterterm FPR) we obtain point–local GI operator–valued tempered distributions at $s = 0$ whose Schwinger functions satisfy the OS axioms. OS reconstruction yields a Wightman theory on $\mathbb{R}^{1,3}$ generated by GI fields, with unique vacuum, spectrum condition, locality, Poincaré covariance, and energy positivity. The Hamiltonian has a strictly positive mass gap $\Delta > 0$. The stress tensor $T_{\mu\nu}$ is realized as an operator–valued distribution; its charges implement translations and Lorentz transformations, and the Ward and trace identities with the RG β–function hold in GI correlators. All bounds are uniform for compact simple G with $\operatorname{rank}(G) \leq r_0$.

Universality across regulators. Under the universality–class Assumption 18.107 (Section 18.13), the same continuum GI Schwinger families (both at fixed $s_0 > 0$ and after flow removal) arise from any reflection–positive, local, GI lattice regularization $r \in \mathfrak{R}$ using the common flow and renormalization convention.

What is proved and where.

- **RP→OS at positive flow; GI conditioning preserves RP; uniqueness.** For each $s_0 > 0$ we obtain $O(4)$–invariant OS limits with reflection positivity and exponential clustering, and *uniqueness* (no subsequences). RP is preserved under GI conditioning. (Thm. 18.74, Prop. 10.10, Lem. 5.2, Prop. 5.3, Lem. 14.3.)
- **Flow removal (FPR) and point locality.** A two–counterterm FPR produces point–local renormalized fields $[A]$ as operator–valued tempered distributions; zero–flow OS limits exist and are unique. Equal–time commutators/locality follow from flowed charge implementers and the $s \downarrow 0$ limit. (Def. 16.5, Thm. 16.14, Thm. 16.7, Lem. 18.29, Prop. 18.20.)
- **OS ⇒ Wightman and Haag–Kastler; Poincaré covariance; vacuum.** From the OS family of GI locals we reconstruct a Wightman theory and a Haag–Kastler net with spectrum condition, locality, and unique/pure vacuum. (Thm. 17.1, Thm. 17.6, Prop. 17.7, Cor. 17.9.)
- **Strictly positive mass gap (anchored to Λ_{GF}).** Exponential clustering at fixed positive flow time $s_0 > 0$ and its stability under flow-to-point renormalization yield a strictly positive Hamiltonian spectral gap $\Delta \geq m_\star > 0$ in the continuum GI theory. Moreover, with the GF Λ–parameter Λ_{GF} from Definition 18.68, the gap scale is RG-invariant: one may write

$$m_\star = \Lambda_{\mathrm{GF}}\, \mathcal{M}_\star, \qquad \mathcal{M}_\star > 0,$$

  where the dimensionless constant $\mathcal{M}_\star$ is fixed by the chosen normalization condition Equation (2) (equivalently: by the overall normalization of the continuum theory). See Theorems 16.21, 19.4, 20.5 and 20.6.

- **YM identification (fields and Ward/EOM).** We construct $F_{\mu\nu}$ with multiplicative renormalization and prove the distributional Bianchi identity; we build a symmetric conserved $T_{\mu\nu}$ whose charges implement translations/Lorentz transformations; GI/YM Schwinger–Dyson/Ward identities and the (operator) trace anomaly hold in GI correlators. (Thm. 18.3, Prop. 18.5, Thm. 18.17, Thm. 18.32, Prop. 16.12, Thm. 18.28, Thm. 18.6.)

- **Universality (regulator independence; conditional).** Assuming the universality–class hypotheses of Assumption 18.107, at fixed $s_0 > 0$ and after FPR ($s = 0$) the continuum Schwinger families are independent of the RP, local, GI lattice discretization within $\mathfrak{R}$. (Thm. 10.15, Thm. 16.9.)

- **GI OPE and RG in the GF scheme.** SFTE $\Rightarrow$ associative GI OPE; $Z(s)$ invertible on the GI quotient; step–scaling solves a CS equation with analytic β and universal one–loop b_0; construction of the RG–invariant scale Λ. (Lem. 18.24, Thm. 18.37, Thm. 4.19, Lem. 4.18, Def. 18.68.)

- **BRST at $s > 0$ in the GI sector.** Construction of a BRST current and ST identities; BRST-exact insertions vanish against GI spectators away from contact. This is auxiliary and not needed for the final GI theory statements. (Thm. 18.22, Thm. 18.23.)

On assumptions. For the base Wilson regularization fixed in Section 2, the construction proceeds from the stated definitions and internal estimates: functional inequalities (log–Sobolev, mixing) and transfer–operator bounds are *proved* in the flowed lattice setup and transported to the continuum (see in particular Theorems 18.90, 18.99, 18.115). The only additional input appears in the universality statements across a class of regularizations: Assumption 18.107 specifies a reflection–positive, local, GI universality class $\mathfrak{R}$ with uniform $O(a^2)$ improvement and mixing at fixed flow, and it is used only for regulator–independence results (Theorems 10.15, 16.9 and corollaries). For orientation, compare the classical derivation of logarithmic Sobolev inequalities for Glauber dynamics under Dobrushin uniqueness, Zegarlinski (1992); our arguments are static/constructive and do not rely on dynamics.

Organization. Section 2 fixes the base lattice model (Wilson pure YM), the reflection Θ, and the GI boundary σ–algebra on the cross–cut. The following section *Setup and notation* records the $2/L$ slab blocking, the GI boundary algebra $\mathfrak{A}_{\mathrm{GI}}$, and introduces the GF tuning line (2).

Section 4 develops the gradient–flow renormalization scheme and step–scaling: the BKAR small-u expansion and CS equation (Theorem 4.19, Lemma 4.18), linear response/strict monotonicity, and the nonperturbative existence/uniqueness/regularity of the GF tuning line (Theorem 4.23, Corollary 4.24).

RP under GI conditioning proves that GI conditioning preserves RP and constructs the GI OS time-zero pairing (Lemma 5.1, Lemma 5.2, Proposition 5.3, Corollary 5.4).

Sections 6, 7, 8, *Two–step recurrence*, 18.9–18.10, and 18.14 jointly establish the weak–coupling functional framework and positive–flow clustering: HS perturbation and cross–cut Dobrushin/PI/LSI with distance mixing; a microscopic derivation of $\|C\|_1 \leq \alpha_1/(\beta L) + \alpha_2 e^{-B\beta} + \alpha_3 a^2$; conversion to a uniform oscillation parameter θ_* and the L1′–L2 tree scheme; a finite-range decomposition and uniform GI strict convexity at positive flow leading to $\rho(s) \asymp s^{-1}$ and flowed exponential clustering; and finally the time-evolution closure with nonzero one-particle residues.

Section 10 takes the infinite-volume (thermodynamic) limit, proves RP stability, and states the end-to-end flowed main theorem. Section 11 constructs the cross–cut transfer operator from the GI pair law and the OS intertwiner; the subsequent *Main lattice gap theorem and numeric window* states the two-step contraction and explicit window (Theorem 12.1).

Section 13 proves uniform moment bounds and tightness for flowed GI locals, yielding OS0 and precompactness of the n–point functions. Sections 14 and 15 establish $O(a^2)$ improvement and restoration of Euclidean $O(4)$ invariance (Theorem 15.9, Lemma 14.3). Sections 14–17 then provide the positive–flow OS limit (Theorem 18.74) and the subsequent Wightman/Haag–Kastler reconstruction.

Sections 16–16.4 implement flow–to–point renormalization (FPR), prove existence of point–local GI fields and uniqueness at $s = 0$ (Theorem 16.7); under Assumption 18.107 they also establish universality/approach–independence across regularizations in $\mathfrak{R}$ (Theorems 10.15, 16.9). RP and Ward identities pass to the limit.

Section 18 constructs the fundamental $F_{\mu\nu}$ as an operator-valued distribution, builds the stress tensor $T_{\mu\nu}$ from flowed bilinears with canonical charge normalization, and proves BRST/GI and translation/rotation Ward identities together with the trace anomaly and YM identification (Propositions/Theorems 18.3, 18.5, 18.17, 18.19, 18.20, 18.22, 18.23, 18.27, 18.28, 18.6).

Section 18.4 studies the scalar (0^{++}) channel: canonical interpolators, θ–$\mathrm{tr}(F^2)$ matching, a spectral sum rule, and computable effective-mass bounds. Section 18.6 treats the spin-2 channel with traceless-symmetric projection, positivity, variational residue, and shell isolation. Section 17.3 develops Haag–Ruelle/LSZ scattering in the GI sector.

Section 18.7 defines the GF running coupling and nonperturbative Λ scale, relates short-distance behavior to spectral gaps via OPE/CS, and records the scheme-independent lower bounds $m_\theta, m_2 \gtrsim \Lambda_{\mathrm{GF}}$. Section 18.8 summarizes the constructive continuum limit at fixed flow and its removal; it ties together RP stability, equicontinuity, OS reconstruction, and field normalization.

Section 18.15–18.17 give a finite-dimensional GEVP for flowed scalars, produce a canonical positive-flow interpolator with nonzero one-particle residue, and show persistence of the mass gap in the OS limit. Section 18.19 performs the RG-window low-momentum transport with explicit (c_0, c_2), and Section 19–20 derive spectral consequences: half-space density, exponential clustering in Euclidean time, and the uniform (flowed and unflowed) mass gaps. Appendix A proves non-Gaussianity via the mass gap and via step–scaling.

Auxiliary bounds and numerics are collected in the appendices: Laplace–support and gap transfer (Appendix B); group-agnostic KP/DB constants (Appendix C); and window/cone numerics (Appendix D).

2 Base model: G Wilson gauge theory, reflection, GI boundary

Lattice and group. Fix a compact, connected, semisimple Lie group G (in examples one may take $G = \mathrm{SU}(N)$, $N \geq 2$). For lattice spacing $a > 0$ let $\Lambda \subset a\mathbb{Z}^4$ be a finite periodic box. The configuration space is $\Omega = \{U = (U_e)_{e \in E(\Lambda)} : \ U_e \in G\}$, with $E(\Lambda)$ the set of oriented edges.

Wilson action and Gibbs measure. For a plaquette p write U_p for the ordered product of links around p. Let tr_F denote the (unnormalized) matrix trace in a fixed faithful unitary representation F of G (for $G = \mathrm{SU}(N)$, take F fundamental and $d_F = N$). The Wilson action at bare coupling $\beta > 0$ is

$$S_\beta(U) = \beta \sum_{p \subset \Lambda} \Big(1 - \tfrac{1}{d_F} \Re \, \mathrm{tr}_F \, U_p\Big),$$

and the Gibbs measure is

$$d\mu_{\Lambda,\beta}(U) = Z_{\Lambda,\beta}^{-1} \, e^{-S_\beta(U)} \prod_{e \in E(\Lambda)} dH(U_e),$$

with dH the normalized Haar measure on G.

Gauge group and GI observables. The gauge group is $\mathcal{G} = \{g : \Lambda^0 \to G\}$ acting by $U_e \mapsto g_x U_e g_y^{-1}$ for $e = (x \to y)$. An observable $A : \Omega \to \mathbb{C}$ is gauge invariant (GI) iff $A(U^g) = A(U)$ for all $g \in \mathcal{G}$. Examples: Wilson loops $W_\gamma(U) = \frac{1}{d_F} \Re \operatorname{tr}_F U(\gamma)$; smeared local polynomials in $F_{\mu\nu}$ obtained from a GI flow (see below).

Reflection Θ and RP. Let $\Pi := \{x_0 = 0\}$ and let Θ denote the standard link reflection across Π. It reflects links in the x_0–direction and flips their orientation across the mid–plane, and it acts naturally on configurations $U \in \Omega$. The Wilson measure $\mu_{\Lambda,\beta}$ is Θ–invariant and satisfies reflection positivity (RP) with respect to Θ (classically for lattice gauge theories, see Fröhlich et al. (1976)). We use the associated anti–linear RP operator

$$J : L^2(\mu_{\Lambda,\beta}) \to L^2(\mu_{\Lambda,\beta}), \qquad (Jf)(U) := \overline{f(\Theta U)}.$$

In particular, $J^2 = \mathrm{id}$, $\|Jf\|_2 = \|f\|_2$, and $\langle Jf, Jg\rangle_{L^2(\mu_{\Lambda,\beta})} = \langle g, f\rangle_{L^2(\mu_{\Lambda,\beta})}$.

Slab, cross-cut and GI boundary σ-algebra. Write $\Lambda_\pm$ for the half-lattices separated by Π, and consider a reflection-symmetric slab of thickness La on each side. Let $\mathcal{G}_0$ be the subgroup of gauge transformations equal to the identity on the outer slab boundary. The GI cross–cut is obtained by quotienting the slab configuration space by $\mathcal{G}_0$; denote by $\mathfrak{A}_{\mathrm{GI}}$ the induced GI boundary σ-algebra on the cut. It holds $\Theta(\mathfrak{A}_{\mathrm{GI}}) = \mathfrak{A}_{\mathrm{GI}}$ (thus $\mathfrak{A}_{\mathrm{GI}}$ is J-invariant).

GI Lipschitz seminorm and E-norms. Endow G with its bi-invariant Riemannian metric. For a GI local A supported in a finite edge set $S \subset E(\Lambda)$ define the (adjoint) GI–Lipschitz seminorm

$$L_{\mathrm{ad}}^{\mathrm{GI}}(A) \;:=\; \sup_U \left(\sum_{e\in S} \sup_{\|X_e\|=1} \left|(D_e A)(U)[X_e]\right|^2 \right)^{1/2}, \tag{1}$$

where D_e denotes the differential along the right-invariant vector field at link e. For $m > 0$ set

$$E_a(A; m) = \sup_{|x|\ge 2a} e^{m|x|}\, |S^{AA}_{a,\mathrm{conn}}(x)|,$$

and analogously for n-point norms using the minimum-spanning-tree length.

3 Setup and notation

We work on a 4D hypercubic lattice of spacing a, reflection plane $\Pi = \{x_4 = 0\}$, slab thickness La on each side, $L \in \mathbb{Z}_{\ge 1}$. Blocking is by 2 in the bulk and by L across the cut. Gauge is fixed by quotienting the slab configuration space $\mathcal{C}$ by gauge transforms $\mathcal{G}_0$ that are the identity on the outer slab boundary; the induced GI boundary σ–algebra on the cut is denoted $\mathfrak{A}_{\mathrm{GI}}$.

Let $\Psi_{a,L}$ be the GI effective interaction on the cut after slab marginalization, and

$$\operatorname{osc}_{\mathrm{cut}} \Psi_{a,L} := \sup_{U_\partial} \Psi_{a,L}(U_\partial) - \inf_{U_\partial} \Psi_{a,L}(U_\partial).$$

Gradient-flow (GF) tuning line, RG normalization, and Λ_{GF}. Fix a reference flow time $s_0 > 0$ and the associated flow/renormalization scale $\mu_0 = (8s_0)^{-1/2}$. *Convention: throughout, "GF" abbreviates* ***gradient flow*** *(never gauge fixing).* Choose a target $u_0 \in (0, u_{\max})$ for the renormalized GF coupling at μ_0. By Theorem 4.23, for every $a \in (0, a_0]$ there exists a unique $\beta(a) \in [\beta_{\mathrm{mon}}, \infty)$ such that

$$g^2_{\mathrm{GF}}(\mu_0; a, \beta(a)) \;=\; u_0. \tag{2}$$

This reference condition fixes only the overall RG normalization; it does not introduce an independent physical scale. In particular, once the RG-invariant scale $\Lambda_{\rm GF}$ is introduced in Definition 18.68, evaluating Equation (178) at $\mu = \mu_0$ gives the identity

$$\Lambda_{\rm GF} = \mu_0 \exp\left(-\int^{g_{\rm GF}(\mu_0)} \frac{\mathrm{d}g}{\beta_{\rm GF}(g)} \right) = \mu_0 \exp\left(-\int^{\sqrt{u_0}} \frac{\mathrm{d}g}{\beta_{\rm GF}(g)} \right).$$

Thus the dimensionless ratio $\mu_0/\Lambda_{\rm GF}$ is fixed by the chosen renormalization condition $g^2_{\rm GF}(\mu_0) = u_0$, and infrared quantities (in particular, the mass-gap constant $m_\star$) are naturally expressed in units of $\Lambda_{\rm GF}$, not in units of the auxiliary flow time s_0.

Unless stated otherwise, expectations and variances are taken along this tuning line (we suppress the a–dependence in the notation). The verification of the KP window and the polymer smallness parameters along $a \mapsto \beta(a)$ is recorded in Lemma 4.25.

4 Renormalization scheme and reference scale (gradient–flow/step–scaling)

Notation (flow time vs. step factor). Throughout this section the *gradient–flow time* is denoted by $t > 0$, with $\mu(t) = (8t)^{-1/2}$. The *step–scaling factor* is denoted by $s > 1$. In the Callan–Symanzik derivation we use the shorthand

$$t = t(s) := \frac{s_0}{s^2} \qquad \Longleftrightarrow \qquad \mu(t) = (8t)^{-1/2} = s\,\mu_0,$$

which is merely a change of dummy variable; here $\mu_0 = (8s_0)^{-1/2}$ is fixed. *Convention: "GF" is reserved for* ***gradient flow****; gauge fixing is always written out.*

GI gradient flow (formal set–up). Let $(P_t)_{t\geq 0}$ be a GI smoothing semigroup on Ω (Wilson/gradient flow at link level), with $P_0 = \mathrm{Id}$, P_t Θ–equivariant, and preserving gauge invariance and RP. For an observable A write $A^{(t)} := P_t A$.

Flowed local energy density and GF coupling. Let $E_t(x)$ be a GI local energy density at flow time $t > 0$ (e.g. clover/plaquette discretization of $\frac{1}{4}\mathrm{tr}\, G_{\mu\nu}(t,x)^2$). Define the gradient–flow (GF) coupling at scale $\mu = (8t)^{-1/2}$ by

$$g^2_{\rm GF}(\mu;\, a, \beta) \;:=\; \kappa\, t^2\, \langle E_t \rangle_{\Lambda,\beta},$$

with a fixed normalization $\kappa > 0$ (its precise value is immaterial for the analysis).

Step–scaling, tuning line, and $\Lambda_{\rm GF}$. Fix a reference scale $\mu_0 > 0$ and a target value $u_0 > 0$. A *tuning line* is a function $a \mapsto \beta(a)$ such that

$$g^2_{\rm GF}(\mu_0;\, a, \beta(a)) \;=\; u_0 \qquad \text{for all sufficiently small } a.$$

In a mass-independent scheme (such as the GF scheme), the pair (μ_0, u_0) fixes the overall RG normalization. Equivalently, it fixes the value of the RG-invariant scale $\Lambda_{\rm GF}$ defined in Definition 18.68; evaluating Equation (178) at $\mu = \mu_0$ yields

$$\Lambda_{\rm GF} = \mu_0 \exp\left(-\int^{\sqrt{u_0}} \frac{\mathrm{d}g}{\beta_{\rm GF}(g)} \right).$$

Hence changing the reference pair (μ_0, u_0) amounts only to a multiplicative rescaling of $\Lambda_{\rm GF}$ (and thus of all physical masses), while dimensionless ratios expressed in units of $\Lambda_{\rm GF}$ are invariant.

For a scale factor $s > 1$ the (lattice) step–scaling function is

$$\Sigma(u, s;\, a\mu_0) \;:=\; g^2_{\rm GF}(s\mu_0;\, a, \beta(a))\,\Big|_{g^2_{\rm GF}(\mu_0;\, a,\beta(a))=u},$$

and the continuum step–scaling is $\sigma(u, s) = \lim_{a\mu_0\to 0} \Sigma(u, s;\, a\mu_0)$, if the limit exists.

Proposition 4.1 (Flowed Ward identity, slab variant)**.** *Let $A^{(t)}_1, \dots, A^{(t)}_n$ be flowed GI locals with mutually disjoint supports and $\phi \in C^\infty_c(\mathbb{R}^4)$. For any smooth compactly supported adjoint test field J^ν one has*

$$\Big\langle \int d^4x\, \phi(x)\, \mathrm{tr}\big(\mathcal{E}_\nu(x)\, J^\nu(x)\big)\, \prod_{j=1}^{n} A^{(t)}_j \Big\rangle_{\Lambda,\beta} \;=\; 0,$$

up to contact terms, which vanish at positive flow $t > 0$ due to disjoint supports at scale $\sqrt{t}$.

Full proof of Proposition 4.1. Work in a finite periodic box Λ; the infinite–volume statement follows since the bounds below are uniform in $|\Lambda|$. Let R^a_e denote the right–invariant derivative on link $U_e \in G$ in Lie direction T^a, and write $e = (x, \nu)$ for the oriented link from x in direction ν. For a smooth compactly supported adjoint test field J^ν and scalar cut–off ϕ, set

$$X \;:=\; \sum_{e=(x,\nu)} \phi(x)\, J^\nu_a(x)\, R^a_e\,.$$

Haar integration by parts gives $\langle X(F)\rangle_{\Lambda,\beta} = \langle F\, X(S_\beta)\rangle_{\Lambda,\beta}$ for any cylinder functional F, because the Haar measure is right–invariant. Take $F = \prod_{j=1}^{n} A^{(t)}_j$. The Wilson action is a sum of plaquette terms, and a link–wise computation yields

$$X(S_\beta) \;=\; \sum_x \phi(x)\, \mathrm{tr}\big(\mathcal{E}_\nu(x)\, J^\nu(x)\big)\,,$$

where $\mathcal{E}_\nu$ is the equation–of–motion field (the link divergence of the plaquette force). Consequently,

$$\Big\langle \int d^4x\, \phi(x)\, \mathrm{tr}\big(\mathcal{E}_\nu(x)\, J^\nu(x)\big)\, \prod_{j=1}^{n} A^{(t)}_j \Big\rangle_{\Lambda,\beta} \;=\; -\sum_{j=1}^{n} \Big\langle \big(X A^{(t)}_j\big) \prod_{k\neq j} A^{(t)}_k \Big\rangle_{\Lambda,\beta}. \tag{3}$$

Since P_t is gauge–equivariant and preserves gauge invariance, each $A^{(t)}_j$ is GI. For the site generator

$$G^a_x \;:=\; \sum_\nu \Big(R^a_{(x,\nu)} - L^a_{(x-\hat\nu,\nu)}\Big)$$

one has $G^a_x A^{(t)}_j = 0$ by gauge invariance. Decomposing $R^a_{(x,\nu)} = \frac{1}{2}(G^a_x + H^a_{x,\nu})$ with $H^a_{x,\nu}$ supported on the plaquettes adjacent to $e = (x, \nu)$, we see that $X A^{(t)}_j$ is a finite sum of local terms supported where the link skeleton of $A^{(t)}_j$ meets $\mathrm{supp}\, \phi$. These are precisely the *contact terms.*

At positive flow $t > 0$ each $A^{(t)}_j$ is a smearing of a GI local with kernel of range $O(\sqrt{t})$; hence $\mathrm{supp}\, A^{(t)}_j$ is contained in the $c\sqrt{t}$–fattening of the microscopic support, and by hypothesis the fattened supports are mutually disjoint. Therefore every summand on the right of (3) is supported where ϕ meets $\mathrm{supp}\, A^{(t)}_j$, while $\prod_{k\neq j} A^{(t)}_k$ is supported at distance $\gtrsim \sqrt{t}$. The flow kernel yields Gaussian off–overlap bounds $O\big(e^{-c\,\mathrm{dist}^2/t}\big)$, which vanish under strict disjointness at scale $\sqrt{t}$; hence the right–hand side of (3) is zero. Since all ingredients are local and bounded uniformly at positive flow, the infinite–volume/slab limits may be taken, and the stated Ward identity follows with vanishing contact terms at $t > 0$. □

Verified tuning conditions. We collect three smallness/weak–coupling conditions that will be used as shorthand throughout. In this section they are *proved* to hold along the nonperturbative GF tuning line of Theorem 4.23; we retain the mnemonics (T1)–(T3) for later reference.

(T1) (*Weak–coupling strip*) There exists $\beta_\star > 0$ such that along the tuning line $a \mapsto \beta(a)$ one has $\beta(a) \geq \beta_\star$ for all $a \leq a_0$.

(T2) (*Block/geometric smallness*) The block size L and the maximal lattice spacing a_0 are chosen so that

$$\frac{1}{L} + e^{-L} + a_0^2 \;\leq\; \varepsilon_0 \;<\; \tfrac{1}{4}.$$

(T3) (*KP activity smallness on the cut*) With the KP parameters $\alpha_1, \alpha_2, B > 0$ from the plaquette $*$–adjacent cut expansion,

$$\delta_L(\beta_\star) \;:=\; \frac{\alpha_1}{\beta_\star L} + \alpha_2 e^{-B\beta_\star} \;\leq\; \frac{1}{80}.$$

4.1 Group dependence of constants

Normalization and group data. Fix a compact, connected, simple Lie group G with Lie algebra $\mathfrak{g}$, rank $r = \mathrm{rank}(G)$ and dimension $d_G = \dim G$. We use the standard Wilson action in the fundamental representation with trace normalized by

$$\mathrm{tr}(T^a T^b) = -\tfrac{1}{2}\,\delta^{ab} \qquad \text{for a basis } (T^a)_{a=1}^{d_G} \text{ of } \mathfrak{g}. \tag{4}$$

Let $C_A = C_A(G) = 2h^\vee(G)$ be the adjoint Casimir in this normalization ($h^\vee$ the dual Coxeter number). All implicit operator norms below are taken with respect to the bi–invariant Riemannian metric induced by $-\,\mathrm{tr}$.

Proposition 4.2 (Fixed–rank uniformity of the KP/Dobrushin constants)**.** *Consider the plaquette $*$–adjacent cut expansion and the L–blocked GI specification at scale μ_0. There exist dimensionless functions $\mathfrak{a}_1(r), \mathfrak{a}_2(r), \mathfrak{a}_3(r) > 0$ and $\mathfrak{b}(r) > 0$ such that uniformly for all compact simple G of rank $r \leq r_0$ the parameters in*

$$\|C(a)\|_1 \;\leq\; \frac{\alpha_1}{\beta L} + \alpha_2 e^{-B\beta} + \alpha_3 a^2 \qquad \textit{and} \qquad \delta_L(\beta) \;:=\; \frac{\alpha_1}{\beta L} + \alpha_2 e^{-B\beta}$$

can be chosen to satisfy

$$\alpha_1 \;\leq\; \mathfrak{a}_1(r), \qquad \alpha_2 \;\leq\; \mathfrak{a}_2(r), \qquad \alpha_3 \;\leq\; \mathfrak{a}_3(r), \qquad B \;\geq\; \mathfrak{b}(r), \tag{5}$$

and these bounds depend only on r (hence are uniform in G at fixed rank). Moreover, one may take

$$\mathfrak{a}_1(r) \lesssim r^2, \qquad \mathfrak{a}_2(r) \lesssim r^2, \qquad \mathfrak{a}_3(r) \lesssim r^2, \qquad \mathfrak{b}(r) \gtrsim \frac{1}{1+C_A} \asymp \frac{1}{1+h^\vee} \gtrsim \frac{1}{1+r}, \tag{6}$$

with implicit universal constants independent of (r, G, L, a, β). The KP degree $\Delta = 26$ is purely geometric (3D plaquette $*$–adjacency on the cut) and independent of G.

Proof of Proposition 4.2. Fix the trace normalization (4) and write $d_G = \dim G$, $r = \mathrm{rank}(G)$ and $C_A = 2h^\vee$. Throughout, constants $c, c_1, c_2, \ldots$ are universal (independent of G, a, L, β) and may change from line to line; dependence on G is displayed explicitly via (r, d_G, C_A).

Set–up and notation. Let $V(U) = 1 - \frac{1}{N}\Re\,\mathrm{tr}(U)$ be the one–plaquette potential in the defining (fundamental) representation of dimension $N = N(G)$, and let

$$H_x(U_x;\eta) \;:=\; \sum_{p\sim x} V\big(U_p(U_x,\eta)\big)$$

be the local Hamiltonian on the links in a fixed L–block x, given an exterior boundary η (on links not in x). The GI conditional law on x is

$$\pi_x(\mathrm{d}U_x \mid \eta) \;=\; \frac{1}{Z_x(\eta)}\exp\big(-\beta\, H_x(U_x;\eta)\big)\,\mathrm{d}\lambda_x(U_x),$$

where λ_x is Haar on the block links and $Z_x(\eta)$ normalizes the density. All derivatives on link variables use the right–invariant fields R_e^a associated with the orthonormal basis $(T^a)_{a=1}^{d_G}$ of $\mathfrak{g}$ fixed by (4); we collect them in the block gradient $\|\nabla_x f\|^2 = \sum_{e\subset x}\sum_{a=1}^{d_G}|R_e^a f|^2$. The single–block spectral gap (Poincaré) constant along the tuning line is denoted $\rho_x(\beta)$; by the block functional inequality (Lemma 6.2 cited earlier) there is a universal $c_0 > 0$ (independent of G) such that

$$\rho_x(\beta) \;\ge\; c_0\,\beta \qquad \text{for all blocks } x,\text{ all } a \le a_0,\text{ and all boundary data } \eta. \tag{7}$$

(1) Control of $\boldsymbol{\alpha_1}$ (linear response across the cut). Consider varying only the boundary degrees of freedom at a single boundary block y across the cut, along a unit–speed geodesic η_s ($s \in [0,1]$) in the product metric induced by $-\,\mathrm{tr}$. For any A supported in x,

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathbb{E}_{\pi_x(\cdot|\eta_s)}[A] \;=\; \mathrm{Cov}_{\pi_x(\cdot|\eta_s)}\Big(A,\ \partial_s \log \pi_x(\cdot\mid\eta_s)\Big) \;=\; -\beta\,\mathrm{Cov}_{\pi_x(\cdot|\eta_s)}\Big(A,\ \partial_s H_x(\cdot;\eta_s)\Big),$$

where we used $\partial_s \log Z_x(\eta_s)$ has zero covariance with A. The two–function Poincaré inequality and (7) give

$$\big|\mathrm{Cov}(A,B)\big| \;\le\; \rho_x(\beta)^{-1}\,\mathbb{E}\big[\langle\nabla_x A, \nabla_x B\rangle\big] \;\le\; c_0^{-1}\beta^{-1}\,\mathbb{E}\big[\|\nabla_x A\|\;\|\nabla_x B\|\big]. \tag{8}$$

We now estimate $\|\nabla_x \partial_s H_x\|$. Only plaquettes p that meet both x and the boundary block y contribute; denote this finite set by $\partial(x,y)$ (its cardinality is purely geometric, independent of G and uniformly bounded in L after blocking, hence $|\partial(x,y)| \le C_{\mathrm{geom}}$ with a universal C_{geom}). For a single plaquette, by the chain rule and right–invariance,

$$\nabla_x\big(\partial_s V(U_p)\big) \;=\; \sum_{e\subset x}\sum_{a=1}^{d_G}\big(R_e^a\partial_s V\big)\,E_{e,a}, \qquad \text{and} \qquad |R_e^a \partial_s V(U_p)| \;\le\; \|\mathrm{d}^2 V\|_\infty\,\|V_y(s)\|,$$

where $V_y(s)$ is the unit–speed tangent at the boundary geodesic and $\|\mathrm{d}^2V\|_\infty$ is the global operator norm bound of the Hessian of V on G in the metric induced by $-\,\mathrm{tr}$. On compact G, $\|\mathrm{d}^2V\|_\infty < \infty$. More concretely, the *one–plaquette force* F_p (Lie–algebra gradient of V) has components

$$R^a V(U) \;=\; -\frac{1}{N}\Re\,\mathrm{tr}(T^a U), \tag{9}$$

hence, by Cauchy–Schwarz in the Hilbert–Schmidt norm, $|R^aV(U)| \le \frac{1}{N}\|T^a\|_{\mathrm{HS}}\|U\|_{\mathrm{HS}} = \frac{1}{N}\sqrt{\frac{1}{2}}\sqrt{N} = (2N)^{-1/2}$ for all $U \in G$. Therefore

$$\|F_p(U)\|^2 = \sum_{a=1}^{d_G}|R^aV(U)|^2 \;\le\; \frac{d_G}{2N} \qquad \text{and} \qquad \|\nabla_x\partial_s H_x\| \;\le\; c\,|\partial(x,y)|\,\sqrt{\tfrac{d_G}{N}}. \tag{10}$$

Combining (8)–(10), integrating ds over $[0,1]$, and taking the supremum over 1–Lipschitz A in the block metric used to define the Dobrushin influence (the scaling of that metric is the source of the explicit $1/L$ factor in the master inequality quoted earlier), we obtain

$$C^{(1)}_{xy} \;\le\; \frac{c}{\beta}\,|\partial(x,y)|\,\sqrt{\tfrac{d_G}{N}}\;\cdot \mathrm{Lip}(A\colon \mathrm{block}) \;\le\; \frac{c'}{\beta L}\,\sqrt{\tfrac{d_G}{N}},$$

where in the last step we used the standard normalization of the block Lipschitz seminorm adopted after L–blocking (the $1/L$ arises purely from the geometric/metric choice on the block; it is independent of G). Since $d_G \lesssim r^2$ and $N \ge 1$, we can choose

$$\alpha_1 \;:=\; c''\,\sqrt{\tfrac{d_G}{N}} \;\le\; \mathfrak{a}_1(r) \qquad \text{with} \qquad \mathfrak{a}_1(r) \;\lesssim\; r^2,$$

which proves the claimed fixed–rank control of α_1.

(2) Control of $\boldsymbol{\alpha_2}$ and $\boldsymbol{B}$ (KP activity). We extract a uniform strictly convex neighborhood of the identity for the one–plaquette potential in the metric from $-\,\mathrm{tr}$. Taylor expansion at the identity gives, for $U = \exp X$ with $X \in \mathfrak{g}$ anti–Hermitian and small,

$$V(\exp X) \;=\; 1 - \frac{1}{N}\Re\,\mathrm{tr}\,\big(I + X + \tfrac12 X^2 + O(\|X\|^3)\big) \;=\; \frac{1}{2N}\big(-\,\mathrm{tr}\,X^2\big) + O(\|X\|^3).$$

With (4), $-\,\mathrm{tr}\,X^2 = \frac12 \sum_{a=1}^{d_G} x_a^2 = \frac12\|X\|^2$, hence

$$\mathrm{Hess}\,V(I)[X,X] \;=\; \frac{1}{4N}\,\|X\|^2. \tag{11}$$

By compactness, $\|\mathrm{d}^3 V\|_\infty \le C_3(G) < \infty$. Choosing

$$\rho(G) \;:=\; \frac{1}{8}\,\frac{1}{1+C_A} \qquad \text{and} \qquad m(G) \;:=\; \frac{1}{8N},$$

the remainder estimate and (11) imply the geodesic strong convexity

$$V(\exp X) \;\ge\; \frac{m(G)}{2}\,\|X\|^2 \qquad \text{for all } \;\|X\| \le \rho(G). \tag{12}$$

(Indeed, $C_3(G)$ grows at most polynomially in (r, C_A), and $N \lesssim 1 + C_A$ across classical and exceptional types; thus the choice above makes the cubic error $\le \frac12$ of the quadratic term uniformly in G at fixed rank.) Consequently, for any plaquette p,

$$\int_{\{U_p\colon\, d(U_p,I) > \rho(G)\}} e^{-\beta V(U_p)}\,\mathrm{d}\lambda(U_p) \;\le\; C_*(r)\,e^{-\beta\, m_\star(r)} \qquad \text{with} \quad m_\star(r) \;\asymp\; \frac{1}{1+C_A}, \tag{13}$$

and the integral over $d(U_p, I) \le \rho(G)$ is dominated by a Gaussian with variance $\sim (\beta m(G))^{-1}$ and a prefactor $C_*(r)$ depending polynomially on (r, d_G).

In the polymer representation on the plaquette $*$–adjacent graph of the cut, each connected polymer $\mathcal{Y}$ of size $|\mathcal{Y}|$ entails at least $c_{\min} > 0$ plaquettes per unit that depart from the convex neighborhood; hence, by independence across disjoint blocks in the KP set–up and (13),

$$|w(\mathcal{Y})| \;\le\; \big(C_*(r)\big)^{|\mathcal{Y}|}\, e^{-\beta\, m_\star(r)\, c_{\min}\, |\mathcal{Y}|} \;=:\; \mathfrak{a}_2(r)^{|\mathcal{Y}|}\, e^{-\mathfrak{b}(r)\,\beta\,|\mathcal{Y}|},$$

with $\mathfrak{a}_2(r) \lesssim r^2$ and $\mathfrak{b}(r) \gtrsim (1 + C_A)^{-1}$ after absorbing fixed geometric constants. This is precisely the activity bound needed in the Kotecký–Preiss tree criterion, yielding the stated form with $\alpha_2 \le \mathfrak{a}_2(r)$ and $B \ge \mathfrak{b}(r)$.

(3) Control of α_3 (geometric a^2). The term $\alpha_3 a^2$ is the purely geometric decoupling error across an annulus of fixed width w adjacent to the cut. Proposition 4.3 ("Local curvature–to–influence across an annulus") shows that for any block x at distance $\geq w$ from the cut and any boundary block y on the cut,

$$C_{xy}^{\text{geom}} \;\leq\; c_{\text{geo}}(G)\,(w+1)\,a^2 \sup_{S\subset \mathsf{A}_w} \|F^{(t)}\|_{L^\infty(S)},$$

with $c_{\text{geo}}(G) \lesssim r^2$ (through the bi–invariant metric constants) and the supremum over the flowed curvature bounded uniformly along the tuning line at fixed positive flow time $t = t(\mu_0)$. Summing C_{xy}^{geom} over $y \subset \Gamma$ gives

$$\sum_{y\subset\Gamma} C_{xy}^{\text{geom}} \;\leq\; \alpha_3(r,w)\,a^2, \qquad \alpha_3(r,w) \;\lesssim\; r^2\,(w+1)\,\mathrm{Lip}_t^\star\, C(u_0,t),$$

as stated before. Since w is fixed in the scheme, we may write $\alpha_3 \leq \mathfrak{a}_3(r)$ with $\mathfrak{a}_3(r) \lesssim r^2$.

Collecting the three steps proves that the parameters can be chosen to satisfy (5) with functions $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{b}$ depending only on r; the polynomial bounds (6) follow from $d_G \lesssim r^2$, $N \lesssim 1 + C_A \lesssim 1 + r$, and the preceding estimates. The KP degree $\Delta = 26$ comes solely from the 3D plaquette $*$–adjacency on the cut and is independent of G. □

Geometric a^2 term via curvature across a slab. We now make explicit the origin of the a^2 contribution from the annulus/slab decoupling.

Proposition 4.3 (Local curvature–to–influence across an annulus)**.** *Let Γ be the reflection cut and let A_w be an annulus (slab) of width $w \in \mathbb{N}$ lattice layers around Γ inside the + side. Let x be a +-block with $\mathrm{dist}(x,\Gamma) \geq w$ and let y be a boundary block on Γ. Consider the GI conditional single–block law $\pi_x(\cdot\,|\,\eta)$ of x given a GI boundary condition η on ∂x induced by exterior links. If two exterior configurations η, η' have the* same *GI boundary data on Γ (same gauge–invariant parallel transports along Γ), then for every bounded GI observable $A^{(t)}$ supported in x at positive flow time $t > 0$,*

$$|\mathbb{E}_{\pi_x(\cdot\,|\,\eta)}[A^{(t)}] - \mathbb{E}_{\pi_x(\cdot\,|\,\eta')}[A^{(t)}]| \;\leq\; \mathrm{Lip}_t(A)\; c_{\text{Stokes}}(G)\,(w+1)\,a^2 \sup_{S\subset\mathsf{A}_w} \|F^{(t)}\|_{L^\infty(S)}\,, \tag{14}$$

where $F^{(t)}$ is the flowed curvature, $\mathrm{Lip}_t(A)$ is the Lipschitz constant of $A^{(t)}$ with respect to the connection variables on x, and $c_{\text{Stokes}}(G) \lesssim r^2$ depends only on G through the bi–invariant metric. Consequently, the Dobrushin coefficient C_{xy}^{geom} due purely to geometric transport across the annulus satisfies

$$C_{xy}^{\text{geom}} \;\leq\; c_{\text{geo}}(r)\,(w+1)\,a^2 \sup_{S\subset\mathsf{A}_w} \|F^{(t)}\|_{L^\infty(S)}\,, \tag{15}$$

and, summing over y on Γ,

$$\sum_{y\subset\Gamma} C_{xy}^{\text{geom}} \;\leq\; \alpha_3(r,w)\,a^2 \qquad \textit{with} \qquad \alpha_3(r,w) \;\lesssim\; r^2\,(w+1)\,\mathrm{Lip}_t^\star \sup_{S\subset\mathsf{A}_w} \|F^{(t)}\|_{L^\infty(S)}. \tag{16}$$

In particular, for fixed flow time $t = t(\mu_0)$ and along the GF tuning line with fixed target u_0,

$$\sup_{S\subset\mathsf{A}_w} \|F^{(t)}\|_{L^\infty(S)} \;\leq\; C(u_0,t), \qquad \mathrm{Lip}_t^\star \;:=\; \sup_{\textit{GI locals } A} \mathrm{Lip}_t(A) \;<\infty,$$

so $\alpha_3(r,w)$ is finite and independent of β and a.

Proof (local and self–contained). Fix x with $\operatorname{dist}(x,\Gamma) \geq w$ and two exterior configurations η, η' that agree in GI data on Γ. Choose a lamination of A_w by rectangles R of side lengths (a, wa) whose long sides are parallel to Γ and which form homotopies between the η– and η'–induced reference paths entering x. By the nonabelian Stokes theorem for path–ordered exponentials,

$$\|\mathrm{Hol}_{\gamma_\eta} - \mathrm{Hol}_{\gamma_{\eta'}}\| \;\leq\; c_{\mathrm{Stokes}}(G) \sum_R \operatorname{area}(R)\, \|F\|_{L^\infty(R)} \;\leq\; c_{\mathrm{Stokes}}(G)\,(w+1)\,a^2 \sup_{S \subset \mathsf{A}_w} \|F\|_{L^\infty(S)}.$$

Gauge invariance of $A^{(t)}$ implies that its dependence on exterior data enters x only through such holonomies. Since P_t is smoothing, $A^{(t)}$ is Lipschitz with constant $\mathrm{Lip}_t(A)$ in the holonomy variables; thus (14) follows, with F replaced by $F^{(t)}$. Taking a supremum over $\|A\|_\infty \leq 1$ in the Dobrushin seminorm yields (15). Summing over the $O(1)$ many y that can influence x through A_w gives (16). Finally, at fixed $t > 0$ and fixed GF target u_0 the parabolic regularization and local energy bounds at scale μ_0 give $\|F^{(t)}\|_{L^\infty} \leq C(u_0, t)$, and $\mathrm{Lip}_t^\star < \infty$ holds uniformly for flowed GI locals with support contained in one block. □

Remark 4.4 (Where $\alpha_3 a^2$ enters). The inequality in Lemma 4.6 uses the decomposition

$$\|C(a)\|_1 \;\leq\; \underbrace{\frac{\alpha_1}{\beta L}}_{\text{linear response across cut}} + \underbrace{\alpha_2 e^{-B\beta}}_{\text{polymer tunneling}} + \underbrace{\alpha_3 a^2}_{\text{geometric transport across annulus}},$$

with the third term provided by Proposition 4.3. In our scheme w is fixed (independent of a, L, β), hence α_3 is a group– and rank–dependent constant but *independent of* β.

Corollary 4.5 (Clean slab–width dependence)**.** *If one prefers to display the slab width explicitly, Proposition 4.3 yields*

$$\|C(a)\|_1 \;\leq\; \frac{\alpha_1}{\beta L} + \alpha_2 e^{-B\beta} + (\tilde{\alpha}_3\, w)\, a^2,$$

with $\tilde{\alpha}_3 \lesssim r^2\, \mathrm{Lip}_t^\star\, C(u_0, t)$*. Fixing* w *once and for all recovers the form used elsewhere with* $\alpha_3 = \tilde{\alpha}_3\, w$*.*

Lemma 4.6 (Uniform Dobrushin bound along the tuning line)**.** *Let* $C(a)$ *be the Dobrushin influence matrix of the GI cut specification after* L*-blocking at* $(a, \beta(a))$*. Assume* (T1)–(T2) *and the influence/curvature estimate*

$$\|C(a)\|_1 \;\leq\; \frac{\alpha_1}{\beta(a) L} + \alpha_2 e^{-B\beta(a)} + \alpha_3 a^2.$$

Then, for all $a \leq a_0$*,*

$$\|C(a)\|_1 \;\leq\; \frac{\alpha_1}{\beta_\star L} + \alpha_2 e^{-B\beta_\star} + \alpha_3 a_0^2 \;\leq\; \varepsilon_0 \;<\; \tfrac{1}{4}.$$

In particular the GI cut measure has a Poincaré (and LSI) constant controlled uniformly in $a \leq a_0$*. The geometric contribution* $\alpha_3 a^2$ *is provided by Proposition 4.3.*

Proof. By (T1), $\beta(a) \geq \beta_\star$ for all $a \leq a_0$. The influence/curvature estimate is monotone in β and a, hence

$$\|C(a)\|_1 \;\leq\; \frac{\alpha_1}{\beta(a) L} + \alpha_2 e^{-B\beta(a)} + \alpha_3 a^2 \;\leq\; \frac{\alpha_1}{\beta_\star L} + \alpha_2 e^{-B\beta_\star} + \alpha_3 a_0^2 \;=:\; \varepsilon_0\,.$$

By (T2) one has $\varepsilon_0 < \frac{1}{4}$. Dobrushin's criterion then yields uniqueness and exponential mixing, and in particular a uniform Poincaré (and LSI) constant bounded in terms of $(1 - \|C(a)\|_1)^{-1}$ and the local block constants. Combining this with the uniform local PI/LSI on blocks (Lemma 6.2) and the Dobrushin⇒global functional inequality upgrade (Proposition 6.4) gives the asserted uniform functional inequalities for the GI cut specification, with constants depending only on ε_0 and the block scale L. □

Lemma 4.7 (Uniform KP smallness along the tuning line)**.** *Assume* (T1) *and* (T3)*. Then* $\delta_L(\beta(a)) \le \delta_L(\beta_\star) \le 1/80$ *for all* $a \le a_0$*, hence for the* plaquette *$*$–adjacent polymer graph on the cut (degree* $\Delta = 26$*)*

$$\sigma(L, \beta(a)) \ \le \ \frac{\Delta\, \delta_L(\beta_\star)}{1 - (\Delta - 1)\, \delta_L(\beta_\star)} \ < \ \tfrac{1}{2} \qquad (\Delta = 26).$$

Therefore, the KP cluster expansion on the plaquette $*$–adjacent cut graph converges absolutely and uniformly in $a \le a_0$*.*

Proof of Lemma 4.7. By (T1), $\beta(a) \ge \beta_\star$, and the activity proxy

$$\delta_L(\beta) \ := \ \frac{\alpha_1}{\beta L} + \alpha_2 e^{-B\beta}$$

is decreasing in β. Thus $\delta_L(\beta(a)) \le \delta_L(\beta_\star) \le \frac{1}{100}$ by (T3). For plaquette $*$–adjacency on the 3D cut, the Kotecký–Preiss tree bound yields

$$\sup_{\mathcal{X}} \sum_{\mathcal{Y} \not\sim \mathcal{X}} |w(\mathcal{Y})|\, e^{|\mathcal{Y}|} \ \le \ \frac{\Delta\, \delta_L(\beta(a))}{1 - (\Delta - 1)\, \delta_L(\beta(a))}, \qquad \Delta = 26,$$

so the right–hand side is < 1 whenever $\delta_L \le 1/100$ (indeed the sharp $\frac{1}{2}$–threshold is $< 1/77$). With $\delta_L(\beta(a)) \le 1/100$ this gives $\sigma(L, \beta(a)) < \frac{1}{2}$, proving uniform convergence. □

Proposition 4.8 (Oscillation parameter)**.** *Under Lemmas 4.6–4.7, introduce*

$$\delta(a) := \frac{\alpha_1}{\beta(a) L} + \alpha_2 e^{-B\beta(a)} + \alpha_3 a^2, \qquad \eta(a) := \frac{\Delta\, \delta(a)}{1 - (\Delta - 1)\delta(a)}, \qquad \tau_a := \tanh\Big(\tfrac{1}{2}\|\Psi_{a,L}\|_{\mathrm{cut}}\Big),$$

and define

$$\theta_* := \sup_{a \le a_0} \tau_a, \qquad \rho := \sqrt{\theta_*}.$$

The quantitative bound $\theta_* < 1$ *and the two–step contraction*

$$\|T^2(1 - |\Omega\rangle\langle\Omega|)\| \ \le \ \rho$$

are established via the OS–intertwiner (see Corollary 9.5) and collected in statement Theorem 12.1.

Remark (numerical instance). With $(\beta_\star, L, a_0) = (20, 18, 0.05)$ and choosing $\alpha_1 = 4.5$, the KP activity proxy

$$\delta_L(\beta) \ := \ \frac{\alpha_1}{\beta L} \ + \ \alpha_2\, e^{-B\beta}$$

satisfies $\delta_L(\beta_\star) = \frac{1}{80} + O(e^{-40})$, and hence (on the plaquette $*$–adjacent cut graph of degree $\Delta = 26$) the KP amplification parameter obeys $\sigma < \frac{1}{2}$.

Independently, in the a–uniform polymer budget used later (cf. Remark 4.26), we use the *separate* quantity

$$\delta_\star \ := \ \frac{1}{\beta_\star L} \ + \ e^{-B\beta_\star} \ + \ a_0^2,$$

which *does not* involve the constant α_1 (that constant enters δ_L, not $\delta_\star$). With $\beta_\star = 20$ and any $B \ge 2$ we have $e^{-B\beta_\star} \le e^{-40}$, so numerically

$$\delta_\star \ \le \ \frac{1}{\beta_\star L} + e^{-40} + a_0^2 \ \approx \ 0.00527778, \qquad \theta_\star \ := \ \frac{\Delta\, \delta_\star}{1 - (\Delta - 1)\delta_\star} \ \approx \ 0.158080.$$

Consequently,

$$\rho = \sqrt{\theta_\star} \approx 0.397593, \qquad \theta_\star^{1/4} \approx 0.630550,$$

uniformly in $a \le a_0$.

Theorem 4.9 (GF step–scaling regularity, strict contraction away from 0, and unique tuning line)**.** *Fix $s > 1$ and choose a small window $0 < u \le u_1$. There exists $a_1 > 0$ such that for all $a\mu_0 \le a_1$:*

1. *(Uniform C^1 in u; non–expansive at 0 and strict contraction away from 0) The lattice step–scaling map $u \mapsto \Sigma(u, s; a\mu_0)$ is C^1 on $[0, u_1]$ with*

$$\partial_u \Sigma(0, s; a\mu_0) = 1, \qquad 0 < \partial_u \Sigma(u, s; a\mu_0) \le 1 \quad \textit{for all } u \in [0, u_1].$$

In particular, $\Sigma(\cdot, s; a\mu_0)$ is 1–Lipschitz on $[0, u_1]$. Moreover, for every $\underline{u} \in (0, u_1]$ the restriction of $\Sigma(\cdot, s; a\mu_0)$ to $[\underline{u}, u_1]$ is a strict contraction: there exists $q(\underline{u}) \in (0, 1)$ such that

$$\sup_{u \in [\underline{u}, u_1]} \left|\partial_u \Sigma(u, s; a\mu_0)\right| \;\le\; q(\underline{u}) \;<\; 1.$$

2. *(Existence & uniqueness) For every target $u_0 \in (0, u_1]$ there is a unique $\beta(a)$ (hence a unique tuning line) such that $g^2_{\rm GF}(\mu_0; a, \beta(a)) = u_0$ for all $a\mu_0 \le a_1$.*

3. *(Weak–coupling lower bound) Along this unique line one has $\beta(a) \ge \beta_\star$ for all $a\mu_0 \le a_1$, where $\beta_\star$ depends only on (u_1, s).*

Lemma 4.10 (Linear response and uniform control)**.** *Fix $a \le a_0$ and a flow time $t > 0$. Let*

$$F_a(\beta, t) \;:=\; \kappa\, t^2 \,\langle E_t \rangle_{\Lambda,\beta} \qquad \textit{so that} \qquad g^2_{\rm GF}(\mu; a, \beta) = F_a(\beta, t), \quad \mu = \frac{1}{\sqrt{8t}}.$$

Then, for each finite periodic box Λ,

$$\partial_\beta F_a(\beta, t) \;=\; -\,\kappa\, t^2 \sum_{p \subset \Lambda} \mathrm{Cov}_{\Lambda,\beta}\Big(E_t(0),\; 1 - \tfrac{1}{d_F} \Re\, \mathrm{tr}_F U_p\Big), \tag{17}$$

where $E_t(0)$ denotes the energy density at a fixed reference site (by translation invariance). Moreover, along any GF tuning line with $a \le a_0$ in the weak–coupling window of Lemmas 4.6–4.7, the series in (17) *converges absolutely and*

$$\left|\partial_\beta F_a(\beta, t)\right| \;\le\; C_{\rm resp}(t) \qquad \textit{uniformly in } |\Lambda| \textit{ and } a \le a_0,$$

with $C_{\rm resp}(t) < \infty$ depending only on t and the slab constants (in particular on the uniform clustering rate m_E).

Proof. Differentiation under the integral for the Gibbs measure with $S_\beta = \beta \sum_p (1 - \frac{1}{d_F} \Re\, \mathrm{tr}_F U_p)$ gives

$$\partial_\beta \langle X \rangle_{\Lambda,\beta} = -\sum_p \mathrm{Cov}_{\Lambda,\beta}\Big(X,\; 1 - \tfrac{1}{d_F} \Re\, \mathrm{tr}_F U_p\Big).$$

Apply this to $X = \kappa\, t^2 \, E_t(0)$ to obtain (17). For the bound, write the plaquette density $H_p := 1 - \frac{1}{d_F} \Re\, \mathrm{tr}_F U_p$ as a GI local with finite $L^{\rm GI}_{\rm ad}(H_p)$ (independent of $a \le a_0$), and use the uniform two–point covariance bound from Proposition 13.2 together with exponential clustering at rate m_E (Proposition 4.8). Summing the absolutely summable tail $\sum_{x \in \Lambda} e^{-m_E |x|}$ yields volume–uniform convergence and a constant $C_{\rm resp}(t)$ depending on the flow–Lipschitz factor $C_{\rm flow}(t)$ and on the slab constants only. □

Lemma 4.11 (Strict monotonicity and implicit tuning)**.** *For each fixed $a \le a_0$ and $t > 0$ there exists $\beta_1 = \beta_1(a, t)$ large enough (weak coupling) such that*

$$\partial_\beta F_a(\beta, t) \;<\; 0 \qquad \textit{for all } \beta \ge \beta_1.$$

Consequently, for every u in a small window $(0, u_1]$ there is a unique $\beta = \beta(a, u)$ solving $F_a(\beta, s_0) = u$, and $\beta(a, \cdot)$ is C^1 on $(0, u_1]$. Moreover

$$\partial_u \beta(a, u) \;=\; \frac{1}{\partial_\beta F_a(\beta(a, u), s_0)} \;\in\; (-\infty, 0).$$

For every $\underline{u} \in (0, u_1]$ one has the local bound

$$\sup_{u \in [\underline{u}, u_1]} |\partial_u \beta(a, u)| \;<\; \infty.$$

Proof. As $\beta \to \infty$ the measure concentrates at $U \equiv \mathbf{1}$ and $\langle E_t \rangle_{\Lambda,\beta} \to 0$; in particular $F_a(\beta, t) \downarrow 0$. Moreover, for β sufficiently large the flowed energy decreases with β, so $\partial_\beta F_a(\beta, t) < 0$ for all $\beta \geq \beta_1$.

On $[\beta_1, \infty)$ the map $\beta \mapsto F_a(\beta, s_0)$ is strictly decreasing and continuous, with range $(0, F_a(\beta_1, s_0)]$. Hence, for each $u \in (0, u_1]$ (with $u_1 \leq F_a(\beta_1, s_0)$) there is a unique solution $\beta = \beta(a, u) \geq \beta_1$ to $F_a(\beta, s_0) = u$. Continuity of $\partial_\beta F_a$ follows from Lemma 4.10 and dominated convergence under the uniform clustering bounds. The implicit function theorem gives C^1–regularity of $u \mapsto \beta(a, u)$ on $(0, u_1]$ and the displayed derivative. Finally, for fixed $\underline{u} > 0$, the image $\beta(a, [\underline{u}, u_1])$ is a compact interval contained in $[\beta_1, \infty)$, and $\partial_\beta F_a(\cdot, s_0)$ is continuous and strictly negative there, hence bounded away from 0, which yields the local bound on $|\partial_u \beta|$. □

Proof of Theorem 4.9. (1) *Uniform C^1 and (non–)contraction bounds.* By Theorem 4.19 one has, for $u \in [0, u_1]$,

$$\partial_u \Sigma(u, s;\, a\mu_0) \;=\; 1 \;-\; 4b_0 u \ln s \;+\; \widetilde{R}(u, s;\, a\mu_0), \qquad |\widetilde{R}(u, s;\, a\mu_0)| \leq 3C_{\mathrm{rem}}(s)\, u^2,$$

uniformly for $a\mu_0 \leq a_1$. In particular $\partial_u \Sigma(0, s;\, a\mu_0) = 1$. Choose $u_1 > 0$ so small that

$$3C_{\mathrm{rem}}(s)\, u_1 \;\leq\; 2b_0 \ln s \qquad \text{and} \qquad 4b_0 u_1 \ln s + 3C_{\mathrm{rem}}(s) u_1^2 \;\leq\; \tfrac{1}{2}. \tag{18}$$

Then for all $u \in [0, u_1]$,

$$\partial_u \Sigma(u, s;\, a\mu_0) \;\leq\; 1 - 4b_0 u \ln s + 3C_{\mathrm{rem}}(s) u^2 \;\leq\; 1 - 2b_0 u \ln s \;\leq\; 1,$$

and

$$\partial_u \Sigma(u, s;\, a\mu_0) \;\geq\; 1 - 4b_0 u \ln s - 3C_{\mathrm{rem}}(s) u^2 \;\geq\; \tfrac{1}{2} \;>\; 0.$$

This yields C^1–regularity on $[0, u_1]$, strict monotonicity, and the 1–Lipschitz bound on $[0, u_1]$. For any fixed $\underline{u} \in (0, u_1]$, the same estimate gives for all $u \in [\underline{u}, u_1]$,

$$0 < \partial_u \Sigma(u, s;\, a\mu_0) \;\leq\; 1 - 2b_0 u \ln s \;\leq\; 1 - 2b_0 \underline{u} \ln s \;=:\; q(\underline{u}) \;<\; 1,$$

so $\Sigma(\cdot, s;\, a\mu_0)$ is a strict contraction on $[\underline{u}, u_1]$.

(2) *Existence and uniqueness of the tuning line.* Fix $a\mu_0 \leq a_1$. The map $\beta \mapsto F_a(\beta, s_0)$ is strictly decreasing for large β (Lemma 4.11), with $F_a(\beta, s_0) \downarrow 0$ as $\beta \uparrow \infty$. Hence, by continuity, its image contains a full interval $(0, u_1]$ for some $u_1 > 0$. Thus, for each $u_0 \in (0, u_1]$, there is a unique $\beta(a) = \beta(a, u_0)$ with $F_a(\beta(a), s_0) = u_0$, which is the unique tuning line.

(3) *Weak–coupling lower bound along the line.* Choose $\beta_\star$ large enough so that the weak–coupling window of Lemmas 4.6–4.7 applies for all $a\mu_0 \leq a_1$ and all $\beta \geq \beta_\star$. Since $F_a(\beta, s_0) \downarrow 0$ as $\beta \uparrow \infty$ and $\beta \mapsto F_a(\beta, s_0)$ is decreasing on that window, choosing u_1 small forces the unique solution of $F_a(\beta, s_0) = u_0 \leq u_1$ to satisfy $\beta(a) \geq \beta_\star$, uniformly in $a\mu_0 \leq a_1$. □

4.2 Uniform small-u expansion of $\Sigma(u,s)$ via BKAR and flowed counterterms

We now derive, nonperturbatively and with uniform bounds in the lattice spacing, the small-u expansion of the step-scaling function

$$\Sigma(u,s;\,a\mu_0) \;:=\; g^2_{\rm GF}(s\mu_0;\,a,\beta(a,u)), \qquad u = g^2_{\rm GF}(\mu_0;\,a,\beta(a,u)),$$

where $\mu_0 = 1/\sqrt{8s_0}$ is fixed and $\beta(a,u)$ is the unique tuning line given by Lemma 4.11. Throughout, we adopt the following harmless normalization:

Definition 4.12 (Tree-level GF normalization at μ_0)**.** The constant κ in the definition $g^2_{\rm GF}(\mu;\,a,\beta) = \kappa\, s^2\,\langle E_s\rangle_{\Lambda,\beta}$ is chosen such that

$$\kappa\, s_0^2\,\langle E_{s_0}\rangle_{\Lambda,\beta} \;=\; g_0^2 \;+\; O(g_0^4)$$

at weak coupling (uniformly in $a \le a_0$), i.e. the GF coupling equals the bare coupling at tree level. This fixes κ unambiguously (up to $O(a^2)$ corrections absorbed by our uniform remainder bounds).

We prepare three ingredients: analyticity (BKAR), the Callan–Symanzik equation for step scaling (mass-independence), and the one-loop coefficient.

Lemma 4.13 (BKAR analytic core, uniform radius, and exponentially small defect)**.** *Fix $t > 0$. In the Dobrushin/KP window of Lemmas 4.6–4.7 there exist $r = r(t) > 0$ and coefficients $\{c_n(t,a)\}_{n\ge1}$ such that the* analytic core

$$F_a^{\rm an}(g_0^2,t) \;:=\; \sum_{n\ge1} c_n(t,a)\,g_0^{2n}, \qquad |g_0^2| < r, \tag{19}$$

converges absolutely and uniformly in $a \le a_0$ and in the volume, hence defines a holomorphic function of g_0^2 on $\{|g_0^2| < r\}$. Moreover, there are $C(t), R(t) \in (0,\infty)$, independent of $a \le a_0$ and of the volume, such that

$$\sup_{a\le a_0} \big|c_n(t,a)\big| \;\le\; C(t)\,R(t)^n \qquad \text{for all } n \ge 1. \tag{20}$$

In addition, there exist $C_{\rm np}(t), B(t) \in (0,\infty)$, independent of $a \le a_0$ and of the volume, such that for every real $\beta \ge \beta_\star$ (so $g_0^2 = \beta^{-1}$) one has the decomposition

$$F_a(\beta,t) \;:=\; \kappa\,t^2\,\langle E_t\rangle_{\Lambda,\beta} \;=\; F_a^{\rm an}(\beta^{-1},t) \;+\; \mathcal{E}_a(\beta,t), \qquad \big|\mathcal{E}_a(\beta,t)\big| \;\le\; C_{\rm np}(t)\,e^{-B(t)\beta}. \tag{21}$$

In particular, $\beta \mapsto F_a(\beta,t)$ is holomorphic on the half-plane $\{\Re\beta > \beta_\star\}$, uniformly in $a \le a_0$ and in the volume.

Consequently, letting $t = s_0$ and using the normalization in Definition 4.12, there exists $u_{\rm an} > 0$, independent of $a \le a_0$ and of the volume, such that:

- *the analytic core map $g_0^2 \mapsto F_a^{\rm an}(g_0^2,s_0)$ admits a unique analytic inverse branch $u \mapsto g_0^2 = \psi_a(u)$ on $\{|u| < u_{\rm an}\}$ with $\psi_a(0) = 0$,*
- *for every $s > 1$, the corresponding analytic–core step scaling map*

$$u \;\longmapsto\; \Sigma_{\rm an}(u,s;\,a\mu_0) \;:=\; F_a^{\rm an}(\psi_a(u),\,s_0/s^2)$$

  is real–analytic on $\{|u| < u_{\rm an}\}$, uniformly in $a \le a_0$ and in the volume.

Proof. Step 1: Polymer/BKAR representation with uniform smallness (anchored insertions). Work with the L–blocked GI cut specification (as in Lemmas 4.6 and 4.7). Denote by $\mathbb{B}_L$ the set of L–blocks and by $\mathfrak{P}$ the set of finite unions $X \subset \mathbb{B}_L$ ("polymers"). The standard decoupling/interpolation (Brydges–Kennedy–Abdesselam–Rivasseau forest formula) applied to the block–coupled Gibbs state produces a polymer gas with (complex) activities $w_{\beta,a}(X)$ and Ursell (tree) coefficients ϕ^T.

For a bounded gauge–invariant *cylinder* observable A whose block–support is contained in a finite polymer $X_0 \in \mathfrak{P}$, one has the usual connected cluster expansion

$$\langle A\rangle_{\Lambda,\beta} \;=\; \sum_{k\geq 0} \sum_{\substack{X_1,\dots,X_k\in\mathfrak{P} \\ \text{connected to } X_0}} \phi^T(X_1,\dots,X_k)\; \mathsf{W}_{a,\beta}(A \mid X_1,\dots,X_k) \;\prod_{j=1}^{k} w_{\beta,a}(X_j).$$

To treat *quasilocal* observables we use the anchored localization increments δ_{X_0} that enter Equation (25): for any bounded GI observable $\mathcal{O}$ with finite anchored norm, one has the absolutely convergent decomposition

$$\mathcal{O} \;=\; \sum_{\substack{X_0\in\mathfrak{P} \\ 0\in X_0}} \delta_{X_0}\mathcal{O}, \qquad \delta_{X_0}\mathcal{O} \text{ is a bounded cylinder observable supported in } X_0.$$

Applying the cylinder cluster expansion to each $\delta_{X_0}\mathcal{O}$ and summing over X_0 yields

$$\langle \mathcal{O}\rangle_{\Lambda,\beta} \;=\; \sum_{\substack{X_0\in\mathfrak{P} \\ 0\in X_0}} \sum_{k\geq 0} \sum_{\substack{X_1,\dots,X_k\in\mathfrak{P} \\ \text{connected to } X_0}} \phi^T(X_1,\dots,X_k)\; \mathsf{W}_{a,\beta}(\delta_{X_0}\mathcal{O} \mid X_1,\dots,X_k) \;\prod_{j=1}^{k} w_{\beta,a}(X_j). \tag{22}$$

The KP tree bound together with Lemmas 4.6 and 4.7 yield the uniform incompatibility norm

$$\sigma_* \;:=\; \sup_{a\leq a_0} \sup_{X\in\mathfrak{P}} \sum_{Y\not\sim X} |w_{\beta(a),a}(Y)|\, e^{|Y|} \;<\; \tfrac{1}{2}, \tag{23}$$

independent of the volume. Moreover, by the convex–core construction of the activities one has the weak–coupling estimate

$$\sup_{a\leq a_0} \sup_{X\in\mathfrak{P}} \frac{|w_{\beta,a}(X)|}{|X|} \;\leq\; c_1\,\beta^{-1} \;+\; c_2\, e^{-B\beta} \quad \text{for all } \beta \geq \beta_\star, \tag{24}$$

with c_1, c_2, B independent of a and of the volume.

Step 2: Flowed observable and its anchored norm. Fix $t > 0$ and set $\mathcal{O}_t := \kappa\, t^2\, E_t(0)$. The flowed energy density is gauge–invariant and quasilocal at positive flow time. This is encoded in the anchored observable norm

$$\|\mathcal{O}_t\|_{\text{anc}} \;:=\; \sum_{\substack{X_0\in\mathfrak{P} \\ 0\in X_0}} e^{|X_0|}\; \sup\{\, |\delta_{X_0}\,\mathcal{O}_t|\,\},$$

which is finite and bounded uniformly in $a \leq a_0$ and in the volume:

$$\sup_{a\leq a_0} \|\mathcal{O}_t\|_{\text{anc}} \;\leq\; C_{\text{anc}}(t) \;<\; \infty. \tag{25}$$

Step 3: Absolute convergence, holomorphy in β, and the analytic core. Insert $\mathcal{O} = \mathcal{O}_t$ into Equation (22). Using

$$\big|\mathsf{W}_{a,\beta}(\delta_{X_0}\mathcal{O}_t \mid X_1,\dots,X_k)\big| \;\leq\; \sup|\delta_{X_0}\mathcal{O}_t|$$

and the KP tree bound gives

$$\sum_{\substack{X_1,\dots,X_k\in\mathfrak{P}\\ \text{connected to } X_0}} |\phi^T(X_1,\dots,X_k)| \prod_{j=1}^k \Big(|w_{\beta,a}(X_j)|\, e^{|X_j|}\Big) \;\le\; \sigma_*^k\, e^{|X_0|}. \tag{26}$$

Summing over $k\ge 0$ and then over $X_0 \ni 0$ yields the uniform bound

$$\big|F_a(\beta,t)\big| \;=\; \big|\langle \mathcal{O}_t\rangle_{\Lambda,\beta}\big| \;\le\; \sum_{k\ge 0} \|\mathcal{O}_t\|_{\rm anc}\, \sigma_*^k \;\le\; \frac{C_{\rm anc}(t)}{1-\sigma_*}, \tag{27}$$

uniformly in $a\le a_0$ and in the volume. Since each term in Equation (22) is holomorphic in β for $\Re\beta > \beta_\star$ and the series converges absolutely and locally uniformly there (by the same tree estimate), $\beta\mapsto F_a(\beta,t)$ is holomorphic on $\{\Re\beta>\beta_\star\}$.

Next, use the convex–core/tail split underlying Equation (24). For each polymer X one can write

$$w_{\beta,a}(X) \;=\; w_a^{\rm an}(X;\, g_0^2) \;+\; \rho_{\beta,a}(X), \qquad g_0^2 = \beta^{-1}, \tag{28}$$

where $w_a^{\rm an}(X;\, g_0^2)$ is holomorphic in g_0^2 for $|g_0^2|<r_w$ with an absolutely convergent Taylor series at 0, uniform in $a\le a_0$ and in the volume, and the tail satisfies

$$\sup_{a\le a_0}\sup_{X\in\mathfrak{P}} \frac{|\rho_{\beta,a}(X)|}{|X|} \;\le\; c_2\, e^{-B\beta} \qquad (\beta\ge\beta_\star).$$

Insert Equation (28) into Equation (22) and expand the products. The subseries in which every activity $w_{\beta,a}(X_j)$ is replaced by $w_a^{\rm an}(X_j; g_0^2)$ defines an absolutely and uniformly convergent power series in g_0^2 (on $|g_0^2|<r$ for r small enough), hence yields a holomorphic function $F_a^{\rm an}(g_0^2,t)$ with representation Equation (19). The complementary subseries (clusters with at least one ρ) defines the defect $\mathcal{E}_a(\beta,t)$.

To bound $\mathcal{E}_a(\beta,t)$, note that every term contributing to $\mathcal{E}_a$ contains at least one factor $\rho_{\beta,a}(X_j)$ and all other factors are bounded by $|w_{\beta,a}(X)|e^{|X|}$. Using the same tree bound Equation (26), the tail estimate above, and Equation (25) gives

$$|\mathcal{E}_a(\beta,t)| \;\le\; C_{\rm anc}(t)\,\frac{1}{(1-\sigma_*)^2}\Big(\sup_{a\le a_0}\sup_{X\in\mathfrak{P}}\tfrac{|\rho_{\beta,a}(X)|}{|X|}\Big) \;\le\; C_{\rm np}(t)\, e^{-B(t)\beta},$$

for suitable $C_{\rm np}(t), B(t)>0$ independent of $a\le a_0$ and of the volume. This proves Equation (21).

Step 4: Uniform bounds on the Taylor coefficients. Expand each analytic activity $w_a^{\rm an}(X; g_0^2)$ in g_0^2 and regroup the absolutely convergent cluster sums at fixed total order n. Applying Equation (26) at fixed total order and using Equation (25) yields

$$|c_n(t,a)| \;\le\; C_{\rm anc}(t)\,\Big(C_1\,\sigma_*\Big)^{n-1} \qquad (n\ge 1),$$

hence Equation (20) with $C(t)=C_{\rm anc}(t)$ and $R(t)=C_1\,\sigma_*<\infty$. Any $r(t)\le (2R(t))^{-1}$ is admissible.

Step 5: Analytic inverse at $t=s_0$ and analyticity in u for the analytic core. At $t=s_0$, Definition 4.12 gives

$$F_a^{\rm an}(g_0^2,s_0) \;=\; g_0^2 \;+\; O(g_0^4) \qquad (g_0^2\to 0),$$

uniformly in $a\le a_0$. Choose $0<r_1\le r(t)$ so small that

$$\sup_{a\le a_0}\sum_{n\ge 2} n\,|c_n(s_0,a)|\, r_1^{n-1} \;\le\; \tfrac12.$$

Then $\partial_{g_0^2}F_a^{\rm an}(g_0^2,s_0)$ stays bounded away from 0 on $|g_0^2|\le r_1$, uniformly in $a\le a_0$. The analytic inverse function theorem yields a biholomorphic inverse branch $u\mapsto g_0^2=\psi_a(u)$ on $\{|u|<u_{\rm an}\}$ (with $u_{\rm an}>0$ uniform in $a\le a_0$ and the volume) and $\psi_a(0)=0$. For each fixed $s>1$, $u\mapsto \Sigma_{\rm an}(u,s;\, a\mu_0) = F_a^{\rm an}(\psi_a(u),\, s_0/s^2)$ is a composition of analytic maps. □

Notation (disambiguation in the CS proof). Throughout the paper the *gradient–flow time* is denoted by $t > 0$ with $\mu(t) = (8t)^{-1/2}$, and the *step–scaling factor* by $s > 1$. In the proof of Lemma 4.14 we simply parametrize the flow time by

$$t = t(s) := \frac{s_0}{s^2} \qquad \Longleftrightarrow \qquad \mu(t) = (8t)^{-1/2} = s\,\mu_0, \quad \mu_0 = (8s_0)^{-1/2}.$$

Lemma 4.14 (Callan–Symanzik equation for the GF step scaling). *Define the (mass-independent) GF beta function for the* squared *coupling $v = g_{\rm GF}^2$ by*

$$\beta_{\rm GF}(v) \;:=\; \Big(\mu\,\partial_\mu g_{\rm GF}^2(\mu;\,a,\beta)\Big)\Big|_{\substack{\text{fixed bare }(a,\beta)\\ g_{\rm GF}^2(\mu;\,a,\beta)=v}}.$$

Then, for every fixed lattice spacing $a \le a_0$ and for all u in the small tuning window where $u \mapsto \beta(a,u)$ is defined (e.g. by Lemma 4.11 / Theorem 4.9), the step-scaling function $\Sigma(u,s;\,a\mu_0)$ satisfies the autonomous ODE

$$\partial_{\ln s}\,\Sigma(u,s;\,a\mu_0) \;=\; \beta_{\rm GF}(\Sigma(u,s;\,a\mu_0)), \qquad \Sigma(u,1;\,a\mu_0) = u. \tag{29}$$

Proof. Let $t(s) := s_0/s^2$ so that $\mu(t) = (8t)^{-1/2}$ and $\mu = s\,\mu_0$ iff $t = t(s)$. With (a,u) fixed,

$$\Sigma(u,s;\,a\mu_0) \;=\; g_{\rm GF}^2(\mu;\,a,\beta(a,u))\Big|_{\mu=s\mu_0} \;=\; F_a(\beta(a,u),\,t(s)).$$

Differentiate with respect to $\ln s$ and use $\mu\,\partial_\mu = -2t\,\partial_t$:

$$\partial_{\ln s}\Sigma \;=\; \frac{\mathrm{d}}{\mathrm{d}\ln s} F_a(\beta(a,u),t(s)) \;=\; \Big(-2t\,\partial_t F_a(\beta(a,u),t)\Big)_{t=t(s)} \;=\; \Big(\mu\,\partial_\mu g_{\rm GF}^2(\mu;\,a,\beta)\Big)_{\substack{\mu=s\mu_0\\ \beta=\beta(a,u)}}.$$

Since $v := \Sigma(u,s;\,a\mu_0) = g_{\rm GF}^2(\mu;\,a,\beta)$ at $\mu = s\mu_0$ with bare (a,β) fixed, the last expression equals $\beta_{\rm GF}(v)$ by definition, which yields (29). The initial condition at $s = 1$ is immediate. ☐

Theorem 4.15 (GF beta function: holomorphy in β and uniform small-coupling control). *Work in the Dobrushin/KP window of Lemmas 4.6–4.7 and fix $a_0 > 0$ and a positive flow time $t > 0$.*

(i) (Holomorphy in β; convergent core expansion with uniform bounds) *There exists $\beta_\sharp = \beta_\sharp(t) \ge \beta_\star$ such that for every finite volume Λ and $a \le a_0$ the map $\beta \mapsto F_a(\beta,t) = \kappa\,t^2\,\langle E_t\rangle_{\Lambda,\beta}$ is holomorphic on the half-plane $\{\Re\beta > \beta_\sharp\}$, uniformly in $a \le a_0$ and in the volume. Moreover, for real $\beta \ge \beta_\sharp$ one has*

$$F_a(\beta,t) \;=\; \sum_{n\ge1} c_n(t,a)\,\beta^{-n} \;+\; \mathcal{E}_a(\beta,t), \qquad |\mathcal{E}_a(\beta,t)| \;\le\; C_{\rm np}(t)\,e^{-B(t)\beta}, \tag{30}$$

where the coefficients satisfy

$$\sup_{a\le a_0} |c_n(t,a)| \;\le\; C(t)\,R(t)^n \qquad (n \ge 1),$$

for some $C(t), R(t) < \infty$ independent of $a \le a_0$ and of the volume. (This is Lemma 4.13 in the present notation.)

(ii) (Implicit tuning line at the reference flow s_0; smoothness in u) *Set $t = s_0$ and write $u := F_a(\beta,s_0)$. There exists $u_1 > 0$, independent of $a \le a_0$ and of the volume, such that: for every $u_0 \in (0,u_1]$ and every $a \le a_0$ there is a unique $\beta(a,u_0) \ge \beta_\sharp$ solving $F_a(\beta(a,u_0),s_0) = u_0$. Moreover, the map $u \mapsto \beta(a,u)$ is C^1 on $(0,u_1]$ (indeed C^∞ there), and for every $\underline{u} \in (0,u_1]$ it is real–analytic on $[\underline{u},u_1]$, uniformly in $a \le a_0$ and in the volume.*

For every $s > 1$, the lattice step–scaling map

$$u \;\longmapsto\; \Sigma(u,s;\,a\mu_0) \;=\; F_a(\beta(a,u),\,s_0/s^2)$$

is C^1 on $(0,u_1]$ (indeed C^∞ there), and real–analytic on $[\underline{u},u_1]$ for every $\underline{u} > 0$.

(iii) (Callan–Symanzik ODE) *For all $u \in (0, u_1]$, the step–scaling function solves the autonomous Callan–Symanzik equation*

$$\partial_{\ln s}\Sigma(u, s;\, a\mu_0) \;=\; \beta_{\rm GF}(\Sigma(u, s;\, a\mu_0)), \qquad \Sigma(u, 1;\, a\mu_0) = u,$$

where $\beta_{\rm GF}$ is defined in Lemma 4.14.

Proof. Item (i) is Lemma 4.13 rewritten with $\beta^{-n} = g_0^{2n}$ on the real axis and with the explicit nonperturbative defect $\mathcal{E}_a(\beta, t) = O(e^{-B(t)\beta})$.

For (ii), strict monotonicity $\partial_\beta F_a(\beta, s_0) < 0$ for $\beta \geq \beta_\sharp$ and the implicit function construction of $u \mapsto \beta(a, u)$ are as in Lemma 4.11, with uniformity in $a \leq a_0$ coming from the a–uniform KP/Dobrushin bounds. Real–analyticity on $[\underline{u}, u_1]$ follows because on that interval $\beta(a, u)$ stays in a compact subset of $(\beta_\sharp, \infty)$ and $\beta \mapsto F_a(\beta, t)$ is real–analytic for $\beta > \beta_\sharp$ (restriction of holomorphy in β).

Item (iii) is Lemma 4.14. □

Remark 4.16 (Where the analyticity comes from and what the "radius" means). The complex analyticity statements available from the BKAR/KP expansion are naturally formulated in the *bare inverse coupling* β (holomorphy on a half–plane $\{\Re\beta > \beta_\sharp\}$) together with a *convergent core series* in β^{-1} and an exponentially small defect $\mathcal{E}_a(\beta, t) = O(e^{-B(t)\beta})$ as in (30). The constants depend only on the anchored observable norm of $t^2 E_t$ (finite by flow locality), the KP incompatibility norm $\sigma_* < \frac{1}{2}$ (uniform in $a \leq a_0$), and the coefficient majorant $R(t)$.

At the level of the renormalized coupling u, one should therefore interpret "analytic in u near 0" as: real–analytic on every $[\underline{u}, u_1]$ with $\underline{u} > 0$, and C^∞ on $(0, u_1]$, with nonperturbative corrections behaving like $e^{-B/u}$ as $u \downarrow 0$.

Remark 4.17 (Minimal wording if one prefers to avoid complex domains). If one elects not to use complex analyticity, the conclusions remain valid in the following form (sufficient for the subsequent arguments): on a small *real* interval $(0, u_1]$, the maps $u \mapsto \beta(a, u)$ and $u \mapsto \Sigma(u, s;\, a\mu_0)$ are C^∞ uniformly in $a \leq a_0$, and for every $\underline{u} > 0$ they are real–analytic on $[\underline{u}, u_1]$. The Callan–Symanzik equation holds with $\beta_{\rm GF}$ real–analytic on $(0, u_1)$ and with a continuous extension at $u = 0$ given by $\beta_{\rm GF}(0) = 0$.

Lemma 4.18 (One-loop coefficient and scheme-independence)**.** *Let C_A be the adjoint quadratic Casimir (for $SU(N)$, $C_A = N$). In any mass-independent scheme written in terms of $v = g^2$ one has*

$$\beta_{\rm scheme}(v) \;=\; -\,2\,b_0\,v^2 \;+\; O(v^3), \qquad b_0 = \frac{11\,C_A}{48\pi^2} > 0.$$

In particular, the GF beta function satisfies

$$\beta_{\rm GF}(v) \;=\; -\,2\,b_0\,v^2 \;+\; B_{\rm GF}(v),$$

where there exist $u_1 > 0$ and $C_B < \infty$, independent of $a \leq a_0$ and the volume, such that

$$|B_{\rm GF}(v)| \;\leq\; C_B\,v^3 \qquad \textit{for all } v \in [0, u_1].$$

Moreover, $B_{\rm GF}$ is real–analytic on $(0, u_1)$ (uniformly on $[\underline{v}, u_1]$ for any $\underline{v} > 0$).

Proof. The universality of the one-loop coefficient in any mass-independent scheme is standard.

For the GF scheme, at fixed bare (a, β) and positive flow time t one has the weak-coupling expansion

$$\kappa\,t^2\langle E_t\rangle \;=\; g_0^2 \;+\; g_0^4\Big(c_1 + 2b_0\ln(\mu\sqrt{8t})\Big) \;+\; O(g_0^6) \;+\; O(e^{-B\beta}), \qquad g_0^2 = \beta^{-1},$$

with c_1 finite (scheme-dependent). The $O(g_0^6)$ term is the convergent core contribution controlled by the BKAR/KP bounds, and the $O(e^{-B\beta})$ term is the nonperturbative defect from Lemma 4.13. Differentiating with respect to $\ln\mu$ at fixed bare (a,β) gives

$$\mu\,\partial_\mu g_{\mathrm{GF}}^2(\mu;\,a,\beta) \;=\; -\,2b_0\,g_0^4 \;+\; O(g_0^6) \;+\; O(e^{-B\beta}).$$

Now impose the renormalization condition $g_{\mathrm{GF}}^2(\mu;\,a,\beta)=v$ (so $v\to 0$ forces $\beta\to\infty$ in the weak-coupling window), and use $v=g_0^2+O(g_0^4)+O(e^{-B\beta})$ to rewrite the right-hand side as

$$\beta_{\mathrm{GF}}(v) \;=\; -2b_0\,v^2 \;+\; O(v^3) \;+\; O(e^{-B/v}).$$

Shrinking $u_1>0$ if needed, the nonperturbative term satisfies $e^{-B/v}\le v^3$ for all $v\in(0,u_1]$, so it is absorbed into the $C_B v^3$ bound. Real–analyticity on $(0,u_1)$ follows from the real–analytic dependence in β for $\beta>\beta_\sharp$ (Theorem 4.15(i)) combined with the implicit dependence $\beta=\beta(v)$ away from $v=0$. □

We can now state and prove the uniform small-u expansion for step scaling.

Theorem 4.19 (Uniform small-u expansion of Σ)**.** *Fix $s>1$. There exist $a_1>0$, $u_1>0$, and $C_{\mathrm{rem}}(s)<\infty$ such that for all $a\mu_0\le a_1$, all $u\in[0,u_1]$, and all volumes,*

$$\Sigma(u,s;\,a\mu_0) \;=\; u \;-\; 2\,b_0\,u^2\ln s \;+\; R(u,s;\,a\mu_0), \qquad |R(u,s;\,a\mu_0)| \;\le\; C_{\mathrm{rem}}(s)\,u^3, \tag{31}$$

with $b_0=\frac{11C_A}{48\pi^2}$. Moreover,

$$\partial_u\Sigma(u,s;\,a\mu_0) \;=\; 1 \;-\; 4\,b_0\,u\ln s \;+\; \widetilde{R}(u,s;\,a\mu_0), \qquad |\widetilde{R}(u,s;\,a\mu_0)| \;\le\; 3\,C_{\mathrm{rem}}(s)\,u^2, \tag{32}$$

with the same constants, all independent of $a\le a_0$ and the volume.

Proof of Theorem 4.19. By Lemma 4.14, Σ solves the autonomous ODE $\partial_{\ln s}\Sigma=\beta_{\mathrm{GF}}(\Sigma)$, $\Sigma(u,1)=u$. By Lemma 4.18, write

$$\beta_{\mathrm{GF}}(v) \;=\; -2b_0v^2+B_{\mathrm{GF}}(v), \qquad |B_{\mathrm{GF}}(v)|\le C_Bv^3 \quad (0\le v\le u_1).$$

Fix $s>1$ and integrate the ODE on $\tau=\ln s\in[0,\ln s]$:

$$\Sigma(u,s) \;=\; u \;+\; \int_0^{\ln s}\beta_{\mathrm{GF}}(\Sigma(u,e^\tau))\,d\tau.$$

Choose $u_1>0$ small enough that the solution stays in $[0,2u]$ on $\tau\in[0,\ln s]$ for all $u\in[0,u_1]$ (e.g. by a differential inequality using $|\beta_{\mathrm{GF}}(v)|\le 3b_0v^2$ on $[0,2u_1]$). Then

$$\Sigma(u,s) \;=\; u \;-\; 2b_0\,u^2\ln s \;+\; R(u,s),$$

where

$$R(u,s) \;=\; \int_0^{\ln s}\Big(-2b_0\big(\Sigma(u,e^\tau)^2-u^2\big) \;+\; B_{\mathrm{GF}}(\Sigma(u,e^\tau))\Big)\,d\tau.$$

Using $\Sigma(u,e^\tau)\le 2u$ and the bound on B_{GF} gives

$$\int_0^{\ln s}|B_{\mathrm{GF}}(\Sigma(u,e^\tau))|\,d\tau \;\le\; C_B(2u)^3\ln s \;=\; O(u^3).$$

Moreover, the term $\Sigma^2-u^2=(\Sigma-u)(\Sigma+u)$ is $O(u)\cdot O(u^2)$ after one Grönwall/Duhamel iteration of the ODE, hence integrates to $O(u^3)$ with a constant depending only on s. This proves (31) with a finite $C_{\mathrm{rem}}(s)$ independent of $a\le a_0$ and the volume.

Differentiating the ODE w.r.t. u gives

$$\partial_{\ln s}\,\partial_u\Sigma(u,s) \;=\; \beta'_{\mathrm{GF}}(\Sigma(u,s))\,\partial_u\Sigma(u,s), \qquad \partial_u\Sigma(u,1)=1.$$

Since $\beta'_{\mathrm{GF}}(v)=-4b_0v+O(v^2)$ with $O(v^2)$ bounded on $[0,2u_1]$, the same argument yields (32) with $|\widetilde{R}|\le 3C_{\mathrm{rem}}(s)u^2$ after shrinking u_1 if needed. □

Remark 4.20 (Recovery of Proposition A.3 *(one–loop universality of b_0)*). Equation (31) implies, in particular, $\sigma(u,s)=\lim_{a\mu_0\to 0}\Sigma(u,s;\,a\mu_0)=u-2b_0u^2\ln s+O(u^3)$ with the universal $b_0>0$. This recovers Proposition A.3.

4.3 Nonperturbative existence and regularity of the GF tuning line

We now *prove* the existence (and regularity) of a gauge–invariant gradient–flow (GF) tuning line $a \mapsto \beta(a)$ that fixes the renormalized GF coupling at a reference scale $\mu_0 = 1/\sqrt{8s_0}$:

$$g^2_{\mathrm{GF}}(\mu_0;\, a, \beta(a)) \;=\; u_0.$$

This removes the only remaining hypothesis in §4 and makes the continuum statements unconditional within our weak–coupling window.

Lemma 4.21 (Uniform weak–coupling analyticity and expansion of the flowed energy)**.** *Fix $s_0 > 0$ and $a_0 > 0$. There exists $\beta_\sharp \ge \beta_\star$ and constants $c_1(s_0) > 0$, $C_2(s_0) < \infty$ (independent of $a \le a_0$ and of the volume) such that, for all $\beta \ge \beta_\sharp$:*

(i) The map $\beta \mapsto \langle E_{s_0}\rangle_{\Lambda,\beta}$ (and its infinite–volume limit) is real–analytic on $(\beta_\sharp, \infty)$.

(ii) One has the uniform expansion

$$\Big|\, \langle E_{s_0}\rangle_{\Lambda,\beta} \;-\; \frac{c_1(s_0)}{\beta} \,\Big| \;\le\; \frac{C_2(s_0)}{\beta^2}, \qquad \Big|\, \partial_\beta \langle E_{s_0}\rangle_{\Lambda,\beta} \;+\; \frac{c_1(s_0)}{\beta^2} \,\Big| \;\le\; \frac{2C_2(s_0)}{\beta^3}. \tag{33}$$

Proof. We work at fixed positive flow $s_0 > 0$.

(i) Real-analyticity in β. By Theorem 4.15(i) (equivalently Lemma 4.13 at $t = s_0$), the function

$$F_a(\beta, s_0) \;:=\; \kappa\, s_0^2\, \langle E_{s_0}\rangle_{\Lambda,\beta}$$

is holomorphic on the half-plane $\{\Re\beta > \beta_\sharp\}$ (uniformly in $a \le a_0$ and in the volume). Restricting to real $\beta > \beta_\sharp$ yields that $\beta \mapsto \langle E_{s_0}\rangle_{\Lambda,\beta}$ is real–analytic on $(\beta_\sharp, \infty)$.

(ii) Uniform $1/\beta$ expansion. Expand the Wilson action near the identity (convex core) and write the interacting measure as a perturbation of the strictly log–concave Gaussian core given by the quadratic approximation (Lemma 7.3). At positive flow time s_0, the observable $E_{s_0}(0)$ is gauge invariant and *quasilocal*; in particular the anchored localization of $\kappa\, s_0^2\, E_{s_0}(0)$ has summable tails uniformly in $a \le a_0$, i.e.

$$\sup_{a \le a_0} \; \big\|\kappa\, s_0^2\, E_{s_0}(0)\big\|_{\mathrm{anc}} \;<\; \infty,$$

as in Equation (25) (with $t = s_0$), which ultimately rests on the explicit GI–Lipschitz decay from Lemma 13.6 and Theorem 18.11.

The Gaussian expectation of E_{s_0} yields the leading term $c_1(s_0)/\beta$ with $c_1(s_0) > 0$. The interacting corrections are given by absolutely convergent connected cluster integrals whose absolute value is $O(\beta^{-2})$ uniformly in $a \le a_0$ (KP activity bound $\delta_L(\beta) = O(1/(\beta L) + e^{-B\beta})$ together with the anchored bound above). This gives the first estimate in Equation (33).

Differentiation in β acts by insertion of the centered energy density $\sum_p V(U_p)$; the same BKAR/KP bounds (termwise differentiation in an absolutely convergent series) yield the second estimate in Equation (33). All constants are uniform in $a \le a_0$ by the a–uniform Dobrushin/KP bounds and the fixed flow range s_0. □

Proposition 4.22 (Strict monotonicity at large β)**.** *With s_0 and $\beta_\sharp$ as in Lemma 4.21, there exists $\beta_{\mathrm{mon}} \ge \beta_\sharp$ such that, for all $\beta \ge \beta_{\mathrm{mon}}$ and all $a \le a_0$,*

$$\partial_\beta \, \langle E_{s_0}\rangle_\beta \;\le\; -\frac{c_1(s_0)}{2\,\beta^2} \;<\; 0.$$

Proof. By the second estimate in (33),

$$\partial_\beta \langle E_{s_0}\rangle_\beta \;=\; -\frac{c_1(s_0)}{\beta^2} \;+\; R(\beta), \qquad |R(\beta)| \;\le\; \frac{2C_2(s_0)}{\beta^3}.$$

Choose $\beta_{\rm mon} \ge \beta_\sharp$ so large that $\frac{2C_2(s_0)}{\beta_{\rm mon}} \le \frac{1}{2}c_1(s_0)$. Then for all $\beta \ge \beta_{\rm mon}$, $\partial_\beta\langle E_{s_0}\rangle_\beta \le -\frac{c_1(s_0)}{2\beta^2} < 0$, uniformly in $a \le a_0$. ☐

Theorem 4.23 (Existence, uniqueness, and regularity of the GF tuning line)**.** *Fix $s_0 > 0$ and pick any target $u_0 \in (0, u_{\rm max})$ with*

$$u_{\rm max} \;:=\; \frac{\kappa\, s_0^2}{2}\,\frac{c_1(s_0)}{\beta_{\rm mon}}\,.$$

Then there exists a unique function $\beta(\cdot)$ defined on $(0, a_0]$ with values in $[\beta_{\rm mon}, \infty)$ such that

$$g^2_{\rm GF}(\mu_0;\, a, \beta(a)) \;=\; \kappa\, s_0^2\, \langle E_{s_0}\rangle_{\beta(a)} \;=\; u_0 \qquad \textit{for all } a \in (0, a_0]. \tag{34}$$

Moreover, $\beta(a)$ is continuous on $(0, a_0]$ and locally Lipschitz; in particular it is bounded below by $\beta_{\rm mon}$ and satisfies the weak–coupling window assumed in §4.

Proof. Fix $a \in (0, a_0]$. By Lemma 4.21, $\beta \mapsto \langle E_{s_0}\rangle_\beta$ is continuous on $[\beta_{\rm mon}, \infty)$, tends to 0 as $\beta \to \infty$, and is strictly decreasing there by Proposition 4.22. At $\beta = \beta_{\rm mon}$ we have

$$\kappa s_0^2\, \langle E_{s_0}\rangle_{\beta_{\rm mon}} \;\ge\; \kappa s_0^2 \Big(\frac{c_1(s_0)}{\beta_{\rm mon}} - \frac{C_2(s_0)}{\beta_{\rm mon}^2}\Big) \;\ge\; \frac{\kappa s_0^2}{2}\,\frac{c_1(s_0)}{\beta_{\rm mon}} \;=\; u_{\rm max},$$

after increasing $\beta_{\rm mon}$ if needed to ensure $C_2(s_0)/(\beta_{\rm mon} c_1(s_0)) \le \frac{1}{2}$. Hence the range of $g^2_{\rm GF}(\mu_0; a, \beta)$ on $[\beta_{\rm mon}, \infty)$ contains the whole interval $(0, u_{\rm max})$. By the intermediate value theorem and strict monotonicity, there is a unique $\beta(a) \in [\beta_{\rm mon}, \infty)$ solving (34).

To see that $a \mapsto \beta(a)$ is continuous (indeed locally Lipschitz), note that E_{s_0} is a finite–range flowed local and its expectation is jointly continuous in (a, β) under our uniform Dobrushin/KP bounds (uniform L^p controls and dominated convergence; see Proposition 13.2). Furthermore, on $[\beta_{\rm mon}, \infty)$, $\partial_\beta g^2_{\rm GF}(\mu_0; a, \beta) = \kappa s_0^2\, \partial_\beta \langle E_{s_0}\rangle_\beta$ is uniformly bounded away from 0 by Proposition 4.22 (indeed $\le -\kappa s_0^2 c_1(s_0)/(2\beta_{\rm mon}^2)$). The implicit function theorem (or quantitative monotone–inverse bound) then yields local Lipschitz continuity of $\beta(a)$. ☐

Corollary 4.24 (Removal of the tuning hypothesis)**.** *All results in §4 that were stated "along a tuning line" now hold with the tuning line $a \mapsto \beta(a)$ supplied by Theorem 4.23, with $\beta(a) \ge \beta_{\rm mon} \ge \beta_\star$ for all $a \le a_0$. In particular, Lemmas 4.6–4.7 and Proposition 4.8 apply uniformly along this nonperturbative tuning line.*

Proof of Corollary 4.24. By Theorem 4.23 there exists a unique tuning line $a \mapsto \beta(a) \in [\beta_{\rm mon}, \infty)$ with $g^2_{\rm GF}(\mu_0;\, a, \beta(a)) = u_0$ for all $a \in (0, a_0]$. In particular $\beta(a) \ge \beta_{\rm mon} \ge \beta_\star$, so (T1) holds along this line. The choices of L and a_0 already ensure (T2), and (T3) concerns fixed KP parameters, independent of a. Therefore Lemmas 4.6–4.7 apply uniformly along $a \mapsto \beta(a)$, and Proposition 4.8 follows uniformly as well. All statements in §4 that were conditional on the existence of a tuning line therefore hold *along* the line produced by Theorem 4.23. ☐

Lemma 4.25 (Verification of (T1)–(T3) along the GF tuning line)**.** *Fix $s_0 > 0$ and let $\beta(\cdot)$ be the unique GF tuning line from Theorem 4.23 at target $u_0 \in (0, u_{\rm max})$. Then, after fixing $L \in \mathbb{Z}_{\ge 1}$ and $a_0 > 0$ with* (T2)*, there exists a choice of $u_0 \in (0, u_{\rm max})$ (depending only on s_0, L, a_0 and the KP parameters) for which* (T1)–(T3) *all hold. In particular, the constants $\beta_\star$, L, a_0 and ε_0 may be fixed once and for all, and every statement below that cites* (T1)–(T3) *can be read as invoking this lemma rather than an external hypothesis.*

Proof. (T1). By Theorem 4.23 there is a unique $\beta(a) \in [\beta_{\mathrm{mon}}, \infty)$ solving $g^2_{\mathrm{GF}}(\mu_0; a, \beta(a)) = u_0$ for every $a \in (0, a_0]$. Hence (T1) holds with $\beta_\star := \beta_{\mathrm{mon}}$ (independent of a).

(T2). This is a choice, not an assumption: pick any L and a_0 satisfying $L^{-1} + e^{-L} + a_0^2 \leq \varepsilon_0 < 1/4$. For concreteness, $L = 18$ and $a_0 = 0.05$ give $L^{-1} + e^{-L} + a_0^2 \approx 0.0580556 < 1/4$.

(T3). The map $\beta \mapsto \delta_L(\beta) = \alpha_1/(\beta L) + \alpha_2 e^{-B\beta}$ is strictly decreasing. Let $\beta_{\mathrm{KP}} = \beta_{\mathrm{KP}}(L)$ be any value with $\delta_L(\beta_{\mathrm{KP}}) \leq 1/80$ (existence follows by monotonicity). Set

$$\beta_\star \;:=\; \max\{\beta_{\mathrm{mon}},\, \beta_{\mathrm{KP}}\}.$$

By Lemma 4.21 there exist $c_1(s_0) > 0$ and $C_2(s_0) < \infty$ such that

$$\Big|\, \langle E_{s_0}\rangle_\beta - \frac{c_1(s_0)}{\beta} \,\Big| \;\leq\; \frac{C_2(s_0)}{\beta^2} \qquad \text{for all } \beta \geq \beta_\sharp,$$

uniformly in $a \leq a_0$ and the volume. Choose $\beta_{\mathrm{req}} \geq \max\{\beta_\star, \beta_\sharp\}$ and define

$$u_{\mathrm{crit}} \;:=\; \kappa\, s_0^2 \left(\frac{c_1(s_0)}{\beta_{\mathrm{req}}} - \frac{C_2(s_0)}{\beta_{\mathrm{req}}^2} \right) \;>\; 0.$$

If we now fix the target coupling to satisfy $0 < u_0 \leq u_{\mathrm{crit}}$, then the monotonicity $\partial_\beta \langle E_{s_0}\rangle_\beta < 0$ (Proposition 4.22) implies

$$g^2_{\mathrm{GF}}(\mu_0; a, \beta_{\mathrm{req}}) \;\geq\; u_0 \qquad \Longrightarrow \qquad \beta(a) \;\geq\; \beta_{\mathrm{req}} \;\geq\; \beta_\star$$

for all $a \leq a_0$. Consequently $\delta_L(\beta(a)) \leq \delta_L(\beta_\star) \leq 1/80$ uniformly in a, i.e. (T3) holds. This completes the verification. □

Remark 4.26 (Explicit admissible window). With the constants entering the cut–KP bound from Appendix C (plaquette $*$–adjacency, degree $\Delta = 26$), one admissible choice is

$$(\beta_\star, L, a_0) = (20, 18, 0.05), \qquad \delta_L(\beta_\star) \leq \tfrac{1}{80}, \qquad \varepsilon_0 = \tfrac{1}{L} + e^{-L} + a_0^2 \approx 0.0580556.$$

In the a–uniform polymer budget used later we also employ

$$\delta_*(a) \;:=\; \frac{1}{\beta_\star L} + e^{-B\beta_\star} + a_0^2,$$

and, with $\beta_\star = 20$ and any $B \geq 2$ (as in Appendix C), we have $e^{-B\beta_\star} \leq e^{-40}$, so numerically $\delta_*(a) \lesssim \frac{1}{\beta_\star L} + e^{-40} + a_0^2 \approx 0.0052778$. These numerics are recorded for orientation; the proof of Lemma 4.25 does not rely on any particular values.

5 RP under GI conditioning (anti–linear J)

Let $(\Omega, \mathfrak{A}, \mu)$ be a probability space, $\Theta : \Omega \to \Omega$ an involutive reflection with $\mu \circ \Theta^{-1} = \mu$, and let $\mathfrak{A}_\pm, \mathfrak{A}_0 \subset \mathfrak{A}$ be the σ–algebras of observables localized in $\{x_0 \gtrless 0\}$ and on the reflection hyperplane, respectively, with $\Theta(\mathfrak{A}_+) = \mathfrak{A}_-$, $\Theta(\mathfrak{A}_0) = \mathfrak{A}_0$. *Convention:* we include the time–zero algebra in both halves, i.e. $\mathfrak{A}_0 \subset \mathfrak{A}_\pm$.

We assume *reflection positivity (RP)* in the standard Osterwalder–Schrader form:

$$\langle JF, F\rangle_{L^2(\mu)} \;=\; \int \overline{F \circ \Theta}\, F \,\mathrm{d}\mu \;\geq\; 0 \qquad \text{for all } F \in L^2(\mu) \text{ with } F\ \mathfrak{A}_+\text{–measurable}, \tag{35}$$

where $J : L^2(\mu) \to L^2(\mu)$ is the anti–linear isometry

$$(Jf)(\omega) \;:=\; \overline{f(\Theta\omega)} \qquad (J^2 = \mathrm{id},\ \langle Jf, Jg\rangle = \langle g, f\rangle). \tag{36}$$

Gauge–invariant boundary algebra. Let $\mathfrak{A}_{\mathrm{GI}} \subset \mathfrak{A}_0$ be a reflection–invariant σ–subalgebra encoding the *gauge–invariant* (GI) boundary data at time 0, i.e. $\Theta(\mathfrak{A}_{\mathrm{GI}}) = \mathfrak{A}_{\mathrm{GI}}$. Denote by

$$P \ := \ \mathbb{E}[\,\cdot\,|\,\mathfrak{A}_{\mathrm{GI}}] \ : \ L^2(\mu) \longrightarrow L^2(\mu) \tag{37}$$

the orthogonal projection (conditional expectation) onto $L^2(\mathfrak{A}_{\mathrm{GI}}, \mu)$.

Lemma 5.1 (Compatibility: J preserves $L^2(\mathfrak{A}_{\mathrm{GI}})$ and commutes with P)**.** *Assume* $\Theta(\mathfrak{A}_{\mathrm{GI}}) = \mathfrak{A}_{\mathrm{GI}}$ *and* μ *is* Θ*–invariant. Then* $J(L^2(\mathfrak{A}_{\mathrm{GI}})) \subset L^2(\mathfrak{A}_{\mathrm{GI}})$ *and*

$$JP \ = \ PJ \qquad \textit{on } L^2(\mu). \tag{38}$$

Proof. If g is $\mathfrak{A}_{\mathrm{GI}}$–measurable, then $g \circ \Theta$ is also $\mathfrak{A}_{\mathrm{GI}}$–measurable, hence $Jg = \overline{g \circ \Theta} \in L^2(\mathfrak{A}_{\mathrm{GI}})$. Thus J preserves $L^2(\mathfrak{A}_{\mathrm{GI}})$. The orthogonal projection P is characterized by $\langle Pf, h\rangle = \langle f, h\rangle$ for all $h \in L^2(\mathfrak{A}_{\mathrm{GI}})$. Using that J is anti–unitary with $J^2 = \mathrm{id}$ and that $J(L^2(\mathfrak{A}_{\mathrm{GI}})) = L^2(\mathfrak{A}_{\mathrm{GI}})$, for any $f \in L^2(\mu)$ and any $h \in L^2(\mathfrak{A}_{\mathrm{GI}})$,

$$\langle JPf, h\rangle = \langle Pf, Jh\rangle = \langle f, Jh\rangle = \langle PJf, h\rangle.$$

Since h ranges over a dense set in the range of P, we conclude $JPf = PJf$. □

Lemma 5.2 (RP preserved by GI conditioning)**.** *If* (35) *holds,* μ *is* Θ*–invariant and* $\Theta(\mathfrak{A}_{\mathrm{GI}}) = \mathfrak{A}_{\mathrm{GI}}$*, then for every* $\mathfrak{A}_+$*–measurable* F*,*

$$\big\langle J\,\mathbb{E}[F\,|\,\mathfrak{A}_{\mathrm{GI}}],\ \mathbb{E}[F\,|\,\mathfrak{A}_{\mathrm{GI}}]\big\rangle \ \geq \ 0. \tag{39}$$

Proof. *Since* $\mathfrak{A}_{\mathrm{GI}} \subset \mathfrak{A}_0 \subset \mathfrak{A}_+$ *and* PF *is* $\mathfrak{A}_{\mathrm{GI}}$*–measurable,* PF *is* $\mathfrak{A}_+$*–measurable. Applying* (35) *with* $H := PF$ *gives* $\langle JPF, PF\rangle \geq 0$*. (The commutation* $JP = PJ$ *from Lemma 5.1 will be used below for OS–pairing identities.)*

The previous lemma has the following standard matrix (Gram)–positivity consequence.

Proposition 5.3 (Matrix RP after GI conditioning)**.** *Let* $F_1, \ldots, F_n$ *be* $\mathfrak{A}_+$*–measurable. Then the* $n \times n$ *matrix*

$$M_{ij} \ := \ \big\langle JPF_i,\ PF_j\big\rangle$$

is Hermitian positive semidefinite. Equivalently,

$$\sum_{i,j=1}^{n} \overline{c_i}\, c_j \,\big\langle JPF_i,\ PF_j\big\rangle \ \geq \ 0 \qquad \textit{for all } (c_1, \ldots, c_n) \in \mathbb{C}^n.$$

Proof. Apply Lemma 5.2 to $F = \sum_j c_j F_j$ and use polarization. □

Corollary 5.4 (GI RP seminorm and OS pre–Hilbert space)**.** *Define, for* $\mathfrak{A}_+$*–measurable* F, G*,*

$$\langle F, G\rangle_{\mathrm{GI}} \ := \ \big\langle JPF,\ PG\big\rangle, \qquad \|F\|_{\mathrm{GI}}^2 := \langle F, F\rangle_{\mathrm{GI}}.$$

Then $\langle \cdot, \cdot\rangle_{\mathrm{GI}}$ *is a positive semidefinite Hermitian form on* $\{F : \ F\ \mathfrak{A}_+\text{–measurable}\}$*. Modding out the null space* $\mathcal{N}_{\mathrm{GI}} = \{F : \|F\|_{\mathrm{GI}} = 0\}$ *and completing yields a Hilbert space* $\mathcal{H}_+^{(\mathrm{GI})}$*, canonically isometric to the RP time–zero Hilbert space built from the GI boundary algebra. Moreover,*

$$\big|\langle F, G\rangle_{\mathrm{GI}}\big| \ \leq \ \|PF\|_2\,\|PG\|_2 \ \leq \ \|F\|_2\,\|G\|_2. \tag{40}$$

Proof. Positivity follows from Proposition 5.3. The Cauchy–Schwarz bound (40) is the L^2 Cauchy–Schwarz inequality together with $\|Jh\|_2 = \|h\|_2$ and the contractivity $\|P\|_{2\to 2} = 1$. □

Remark 5.5 (Monotonicity under enlarging the boundary σ–algebra)*.* If $\mathfrak{A}_{\mathrm{GI}} \subset \mathfrak{B} \subset \mathfrak{A}_0$ are reflection–invariant σ–algebras with projections $P_{\mathrm{GI}}, P_{\mathfrak{B}}$, then

$$\|F\|_{\mathrm{GI}}^2 = \langle JP_{\mathrm{GI}}F, P_{\mathrm{GI}}F\rangle \;=\; \|P_{\mathrm{GI}}F\|_2^2 \;\le\; \|P_{\mathfrak{B}}F\|_2^2 \;=\; \langle JP_{\mathfrak{B}}F, P_{\mathfrak{B}}F\rangle,$$

since for h measurable w.r.t. $\mathfrak{A}_0$ one has $\langle Jh, h\rangle = \|h\|_2^2$, and $P_{\mathfrak{B}}$ is the L^2–orthogonal projection onto a larger subspace. Thus refining the boundary information can only *increase* the RP seminorm.

GI sufficiency and descent of Markov factorization. We write $\mathfrak{A}_{\pm}^{\mathrm{GI}} \subset \mathfrak{A}_{\pm}$ for the σ–algebras of *gauge–invariant* observables localized in the halves $\{x_0 \gtrless 0\}$ (so $\Theta(\mathfrak{A}_{+}^{\mathrm{GI}}) = \mathfrak{A}_{-}^{\mathrm{GI}}$).

Lemma 5.6 (GI sufficiency of the time–zero boundary)**.** *Let $(\Omega, \mathfrak{A}, \mu)$ be the underlying probability space and let $\mathcal{G}_0$ (the time–zero gauge group) act on Ω by measurable bijections $\omega \mapsto g\omega$. Assume:*

*(i) (*Measurable action and stability of $\mathfrak{A}_0$*) The action map $(g, \omega) \mapsto g\omega$ is measurable and $g^{-1}\mathfrak{A}_0 = \mathfrak{A}_0$ for all $g \in \mathcal{G}_0$.*

(ii) ($\mathcal{G}_0$–invariance of μ*) The measure is $\mathcal{G}_0$–invariant: $\mu \circ g^{-1} = \mu$ for all $g \in \mathcal{G}_0$.*

Define the fixed–point σ–algebra on the boundary

$$\mathfrak{A}_{\mathrm{GI}} := \mathfrak{A}_0^{\mathcal{G}_0} := \{B \in \mathfrak{A}_0 : \; g^{-1}B = B \;\forall\, g \in \mathcal{G}_0\},$$

and work with its μ–completion.

Then:

*(iii) (*Conditional expectation commutes with the action*) For every $g \in \mathcal{G}_0$ and every $F \in L^1(\mu)$,*

$$\mathbb{E}[F \circ g \mid \mathfrak{A}_0] = (\mathbb{E}[F \mid \mathfrak{A}_0]) \circ g \qquad \mu\text{–a.s.} \tag{41}$$

Equivalently, if $K_0(\omega, \cdot) = \mu(\cdot \mid \mathfrak{A}_0)(\omega)$ is a regular conditional law (which exists in our lattice setting since Ω is standard Borel), it may be chosen $\mathcal{G}_0$–equivariant:

$$K_0(g\omega, B) = K_0(\omega, g^{-1}B) \qquad (B \in \mathfrak{A},\; g \in \mathcal{G}_0), \tag{42}$$

and the same argument applies to any other conditional kernel obtained from μ (e.g. the half–space specifications).

*(iv) (*GI sufficiency*) If F is $\mathcal{G}_0$–invariant (in particular if $F \in L^\infty(\mathfrak{A}_{\pm}^{\mathrm{GI}})$), then $\mathbb{E}[F \mid \mathfrak{A}_0]$ is $\mathcal{G}_0$–invariant and therefore $\mathfrak{A}_{\mathrm{GI}}$–measurable. Equivalently,*

$$\mathbb{E}[F \mid \mathfrak{A}_0] \;\textit{ is }\; \mathfrak{A}_{\mathrm{GI}}\text{–measurable}, \qquad \textit{equivalently} \qquad \mathbb{E}[F \mid \mathfrak{A}_0] = \mathbb{E}[F \mid \mathfrak{A}_{\mathrm{GI}}]. \tag{43}$$

Proof. Step (i): measurability of the $\mathcal{G}_0$–action (lattice Yang–Mills case). In the lattice set–up, Ω is a (finite or countable) product of copies of the compact Lie group G (links), endowed with the product Borel σ–algebra. The gauge action is given coordinatewise by the continuous maps $(g_x, U) \mapsto g_x U$ and $(g_y, U) \mapsto U g_y^{-1}$, hence $(g, \omega) \mapsto g\omega$ is continuous (therefore measurable). By construction, $\mathfrak{A}_0$ is generated by time–zero boundary/cross–cut variables and is stable under $\mathcal{G}_0$.

Step (ii): $\mathcal{G}_0$–invariance of μ and equivariance of conditional kernels. For Wilson–type measures, μ has density proportional to $e^{-S_\beta(U)} \prod_e dH(U_e)$. Haar measure is bi–invariant and the plaquette action S_β is gauge invariant, hence $\mu \circ g^{-1} = \mu$ for all $g \in \mathcal{G}_0$. Since Ω is standard Borel, there exists a regular conditional law $K_0(\omega, \cdot) = \mu(\cdot \mid \mathfrak{A}_0)(\omega)$. Fix $g \in \mathcal{G}_0$ and define

$$K_0^{(g)}(\omega, B) := K_0(g\omega, gB), \qquad gB := \{g\omega' : \; \omega' \in B\}.$$

Using μ–invariance and $g^{-1}\mathfrak{A}_0 = \mathfrak{A}_0$, one checks that $K_0^{(g)}$ is again a regular conditional law of μ given $\mathfrak{A}_0$. By uniqueness of regular conditionals up to μ–null sets, $K_0^{(g)} = K_0$ μ–a.s., which is equivalent to (42). The same disintegration argument applies to any conditional kernel derived from μ (in particular the half–space kernels).

Step (iii): conditional expectation commutes with the action. Let $g \in \mathcal{G}_0$ and $F \in L^1(\mu)$. For any bounded $\mathfrak{A}_0$–measurable H, μ–invariance and $g^{-1}\mathfrak{A}_0 = \mathfrak{A}_0$ yield

$$\begin{aligned}\int H\,\mathbb{E}[F \circ g \mid \mathfrak{A}_0]\,d\mu &= \int H\,(F \circ g)\,d\mu = \int (H \circ g^{-1})\,F\,d\mu \\ &= \int (H \circ g^{-1})\,\mathbb{E}[F \mid \mathfrak{A}_0]\,d\mu = \int H\,(\mathbb{E}[F \mid \mathfrak{A}_0] \circ g)\,d\mu.\end{aligned}$$

Since this holds for all bounded $H \in L^\infty(\mathfrak{A}_0)$, the conditional expectations agree μ–a.s., proving (41). (Equivalently, one may deduce (41) from (42) by integrating against K_0.)

Step (iv): GI sufficiency. If F is $\mathcal{G}_0$–invariant, then (41) gives $\mathbb{E}[F \mid \mathfrak{A}_0] = \mathbb{E}[F \mid \mathfrak{A}_0] \circ g$ for all $g \in \mathcal{G}_0$. Hence $\mathbb{E}[F \mid \mathfrak{A}_0]$ is $\mathfrak{A}_0$–measurable and $\mathcal{G}_0$–invariant. For any Borel set $I \subset \mathbb{R}$, the level set $\{\mathbb{E}[F \mid \mathfrak{A}_0] \in I\}$ is $\mathfrak{A}_0$–measurable and invariant (up to null sets), hence belongs to the μ–completion of $\mathfrak{A}_0^{\mathcal{G}_0} = \mathfrak{A}_{\mathrm{GI}}$. Therefore $\mathbb{E}[F \mid \mathfrak{A}_0]$ is $\mathfrak{A}_{\mathrm{GI}}$–measurable and the tower property yields $\mathbb{E}[F \mid \mathfrak{A}_0] = \mathbb{E}[F \mid \mathfrak{A}_{\mathrm{GI}}]$, i.e. (43). Applying this to $F \in L^\infty(\mathfrak{A}_\pm^{\mathrm{GI}})$ proves the claimed GI sufficiency. □

Lemma 5.7 (Descent of conditional independence along a sufficient boundary)**.** *Assume $\mathfrak{A}_+$ and $\mathfrak{A}_-$ are conditionally independent given $\mathfrak{A}_0$. Let $\mathfrak{B} \subset \mathfrak{A}_0$ be a sub–σ–algebra such that for every bounded $\mathfrak{A}_\pm$–measurable H one has $\mathbb{E}[H \mid \mathfrak{A}_0]$ $\mathfrak{B}$–measurable. Then $\mathfrak{A}_+$ and $\mathfrak{A}_-$ are conditionally independent given $\mathfrak{B}$ and, for all $F \in L^\infty(\mathfrak{A}_+)$ and $G \in L^\infty(\mathfrak{A}_-)$,*

$$\mathbb{E}[FG \mid \mathfrak{B}] \;=\; \mathbb{E}[F \mid \mathfrak{B}]\;\mathbb{E}[G \mid \mathfrak{B}]. \tag{44}$$

Proof. By the tower property and the Markov property across $\mathfrak{A}_0$,

$$\mathbb{E}[FG \mid \mathfrak{B}] \;=\; \mathbb{E}\big[\mathbb{E}[F \mid \mathfrak{A}_0]\;\mathbb{E}[G \mid \mathfrak{A}_0] \mid \mathfrak{B}\big].$$

If both inner factors are $\mathfrak{B}$–measurable, the right–hand side equals $\mathbb{E}[F \mid \mathfrak{B}]\;\mathbb{E}[G \mid \mathfrak{B}]$, proving (44). □

Proposition 5.8 (GI Markov property on the GI sector)**.** *Assume the Markov property across the reflection hyperplane ($\mathfrak{A}_+ \perp\!\!\!\perp \mathfrak{A}_- \mid \mathfrak{A}_0$) and Lemma 5.6. Then $\mathfrak{A}_+^{\mathrm{GI}}$ and $\mathfrak{A}_-^{\mathrm{GI}}$ are conditionally independent given $\mathfrak{A}_{\mathrm{GI}}$ and, for all $F \in L^2(\mathfrak{A}_+^{\mathrm{GI}})$, $G \in L^2(\mathfrak{A}_-^{\mathrm{GI}})$,*

$$\begin{aligned}&\mathbb{E}[FG \mid \mathfrak{A}_{\mathrm{GI}}] = \mathbb{E}[F \mid \mathfrak{A}_{\mathrm{GI}}]\,\mathbb{E}[G \mid \mathfrak{A}_{\mathrm{GI}}], \\ &\qquad \langle JG,\, F\rangle = \langle J\,\mathbb{E}[F \mid \mathfrak{A}_{\mathrm{GI}}],\; \mathbb{E}[G \mid \mathfrak{A}_{\mathrm{GI}}]\rangle = \langle JPF,\, PG\rangle.\end{aligned} \tag{45}$$

Proof. Apply Lemma 5.7 with $\mathfrak{B} = \mathfrak{A}_{\mathrm{GI}}$ and use (43). The identity for the OS pairing follows from $\langle JG, F\rangle = \mathbb{E}(\overline{G \circ \Theta}\,F)$ and the first equality in (45). The last equality is the definition of $P = \mathbb{E}[\cdot \mid \mathfrak{A}_{\mathrm{GI}}]$ together with $JP = PJ$. □

Lemma 5.9 (Factorization and conditional independence)**.** *Assume, in addition, the (standard) Markov property across the reflection hyperplane: $\mathfrak{A}_+$ and $\mathfrak{A}_-$ are conditionally independent given $\mathfrak{A}_0$. Then for F $\mathfrak{A}_+$–measurable and G $\mathfrak{A}_-$–measurable,*

$$\langle JG, F\rangle \;=\; \big\langle J\,\mathbb{E}[F \mid \mathfrak{A}_0],\; \mathbb{E}[G \mid \mathfrak{A}_0]\big\rangle. \tag{46}$$

Amendment to Lemma 5.9. *The statement and proof up to* (46) *are unchanged.* For the GI pairing, use Proposition 5.8: if $F \in L^2(\mathfrak{A}_+^{\mathrm{GI}})$ and $G \in L^2(\mathfrak{A}_-^{\mathrm{GI}})$, then $\langle JG, F\rangle = \langle JPF,\; PG\rangle$.

Proof. By conditional independence,

$$\mathbb{E}\big[\overline{G\circ\Theta}\,F\big] = \mathbb{E}\big[\mathbb{E}(\overline{G\circ\Theta}\mid\mathfrak{A}_0)\,\mathbb{E}(F\mid\mathfrak{A}_0)\big].$$

Since Θ fixes $\mathfrak{A}_0$, $\mathbb{E}(\overline{G\circ\Theta}\mid\mathfrak{A}_0) = \overline{\mathbb{E}(G\mid\mathfrak{A}_0)\circ\Theta}$, which yields (46). □

Remark 5.10 (Bridge to the cross–cut transfer operator). To avoid duplication with Section 11, we refrain here from introducing the pair law on the GI boundary and the associated correlation/transfer operators. Section 11 realizes the bounded positive form $(f,g)\mapsto\langle Jf,g\rangle$ on $L^2(\mathfrak{A}_{\rm GI},\mu)$ as a symmetric integral operator induced by the GI pair law across the cut and proves the full OS–intertwiner identity there. The results of the present section provide the input (RP under GI conditioning and the Markov factorization) used in that construction.

6 Dobrushin/Holley–Stroock and the slab constants

We index the GI cut blocks by a finite set $\mathcal{I}$ (face/edge/vertex adjacency on $L\mathbb{Z}^3$). For $x\in\mathcal{I}$ let $\mathfrak{F}_{x^c}$ be the σ–algebra generated by all blocks $\neq x$ and write

$$\mathbb{E}_x[f] \;:=\; \mathbb{E}[f\,|\,\mathfrak{F}_{x^c}], \qquad \mathrm{Var}_x(f) \;:=\; \mathbb{E}\big[\,(f-\mathbb{E}_x f)^2\,\big|\,\mathfrak{F}_{x^c}\big].$$

We also use the block GI-adjoint Lipschitz seminorm (right-invariant gradient restricted to block x):

$$L^{\rm GI}_{{\rm ad},x}(f) \;:=\; \sup_U \left(\sum_{e\subset\text{block }x}\ \sup_{\|X_e\|=1} \big|(D_e f)(U)[X_e]\big|^2\right)^{1/2},$$

so that $L^{\rm GI}_{\rm ad}(f)^2 = \sum_{x\in\mathcal{I}} L^{\rm GI}_{{\rm ad},x}(f)^2$ whenever f is supported on $\bigcup_x$.

6.1 Holley–Stroock (HS) perturbation and local Poincaré constant

Lemma 6.1 (Holley–Stroock Perturbation)**.** *Let μ_0 and μ be probability measures on a smooth manifold with $d\mu = Z^{-1}e^h\,d\mu_0$. If $\mathrm{osc}(h) := \sup h - \inf h \le \delta$ and μ_0 satisfies a Poincaré inequality*

$$\mathrm{Var}_{\mu_0}(f) \;\le\; C_0 \int \|\nabla f\|^2\,d\mu_0 \qquad (\forall f\in C^1),$$

then μ satisfies

$$\mathrm{Var}_{\mu}(f) \;\le\; e^{2\delta}\,C_0 \int \|\nabla f\|^2\,d\mu \qquad (\forall f\in C^1).$$

Proof. Since $e^{-\delta} \le e^h \le e^{\delta}$, we have $e^{-\delta}\,d\mu_0 \le Z\,d\mu \le e^{\delta}\,d\mu_0$, hence $\|g\|^2_{L^2(\mu)} \le e^{\delta}Z^{-1}\|g\|^2_{L^2(\mu_0)}$ and $\int\|\nabla f\|^2 d\mu_0 \le e^{\delta}Z\int\|\nabla f\|^2 d\mu$. Apply the Poincaré inequality for μ_0 to $f-\mathbb{E}_\mu f$ and use the two-sided L^2 comparison. □

Lemma 6.2 (Block–HS: uniforme lokale Poincaré-Konstante)**.** *Uniformly in the boundary condition on $\mathfrak{F}_{x^c}$ there exists a constant*

$$C_{\rm PI,loc} \;=\; \frac{C_{\rm db}}{\beta\,\kappa_G}\,e^{2\delta_{\rm loc}}$$

(depending only on geometry, not on the volume) such that, for every $x\in\mathcal{I}$ and C^1 functional f,

$$\mathrm{Var}_x(f) \;\le\; C_{\rm PI,loc}\,\mathbb{E}_x\big[\,\|\nabla_x f\|^2\,\big] \;\le\; C_{\rm PI,loc}\,\big(L^{\rm GI}_{{\rm ad},x}(f)\big)^2,$$

with $\|\nabla_x f\|^2 = \sum_{e\subset x}\sup_{\|X_e\|=1}|(D_e f)[X_e]|^2$. Here κ_G is from Lemma 7.3, $C_{\rm db}\ge 1$ collects the deterministic plaquette-to-link/GI-quotient Lipschitz factors, and $\delta_{\rm loc} = O(a^2)+O(e^{-B\beta})$ bounds the oscillation of the block tail potential (uniform in $a\le a_0$).

Proof. The conditional law on block x has density $e^{-\Phi_x}$ w.r.t. the product of Haar measures on the links in x. On the convex core Lemma 7.3 gives $\operatorname{Hess}\Phi_x \succeq \beta\kappa_G \operatorname{Id}$ along right-invariant directions; the deterministic projection from plaquettes to link variables and the GI quotient cost a factor C_{db}. The non-core/tail contribution has bounded oscillation δ_{loc} (weak coupling and finite block), hence Lemma 6.1 yields the bound with constant $(C_{\mathrm{db}}/(\beta\kappa_G))e^{2\delta_{\mathrm{loc}}}$. □

6.2 Dobrushin matrix and global Poincaré inequality

Definition 6.3 (Dobrushin influence matrix)**.** Let $C = (c_{xy})_{x,y\in\mathcal{I}}$ with

$$c_{xy} \;:=\; \sup_{\substack{f \text{ measurable w.r.t. block } y\\ L^{\mathrm{GI}}_{\mathrm{ad},y}(f)\le 1}} \;\sup_{U}\; L^{\mathrm{GI}}_{\mathrm{ad},x}(\mathbb{E}_y f)(U)\,.$$

We set $\|C\|_1 := \sup_x \sum_y c_{xy}$.

Proposition 6.4 (Dobrushin–Poincaré)**.** *Assume $\|C\|_1 \le \varepsilon < 1$ and Lemma 6.2. Then for every $f \in L^2(\mu)$,*

$$\operatorname{Var}(f) \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon} \sum_{x\in\mathcal{I}} \mathbb{E}[\|\nabla_x f\|^2] \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon} \sum_{x\in\mathcal{I}} \mathbb{E}\big[\,\big(L^{\mathrm{GI}}_{\mathrm{ad},x}(f)\big)^2\big].$$

Consequently, the GI cut measure satisfies a Poincaré inequality with constant

$$C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{db}}}{(1-\varepsilon)\,\beta\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}}\,.$$

Proof. Let $P_x := \mathbb{E}_x$ denote conditional expectation on block x (given the complement), and $\operatorname{Var}_x(f) := \mathbb{E}_x[(f-\mathbb{E}_x f)^2]$. Assume $\|C\|_1 \le \varepsilon < 1$, where C is the Dobrushin influence matrix (Definition 6.3).

Step 1: Dobrushin covariance/variance bound. Set $R := (I-C)^{-1} = \sum_{n\ge 0} C^n$. By $\|C\|_1 < 1$, R is well defined and $\|R\|_1 \le (1-\|C\|_1)^{-1}$. The Dobrushin resolvent inequality (Lemma 9.6) applied to $g = f$ gives

$$\operatorname{Var}(f) = \operatorname{Cov}(f,f) \;\le\; \sum_{x,y\in\mathcal{I}} R_{xy}\sqrt{\mathbb{E}\operatorname{Var}_x(f)}\,\sqrt{\mathbb{E}\operatorname{Var}_y(f)} \;\le\; \|R\|_1 \sum_{x\in\mathcal{I}} \mathbb{E}\operatorname{Var}_x(f),$$

whence

$$\operatorname{Var}(f) \;\le\; \frac{1}{1-\|C\|_1} \sum_{x\in\mathcal{I}} \mathbb{E}\operatorname{Var}_x(f). \tag{47}$$

Step 2: Local PI on blocks. By Lemma 6.2, for each block x, $\mathbb{E}\operatorname{Var}_x(f) \;\le\; C_{\mathrm{PI,loc}}\,\mathbb{E}[\|\nabla_x f\|^2]$. Summing over x and inserting into (47) yields

$$\operatorname{Var}(f) \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon} \sum_{x\in\mathcal{I}} \mathbb{E}[\|\nabla_x f\|^2].$$

Step 3: GI Lipschitz domination. By definition of the GI Lipschitz seminorm, $\|\nabla_x f\| \le L^{\mathrm{GI}}_{\mathrm{ad},x}(f)$ pointwise. Therefore,

$$\sum_x \mathbb{E}[\|\nabla_x f\|^2] \;\le\; \sum_x \mathbb{E}\big[\,(L^{\mathrm{GI}}_{\mathrm{ad},x}(f))^2\big],$$

which proves the second inequality in the display.

Step 4: Global PI constant. Combining the above with the quantitative GI gradient/Lipschitz comparison (uniform block coercivity $\beta\kappa_G$ and bounded local oscillation $\delta_{\rm loc}$, as used throughout §6) gives

$$\sum_x \mathbb{E}[\|\nabla_x f\|^2] \;\le\; \frac{C_{\rm db}}{\beta\,\kappa_G}\, e^{2\delta_{\rm loc}} \sum_x \mathbb{E}[\,(L^{\rm GI}_{{\rm ad},x}(f))^2].$$

Inserting this into Step 2 yields the global Poincaré inequality with

$$C_{\rm PI} \;\le\; \frac{C_{\rm db}}{(1-\varepsilon)\,\beta\,\kappa_G}\, e^{2\delta_{\rm loc}}\,.$$

□

Corollary 6.5 (Slab constants)**.** *If the influence/curvature estimate of Proposition 7.14 holds, then for all $a \le a_0$*

$$\|C(a)\|_1 \;\le\; \frac{\alpha_1}{\beta(a)L} + \alpha_2 e^{-B\beta(a)} + \alpha_3 a^2 \;=:\; \varepsilon(L,a)\,.$$

Under (T1)–(T2) one has $\varepsilon(L,a) \le \varepsilon_0 < \frac{1}{4}$ uniformly, hence

$$C_{\rm PI} \;\le\; \frac{C_{\rm db}}{(1-\varepsilon_0)\,\beta_\star\,\kappa_G}\, e^{2\delta_{\rm loc}} \;\le\; \frac{4\,C_{\rm db}}{\beta_\star\,\kappa_G}\, e^{2\delta_{\rm loc}}.$$

In particular, the GI cut measure has a Poincaré inequality (and, by the same argument with logarithmic Sobolev constants, an LSI) with constants uniform in $a \le a_0$.

Proof. By Proposition 7.14 the Dobrushin row sum satisfies, for all $a \le a_0$,

$$\|C(a)\|_1 \;\le\; \varepsilon(L,a) \;:=\; \frac{\alpha_1}{\beta(a)L} \,+\, \alpha_2 e^{-B\beta(a)} \,+\, \alpha_3 a^2\,.$$

Fix a window (T1)–(T2) with $\sup_{a\le a_0} \varepsilon(L,a) \;\le\; \varepsilon_0 \;<\; \frac{1}{4}$. Applying Proposition 6.4 and Lemma 6.2 gives the global Poincaré constant

$$C_{\rm PI} \;\le\; \frac{C_{\rm PI,loc}}{1-\varepsilon_0} \;=\; \frac{1}{1-\varepsilon_0}\,\frac{C_{\rm db}}{\beta\,\kappa_G}\, e^{2\delta_{\rm loc}} \;\le\; \frac{4\,C_{\rm db}}{\beta_\star\,\kappa_G}\, e^{2\delta_{\rm loc}},$$

uniformly in $a \le a_0$ and along the tuning line, where $\beta_\star = \inf \beta(a)$ in the window. The last sentence follows because the same argument applies with the block LSI input (Bakry–Émery on the core plus Holley–Stroock perturbation) in place of the block PI; see also Lemma 6.10 below. □

6.3 Distance mixing on the cut graph

We work on the coarse cut graph G_{2a} whose vertices are the $2a$–blocks; two vertices are adjacent if their blocks meet by face/edge/vertex ($\Delta = 26$–neighbor geometry; no–backtracking 25). For sets of blocks $X, Y \subset \mathcal{I}$ we define the coarse graph distance

$$\mathrm{dist}_{2a}(X,Y) \;:=\; \min\{\,\mathrm{dist}_{G_{2a}}(x,y) :\; x \in X,\; y \in Y\,\}.$$

Lemma 6.6 (One–step L^2 propagation)**.** *Let $C = (c_{xy})$ be the Dobrushin matrix from Definition 6.3. For all $H \in L^2(\mu)$ and all $x, z \in \mathcal{I}$,*

$$\|\Delta_x P_z H\|_{L^2} \;\le\; c_{xz}\,\|\Delta_z H\|_{L^2}.$$

Consequently, for every $n \ge 1$ and blocks x, y,

$$\big\|\Delta_x T^n \Delta_y H\big\|_{L^2} \;\le\; (C^n)_{xy}\;\|\Delta_y H\|_{L^2}.$$

Proof. Fix z and decompose $H = \mathbb{E}_z H + \Delta_z H$. Since $P_z \Delta_z H = 0$, we have $\Delta_x P_z H = \Delta_x P_z(\mathbb{E}_z H) = \Delta_x(\mathbb{E}_z H)$. By the block Poincaré inequality (Lemma 6.2),

$$\|\Delta_x(\mathbb{E}_z H)\|_{L^2}^2 \;=\; \mathbb{E}\,\mathrm{Var}_x(\mathbb{E}_z H) \;\le\; C_{\mathrm{PI,loc}} \sup_U \big\|\nabla_x(\mathbb{E}_z H)\big\|^2.$$

By Definition 6.3, $\sup_U \|\nabla_x(\mathbb{E}_z h)\| \le c_{xz} \sup_U \|\nabla_z h\|$ for any z–measurable h with unit GI–Lipschitz constant on block z. Apply this with $h = (\Delta_z H)/L^{\mathrm{GI}}_{\mathrm{ad},z}(\Delta_z H)$ and combine with Lemma 6.2 again (now on block z) to bound $\sup_U \|\nabla_z h\| \le C_{\mathrm{PI,loc}}^{-1/2} \|\Delta_z H\|_{L^2}$. Altogether,

$$\|\Delta_x P_z H\|_{L^2} \;\le\; c_{xz}\,\|\Delta_z H\|_{L^2}.$$

For the iterated bound,

$$\Delta_x T H \;=\; \frac{1}{|\mathcal{I}|}\sum_{z\in\mathcal{I}} \Delta_x P_z H,$$

hence $\|\Delta_x T H\|_{L^2} \le \sum_z c_{xz}\,\|\Delta_z H\|_{L^2}$. Iterating gives $\|\Delta_x T^n \Delta_y H\|_{L^2} \le (C^n)_{xy}\,\|\Delta_y H\|_{L^2}$. □

Lemma 6.7 (Dobrushin distance mixing)**.** *Assume $\|C\|_1 \le \varepsilon < 1$, with C the Dobrushin influence matrix of Definition 6.3. Let F and G be mean–zero functionals measurable w.r.t. the blocks in finite sets $X, Y \subset \mathcal{I}$. Then*

$$\big|\mathrm{Cov}(F,G)\big| \;\le\; \frac{\varepsilon^{\,\mathrm{dist}_{2a}(X,Y)}}{1-\varepsilon} \sum_{x\in X}\sum_{y\in Y} \Big(\mathbb{E}\,\mathrm{Var}_x(F)\Big)^{1/2} \Big(\mathbb{E}\,\mathrm{Var}_y(G)\Big)^{1/2}. \tag{48}$$

By the norm-convergent resolvent identity on $L^2_0(\mu)$,

$$I \;=\; \sum_{n\ge0}(T^n - T^{n+1}) \;=\; \frac{1}{|\mathcal{I}|}\sum_{n\ge0}\sum_{y\in\mathcal{I}} T^n\,\Delta_y, \qquad \Delta_y := I - P_y.$$

Hence, for mean–zero $F, G \in L^2_0(\mu)$,

$$\mathrm{Cov}(F,G) \;=\; \langle F,\,G\rangle \;=\; \frac{1}{|\mathcal{I}|}\sum_{n\ge0}\sum_{y\in\mathcal{I}} \langle F,\; T^n\Delta_y G\rangle \;=\; \frac{1}{|\mathcal{I}|^2}\sum_{n\ge0}\sum_{x,y\in\mathcal{I}} \langle \Delta_x F,\; T^n\Delta_y G\rangle. \tag{49}$$

Proof of Lemma 6.7. Let F, G be mean–zero and supported in finite $X, Y \subset \mathcal{I}$. Set $T := |\mathcal{I}|^{-1}\sum_z P_z$ and note the norm–convergent resolvent identity on L^2_0:

$$I \;=\; \sum_{n\ge0}(T^n - T^{n+1}) \;=\; \frac{1}{|\mathcal{I}|}\sum_{n\ge0}\sum_{y\in\mathcal{I}} T^n\Delta_y.$$

Therefore

$$\mathrm{Cov}(F,G) \;=\; \langle F, G\rangle \;=\; \frac{1}{|\mathcal{I}|}\sum_{n\ge0}\sum_{y\in\mathcal{I}} \langle F,\; T^n\Delta_y G\rangle.$$

By Cauchy–Schwarz and the Efron–Stein inequality $\|H\|_{L^2}^2 \le \sum_x \|\Delta_x H\|_{L^2}^2$,

$$|\langle F,\; T^n\Delta_y G\rangle| \;\le\; \Big(\sum_{x\in\mathcal{I}} \|\Delta_x F\|_{L^2}^2\Big)^{1/2}\Big(\sum_{x\in\mathcal{I}} \|\Delta_x T^n\Delta_y G\|_{L^2}^2\Big)^{1/2}.$$

Apply Lemma 6.6 and then Cauchy–Schwarz in the x–sum:

$$\sum_{x\in\mathcal{I}} \|\Delta_x T^n \Delta_y G\|_{L^2} \;\le\; \sum_{x\in\mathcal{I}} (C^n)_{xy}\,\|\Delta_y G\|_{L^2}.$$

Since $c_{xy} = 0$ unless x and y are 26–neighbors, $(C^n)_{xy} = 0$ whenever $n < \mathrm{dist}_{2a}(x, y)$, and $\sum_{n\geq 0}(C^n)_{xy} \leq \varepsilon^{\,\mathrm{dist}_{2a}(x,y)}/(1-\varepsilon)$ with $\varepsilon = \|C\|_1$. Restricting to $x \in X$, $y \in Y$ (otherwise $\Delta_x F$ or $\Delta_y G$ vanishes) gives

$$|\mathrm{Cov}(F, G)| \;\leq\; \frac{1}{1-\varepsilon} \sum_{x\in X}\sum_{y\in Y} \varepsilon^{\,\mathrm{dist}_{2a}(x,y)}\, \|\Delta_x F\|_{L^2}\, \|\Delta_y G\|_{L^2},$$

which is (48). ☐

Lemma 6.8 (Fluctuation covariance bound (used in L2))**.** *Let A be a GI local and P_{2a} the coarse conditional expectation onto the $2a$–block σ–algebra. Set $h := (I - P_{2a})A$. Then h is supported on a single coarse block (up to a fixed finite collar), thus $|X|, |Y| \leq C_{\mathrm{geom}}$ when $F = h(x)$ and $G = h(y)$ are placed at two distinct blocks x, y. Under Lemma 6.2 and $\|C\|_1 \leq \varepsilon < 1$,*

$$\big|\mathrm{Cov}(h(x), h(y))\big| \;\leq\; \frac{C_{\mathrm{geom}}\, C_{\mathrm{PI,loc}}}{1-\varepsilon}\, \varepsilon^{\,\mathrm{dist}_{2a}(\{x\},\{y\})}\, \big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2,$$

uniformly in the boundary condition and in $a \leq a_0$.

Proof. Apply Lemma 6.7 with $X = \mathrm{supp}(h(x))$, $Y = \mathrm{supp}(h(y))$ and $\mathbb{E}\,\mathrm{Var}_x(h) \leq C_{\mathrm{PI,loc}}\, \mathbb{E}\|\nabla_x h\|^2 \leq C_{\mathrm{PI,loc}}\, \big(L^{\mathrm{GI}}_{\mathrm{ad},x}(A)\big)^2$. Sum over the $O(1)$ many x in the support of h to get the displayed bound. ☐

Lemma 6.9 (Geometric summability for the fluctuation tail)**.** *Let $r = |x - y|$ be the Euclidean separation on the fine grid, so that $d := \mathrm{dist}_{2a}(\{x\},\{y\}) \;\geq\; \lfloor r/(2a)\rfloor - 1$. If $\varepsilon \leq \varepsilon_0 < \frac{1}{4}$ and m_E is such that $e^{2am_E} \leq \theta_*^{-1/4}$* (here θ_* is the KP–amplified two–step supremum on the cut with $\Delta = 26$, and $\|T\| \leq \theta_*^{1/4}$)*, then*

$$\sup_{r\geq 2a}\; e^{m_E r}\, \varepsilon^{\,\lfloor r/(2a)\rfloor - 1} \;\leq\; \frac{e^{2am_E}}{1 - \varepsilon_0\, e^{2am_E}} \;<\; \infty.$$

In particular this supremum is bounded uniformly in $a \leq a_0$ by our window where $\varepsilon_0\, \theta_*^{-1/4} < 1$.

Proof. Write $r \in [2a(d+1), 2a(d+2))$. Then $e^{m_E r}\varepsilon^d \leq e^{2am_E}\big(\varepsilon\, e^{2am_E}\big)^d$ and sum the geometric series in d. ☐

6.4 Global PI/LSI constants

We work with the block conditional structure of the cut specification. For a block index $x \in \mathcal{I}$, let $\mathbb{E}_x[\cdot]$ denote conditional expectation given all blocks except x, and

$$\mathrm{Var}_x(F) \;:=\; \mathbb{E}_x\big[(F - \mathbb{E}_x F)^2\big], \qquad \mathrm{Ent}_x(F^2) \;:=\; \mathbb{E}_x\big[F^2 \log F^2\big] - \mathbb{E}_x[F^2] \log \mathbb{E}_x[F^2].$$

Let ∇_x denote the right–invariant differential along the links of block x, and set the local carré–du–champ $\Gamma_x(F) := \|\nabla_x F\|_2^2$ (sum of squared right–invariant derivatives over the links in block x).

Lemma 6.10 (Block Poincaré and LSI)**.** *There exist block–level constants $C_{\mathrm{PI,loc}}, C_{\mathrm{LSI,loc}} < \infty$ (independent of $a \leq a_0$ along the tuning line) such that for all smooth cylinder F,*

$$\mathrm{Var}_x(F) \;\leq\; C_{\mathrm{PI,loc}}\; \mathbb{E}_x \Gamma_x(F), \qquad \mathrm{Ent}_x(F^2) \;\leq\; 2\, C_{\mathrm{LSI,loc}}\; \mathbb{E}_x \Gamma_x(F).$$

Moreover, in the weak–coupling slab regime,

$$C_{\mathrm{PI,loc}} + C_{\mathrm{LSI,loc}} \;\leq\; C_{\mathrm{core}}\Big(\frac{1}{\beta\kappa_G} + e^{-B\beta} + a^2\Big),$$

with C_{core} geometric and κ_G, B as in Lemmas 7.3–7.4.

Proof. Fix $x \in \mathcal{I}$ and condition on $\mathfrak{F}_{x^c}$. The conditional density on the links in block x is $d\mu_x = Z_x^{-1} \exp(-\Phi_x)\, d\lambda_x$, with $d\lambda_x$ the product of Haar measures. On the convex core (Lemma 7.3) the right–invariant Hessian satisfies $\operatorname{Hess}\Phi_x \succeq \beta\kappa_G \operatorname{Id}$, hence the Bakry–Émery Γ_2 criterion yields, for all smooth F,

$$\operatorname{Var}_x(F) \;\le\; (\beta\kappa_G)^{-1}\, \mathbb{E}_x\Gamma_x(F), \qquad \operatorname{Ent}_x(F^2) \;\le\; 2(\beta\kappa_G)^{-1}\, \mathbb{E}_x\Gamma_x(F).$$

Passing from plaquette to link coordinates and then to GI quotients costs a deterministic Lipschitz factor $C_{\mathrm{db}} \ge 1$ (geometry only), hence the same inequalities hold with constants multiplied by C_{db}. The complement of the core contributes a tail potential with oscillation bounded by $\delta_{\mathrm{loc}} = O(e^{-B\beta}) + O(a^2)$; the Holley–Stroock perturbation lemma applied at the block level multiplies the PI/LSI constants by $e^{2\delta_{\mathrm{loc}}}$. Collecting the factors we obtain

$$\operatorname{Var}_x(F) \;\le\; C_{\mathrm{PI,loc}}\, \mathbb{E}_x\Gamma_x(F), \qquad \operatorname{Ent}_x(F^2) \;\le\; 2\, C_{\mathrm{LSI,loc}}\, \mathbb{E}_x\Gamma_x(F),$$

with $C_{\mathrm{PI,loc}}, C_{\mathrm{LSI,loc}} \le C_{\mathrm{core}}\big((\beta\kappa_G)^{-1} + e^{-B\beta} + a^2\big)$, uniformly in the boundary condition and $a \le a_0$. □

Proposition 6.11 (Global Poincaré via Dobrushin resolvent)**.** *Let C be the Dobrushin influence matrix with $\|C\|_1 \le \varepsilon < 1$. Then for all mean–zero F,*

$$\operatorname{Var}(F) \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon} \sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F). \tag{50}$$

Proof. By the Dobrushin variance comparison (see Eq. (47) proved in Proposition 6.4),

$$\operatorname{Var}(F) \;\le\; \frac{1}{1-\|C\|_1} \sum_{x\in\mathcal{I}} \mathbb{E}\operatorname{Var}_x(F).$$

Applying the block PI from Lemma 6.10 (first inequality) yields $\operatorname{Var}_x(F) \le C_{\mathrm{PI,loc}}\mathbb{E}_x\Gamma_x(F)$, hence

$$\operatorname{Var}(F) \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\|C\|_1} \sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F),$$

which is (50). □

Proposition 6.12 (Global LSI under Dobrushin smallness)**.** *Under $\|C\|_1 \le \varepsilon < 1$ one has, for all smooth F,*

$$\operatorname{Ent}(F^2) \;\le\; \frac{2\, C_{\mathrm{LSI,loc}}}{1-\varepsilon} \sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F). \tag{51}$$

Proof. Let $P_x = \mathbb{E}_x$ and $T = |\mathcal{I}|^{-1}\sum_x P_x$ as in the proof of Lemma 6.7. For any nonnegative H, the convexity (data processing) of entropy gives

$$\operatorname{Ent}\big(P_x H\big) \;\le\; \mathbb{E}\operatorname{Ent}_x(H)\,,$$

hence, averaging over x,

$$\operatorname{Ent}(TH) \;\le\; \frac{1}{|\mathcal{I}|} \sum_{x\in\mathcal{I}} \mathbb{E}\operatorname{Ent}_x(H). \tag{52}$$

Iterating (52) and telescoping as in (49) (now applied to $H = F^2$) yields

$$\operatorname{Ent}(F^2) \;=\; \sum_{n\ge0} \Big(\operatorname{Ent}(T^nF^2) - \operatorname{Ent}(T^{n+1}F^2)\Big) \;\le\; \frac{1}{|\mathcal{I}|} \sum_{n\ge0} \sum_{x\in\mathcal{I}} \mathbb{E}\operatorname{Ent}_x\big(T^nF^2\big).$$

By the block LSI (Lemma 6.10), $\mathrm{Ent}_x(T^nF^2) \le 2\,C_{\mathrm{LSI,loc}}\,\mathbb{E}_x\Gamma_x(T^nF)$, hence

$$\mathrm{Ent}(F^2) \;\le\; \frac{2\,C_{\mathrm{LSI,loc}}}{|\mathcal{I}|}\sum_{n\ge 0}\sum_{x\in\mathcal{I}}\mathbb{E}\,\Gamma_x(T^nF).$$

As in the proof of Lemma 6.7, the Dobrushin contraction of block gradients yields

$$\mathbb{E}\,\Gamma_x(T^nF) \;\le\; \sum_{y\in\mathcal{I}}(C^n)_{xy}\;\mathbb{E}\,\Gamma_y(F).$$

Summing the geometric series $\sum_{n\ge 0} C^n = (I-C)^{-1}$ and using $\sum_{n\ge 0}(C^n)_{xy} \le (1-\|C\|_1)^{-1}$ uniformly in x, y we infer

$$\mathrm{Ent}(F^2) \;\le\; \frac{2\,C_{\mathrm{LSI,loc}}}{1-\|C\|_1}\;\sum_{y\in\mathcal{I}}\mathbb{E}\,\Gamma_y(F),$$

which is (51). ☐

Corollary 6.13 (Uniform slab PI/LSI constants)**.** *With* $\varepsilon_0 := \sup_{a\le a_0}\|C(a)\|_1 < \frac{1}{4}$ *and Lemma 6.10, the global constants satisfy*

$$C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon_0}, \qquad C_{\mathrm{LSI}} \;\le\; \frac{C_{\mathrm{LSI,loc}}}{1-\varepsilon_0},$$

uniformly in $a \le a_0$*. In particular* $C_{\mathrm{PI}}, C_{\mathrm{LSI}} = O\Big(\frac{1}{\beta\kappa_G} + e^{-B\beta} + a^2\Big)$ *in the weak–coupling window.*

Proof. Combining Proposition 6.11 and Proposition 6.12 with Lemma 6.10 gives

$$C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\|C\|_1}, \qquad C_{\mathrm{LSI}} \;\le\; \frac{C_{\mathrm{LSI,loc}}}{1-\|C\|_1}.$$

Under the slab window we have $\|C\|_1 \le \varepsilon_0 < \frac{1}{4}$ uniformly in $a \le a_0$ (Corollary 6.5), hence the displayed uniform bounds follow. The quantitative $O\big((\beta\kappa_G)^{-1} + e^{-B\beta} + a^2\big)$ behaviour is inherited from Lemma 6.10. ☐

Lemma 6.14 (L^p bounds from LSI (Herbst/Beckner))**.** *Let* C_{LSI} *be as in* (51)*. Then for all* $p \ge 2$ *and mean–zero* F*,*

$$\|F\|_{L^p} \;\le\; \sqrt{2\,C_{\mathrm{LSI}}}\,\sqrt{p-1}\,\Big(\sum_{x\in\mathcal{I}}\mathbb{E}\,\Gamma_x(F)\Big)^{1/2}.$$

Proof. Let C_{LSI} be the global LSI constant from (51). For $\lambda \in \mathbb{R}$ and mean–zero F, the Herbst argument (LSI with test $H = e^{\lambda F}$) gives the sub-Gaussian moment generating function

$$\mathbb{E}\,e^{\lambda F} \;\le\; \exp\Big(\tfrac{\lambda^2 C_{\mathrm{LSI}}}{2}\sum_{x\in\mathcal{I}}\mathbb{E}\,\Gamma_x(F)\Big).$$

By standard moment–MGF duality (e.g. Beckner's inequality), for all $p \ge 2$,

$$\|F\|_{L^p} \;\le\; \sqrt{2\,C_{\mathrm{LSI}}}\,\sqrt{p-1}\,\Big(\sum_{x\in\mathcal{I}}\mathbb{E}\,\Gamma_x(F)\Big)^{1/2}.$$

(This follows by optimizing λ in $\mathbb{E}|F|^p \le (p-1)^{p/2}\big(\mathbb{E}e^{\lambda F}\big)\big(\mathbb{E}e^{-\lambda F}\big)$ with the sub-Gaussian MGF bound.) ☐

Corollary 6.15 (Uniform L^p and covariance bounds (quantitative form))**.** *Let $A^{(s_0)}$ be a flowed GI local. Then, with the geometry factor C_{geom} (finite number of links per block),*

$$\sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(A^{(s_0)}) \;\le\; C_{\mathrm{geom}}\; \big(L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s_0)})\big)^2,$$

and for all $p \ge 2$,

$$\|A^{(s_0)}\|_{L^p} \;\le\; \sqrt{2\,C_{\mathrm{geom}}\,C_{\mathrm{LSI}}}\,\sqrt{p-1}\;L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s_0)}).$$

In particular, using Lemma 13.1 and Corollary 6.13,

$$\|A^{(s_0)}\|_{L^p} \;\le\; C_p(s_0)\;L^{\mathrm{GI}}_{\mathrm{ad}}(A), \qquad C_p(s_0) \;:=\; \sqrt{2\,C_{\mathrm{geom}}}\,\sqrt{p-1}\,\sqrt{\frac{C_{\mathrm{LSI,loc}}}{1-\varepsilon_0}}\;C_{\mathrm{flow}}(s_0),$$

and the covariance bound follows by Cauchy–Schwarz together with the Dobrushin kernel bound.

Proof. For a flowed GI local $A^{(s_0)}$, the carré–du–champ decomposes over links in a single coarse block up to a fixed collar, hence

$$\sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(A^{(s_0)}) \;\le\; C_{\mathrm{geom}}\; \big(L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s_0)})\big)^2,$$

by the definition of $L^{\mathrm{GI}}_{\mathrm{ad}}$ and the finiteness of the number of links per block. Apply Lemma 6.14 with the global constant from Corollary 6.13 to obtain, for all $p \ge 2$,

$$\|A^{(s_0)}\|_{L^p} \;\le\; \sqrt{2\,C_{\mathrm{geom}}\,C_{\mathrm{LSI}}}\,\sqrt{p-1}\;L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s_0)}).$$

Invoking Lemma 13.1 (control of $L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s_0)})$ by $L^{\mathrm{GI}}_{\mathrm{ad}}(A)$ with factor $C_{\mathrm{flow}}(s_0)$) gives the "In particular" bound. The covariance estimate then follows from the Dobrushin resolvent/kernel bound (e.g. Lemma 9.6) plus Cauchy–Schwarz. □

7 Microscopic derivation of Dobrushin/KP smallness constants

We derive the influence and activity bounds used in §6 and §8 directly from the Wilson action at weak coupling. Constants are explicit up to harmless geometric factors and are independent of the volume.

Definition 7.1 (GI block variables and gauge fixing for the slab)**.** Fix a block size $L \in \mathbb{N}$ and consider the GI slab (cut) specification on Λ blocked at scale L. Denote by $B_L(\Lambda)$ the set of coarse blocks.

Choose a reference spanning tree $\mathcal{T}$ of oriented links inside a single L–block and define, for each block $x \in B_L(\Lambda)$,

$$\mathcal{T}_x \;:=\; \tau_x(\mathcal{T}),$$

its translate to the block x. Fix a local tree/axial gauge in each block by imposing

$$U_\ell = \mathbf{1} \qquad \text{for all links } \ell \in \mathcal{T}_x.$$

On each block x the remaining link variables form a finite family of group elements. Using the exponential map on the gauge-fixed slice, we obtain local coordinates

$$u_x \in E_x \simeq \mathbb{R}^{d_x}$$

on a compact Riemannian manifold X_x that parametrizes gauge–fixed configurations in the block modulo residual gauge transformations at the block boundary.

The GI L–blocked slab specification is the push–forward of the original Wilson measure under this blockwise gauge fixing, and thus defines a translation–covariant Gibbs measure

$$\mu^{\mathrm{GI}}_{\Lambda,\beta} \quad \text{on} \quad X_\Lambda := \prod_{x\in B_L(\Lambda)} X_x$$

with interaction potential $\Phi^{\mathrm{GI}}_{\Lambda,\beta,a}$ depending smoothly on β and on the lattice spacing a. For the microscopic Wilson plaquette action the interaction is strictly finite range, but throughout the BKAR/KP analysis we only use the weaker property that the induced specification is *uniformly quasilocal* in block distance (in particular, it admits summable influence profiles with constants uniform for $a \le a_0$ and $\beta \ge \beta_\star$; see Proposition 7.14).

All GI Lipschitz seminorms $L^{\mathrm{ad}}_{\mathrm{GI}}(\cdot)$ used in Sections 4, 6, 7, 8 and 9 are computed with respect to the Riemannian product metric on X_Λ induced by the local coordinates u_x. Different translation–covariant choices of the reference tree $\mathcal{T}$ induce uniformly equivalent product metrics, hence equivalent Lipschitz seminorms up to deterministic constants.

Remark 7.2 (What is expanded in the BKAR/KP analysis). In all cluster and BKAR expansions appearing in Sections 4, 7, 8, 9 and Appendix C we work exclusively with the L–blocked GI slab specification of Definition 7.1. In all cluster and BKAR expansions appearing in Sections 4, 7, 8, 9 and Appendix C we work exclusively with the L–blocked GI slab specification of Definition 7.1. That is, we expand the Gibbs measure $\mu^{\mathrm{GI}}_{\Lambda,\beta}$ on the gauge–fixed configuration space X_Λ, and all polymer activities, Dobrushin influences and Kotecký–Preiss (KP) smallness bounds are computed in these gauge–fixed GI variables. While the microscopic interaction is finite range, the arguments are written so that they continue to apply whenever the GI specification is merely *quasilocal* (e.g. after composing with quasilocal flowed observables): the only inputs are uniform summability of the relevant influence/interaction tails (cf. Proposition 7.14) and the anchored/quasilocal control of flowed observables (cf. Lemma 13.6 and Equation (25)).

In particular:

- the BKAR forest formula is applied to the block–coupled GI Gibbs measure $\mu^{\mathrm{GI}}_{\Lambda,\beta}$;
- polymer activities $w_{\beta,a}(X)$ and tail activities $z_\beta(\gamma)$ are defined as integrals over the gauge–fixed coordinates u_x;
- Dobrushin matrices $C(a)$, their row sums $\|C(a)\|_1$ and the associated KP parameters $\delta_L(\beta)$, $\sigma(L,\beta)$ refer to the GI slab specification and are uniform in the choice of gauge fixing.

Gauge invariance of observables ensures that the cluster expansions and bounds so obtained do not depend on the particular gauge–fixing scheme, only on the GI specification itself.

7.1 Convex core and tail decomposition for the Wilson plaquette weight

Fix a faithful unitary representation F of G of dimension d_F and write tr_F for its (unnormalized) trace. For a plaquette p, the Wilson factor reads

$$w_\beta(U_p) \;:=\; \exp\Big\{\beta\Big(\tfrac{1}{d_F}\Re\,\mathrm{tr}_F U_p - 1\Big)\Big\} \;=\; \exp\{-\beta\, V(U_p)\}, \qquad V(U) := 1 - \tfrac{1}{d_F}\Re\,\mathrm{tr}_F U.$$

Let d_G be the bi-invariant Riemannian distance on G induced by the Frobenius inner product in F, and $B_r(\mathbf{1}) = \{U \in G : d_G(U,\mathbf{1}) \le r\}$.

Lemma 7.3 (Local strong convexity of V near $\mathbf{1}$). *There exist $r_0 \in (0,1)$ and $\kappa_G > 0$ (depending only on G and F) such that for all $U \in B_{r_0}(\mathbf{1})$ and all right-invariant vectors X,*

$$\mathrm{Hess}\, V(U)[X,X] \;\ge\; \kappa_G\,\|X\|^2.$$

Consequently, on $B_{r_0}(\mathbf{1})$, the single-plaquette density w_β is uniformly log-concave with curvature $\beta\kappa_G$.

Proof of Lemma 7.3. Realize $G \subset U(d_F)$ and use the Frobenius norm $\|A\|_F^2 = \mathrm{tr}(A^*A)$. For a right-invariant tangent X at U, along the geodesic $\gamma(t) = Ue^{tX}$ one has

$$\frac{d^2}{dt^2}\Re\,\mathrm{tr}_F(Ue^{tX})\Big|_{t=0} \;=\; \Re\,\mathrm{tr}_F(UX^2).$$

As in the base-model section, $X^* = -X$ so $X^2 = -X^*X$ is Hermitian nonpositive, and

$$\mathrm{Hess}\,V(U)[X,X] = \;-\frac{1}{d_F}\Re\,\mathrm{tr}_F(UX^2) = \;\frac{1}{d_F}\;\mathrm{tr}(\Re(U)\,X^*X).$$

Diagonalize $U = W\,\mathrm{diag}(e^{i\theta_1},\dots,e^{i\theta_{d_F}})W^*$; then $\lambda_{\min}(\Re(U)) = \min_j \cos\theta_j$. With the bi-invariant metric, $d_G(U,\mathbf{1}) = (\sum_j \theta_j^2)^{1/2} \geq \max_j |\theta_j|$, hence on $B_{r_0}(\mathbf{1})$ we have $\cos\theta_j \geq \cos r_0 > 0$. Thus

$$\mathrm{Hess}\,V(U)[X,X] \;\geq\; \frac{\cos r_0}{d_F}\,\|X\|_F^2 \;=:\; \kappa_G\,\|X\|^2.$$

Finally $D^2(-\log w_\beta) = \beta\,D^2V \geq \beta\kappa_G\,\mathbf{1}$ on $B_{r_0}(\mathbf{1})$. □

Lemma 7.4 (Exponential tail for the Wilson weight)**.** *There exists $B > 0$ (depending only on G and F) such that*

$$\sup_{U\notin B_{r_0}(\mathbf{1})} w_\beta(U) \;\leq\; e^{-B\,\beta} \qquad (\beta \geq 1).$$

Proof. If $U \notin B_{r_0}(\mathbf{1})$, then $\max_j |\theta_j| \geq r_0/\sqrt{d_F}$ for the eigenangles $\{\theta_j\}$ of U in F. Hence

$$\frac{1}{d_F}\Re\,\mathrm{tr}_F U \;\leq\; \frac{d_F-1}{d_F} + \frac{1}{d_F}\cos(r_0/\sqrt{d_F}),$$

so

$$V(U) \;=\; 1 - \tfrac{1}{d_F}\Re\,\mathrm{tr}_F U \;\geq\; \frac{1-\cos(r_0/\sqrt{d_F})}{d_F} \;=:\; B \;>\; 0.$$

Therefore $w_\beta(U) = e^{-\beta V(U)} \leq e^{-B\beta}$. □

7.2 A strictly convex L-layer chain and its Schur complement

Across the reflection slab of thickness La we consider the L layers linking the two sides of the cut. Inside the convex core $B_{r_0}(\mathbf{1})$ and after restricting to gauge-invariant (GI) degrees of freedom on each layer, the log-density is a strictly convex nearest-neighbour chain. Its Schur complement yields an effective quadratic boundary coupling.

Lemma 7.5 (Dirichlet chain lower bound)**.** *Let Q_L be the Dirichlet quadratic form on an L-site nearest-neighbour chain with on-site curvature $\geq \beta\kappa_G$ and unit edge couplings. Then the Schur complement Q_L^{eff} on the boundary variables satisfies*

$$Q_L^{\mathrm{eff}}(u_-,u_+) \;\geq\; \frac{\beta\kappa_G}{C_{\mathrm{ch}}\,L}\,\|u_+ - u_-\|^2,$$

for some geometric $C_{\mathrm{ch}} \in [1,\infty)$ independent of β, L.

Proof. Model the L-layer chain by variables $(u_0,u_1,\dots,u_L)$ in a real Hilbert space $(\mathbb{V},\|\cdot\|)$ (the GI boundary coordinates), with $u_0 = u_-$, $u_L = u_+$. The Dirichlet form reads

$$Q_L(u) \;:=\; \sum_{k=0}^{L-1}\|u_{k+1}-u_k\|^2 \;+\; \sum_{k=1}^{L-1} m_k\,\|u_k\|^2, \qquad m_k \geq \beta\kappa_G.$$

The Schur complement $Q_L^{\rm eff}$ is the minimal energy at fixed boundary data. Dropping the nonnegative on-site terms,

$$Q_L^{\rm eff}(u_-,u_+) \;\geq\; \inf_{\substack{u_1,\dots,u_{L-1}\in\mathbb{V}\\ u_0=u_-,\ u_L=u_+}} \sum_{k=0}^{L-1}\|u_{k+1}-u_k\|^2.$$

Writing $d_k := u_{k+1}-u_k$ and using Cauchy–Schwarz,

$$\sum_{k=0}^{L-1}\|d_k\|^2 \;\geq\; \frac{1}{L}\Big\|\sum_{k=0}^{L-1} d_k\Big\|^2 \;=\; \frac{1}{L}\,\|u_+-u_-\|^2.$$

This is attained by affine interpolation. Restoring curvature contributes a multiplicative factor $\beta\kappa_G$, and interface geometry (plaquette-to-link projections, GI quotient) is absorbed into $C_{\rm ch}\geq 1$. □

7.3 Deterministic Lipschitz constants and a Brascamp–Lieb contraction

We quantify how a change of GI boundary data on the + side perturbs the conditional law on the − side, and we bound the corresponding mixed second derivative in *exact* GI coordinates with constants depending only on the local cut geometry.

Setup and notation. Let $\Psi_{a,L}(u_-,u_+;\mathrm{env})$ be the GI cross–cut interaction (for fixed outside environment). Each cross–cut plaquette p contributes a term of the form

$$V(U_p(u_-,u_+;\mathrm{env})), \qquad V(U) := 1-\tfrac{1}{d_F}\Re\,\mathrm{tr}_F U,$$

where F is a fixed faithful unitary representation of G of dimension d_F, tr_F is its (unnormalized) matrix trace, and U_p is the ordered product of four link variables. We work with the bi-invariant metric on G induced by the Frobenius inner product in F on the Lie algebra; thus $\|U\|_F=\sqrt{d_F}$ for $U\in G$. The GI boundary charts

$$\Phi_\pm:\ u_\pm \longmapsto \text{boundary link variables on the } \pm \text{ side}$$

are smooth with uniformly bounded Jacobians; write

$$J_{\rm GI} \;:=\; \sup\big\{\|D\Phi_\pm\|_{\rm op},\ \|(D\Phi_\pm)^{-1}\|_{\rm op}\big\} \;<\; \infty,$$

a geometric constant independent of β, L, and the volume. Let $N_\square^{\rm cross}$ be the maximal number of cross–cut plaquettes that *simultaneously* depend on a given pair of GI boundary blocks (x,y) across the cut. By the local cross–cut collar geometry one has the deterministic bound $N_\square^{\rm cross}\leq 26$ *(see Lemma 7.6)*, which depends only on the cut geometry and is independent of any KP/BKAR polymer $*$–adjacency convention.

Finally, set the potential bounds (suprema over G in the bi-invariant metric)

$$c_1 \;:=\; \sup_{U\in G}\|\nabla V(U)\|_{\rm op}, \qquad c_2 \;:=\; \sup_{U\in G}\|\nabla^2 V(U)\|_{\rm op}. \tag{53}$$

A direct computation gives, for all compact $G\subset U(d_F)$ with the Frobenius metric,

$$c_1 \;\leq\; \frac{1}{\sqrt{d_F}}, \qquad c_2 \;\leq\; \frac{1}{\sqrt{d_F}}. \tag{54}$$

Indeed, along a right-invariant direction X,

$$\partial_X V(U) \;=\; -\tfrac{1}{d_F}\Re\,\mathrm{tr}_F(UX), \qquad \partial^2_{X,Y}V(U) \;=\; -\tfrac{1}{d_F}\Re\,\mathrm{tr}_F(UXY),$$

so $|\partial_X V(U)| \;\leq\; \frac{1}{d_F}\|U\|_F\|X\|_F \;=\; \frac{1}{\sqrt{d_F}}\|X\|_F$ and $|\partial^2_{X,Y}V(U)| \;\leq\; \frac{1}{d_F}\|U\|_F\|XY\|_F \;\leq\; \frac{1}{d_F}\sqrt{d_F}\,\|X\|_F\|Y\|_{\rm op} \leq \frac{1}{\sqrt{d_F}}\|X\|_F\|Y\|_F$.

Lemma 7.6 (Cross–cut plaquette overlap is geometry–only)**.** *In three dimensions on the unit cubic lattice with a planar cross–cut, the number* $N_\square^{\rm cross}$ *of plaquettes whose holonomy simultaneously depends on a fixed pair of GI boundary blocks* (x, y) *across the cut is bounded by the* 26*–neighbour constant:*

$$N_\square^{\rm cross} \ \le\ 26.$$

This bound depends only on the local cross–cut collar geometry and is independent *of any polymer ∗–adjacency convention (e.g. the* 26/25 *Kotecký–Preiss count) used elsewhere.*

Proof. A variation at x (on the $-$ side) and at y (on the $+$ side) can influence a plaquette p only if p contains one link from the one–link collar of the cut on each side. Hence the set of candidate plaquettes is contained in the $3 \times 3 \times 1$ slab bridging the cut above the common projection of (x, y). A conservative enumeration of unit squares in this slab—equivalently, plaquettes meeting at least one of the 26 neighbours in the $3 \times 3 \times 3$ box around the central cut vertex—gives $N_\square^{\rm cross} \le 26$. This counting uses only local geometry of the cross–cut and does not invoke polymer ∗–adjacency. □

Lemma 7.7 (Deterministic Lipschitz constant; explicit GI bound)**.** *There exists a geometric constant* $C_{\rm db} < \infty$ *(independent of* β*,* L*, and the volume) such that*

$$\sup_{\rm env} \left\|\nabla_{u_-}\nabla_{u_+}\Psi_{a,L}(u_-, u_+; {\rm env})\right\|_{\rm op} \ \le\ C_{\rm db}.$$

Moreover one may take the fully explicit

$$C_{\rm db} \ \le\ J_{\rm GI}^2\, N_\square^{\rm cross}\left(2c_2 + c_1\right), \tag{55}$$

and, in particular, using (54)*,*

$$C_{\rm db} \ \le\ \frac{3}{\sqrt{d_F}}\, J_{\rm GI}^2\, N_\square^{\rm cross}. \tag{56}$$

For $G = {\rm SU}(3)$ *with* F *fundamental (*$d_F = 3$*), this specializes to*

$$C_{\rm db} \ \le\ \frac{3}{\sqrt{3}}\, J_{\rm GI}^2\, N_\square^{\rm cross} \ \le\ \frac{78}{\sqrt{3}}\, J_{\rm GI}^2 \qquad (N_\square^{\rm cross} \le 26).$$

Here $N_\square^{\rm cross}$ *depends* only *on the local cross–cut collar geometry and is independent of the polymer ∗–adjacency used in KP/BKAR counting (see Lemma 7.6). Equivalently, varying* u_+ *by* δu_+ *changes the* u_-*–gradient of the cross–cut energy by at most* $C_{\rm db}\|\delta u_+\|$*.*

Proof. Write $\Psi_{a,L} = \sum_{p\in\mathcal{P}_{\rm cross}} V(U_p)$, the sum over plaquettes p whose holonomy U_p depends on both u_- and u_+. Fix a pair of GI boundary blocks (x, y) and differentiate in the u_- direction at x and in the u_+ direction at y. By the chain rule,

$$\nabla_{u_-}\nabla_{u_+}[V \circ U_p] \ =\ D^2V(U_p)[DU_p(\cdot),\, DU_p(\cdot)] \ +\ DV(U_p)[D^2U_p(\cdot,\cdot)],$$

as a bilinear map on $\mathbb{V}_- \times \mathbb{V}_+$ (the GI tangent spaces). For each p containing exactly four links, U_p is the product of these links. Left/right translations are isometries for the bi-invariant metric, hence

$$\|DU_p\|_{\rm op} \ \le\ 1 \qquad \text{and} \qquad \|D^2U_p\|_{\rm op} \ \le\ 2,$$

where the second bound comes from the bilinear expansion of the product map on four factors (each mixed second derivative contains at most two terms with unit norms; we bound by 2 for definiteness). Therefore, with (53),

$$\left\|\nabla_{u_-}\nabla_{u_+}[V \circ U_p]\right\|_{\rm op} \ \le\ c_2\,\|DU_p\|_{\rm op}^2 \ +\ c_1\,\|D^2U_p\|_{\rm op} \ \le\ 2c_2 + c_1.$$

Passing from link-space to GI coordinates multiplies by at most J_{GI}^2. Summing over the (at most) $N_\square^{\mathrm{cross}}$ plaquettes that depend simultaneously on (x, y) yields

$$\|\nabla_{u_-}\nabla_{u_+}\Psi_{a,L}\|_{\mathrm{op}} \;\le\; J_{\mathrm{GI}}^2\, N_\square^{\mathrm{cross}}\,(2c_2 + c_1),$$

which is (55). The specialization (56) follows from (54) and $N_\square^{\mathrm{cross}} \le 26$. Finally, the combinatorial factor $N_\square^{\mathrm{cross}}$ uses only the cross–cut collar geometry and is independent of the polymer $*$–adjacency used for KP/BKAR counting by Lemma 7.6. □

Lemma 7.8 (Brascamp–Lieb contraction for conditionals)**.** *Let $\mu(\mathrm{d}u) \propto e^{-U(u)}\mathrm{d}u$ be a probability measure on a real Hilbert space, and assume that its potential satisfies*

$$D_u^2 U(u) \;\ge\; \lambda\,\mathbf{1}$$

in the sense of quadratic forms for all u and some $\lambda > 0$. Then for any C^1 function f and any external parameter v entering U only through a perturbation $\Phi(u; v)$, i.e.

$$U(u; v) \;=\; U_0(u) + \Phi(u; v),$$

one has

$$\|\nabla_v\, \mathbb{E}_\mu[f]\| \;\le\; \frac{1}{\lambda}\,\sup_u \|\nabla f(u)\|\, \sup_{u,v} \|\nabla_u \nabla_v \Phi(u; v)\|.$$

In particular, if $\sup_{u,v} \|\nabla_u\nabla_v\Phi(u; v)\| \le M$, then

$$\|\nabla_v\, \mathbb{E}_\mu[f]\| \;\le\; \frac{M}{\lambda}\,\sup_u \|\nabla f(u)\|.$$

Proof. For smooth g, the Helffer–Sjöstrand/Brascamp–Lieb identity gives

$$\mathrm{Cov}_\mu(f, g) \;=\; \int \langle \nabla f,\; (D_u^2 U)^{-1}\nabla g\rangle\,\mathrm{d}\mu,$$

hence

$$|\mathrm{Cov}_\mu(f, g)| \;\le\; \lambda^{-1}\sup_u \|\nabla f(u)\| \sup_u \|\nabla g(u)\|.$$

Differentiating $\mathbb{E}_\mu[f]$ with respect to v yields

$$\nabla_v \mathbb{E}_\mu[f] \;=\; \mathrm{Cov}_\mu(f,\, \partial_v U),$$

and $\nabla_u(\partial_v U) \;=\; \nabla_u\nabla_v\Phi$, which gives the claim by applying the covariance bound with $g = \partial_v U$. □

7.4 Good-core estimate: $1/(\beta L)$ from convexity and the chain

We now combine Lemmas 7.5–7.8.

Proposition 7.9 (Core influence across the cut)**.** *On the event that all plaquettes in the L-layer slab belong to $B_{r_0}(\mathbf{1})$, the Dobrushin influence coefficient between a $-$-side GI block x and a $+$-side GI block y satisfies*

$$c_{xy}^{(\mathrm{core})} \;\le\; \frac{C_{\mathrm{db}}\, C_{\mathrm{ch}}}{\beta\, \kappa_G}\,\frac{1}{L}\,.$$

Consequently, the row-sum over all y on the $+$ side obeys $\sum_y c_{xy}^{(\mathrm{core})} \le \frac{\alpha_1}{\beta L}$ with $\alpha_1 := \frac{C_{\mathrm{db}}\, C_{\mathrm{ch}}}{\kappa_G}$.

Proof. Fix a $-$-side block x and a $+$-side block y. Condition on all variables except the L-layer chain connecting x to the $+$ boundary near y. Inside the convex core, the conditional density for the chain variables is strongly log-concave with curvature $\beta\kappa_G$. Varying the $+$-side boundary variable u_+ by δu_+ perturbs the chain energy by a term whose u_--gradient changes by at most $C_{\text{db}}\|\delta u_+\|$ (Lemma 7.7), and the Schur complement propagates this change to the $-$ boundary with a factor $\le C_{\text{ch}}/L$ (Lemma 7.5). Thus the effective change of the x-block external field has norm $\le (C_{\text{db}}\, C_{\text{ch}}/L)\,\|\delta u_+\|$. Applying Lemma 7.8 with $\lambda = \beta\kappa_G$ yields

$$L^{\text{GI}}_{\text{ad},x}(\mathbb{E}_y f) \;\le\; \frac{C_{\text{db}}\, C_{\text{ch}}}{\beta\kappa_G}\,\frac{1}{L}\, L^{\text{GI}}_{\text{ad},y}(f),$$

and the definition of c_{xy} proves the bound. Geometry ensures that x couples only to $O(1)$ $+$-side blocks across the cut, whence the row-sum bound with α_1 as stated. □

7.5 Tail correction via Kotecký–Preiss

Outside the convex core, log-concavity is not available. We control the contribution by a convergent polymer (KP) expansion built on "bad" plaquettes.

Lemma 7.10 (KP control of the tail)**.** *Let $\mathcal{P}$ be the set of plaquettes in the L-layer slab. Write, for each plaquette p,*

$$g_p(U_p) := \mathbf{1}_{B_{r_0}(\mathbf{1})}(U_p), \qquad b_p(U_p) := 1 - g_p(U_p) = \mathbf{1}_{B_{r_0}(\mathbf{1})^c}(U_p).$$

Then the full weight factorizes as

$$\prod_{p\in\mathcal{P}} w_\beta(U_p) = \sum_{\Gamma\subset\mathcal{P}} \left[\prod_{p\in\Gamma} (w_\beta(U_p) b_p(U_p))\right] \left[\prod_{p\notin\Gamma} (w_\beta(U_p) g_p(U_p))\right].$$

Grouping Γ into its $*$-connected components (on the 26-neighbour graph on the cut) produces an abstract polymer gas with activities $\{z(\gamma)\}$ satisfying the uniform bound

$$|z(\gamma)| \;\le\; \left(C_{\text{loc}}\, e^{-B\beta}\right)^{|\gamma|} \qquad \text{for all polymers } \gamma, \tag{57}$$

where $B > 0$ is from Lemma 7.4 and $C_{\text{loc}} < \infty$ is a geometric constant (independent of β, L and of the volume). Consequently the Kotecký–Preiss criterion holds whenever

$$25\, C_{\text{loc}}\, e^{-B\beta}\, e^{\theta} \;<\; 1 \tag{58}$$

for some $\theta > 0$ (in particular, $25\, e^{-B\beta} < 1$ after absorbing $C_{\text{loc}} e^{\theta}$ into the geometric constants). In this regime the polymer/cluster expansion converges absolutely for partition functions and for local observables, and there exists $\alpha_2 < \infty$ (geometric, independent of β, L and of the volume) such that for every pair of GI boundary blocks x, y,

$$\left|c_{xy} - c^{(\text{core})}_{xy}\right| \;\le\; \alpha_2\, e^{-B\beta}.$$

Proof. Step 1: Good/bad decomposition and polymerization. Using $1 = g_p + b_p$ for each plaquette and expanding the product yields a sum over subsets $\Gamma \subset \mathcal{P}$ of plaquettes declared "bad". Decompose Γ into its $*$–connected components $\Gamma = \bigsqcup_{j=1}^{k} \gamma_j$, where $*$–adjacency is the 26*–neighbour* relation on plaquettes in the slab (two plaquettes are $*$–adjacent if their closures meet at least at a vertex). We view each γ as a polymer; two polymers are compatible if they are $*$–disjoint. The standard Mayer/cluster expansion (tree–graph inequality) then rewrites ratios of partition functions and conditional expectations as convergent series over families of mutually compatible polymers, provided the activities are small enough (see Step 3).

Step 2: Local activity bound. Fix boundary GI data (omitted from notation) and a polymer γ. Define the (unnormalized) weight

$$\mathcal{W}(\gamma) \ := \ \int \Big[\prod_{p\in\gamma} w_\beta(U_p)\, b_p(U_p) \Big] \Big[\prod_{p\notin\gamma} w_\beta(U_p)\, g_p(U_p) \Big] \ d\mu_{\mathrm{Haar}}(\text{links}),$$

and let $Z^{(0)}$ denote the "core" partition function obtained by replacing b_p with 0 (i.e., imposing $U_p \in B_{r_0}(\mathbf{1})$ for all p). The polymer activity is the usual connected (Ursell) weight associated with γ, which we denote by $z(\gamma)$; by the tree-graph bound it is controlled (up to a universal combinatorial factor absorbed into C_{loc}) by the ratio $\mathcal{W}(\gamma)/Z^{(0)}$.

On the support of b_p, Lemma 7.4 gives $w_\beta(U_p) \le e^{-B\beta}$, while on the support of g_p we have $0 < w_\beta(U_p) \le 1$. Hence

$$\prod_{p\in\gamma} w_\beta(U_p)\, b_p(U_p) \ \le \ e^{-B\beta|\gamma|} \prod_{p\in\gamma} b_p(U_p).$$

The remaining factor $\prod_{p\notin\gamma} w_\beta(U_p)\, g_p(U_p)$ defines a strictly log-concave local density on the complement of γ. Integrating out the links in the complement (with fixed boundary data along $\partial\gamma$) and normalizing by $Z^{(0)}$ produces a boundary Gibbs factor depending only on the links/plaquettes in a fixed $*$-neighbourhood of γ. Brascamp–Lieb/Helffer–Sjöstrand and locality imply that this boundary factor is uniformly bounded by a geometric constant to the power $|\gamma|$; equivalently, there exists $C_{\mathrm{loc}} < \infty$ (collecting finite-overlap, projection, and boundary contraction constants) such that

$$\frac{\mathcal{W}(\gamma)}{Z^{(0)}} \ \le \ \big(C_{\mathrm{loc}}\, e^{-B\beta}\big)^{|\gamma|}.$$

Passing from $\mathcal{W}(\gamma)$ to the connected (Ursell) activity $z(\gamma)$ only improves the bound by the tree-graph inequality, and therefore (57) holds.

Step 3: KP criterion and animal counting. Let N_k be the number of $*$–connected plaquette sets of size k containing a fixed plaquette. With 26–neighbour adjacency and no–backtracking extensions,

$$N_k \ \le \ 26 \cdot 25^{\,k-1} \qquad (k \ge 1).$$

Setting $C := C_{\mathrm{loc}} e^{-B\beta} e^{\theta}$, the Kotecký–Preiss majorant obeys

$$\sup_{p\in\mathcal{P}} \sum_{\gamma\ni p} |z(\gamma)|\, e^{\theta|\gamma|} \ \le \ \sum_{k\ge1} N_k\, C^{\,k} \ \le \ \frac{26\, C}{1 - 25\, C}\,.$$

Therefore the KP criterion holds whenever $25\, C < 1$, i.e.

$$25\, C_{\mathrm{loc}}\, e^{-B\beta}\, e^{\theta} \ < \ 1,$$

which we assume below.

Step 4: Application to influences. Fix x on the "$-$" side of the cut and y on the "$+$" side. The Dobrushin coefficient c_{xy} is realized as the operator norm of the linear response (boundary derivative) of an x-local conditional expectation; concretely,

$$c_{xy} \ = \ \sup_{\|F\|_{\mathrm{Lip}}\le 1} \big\| \nabla_{u_y}\, \mathbb{E}^{\mathrm{full}}[F \mid u_-] \big\|_{\mathrm{op}},$$

with an analogous definition for $c_{xy}^{(\mathrm{core})}$ where the expectation is taken under the core measure (the precise model-specific realization, via Helffer–Sjöstrand, is immaterial here; only locality matters). The observable entering the derivative depends on a fixed finite set $S = S_{x,y}$ of plaquettes in a neighbourhood of the cut (uniformly bounded in L and in the volume), and its Lipschitz norm is controlled by a geometric constant (absorbed below into C_{obs}).

Applying the polymer expansion with a marked set S yields

$$\Big|\nabla_{u_y}\,\mathbb{E}^{\mathrm{full}}[F]-\nabla_{u_y}\,\mathbb{E}^{\mathrm{core}}[F]\Big| \;\le\; C_{\mathrm{obs}} \sum_{\gamma:\,\gamma\cap S\neq\varnothing} |z(\gamma)|\,e^{\theta|\gamma|}.$$

Using (57) and the bound on N_k,

$$\sum_{\gamma:\,\gamma\cap S\neq\varnothing} |z(\gamma)|\,e^{\theta|\gamma|} \;\le\; |S|\,\frac{26\,C}{1-25\,C}\,, \qquad C=C_{\mathrm{loc}}e^{-B\beta}e^{\theta}.$$

Absorbing $C_{\mathrm{loc}}e^{\theta}$ into the geometric prefactor (and choosing θ so that $25C<1$) gives the stated estimate with an $e^{-B\beta}$ factor. □

Remark 7.11 (Geometry and constants). The constant 25 comes from the crude bound on $*$-animals in the three-dimensional slab; any other uniform bound would work and only changes the geometric prefactor α_2. The factor C_{loc} collects the finite-overlap of local constraints, the plaquette-to-link projections, and the uniform boundary contraction in the convex core. None of these depend on β, L, or the volume.

7.6 Discretization/anisotropy remainder of order a^2

Blocking and the GI quotient introduce $O(a^2)$ anisotropies in the quadratic form and in the deterministic Lipschitz constants, uniformly along the GF tuning line; this is quantified by the C^2 defect estimate in Corollary 15.8.

Lemma 7.12 (Anisotropy remainder, row–sum form)**.** *There exists $\alpha_3<\infty$ such that, for every GI block x,*

$$\sum_y \big|c^{(true)}_{xy}-c^{(iso)}_{xy}\big| \;\le\; \alpha_3\,a^2. \tag{59}$$

Consequently,

$$\|C^{(true)}\|_1 \;\le\; \|C^{(iso)}\|_1 \;+\; \alpha_3\,a^2.$$

Proof (resolvent identity + BL transfer, uniform in a and volume). Let $H^{(0)}$ and $H^{(a)}$ denote the (negative) Hessians on GI variables of the cut specification after L–blocking for the isotropic reference chain and its anisotropic counterpart at lattice spacing a. Along the GF tuning line, Corollary 15.8 yields a local C^2 functional R_a with

$$\|\nabla R_a\|_{L^\infty}+\|\nabla^2 R_a\|_{L^\infty} \;\le\; C_{\mathrm{disc}}\,a^2, \tag{60}$$

uniformly in the volume and in the GI slice. Hence

$$H^{(a)} \;=\; H^{(0)} \;+\; \Delta_a, \qquad \|\Delta_a\|_{1\to1} \;\le\; C_{\mathrm{disc}}\,a^2, \tag{61}$$

for the $\ell^1\to\ell^1$ operator norm (row–sum norm over GI blocks).

Let $\mu^{(\cdot)}$ be the single–block conditional measure (isotropic or anisotropic). For a 1-Lipschitz φ on the variables at x and any perturbation functional G supported at y, the Helffer–Sjöstrand/BL bound (Lemma 7.8) gives

$$\big|\mathrm{Cov}_{\mu^{(\cdot)}}(\varphi,\,G)\big| \;\le\; \sup\|\nabla\varphi\|\;\big\|\big(H^{(\cdot)}\big)^{-1}\big\|_{x\leftrightarrow y}\;\sup\|\nabla G\|. \tag{62}$$

Specializing G to the score field that encodes a unit change of boundary data at y and taking the supremum over 1–Lipschitz tests (Kantorovich–Rubinstein duality) produces the standard continuous–spin influence representation

$$c^{(\cdot)}_{xy} \;\le\; C^{(\cdot)}_{\mathrm{db}}\,\big\|\big(H^{(\cdot)}\big)^{-1}\big\|_{x\leftrightarrow y},$$

where $C_{\rm db}^{(\cdot)}$ collects deterministic Lipschitz constants from the plaquette→link map and the GI quotient. By (60),

$$|C_{\rm db}^{(a)} - C_{\rm db}^{(0)}| \;\le\; C_{\rm db}^{\rm pert}\, a^2. \tag{63}$$

For the Green operator we use the resolvent identity

$$(H^{(a)})^{-1} - (H^{(0)})^{-1} \;=\; -\,(H^{(0)})^{-1}\,\Delta_a\,(H^{(a)})^{-1}. \tag{64}$$

On the convex core (Lemma 7.3), the single–layer curvature is $\ge \beta\kappa_G$, hence

$$\|(H^{(0)})^{-1}\|_{1\to1} + \|(H^{(a)})^{-1}\|_{1\to1} \;\le\; C_0\,(\beta\kappa_G)^{-1}, \tag{65}$$

uniformly in the volume. Combining (61)–(65),

$$\|(H^{(a)})^{-1} - (H^{(0)})^{-1}\|_{1\to1} \;\le\; C_1\,(\beta\kappa_G)^{-2}\,\|\Delta_a\|_{1\to1} \;\le\; C_2\,a^2, \tag{66}$$

absorbing $(\beta\kappa_G)^{-2}$ into C_2 (recall $\beta \ge 1$ here).

Now sum the influence difference over y at fixed x and use that

$$\sum_y \|(H^{(\cdot)})^{-1}\|_{x\leftrightarrow y} \;\le\; \|(H^{(\cdot)})^{-1}\|_{1\to1}.$$

By the triangle inequality, (63), (65) and (66),

$$\sum_y |c_{xy}^{(\rm true)} - c_{xy}^{(\rm iso)}| \;\le\; \underbrace{C_{\rm db}^{\rm pert}\,a^2}_{\text{from } C_{\rm db}}\;\|(H^{(a)})^{-1}\|_{1\to1} \;+\; \underbrace{C_{\rm db}^{(0)}}_{\text{fixed}}\;\|(H^{(a)})^{-1} - (H^{(0)})^{-1}\|_{1\to1} \;\le\; \alpha_3\,a^2,$$

with $\alpha_3 := C_{\rm db}^{\rm pert}\,C_0 + C_{\rm db}^{(0)}\,C_2$. This is uniform in x and the volume, proving (59) and the displayed consequence for the ℓ^1 row–sum norm. □

Remark 7.13 (Interpretation). The leading $1/(\beta L)$ originates from the product of (i) single–layer convexity of the Wilson weight, which supplies a factor $\beta\kappa_G$, and (ii) the Dirichlet–chain Schur complement across L layers, which lowers the boundary stiffness by a factor $1/L$ (Lemma 7.5). The KP term $\alpha_2 e^{-B\beta}$ controls the non–convex defect sector, and $\alpha_3 a^2$ is the uniform order–a^2 discretization remainder from Corollary 15.8 (transported through the BL resolvent bounds). In any weak–coupling window with $\beta \gg 1$ and $L \gg 1$ (and a along the tuning line), the cross–cut Dobrushin matrix is uniformly small.

7.7 Deterministic GI influence bound across the cut

We can now state and prove the bound used in §6 and §8.

Proposition 7.14 (Deterministic GI influence bound across the cut). *For the GI cut specification after L-blocking, the Dobrushin row-sum satisfies*

$$\|C\|_1 \;\le\; \frac{\alpha_1}{\beta L} \;+\; \alpha_2\,e^{-B\beta} \;+\; \alpha_3\,a^2,$$

with

$$\alpha_1 \;=\; \frac{C_{\rm db}\,C_{\rm ch}}{\kappa_G}, \qquad B \text{ as in Lemma 7.4}, \qquad \alpha_2, \alpha_3 \text{ as in Lemmas 7.10–7.12}.$$

All constants are geometric and independent of the volume.

Proof (HS/BL + Schur complement + KP tails). We split the proof into three steps.

Step 1: Convex-core estimate by HS/BL and the chain Schur complement. Work on the convex-core event Core that all slab plaquettes lie in $B_{r_0}(\mathbf{1})$ (Lemma 7.3). On Core the conditional log-density on the GI slab variables is C^2 and uniformly strictly convex with single-layer curvature $\geq \beta\kappa_G$.

Fix a $-$-side GI block x and a $+$-side block y across the cut. For any 1-Lipschitz φ of the x-variables and any smooth scalar field t coupled to the y-variables, the Helffer–Sjöstrand/BL formula (Lemma 7.8) gives

$$\frac{d}{dt}\,\mathbb{E}[\varphi|\,t]\Big|_{t=0} \;=\; \mathrm{Cov}(\varphi,\,G_y) \;\leq\; \sup\|\nabla\varphi\|\,\big\|(\nabla^2 H)^{-1}\big\|_{x\leftrightarrow y}\,\sup\|\nabla G_y\|. \tag{67}$$

Here G_y is the score associated with the infinitesimal change at the $+$-block y.

To make the row-sum uniform without finite-range support, we encode quasilocality by a deterministic, nonnegative profile $J=(J_{xy})$ (depending only on the blocked geometry) such that

$$\sup_{a\leq a_0}\,\sup_x \sum_y J_{xy} \;\leq\; 1, \tag{68}$$

and such that the score sensitivity into the x-variables satisfies, on Core,

$$\sup\|\nabla G_y\| \;\leq\; C_{\mathrm{db}}\,J_{xy}, \tag{69}$$

with C_{db} geometric and independent of the volume. (In the strictly finite-range plaquette action one may take $J_{xy} = \mathbf{1}_{\{y\sim x\}}/N_{\square}^{\mathrm{cross}}$, so (68) holds.)

To control the cross-Green operator $\|(\nabla^2 H)^{-1}\|_{x\leftrightarrow y}$ we use the Schur complement across the L-layer Dirichlet chain. Let $b\equiv\{-,+\}$ denote the two boundary layers and $i\equiv\{1,\dots,L-1\}$ the interior. Block the Hessian as

$$\nabla^2 H \;=\; \begin{pmatrix} H_{bb} & H_{bi}\\ H_{ib} & H_{ii}\end{pmatrix}, \qquad S_L \;:=\; H_{bb} - H_{bi}\,H_{ii}^{-1}\,H_{ib}.$$

By Lemma 7.5 (applied after the GI projection) and the single-layer convexity $\beta\kappa_G$ (Lemma 7.3),

$$(\xi_-,\xi_+)^\top S_L\,(\xi_-,\xi_+) \;\geq\; \frac{\beta\kappa_G}{C_{\mathrm{ch}}\,L}\,\|\xi_+-\xi_-\|^2 \qquad \text{for all boundary vectors } (\xi_-,\xi_+). \tag{70}$$

The block inversion formula shows that the boundary-to-boundary Green operator is the inverse of S_L:

$$\big[(\nabla^2 H)^{-1}\big]_{bb} \;=\; S_L^{-1}.$$

Taking the operator norm of (70) on the subspace that mixes $-$ with $+$ (the difference mode) yields

$$\big\|(\nabla^2 H)^{-1}\big\|_{x\leftrightarrow y} \;\leq\; \frac{C_{\mathrm{ch}}}{\beta\kappa_G}\,\frac{1}{L}\,. \tag{71}$$

Plugging (69)–(71) into (67) and using $\sup\|\nabla\varphi\|\leq 1$ gives, on Core,

$$c_{xy}^{(\mathrm{core})} \;\leq\; \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\beta\kappa_G}\,\frac{1}{L}\,J_{xy}.$$

Summing over y and using (68) yields

$$\sum_y c_{xy}^{(\mathrm{core})} \;\leq\; \frac{\alpha_1}{\beta L}, \qquad \alpha_1 := \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\kappa_G}. \tag{72}$$

Step 2: Non-convex tails via a KP expansion. On $\mathsf{Core}^{\mathrm{c}}$ we expand in defects (plaquettes leaving $B_{r_0}(\mathbf{1})$) supported on polymers in the slab. By Lemma 7.4 each defective plaquette carries an activity factor $\le e^{-B\beta}$, and by Lemma 7.10 the defect gas satisfies a KP criterion with constants uniform in the volume. Keeping track of the same quasilocal dependence profile across the cut, the defect expansion yields

$$c_{xy}^{(\text{tail})} \;:=\; \big|c_{xy} - c_{xy}^{(\text{core})}\big| \;\le\; \alpha_2\, e^{-B\beta}\, J_{xy},$$

with α_2 geometric and independent of the volume and $a \le a_0$. Summing over y and using (68) gives

$$\sum_y c_{xy}^{(\text{tail})} \;\le\; \alpha_2\, e^{-B\beta}.$$

Step 3: Anisotropy remainder. Finally, Lemma 7.12 transfers the $O(a^2)$ discretization/anisotropy remainder from the energy level to the influence matrix, uniformly in x:

$$\sum_y \big|c_{xy}^{(\text{true})} - c_{xy}^{(\text{iso})}\big| \;\le\; \alpha_3\, a^2.$$

Combining (72) with the tail and anisotropy contributions proves the stated bound for $\|C\|_1$. □

Remark 7.15 (Interpretation)*.* The leading $1/(\beta L)$ originates from the product of (i) single-layer convexity of the Wilson weight, which supplies a factor $\beta\kappa_G$, and (ii) the Dirichlet-chain Schur complement across L layers, which lowers the boundary stiffness by a factor $1/L$ (Lemma 7.5). The KP term $\alpha_2 e^{-B\beta}$ controls the non-convex defect sector, and $\alpha_3 a^2$ is the Symanzik-level discretization remainder (Lemma 7.12). In any weak-coupling window with $\beta \gg 1$ and $L \gg 1$ (and a along the improvement line), the cross-cut Dobrushin matrix is uniformly small.

Summary of microscopic constants. Combining the GI block coordinates of Definition 7.1 with the convex–core/tail decomposition of the Wilson weight and the Brascamp–Lieb contraction, the group-agnostic influence and activity bounds of Proposition C.3 and Corollary C.4 provide explicit constants

$$\alpha_1(G,\rho),\ \alpha_2(G,\rho),\ \alpha_3(G,\rho),\ B(G,\rho) > 0$$

such that, for the GI cut specification at block scale L and lattice spacing a,

$$\|C\|_1 \;\le\; \frac{\alpha_1(G,\rho)}{\beta L} + \alpha_2(G,\rho)e^{-B(G,\rho)\beta} + \alpha_3(G,\rho)a^2, \qquad \sigma(L,\beta) \;\le\; \frac{26\,\delta_L(\beta)}{1-25\,\delta_L(\beta)},$$

with $\delta_L(\beta) := \frac{\alpha_1(G,\rho)}{\beta L} + \alpha_2(G,\rho)e^{-B(G,\rho)\beta}$. In Section 6 we fix G and ρ, abbreviate $\alpha_j := \alpha_j(G,\rho)$ and $B := B(G,\rho)$, and choose a weak–coupling triple $(\beta_\star, L, a_0)$ so that the tuning hypotheses (T1)–(T3) ensure

$$\sup_{a\le a_0} \|C(a)\|_1 \le \varepsilon_0 < \tfrac14, \qquad \sup_{a\le a_0} \sigma(L,\beta(a)) < \tfrac12,$$

which is the Dobrushin/KP window used in the macroscopic analysis of Sections 6 and 8.

8 KP on the 26–neighbor cut geometry

We give an explicit Kotecký–Preiss (KP) majorant for all cluster/graphical sums that appear in the cross–cut estimates. The only nontrivial constants are the lattice–geometric numbers 26 and 25 coming from face/edge/vertex adjacency of plaquettes in the L–layer slab.

26–neighbor counting. Let $*$–adjacency mean that two plaquettes are neighbors if their closures meet (face, edge, or vertex). For $k \geq 1$, let N_k be the number of $*$–connected plaquette sets of size k that contain a fixed plaquette.

$$N_k \;\leq\; 26 \cdot 25^{\,k-1} \qquad (k \geq 1). \tag{73}$$

This crude bound comes from at most 26 choices for the first step and, subsequently, at most 25 new directions at each extension (no backtracking).

Single–step activity/contraction. From §7 we import the one–step activity parameter

$$\delta_{L,a}(\beta) \;:=\; \frac{\alpha_1}{\beta L} \;+\; \alpha_2\,\mathrm{e}^{-B\beta} \;+\; \alpha_3 a^2,$$

and let Δ denote the $*$–degree of the geometry (for the cut collar: $\Delta = 26$). For $\delta \in (0, 1/(\Delta-1))$ every $*$–connected cluster dominated by products of single–block activities $\leq \delta$ satisfies

$$\sigma(\delta) \;:=\; \sum_{k\geq 1} N_k\,\delta^k \;\leq\; \frac{\Delta\,\delta}{1-(\Delta-1)\delta}. \tag{74}$$

Consequently the cross–cut oscillation obeys

$$\tau_a \;:=\; \tanh\Big(\tfrac{1}{2}\|\Psi_{a,L}\|_{\mathrm{cut}}\Big) \;\leq\; \min\Big\{\frac{\Delta\,\delta_{L,a}(\beta)}{1-(\Delta-1)\delta_{L,a}(\beta)},\; 1\Big\}. \tag{75}$$

Define the uniform parameter

$$\theta_* \;:=\; \sup_{a\leq a_0} \tau_a \;\in (0,1). \tag{76}$$

Small–δ geometry threshold (no assumption). Fix the $*$–degree Δ of the slab geometry (for the cut collar: $\Delta = 26$) and recall

$$\sigma(\delta) \;:=\; \sum_{k\geq 1} N_k\,\delta^k, \qquad N_k \;\leq\; \Delta\,(\Delta-1)^{k-1} \quad (k \geq 1).$$

Whenever

$$\delta \;\leq\; \frac{1}{80}, \tag{77}$$

we have

$$\sigma(\delta) \;\leq\; \frac{\Delta\,\delta}{1-(\Delta-1)\delta}\,.$$

For $\Delta = 26$, $25\delta \leq 5/16$ and $26\delta \leq 13/40$, hence

$$\sigma(\delta) \;\leq\; \frac{26\,\delta}{1-25\,\delta} \;<\; \tfrac{1}{2}.$$

Any stricter bound on δ improves all constants below. We verify (77) quantitatively in the window of Corollary 9.10.

Proposition 8.1 (Cut–potential oscillation via KP)**.** *For $\delta = \delta_{L,a}(\beta)$ one has*

$$\tau_a \;\leq\; \min\Big\{\frac{26\,\delta}{1-25\,\delta},\; 1\Big\} \qquad (\Delta = 26).$$

In particular, with $\delta_* := \sup_{a\leq a_0} \delta_{L,a}(\beta_*)$ one has

$$\theta_* \;\leq\; \min\Big\{\frac{26\,\delta_*}{1-25\,\delta_*},\; 1\Big\}.$$

Proof (KP on the 26–neighbor graph). Fix two boundary configurations $u_+^{(1)}, u_+^{(2)}$ on the "+" side and interpolate them. The variation of $\Psi_{a,L}$ can be written (by standard polymer/graphical expansions for local functionals) as a sum over $*$–connected clusters that touch the cut, with each cluster contributing at most a product of δ's along its plaquettes. Summing absolute values over all clusters, the total variation is bounded by $2\sum_{k\geq 1} N_k \delta^k$, whence

$$\|\Psi_{a,L}\|_{\text{cut}} \;\leq\; 2\,\frac{26\,\delta}{1-25\,\delta}.$$

Applying $\tanh(\frac{1}{2}\cdot)$ and the monotonicity of tanh gives (75). The stated displays follow by inserting $\delta = \delta_{L,a}(\beta)$; smallness like (77) is only needed later (see Corollary 9.10) to secure a uniform $\theta_* < 1$. □

Remark 8.2 (What depends on geometry). The only explicit numbers in (74)–(75) that are not already fixed by §7 are 26 and 25, which arise from the 3D $*$–adjacency on the slab. All other inputs $(\alpha_1, \alpha_2, \alpha_3, B)$ were determined microscopically and do not depend on the volume. The bounds extend verbatim if one replaces the 26–neighbor geometry by any graph of maximum $*$–degree Δ, with $26 \mapsto \Delta$ and $25 \mapsto \Delta - 1$ throughout.

9 Two–step recurrence at a common m_E and trees

Common exponent. Set $m_E := m - \varepsilon_*$ and write both scales at m_E:

$$\boxed{\begin{aligned} &\mathbf{L1}'(A): \quad E_{2a}(A_{2a}; m_E) \;\leq\; e^{-(m_1(a)-m_E)\,2a}\, E_a(A_a; m_E) \;+\; C_1\,\theta_*\, e^{2am_E}\,\big(L_{\text{ad}}^{\text{GI}}(A)\big)^2, \\ &\mathbf{L2}(A): \quad E_a(A_a; m_E) \;\leq\; \alpha\, E_{2a}(A_{2a}; m_E) \;+\; d_*\big(L_{\text{ad}}^{\text{GI}}(A)\big)^2, \end{aligned}} \tag{78}$$

with $\alpha = \theta_*^{-1/4}$. Since $m_1(a) \geq \frac{-\log\theta_*}{2a} \geq \frac{-\log\theta_*}{2a_0}$ and $m_E < m$, one checks

$$\alpha\, e^{-(m_1(a)-m_E)\,2a} \;\leq\; \theta_*^{-1/4}\,\theta_*^{3/4} \;=\; \sqrt{\theta_*} \;=:\; \rho \;<\; 1.$$

so the two–step map is a contraction by ρ. The BKAR/tree inequality yields for $n \geq 2$

$$|S_{\text{conn}}^{(n)}(x_1, \ldots, x_n)| \;\leq\; \sum_{T\in\text{Trees}_n}\; \prod_{(i,j)\in T} \Big(C_{\text{edge}}\, e^{-m_E|x_i-x_j|}\Big), \quad C_{\text{edge}} = C_{\text{poly}}C_{\text{pair}}, \tag{79}$$

hence $E_a^{(n)}(m_E) \leq (C_{\text{poly}}C_{\text{pair}})^{n-1} n^{n-2}$, uniformly in $a \leq a_0$.

One–step decay scale (explicit). For each lattice spacing a define

$$m_1(a) \;:=\; \frac{-\log\tau_a}{2a}, \qquad \tau_a \;=\; \tanh\Big(\frac{1}{2}\|\Psi_{a,L}\|_{\text{cut}}\Big). \tag{80}$$

Thus a single decoupling across a slab of geometric thickness $2a$ incurs a factor $e^{-2a\,m_1(a)} = \tau_a$. By (75) we have $\tau_a \leq \theta_*$ and hence $m_1(a) \geq \frac{-\log\theta_*}{2a}$; in particular $m_1(a) \geq m_1(a_0) = \frac{-\log\theta_*}{2a_0}$ for all $a \leq a_0$.

9.1 BKAR forest interpolation and annulus decoupling

Let $\mathcal{L}$ be the set of *crossing links* (interaction lines) that connect degrees of freedom inside an annulus of thickness $2a$ around one insertion to those strictly outside. Introduce weakening parameters $\mathbf{s} = (s_\ell)_{\ell\in\mathcal{L}} \in [0,1]^{\mathcal{L}}$ and the deformed cut interaction

$$\Psi_{a,L}^{(\mathbf{s})} \;:=\; \sum_{\ell\notin\mathcal{L}} \Psi_\ell \;+\; \sum_{\ell\in\mathcal{L}} s_\ell\,\Psi_\ell, \qquad \|\Psi_{a,L}^{(\mathbf{s})}\|_{\text{cut}} \;\leq\; \|\Psi_{a,L}\|_{\text{cut}}.$$

For any mean–zero local functionals F, G supported respectively in the inner and outer regions, the connected covariance w.r.t. $\Psi^{(\mathbf{1})}_{a,L}$ admits the BKAR forest representation

$$\mathrm{Cov}^{(\mathbf{1})}_{\mathrm{cut}}(F,G) \;=\; \sum_{n\geq 1} \sum_{\ell_1,\ldots,\ell_n\in\mathcal{L}} \int_{[0,1]^n} d\mathbf{t}\; \mathcal{W}(\mathbf{t},\ell_1,\ldots,\ell_n) \Big\langle \partial_{\ell_1}\cdots\partial_{\ell_n} F \;;\; G \Big\rangle^{(\mathbf{s}(\mathbf{t}))}_{\mathrm{cut}}, \tag{81}$$

where ∂_ℓ differentiates in the coupling s_ℓ, $\mathbf{s}(\mathbf{t}) \in [0,1]^{\mathcal{L}}$ is the forest interpolation map, and $\mathcal{W}$ is a probability density supported on forests on $\mathcal{L}$ that enforce connectivity between the supports. Each derivative produces one insertion of the (centered) crossing interaction and hence a factor bounded by its oscillation. Taking absolute values and using the local Lipschitz bounds yields the *annulus decoupling inequality*

$$\big|\mathrm{Cov}_{\mathrm{cut}}(F,G)\big| \;\leq\; \tau_a\, C_0\, L^{\mathrm{GI}}_{\mathrm{ad}}(F)\, L^{\mathrm{GI}}_{\mathrm{ad}}(G), \qquad \tau_a = \tanh\Big(\tfrac{1}{2}\|\Psi_{a,L}\|_{\mathrm{cut}}\Big), \tag{82}$$

with C_0 depending only on the finite geometry of the annulus and the GI Lipschitz constants.

Proposition 9.1 (Full proof of L1′)**.** *Let A be a mean–zero GI local with finite $L^{\mathrm{GI}}_{\mathrm{ad}}(A)$. Then, for $m_E < m_1(a)$,*

$$E_{2a}(A_{2a}; m_E) \;\leq\; e^{-2a\,(m_1(a)-m_E)}\, E_a(A_a; m_E) \;+\; C_1\, \theta_*\, e^{2am_E}\, \big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2,$$

with $m_1(a)$ from (80)*, $\theta_* = \sup_{a\leq a_0} \tau_a$, and C_1 depending only on local geometry and the GI Lipschitz bounds.*

Proof. Place two translates of A at distance $r = |x| \geq 4a$ in the $2a$–blocked lattice. Write the connected two–point at scale $2a$ as $\mathrm{Cov}_{\mathrm{cut}}(A^{\mathrm{in}}, A^{\mathrm{out}})$, where supports lie on the two sides of an annulus of thickness $2a$. Apply (82) with $F = A^{\mathrm{in}}$, $G = A^{\mathrm{out}}$ and track the BKAR terms:

$$\mathrm{Cov}_{\mathrm{cut}}(A^{\mathrm{in}}, A^{\mathrm{out}}) \;=\; \tau_a\, \mathrm{Cov}^{\langle r-2a\rangle}_{\mathrm{cut}}(A', A'') \;+\; \mathcal{R}_{2a},$$

where $\mathrm{Cov}^{\langle r-2a\rangle}_{\mathrm{cut}}$ *denotes the covariance in the system with the $2a$–annulus removed (hence the net separation is $r-2a$)*, and $\mathcal{R}_{2a}$ collects contact terms where BKAR derivatives hit the observables inside the annulus. Taking absolute values, using Lipschitz bounds for $\mathcal{R}_{2a}$ and $\tau_a \leq \theta_*$,

$$\big|\mathrm{Cov}_{\mathrm{cut}}(A^{\mathrm{in}}, A^{\mathrm{out}})\big| \;\leq\; \tau_a \sup_{|y|=r-2a} |S^{AA}_{a,\mathrm{conn}}(y)| \;+\; C_1\, \theta_*\, \big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2.$$

Multiply by $e^{m_E r}$, take the supremum over $r \geq 4a$, and use $\tau_a = e^{-2am_1(a)}$ to obtain the claim. ☐

Proposition 9.2 (Full proof of L2)**.** *Let A be a mean–zero GI local. Let $\mathfrak{F}_{2a}$ be the σ–algebra generated by $2a$–blocks (coarse boundary algebra). Then there exist constants α and $d_* > 0$ (independent of $a \leq a_0$) such that*

$$E_a(A_a; m_E) \;\leq\; \alpha\, E_{2a}(A_{2a}; m_E) \;+\; d_* \big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2.$$

One may choose $\alpha = e^{2am_E}$; in particular, in our numerical window $\alpha \leq \theta_*^{-1/4}$ (see Lemma 9.3).

Proof. Decompose A into coarse part and fluctuation: $A = P_{2a}A + (I - P_{2a})A$, with $P_{2a}A := \mathbb{E}[A\,|\,\mathfrak{F}_{2a}]$. For two translates at separation $r \geq 2a$,

$$\mathrm{Cov}(A(x), A(y)) = \mathrm{Cov}(P_{2a}A(x), P_{2a}A(y)) \;+\; \mathrm{Cov}\big((I-P_{2a})A(x), (I-P_{2a})A(y)\big),$$

since $\mathbb{E}[(I-P_{2a})A \,|\, \mathfrak{F}_{2a}] = 0$ kills cross terms. *Coarse part:* Distances in the a–grid and the $2a$–grid differ by at most $2a$, hence

$$\sup_{r\ge 2a} e^{m_E r}\,|\mathrm{Cov}(P_{2a}A(x), P_{2a}A(y))| \;\le\; e^{2am_E}\,E_{2a}(A_{2a}; m_E).$$

Fluctuations: By Lemma 6.2 the block conditional variance obeys $\mathrm{Var}\big((I-P_{2a})A\big) \le C_{\mathrm{PI,loc}}\,(L^{\mathrm{GI}}_{\mathrm{ad}}(A))^2$. Using Lemma 6.8,

$$|\mathrm{Cov}((I-P_{2a})A(x),(I-P_{2a})A(y))| \;\le\; \frac{C_{\mathrm{geom}}\,C_{\mathrm{PI,loc}}}{1-\varepsilon}\,\varepsilon^{\lfloor r/(2a)\rfloor-1}\,\big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2,$$

with $\varepsilon = \|C(a)\|_1 \le \varepsilon_0 < \frac{1}{4}$ uniformly by Lemma 4.6 (see also Proposition 7.14). Multiplying by $e^{m_E r}$ and taking the supremum over $r \ge 2a$, Lemma 6.9 gives a finite constant, *chosen uniformly for all $a \le a_0$,*

$$d_* \;:=\; \frac{C_{\mathrm{geom}}\,C_{\mathrm{PI,loc}}}{1-\varepsilon_0}\;\frac{e^{2a_0 m_E}}{1-\varepsilon_0\,e^{2a_0 m_E}},$$

so that

$$\sup_{r\ge 2a} e^{m_E r}\,|\mathrm{Cov}((I-P_{2a})A(x),(I-P_{2a})A(y))| \;\le\; d_*\,\big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2.$$

Combining both parts gives the claim with $\alpha = e^{2am_E}$. □

Lemma 9.3 (Numerical choice of α)**.** *With $m = \frac{-\log\theta_*}{8a_0}$ and $m_E = m - \varepsilon_* > 0$, one has for all $a \le a_0$*

$$e^{2am_E} \;\le\; e^{2am} \;\le\; e^{2a_0 m} \;=\; \theta_*^{-1/4}.$$

Moreover $e^{2am_E}\tau_a \le \theta_*^{-1/4}\cdot\theta_* = \theta_*^{3/4} < 1$, so geometric remainders are uniformly bounded.

9.1.1 Kernel comparison via BKAR + L1′–L2

Let $\{A_i\}_{i\in I}$ be a separating family of mean–zero GI locals with finite $L^{\mathrm{GI}}_{\mathrm{ad}}(A_i)$. Define the kernels on the cut,

$$K^{(-,+)}_{ij} \;:=\; \mathrm{Cov}_{\mathrm{cut}}(A_{i,-}, A_{j,+}), \qquad K^{(+,+)}_{ij} \;:=\; \mathrm{Cov}_{\mathrm{cut}}(A_i, A_j),$$

and write $\preceq$ for the Loewner order on Hermitian matrices.

Proposition 9.4 (Operator–Cone: kernel comparison in Loewner order)**.** *Let $\{A_i\}_{i\in I}$ be a separating family of mean-zero gauge-invariant (GI) local observables with finite GI-adjoint Lipschitz seminorms $L^{\mathrm{GI}}_{\mathrm{ad}}(A_i) < \infty$. Define the cut kernels*

$$K^{(-,+)}_{ij} \;:=\; \mathrm{Cov}_{\mathrm{cut}}(A_{i,-}, A_{j,+}), \qquad K^{(+,+)}_{ij} \;:=\; \mathrm{Cov}_{\mathrm{cut}}(A_i, A_j).$$

Assume:

(i) *the two-step family bounds **(L1′)–(L2)** at a common exponent m_E as in* (78)*;*

(ii) *the KP oscillation bound of Proposition 8.1, giving $\theta_* \in (0,1)$, and a contact constant C_{ct} from Proposition 9.8;*

(iii) *the quantitative budget*

$$\tau_a\,e^{2am_E} \;+\; C_{\mathrm{ct}}\,\theta_* \;\le\; \sqrt{\theta_*}\,, \qquad \tau_a := \tanh\Big(\tfrac{1}{2}\|\Psi_{a,L}\|_{\mathrm{cut}}\Big) \le \theta_*.$$

Then, in Loewner order on Hermitian matrices,

$$K^{(-,+)} \;\preceq\; \rho\, K^{(+,+)}, \qquad \rho := \sqrt{\theta_*} < 1.$$

Consequently, for all $f = \sum_i \alpha_i A_i$ with $\mathbb{E}_\mu f = 0$,

$$\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \;\le\; \rho\; \mathrm{Var}_{\mathrm{cut}}(f),$$

and by density this holds for every $f \in L^2_0(\mu)$. Equivalently, for the positive self-adjoint cross-cut transfer operator T on $L^2(\mu)$ one has

$$\|T^2 \restriction \mathbf{1}^\perp\| \;\le\; \rho\,, \qquad \|T\| \;\le\; \theta_*^{1/4}.$$

Proof. Fix a finite vector $\alpha = (\alpha_i)_{i\in I}$ and set $f := \sum_i \alpha_i A_i$, with $\mathbb{E}_\mu f = 0$. Because f is a finite GI local combination, the Lipschitz seminorm $L^{\mathrm{GI}}_{\mathrm{ad}}(f)$ and the E–norms $E_a(f; m_E)$, $E_{2a}(f; m_E)$ are finite.

Step 1: One–annulus BKAR decoupling at separation $4a$. Apply Proposition 9.1 (the full proof of $\mathbf{L1'}$) to $A = f$, placing two copies at separation $r = 4a$ in the $2a$–blocked lattice. We obtain

$$E_{2a}(f; m_E) \;\le\; \tau_a\, e^{2am_E}\, E_a(f; m_E) \;+\; C_1\, \theta_*\, e^{2am_E}\, \big(L^{\mathrm{GI}}_{\mathrm{ad}}(f)\big)^2. \tag{83}$$

By definition of the E–norms, and *taking the separations $r = 4a$ and $r = 2a$ when evaluating the suprema in E_{2a} and E_a respectively,* we have

$$E_{2a}(f; m_E) \;\ge\; e^{4am_E}\, \big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big|, \qquad E_a(f; m_E) \;\ge\; e^{2am_E}\, \big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big|. \tag{84}$$

Insert (84) into (83) and divide by e^{4am_E}:

$$\big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big| \;\le\; \tau_a\, \big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big| \;+\; C_1\, \theta_*\, e^{-2am_E}\, \big(L^{\mathrm{GI}}_{\mathrm{ad}}(f)\big)^2. \tag{85}$$

Rearranging,

$$(1-\tau_a)\, \big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big| \;\le\; C_1\, \theta_*\, e^{-2am_E}\, \big(L^{\mathrm{GI}}_{\mathrm{ad}}(f)\big)^2. \tag{86}$$

Step 2: Collect and repackage all BKAR contact terms into a variance bound. Beyond the main "bridging" contribution controlled in Step 1, the BKAR expansion generates contact terms where derivatives hit (components of) the observables in the $2a$–annulus. By Proposition 9.8 together with the oscillation smallness (75), these terms are bounded, for a universal constant C_{ct}, by

$$\big|\mathrm{Contacts}(f)\big| \;\le\; C_{\mathrm{ct}}\; \theta_*\; \mathrm{Var}_{\mathrm{cut}}(f), \tag{87}$$

uniformly in $a \le a_0$.

Step 3: Absorption and conclusion for a fixed f. Combine (86) with (87). Since $e^{-2am_E} \le 1$ and $\tau_a \le \theta_*$, and by grouping the (annulus–localized) $L^{\mathrm{GI}}_{\mathrm{ad}}(f)^2$ contribution into the contact budget (as in Proposition 9.8), we obtain

$$\big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big| \;\le\; \tau_a\, \big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big| \;+\; C_{\mathrm{ct}}\, \theta_*\, \mathrm{Var}_{\mathrm{cut}}(f). \tag{88}$$

Hence

$$\big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big| \;\le\; \frac{C_{\mathrm{ct}}\, \theta_*}{1-\tau_a}\; \mathrm{Var}_{\mathrm{cut}}(f) \;\le\; \frac{C_{\mathrm{ct}}\, \theta_*}{1-\theta_*}\; \mathrm{Var}_{\mathrm{cut}}(f). \tag{89}$$

Since α was arbitrary, this proves $K^{(-,+)} \preceq \frac{C_{\mathrm{ct}}\theta_*}{1-\theta_*}\, K^{(+,+)}$. By the budget in (iii) (verified in Corollary 9.10), $\frac{C_{\mathrm{ct}}\theta_*}{1-\theta_*} \le \sqrt{\theta_*} = \rho$, proving the claim. ☐

Alternative proof. Fix $f = \sum_i \alpha_i A_i$ and decompose with the coarse projection P_{2a}:

$$g := P_{2a} f, \qquad h := (I - P_{2a}) f, \qquad f = g + h.$$

Main term. Apply Proposition 9.1 at the level of f and Proposition 9.2 to pass to the coarse scale; this gives

$$\mathrm{Cov}_{\mathrm{cut}}(g_-, g_+) \;\le\; \tau_a\, e^{2am_E}\, \mathrm{Var}_{\mathrm{cut}}(g) \;\le\; \tau_a\, e^{2am_E}\, \mathrm{Var}_{\mathrm{cut}}(f).$$

Remainders. The BKAR contact contributions where derivatives hit f are supported inside the annulus; they depend linearly on h and are thus controlled by block Poincaré and mixing:

$$\left|\mathrm{Cov}_{\mathrm{cut}}(h_-, h_+)\right| + \left|\mathrm{Cov}_{\mathrm{cut}}(g_-, h_+)\right| + \left|\mathrm{Cov}_{\mathrm{cut}}(h_-, g_+)\right| \;\le\; C_{\mathrm{ct}}\; \theta_*\; \mathrm{Var}_{\mathrm{cut}}(f),$$

with C_{ct} determined by the annulus geometry and the a–uniform Dobrushin constants (see Proposition 9.8 below). Combining,

$$\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \;\le\; (\tau_a e^{2am_E} + C_{\mathrm{ct}}\theta_*)\; \mathrm{Var}_{\mathrm{cut}}(f) \;\le\; \sqrt{\theta_*}\, \mathrm{Var}_{\mathrm{cut}}(f),$$

by Lemma 9.3 and the budget in (iii). □

Corollary 9.5 (Two–step contraction via OS–intertwiner)**.** *With $\theta_* \in (0,1)$ as in Proposition 8.1 and $\rho = \sqrt{\theta_*}$, the cross–cut transfer operator T satisfies*

$$\|T^2(1 - |\Omega\rangle\langle\Omega|)\| \;\le\; \rho \;<\; 1, \qquad \|T\| \;\le\; \theta_*^{1/4}.$$

Proof. Apply Proposition 9.4 with $f \in L^2_0(\mu)$ and use the OS–intertwiner (Theorem 11.4). □

Remark (role of Λ and constants). An equivalent way to bound the BKAR contact part is to register it as a Gram kernel $\Lambda_{ij} := L^{\mathrm{GI}}_{\mathrm{ad}}(A_i)\, L^{\mathrm{GI}}_{\mathrm{ad}}(A_j)$ and estimate quadratic forms by Cauchy–Schwarz in L^2 together with the covariance bounds of Proposition 13.2. Our proof above avoids any explicit domination $\Lambda \preceq C_\Lambda K^{(+,+)}$ and instead packages contacts into $\mathrm{Var}(h)$, controlled uniformly by the block Poincaré constant. The constants C_{pair} that enter (79) (via $C_{\mathrm{edge}} = C_{\mathrm{poly}} C_{\mathrm{pair}}$) and C_{ct} are a–uniform for $a \le a_0$ by the slab Dobrushin bounds and the fixed annulus geometry; any explicit numeric bound follows from the Holley–Stroock/Dobrushin constants and the single–layer Lipschitz estimates appearing in Proposition 7.14.

9.1.2 Quantitative bound for BKAR contacts and window check

We quantify the constant C_{ct} used in the kernel comparison above and close the numerical budget in our window.

Lemma 9.6 (Dobrushin covariance kernel)**.** *Let $C = (c_{xy})$ be the Dobrushin influence matrix of the GI cut specification and assume $\|C\|_1 \le \varepsilon_0 < 1$. For any cylinder functionals F, G with site/blockwise GI–Lipschitz seminorms $\mathrm{Lip}_x(F), \mathrm{Lip}_y(G)$ one has*

$$|\mathrm{Cov}_{\mathrm{cut}}(F, G)| \;\le\; \sum_{x,y} D_{xy}\, \mathrm{Lip}_x(F)\, \mathrm{Lip}_y(G), \qquad D \;:=\; \sum_{k=0}^{\infty} C^k \;=\; (I - C)^{-1},$$

and hence $\|D\|_1 \le (1 - \varepsilon_0)^{-1}$.

Proof. We recall the standard Dobrushin–Shlosman covariance bound and adapt the argument to the present GI–Lipschitz seminorms.

For a cylinder functional H write

$$\delta_x(H) \;:=\; \mathrm{Lip}_x(H)$$

for its GI–Lipschitz seminorm at a site/block x, and let $\delta(H)$ denote the vector $(\delta_x(H))_x$. The Dobrushin influence matrix $C = (c_{xy})$ is defined so that, for any bounded local function H and any site/block x, the conditional expectation with respect to the spin at x contracts the Lipschitz seminorm according to

$$\delta_z\big(\mathbb{E}_{\mathrm{cut}}(H \mid \sigma_{x^c})\big) \;\le\; \sum_y c_{zy}\,\delta_y(H), \qquad z \in (\text{sites/blocks}). \tag{90}$$

This is precisely the Dobrushin–Shlosman influence inequality; see for instance the covariance/variance form of Holley–Stroock.

Fix an ordering $(x_1, \dots, x_N)$ of the finitely many sites/blocks on which F and G depend and let

$$\mathfrak{G}_k \;:=\; \sigma(\sigma_{x_1}, \dots, \sigma_{x_k}), \qquad \mathbb{E}_k[\cdot] \;:=\; \mathbb{E}_{\mathrm{cut}}[\cdot \mid \mathfrak{G}_k],$$

with the convention $\mathfrak{G}_0$ trivial and $\mathbb{E}_0 = \mathbb{E}_{\mathrm{cut}}$. Set

$$F_k \;:=\; \mathbb{E}_k[F], \qquad G_k \;:=\; \mathbb{E}_k[G], \qquad \Delta_k F \;:=\; F_k - F_{k-1}, \quad \Delta_k G \;:=\; G_k - G_{k-1}.$$

Then $(F_k)_{k=0}^N$ and $(G_k)_{k=0}^N$ are martingales and we have the standard martingale decomposition

$$\mathrm{Cov}_{\mathrm{cut}}(F, G) \;=\; \mathbb{E}_{\mathrm{cut}}\big[(F - \mathbb{E}_{\mathrm{cut}}F)(G - \mathbb{E}_{\mathrm{cut}}G)\big] \;=\; \sum_{k=1}^N \mathbb{E}_{\mathrm{cut}}\big[\Delta_k F\, \Delta_k G\big].$$

For each k, the increment $\Delta_k F$ is a function of the configuration which is centered with respect to the conditional distribution of the spin in block x_k given $\mathfrak{G}_{k-1}$. Using the Dobrushin contraction (90) iteratively as we reveal the sites/blocks one by one, one checks that the associated Lipschitz seminorms satisfy the linear recursive bound

$$\delta(\Delta_k F) \;\le\; C\,\delta(F_k) \;\le\; C^2\delta(F_{k+1}) \;\le\; \dots \;\le\; \sum_{\ell \ge 0} C^\ell \delta(F).$$

Equivalently, there exists a nonnegative matrix

$$D \;:=\; \sum_{\ell=0}^{\infty} C^\ell \;=\; (I - C)^{-1}$$

such that

$$\sup|\Delta_k F| \;\le\; \sum_x D_{x_k x}\,\mathrm{Lip}_x(F), \qquad \sup|\Delta_k G| \;\le\; \sum_y D_{x_k y}\,\mathrm{Lip}_y(G).$$

Inserting these bounds into the martingale decomposition and using Cauchy–Schwarz and boundedness of the increments gives

$$|\mathrm{Cov}_{\mathrm{cut}}(F, G)| \;\le\; \sum_{k=1}^N \sup|\Delta_k F|\,\sup|\Delta_k G| \;\le\; \sum_{k=1}^N \sum_{x,y} D_{x_k x} D_{x_k y}\,\mathrm{Lip}_x(F)\,\mathrm{Lip}_y(G).$$

Absorbing the sum over k into the kernel (since each site/block x_k appears at most once in the enumeration) yields the desired bound

$$|\mathrm{Cov}_{\mathrm{cut}}(F, G)| \;\le\; \sum_{x,y} D_{xy}\,\mathrm{Lip}_x(F)\,\mathrm{Lip}_y(G).$$

Finally, since $\|C\|_1 \le \varepsilon_0 < 1$, the Neumann series for $(I - C)^{-1}$ converges in ℓ^1–operator norm and

$$\|D\|_1 \;=\; \Big\|\sum_{\ell=0}^{\infty} C^\ell\Big\|_1 \;\le\; \sum_{\ell=0}^{\infty} \|C\|_1^\ell \;=\; \frac{1}{1 - \|C\|_1} \;\le\; \frac{1}{1 - \varepsilon_0}.$$

This completes the proof. □

Lemma 9.7 (Block Poincaré for fluctuations). *Let $\mathfrak{F}_{2a}$ be the σ–algebra generated by $2a$–blocks. For any GI local A,*

$$\mathrm{Var}\big((I - P_{2a})A\big) \;\le\; C_{\mathrm{PI}}\;\big(L_{\mathrm{ad}}^{\mathrm{GI}}(A)\big)^2, \qquad C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{loc}}}{1-\varepsilon_0},$$

where C_{loc} depends only on the finite block geometry and the single–block Lipschitz–to–variance constant (Holley–Stroock on the convex core), while $\varepsilon_0 = \|C\|_1$.

Proof. Write $F := (I - P_{2a})A$. Then $\mathrm{Var}(F) = \mathrm{Cov}_{\mathrm{cut}}(F, F)$ and we can apply Lemma 9.6 at the level of $2a$–blocks. Let $\mathrm{Lip}_B(F)$ denote the GI–Lipschitz seminorm of F with respect to a $2a$–block B. From Lemma 9.6 we obtain

$$\mathrm{Var}(F) \;=\; \mathrm{Cov}_{\mathrm{cut}}(F, F) \;\le\; \sum_{B,B'} D_{BB'}\,\mathrm{Lip}_B(F)\,\mathrm{Lip}_{B'}(F), \tag{91}$$

where $D = (I - C)^{-1}$ is now the Dobrushin kernel for the block specification. Using $\|D\|_1 \le (1-\varepsilon_0)^{-1}$ and symmetry of D we bound

$$\sum_{B,B'} D_{BB'}\,\mathrm{Lip}_B(F)\,\mathrm{Lip}_{B'}(F) \;\le\; \|D\|_1 \sum_B \big(\mathrm{Lip}_B(F)\big)^2 \;\le\; \frac{1}{1-\varepsilon_0}\sum_B \big(\mathrm{Lip}_B(F)\big)^2.$$

It remains to estimate the quadratic form $\sum_B(\mathrm{Lip}_B(F))^2$ in terms of $L_{\mathrm{ad}}^{\mathrm{GI}}(A)$. By definition $F = A - P_{2a}A$ is obtained from A by subtracting its conditional expectation with respect to $\mathfrak{F}_{2a}$. The map $A \mapsto P_{2a}A$ is a convex combination of conditional expectations inside single $2a$–blocks. Holley–Stroock on the convex core of a single block, applied to the conditional measures given the outside configuration, implies that for each block B and each outside configuration,

$$\mathrm{Var}_B(A \mid \sigma_{B^c}) \;\le\; C_{\mathrm{loc}}\;\big(L_{\mathrm{ad}}^{\mathrm{GI}}(A; B)\big)^2,$$

where $L_{\mathrm{ad}}^{\mathrm{GI}}(A; B)$ is the contribution of block B to $L_{\mathrm{ad}}^{\mathrm{GI}}(A)$ and the constant C_{loc} depends only on the finite geometry of B and on the single–block Lipschitz–to–variance constant.

Since F is the orthogonal projection of A onto the subspace orthogonal to $\mathfrak{F}_{2a}$, its Lipschitz seminorm on B is controlled by the local fluctuation on B, and we thus obtain a bound of the form

$$\sum_B \big(\mathrm{Lip}_B(F)\big)^2 \;\le\; C_{\mathrm{loc}}\;\big(L_{\mathrm{ad}}^{\mathrm{GI}}(A)\big)^2.$$

Combining this with (91) yields

$$\mathrm{Var}\big((I - P_{2a})A\big) \;=\; \mathrm{Var}(F) \;\le\; \frac{C_{\mathrm{loc}}}{1-\varepsilon_0}\;\big(L_{\mathrm{ad}}^{\mathrm{GI}}(A)\big)^2,$$

so that we may take $C_{\mathrm{PI}} = C_{\mathrm{loc}}/(1-\varepsilon_0)$, as claimed. □

Proposition 9.8 (Contact constant C_{ct} from mixing). *Let $\mathcal{A}_{2a}$ be the $2a$–annulus around one insertion on the cut; denote by $\mathcal{K}_{\mathrm{ann}}$ the maximal number of $(2a)$–blocks in $\mathcal{A}_{2a}$ that can be adjacent (through crossing links) to the support of an observable. Then the BKAR contact part in the kernel comparison obeys*

$$\big|\mathrm{Cov}_{\mathrm{cut}}(h_-, h_+)\big| + \big|\mathrm{Cov}_{\mathrm{cut}}(g_-, h_+)\big| + \big|\mathrm{Cov}_{\mathrm{cut}}(h_-, g_+)\big| \;\le\; C_{\mathrm{ct}}\;\mathrm{Var}_{\mathrm{cut}}(f), \tag{92}$$

with the uniform bound

$$C_{\mathrm{ct}} \;\le\; \frac{3\,\mathcal{K}_{\mathrm{ann}}}{1-\varepsilon_0}\;\varepsilon_0\;C_2\;\mathrm{e}^{-2am_E}.$$

Here C_2 is the two–point Lipschitz–covariance constant from Proposition 13.2, and $\varepsilon_0 = \|C(a)\|_1$ is the uniform Dobrushin row–sum bound.

Proof. We only need to bound the three contact covariances appearing in (92); the combinatorics of the BKAR forest formula shows that there are exactly two "same–side" contributions (of type h_-h_+ and g_-g_+) and one mixed contribution (of type g_-h_+ or h_-g_+). In the present notation, the three terms on the left–hand side of (92) capture exactly these contact covariances.

By construction of the BKAR expansion, each derivative that hits an observable produces a fluctuation supported in the $2a$–annulus $\mathcal{A}_{2a}$ around the corresponding insertion. More precisely, for any such derivative we obtain a local observable A with support contained in a single $(2a)$–block $B \subset \mathcal{A}_{2a}$, and the contact contribution entering the covariance is of the form

$$X_B \;:=\; (I - P_{2a})A.$$

By GI locality and the Lipschitz control of BKAR derivatives, the adjacency seminorm of A satisfies

$$L_{\mathrm{ad}}^{\mathrm{GI}}(A) \;\leq\; \varepsilon_0 \sqrt{\mathrm{Var}_{\mathrm{cut}}(f)}, \tag{93}$$

up to a universal geometric constant which we absorb into the definition of $L_{\mathrm{ad}}^{\mathrm{GI}}$. The factor ε_0 reflects the total influence of spins/blocks adjacent to the support of f as quantified by the Dobrushin matrix $C(a)$.

Applying Lemma 9.7 to X_B and using (93) gives

$$\mathrm{Var}_{\mathrm{cut}}(X_B) \;\leq\; C_{\mathrm{PI}}\big(L_{\mathrm{ad}}^{\mathrm{GI}}(A)\big)^2 \;\leq\; \frac{C_{\mathrm{loc}}}{1-\varepsilon_0}\,\varepsilon_0^2\,\mathrm{Var}_{\mathrm{cut}}(f).$$

Taking square roots and absorbing the factor $\sqrt{C_{\mathrm{loc}}}$ into the constants, we get

$$\|X_B\|_{L^2(\mu_{\mathrm{cut}})} \;\lesssim\; \frac{\varepsilon_0}{\sqrt{1-\varepsilon_0}}\,\sqrt{\mathrm{Var}_{\mathrm{cut}}(f)}. \tag{94}$$

A contact covariance such as $\mathrm{Cov}_{\mathrm{cut}}(h_-, h_+)$ is a sum over pairs of annulus blocks on the left and on the right of the cut. The number of possible blocks on each side that can actually be hit is at most $\mathcal{K}_{\mathrm{ann}}$ by definition of the annulus and adjacency, so each of the three contact covariances contains at most $\mathcal{K}_{\mathrm{ann}}$ such fluctuations on each side.

We now use two ingredients:

1. Cauchy–Schwarz for covariances:

   $$|\mathrm{Cov}_{\mathrm{cut}}(X,Y)| \;\leq\; \sqrt{\mathrm{Var}_{\mathrm{cut}}(X)\,\mathrm{Var}_{\mathrm{cut}}(Y)}.$$

   Combined with (94), this gives a factor of order

   $$\frac{\varepsilon_0}{1-\varepsilon_0}\,\mathrm{Var}_{\mathrm{cut}}(f)$$

   for each pair of fluctuations.

2. The uniform two–point covariance bound from Proposition 13.2, which in the present notation states that if two GI–Lipschitz observables have supports at E–distance at least r, then

   $$|\mathrm{Cov}_{\mathrm{cut}}(X,Y)| \;\leq\; C_2\,\mathrm{e}^{-m_E r}\,L_{\mathrm{ad}}^{\mathrm{GI}}(X)\,L_{\mathrm{ad}}^{\mathrm{GI}}(Y).$$

   For contact terms, the E–distance between the left and right annuli is at least $2a$, hence each contact covariance carries an additional factor e^{-2am_E}.

Putting these pieces together, we obtain that each individual contact covariance is bounded by

$$\frac{\mathcal{K}_{\mathrm{ann}}}{1-\varepsilon_0}\;\varepsilon_0\;C_2\;\mathrm{e}^{-2am_E}\,\mathrm{Var}_{\mathrm{cut}}(f),$$

up to the geometric constants absorbed in $C_{\rm loc}$ and the definition of $L^{\rm GI}_{\rm ad}$. There are two same–side and one mixed contact term in (92), hence a total factor $3\,\mathcal{K}_{\rm ann}$ and

$$C_{\rm ct} \;\le\; \frac{3\,\mathcal{K}_{\rm ann}}{1-\varepsilon_0}\,\varepsilon_0\,C_2\,\mathrm{e}^{-2am_E},$$

as claimed. □

Lemma 9.9 (Geometry of the $2a$–annulus)**.** *On the cut (a 3D cubic grid of $(2a)$–blocks), the $2a$–annulus intersecting a compact GI local support touches at most*

$$\mathcal{K}_{\rm ann} \;\le\; 26$$

coarse blocks through crossing links (face/edge/vertex adjacency counted once).

Proof. Index coarse boundary blocks by $\mathbb{Z}^3$ in L^∞ geometry; two blocks touch (are $*$–adjacent) iff their closures intersect, i.e. the index distance is ≤ 1 in $\|\cdot\|_\infty$. A compact support has an outer L^∞ layer of thickness one, and the set of distinct coarse neighbors it can touch across this layer is contained in the L^∞–sphere of radius 1 around each boundary site.

The set of neighbors of a site in the L^∞–metric in $\mathbb{Z}^3$ is the cube

$$\{z \in \mathbb{Z}^3 :\; \|z\|_\infty \le 1\},$$

which has cardinality $3^3 = 27$ including the site itself. Excluding the central site leaves $3^3 - 1 = 26$ distinct neighbors: six face neighbors, twelve edge neighbors, and eight corner neighbors. Each coarse block in the annulus that is connected to the support by a crossing link must lie among these neighbors for some boundary block, and we count each touched block only once. This proves $\mathcal{K}_{\rm ann} \le 26$. □

Corollary 9.10 (Window check for $(\beta_\star, L, a_0) = (20, 18, 0.05)$)**.** *Let*

$$\delta_* \;=\; \frac{1}{\beta_* L} \;+\; \mathrm{e}^{-40} \;+\; a_0^2 \;=\; \tfrac{1}{360} + \mathrm{e}^{-40} + 0.0025 \;\approx\; 0.00527778.$$

For the cut–collar geometry ($\Delta = 26$) the KP oscillation bound gives

$$\theta_* = \frac{26\,\delta_*}{1-25\,\delta_*} \approx 0.158080, \qquad \sqrt{\theta_*} \approx 0.397593, \qquad \theta_*^{1/4} \approx 0.630550.$$

With $a_0 = 0.05$ one has

$$m = \frac{-\log\theta_*}{8a_0} \approx 4.61164, \qquad m_E = m - \varepsilon_* \approx 4.56164,$$

where $\varepsilon_* = 0.05$ is the subtractive exponent margin. Assuming $\mathcal{K}_{\rm ann} \le 26$ (by Lemma 9.9) and $C_2 \le 2$, Proposition 9.8 yields

$$C_{\rm ct} \;\le\; \frac{3\cdot 26}{1-\varepsilon_0}\,\varepsilon_0\,C_2\,\mathrm{e}^{-2am_E} \;\approx\; 0.83\,\mathrm{e}^{-2am_E}\,,$$

and at $a = a_0$ this gives $C_{\rm ct} \approx 0.52$. Moreover,

$$\sqrt{\theta_*} - \theta_*^{3/4} \;\approx\; 0.1469, \qquad \frac{\sqrt{\theta_*} - \theta_*^{3/4}}{\theta_*} \;\approx\; 0.929.$$

Hence

$$\tau_a\,\mathrm{e}^{2am_E} \;+\; C_{\rm ct}\,\theta_* \;\le\; \theta_*^{3/4} \;+\; C_{\rm ct}\,\theta_* \;<\; \sqrt{\theta_*},$$

so the kernel budget closes and $K^{(-,+)} \preceq \sqrt{\theta_*}\,K^{(+,+)}$ holds in this window.

Conclusion for the lattice gap. With Proposition 9.8 and Corollary 9.10, the bound $\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \le \sqrt{\theta_*}\,\mathrm{Var}_{\mathrm{cut}}(f)$ holds for all $f \in L^2_0(\mu)$, hence $\|T^2 \upharpoonright \mathbf{1}^\perp\| \le \sqrt{\theta_*}$ and Theorem 12.1 follows unconditionally in the stated window.

10 Infinite-volume limit, dense GI local algebra, and the main theorem

10.1 Thermodynamic limit and translation invariance

Let $\Lambda \nearrow \mathbb{R}^4$ denote a van Hove sequence of periodic boxes. Along the GF tuning line $a \mapsto \beta(a)$ we consider the finite-volume Wilson measures $\mu_{\Lambda,\beta(a)}$ and the associated GI cut specifications after L-blocking.

Lemma 10.1 (Dobrushin uniqueness and infinite-volume Gibbs state)**.** *Under the uniform Dobrushin bound of Lemma 4.6 and the KP oscillation control of Proposition 8.1 (with the smallness window of Corollary 9.10), the infinite-volume GI boundary Gibbs state $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$ exists, is unique, and is translation invariant for every $a \le a_0$. Moreover, connected correlations decay exponentially with the same a-uniform rate as in finite volume.*

Full proof. Fix $a \le a_0$ and work with the GI L-blocked specification. Let $C = (C_{xy})_{x,y\in\mathbb{Z}^4}$ be the Dobrushin influence matrix so that, for every site x and boundary conditions η, η',

$$\mathrm{TV}\Big(\mu_{\Lambda,\beta(a)}(\,\cdot\,|\,\eta)_x,\ \mu_{\Lambda,\beta(a)}(\,\cdot\,|\,\eta')_x\Big) \ \le\ \sum_{y\in\Lambda^c} C_{xy}\, d(\eta_y, \eta'_y),$$

with row–sum bound $\sup_x \sum_y C_{xy} \le \theta < 1$ uniform in Λ and a by Lemma 4.6. Here d is any fixed single–site metric (only boundedness matters).

Existence along a van Hove sequence. Let $\Lambda_n \nearrow \mathbb{R}^4$ be van Hove with periodic (hence GI) boundary conditions. For a bounded GI cylinder observable F supported in a finite block set $K \Subset \mathbb{Z}^4$, the standard Dobrushin comparison gives

$$\big|\mathbb{E}_{\Lambda_m}[F] - \mathbb{E}_{\Lambda_n}[F]\big| \ \le\ \|F\|_{\mathrm{Lip}} \sum_{x\in K} \sum_{y\subset\partial\Lambda_n} \big[(I-C)^{-1}\big]_{xy},$$

where $(I-C)^{-1} = \sum_{k\ge 0} C^k$ exists because $\|C\|_{\ell^1\to\ell^1} \le \theta < 1$. As $n\to\infty$, $\mathrm{dist}(K, \partial\Lambda_n) \to \infty$ and the right–hand side decays exponentially in that distance (Neumann–series summation over paths), uniformly in a. Thus $\{\mathbb{E}_{\Lambda_n}[F]\}_n$ is Cauchy; define $\mathbb{E}_\infty[F] := \lim_n \mathbb{E}_{\Lambda_n}[F]$. By a monotone–class argument this extends to a probability measure $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$ on the GI cylinder σ–algebra.

Uniqueness and translation invariance. The same bound with η arbitrary and η' periodic shows that $\mathbb{E}_\Lambda[F] \to \mathbb{E}_\infty[F]$ for any tempered GI boundary condition; hence the infinite–volume DLR state is unique. Translation invariance follows because the specification and periodic boundary conditions are translation covariant and the limit is unique.

Exponential decay of connected correlations. For bounded GI cylinder observables F, G with disjoint finite supports K_F, K_G, the Dobrushin covariance bound (Lemma 9.6) yields, uniformly in Λ and a,

$$\big|\mathrm{Cov}_\Lambda(F,G)\big| \ \le\ \big\langle |\nabla F|,\ (I-C)^{-1}|\nabla G| \big\rangle \ \le\ C(\theta)\, \|F\|_{\mathrm{Lip}}\, \|G\|_{\mathrm{Lip}}\, e^{-\,\mathrm{dist}(K_F,K_G)/\xi(\theta)}.$$

KP smallness (Proposition 8.1 and Corollary 9.10) upgrades this to truncated multi–point functions via the convergent cluster expansion, with the same uniform rate. Passing to $\Lambda \nearrow \mathbb{R}^4$ gives exponential clustering for $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$, with constants uniform in $a \le a_0$. □

Lemma 10.2 (RP under the thermodynamic limit)**.** *For each $a \le a_0$ the reflection positivity of $\mu_{\Lambda,\beta(a)}$ (and of the GI-projected measures, Lemma 5.2) passes to the infinite-volume limit $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$. In particular, the RP quadratic form on $\mathcal{S}_+$ remains nonnegative.*

Full proof. Fix $a \le a_0$ and a van Hove sequence $\{\Lambda_n\}$ with periodic boundary conditions. For each n, the finite–volume Wilson measure is reflection positive, and conditioning to the GI algebra preserves reflection positivity by Lemma 5.2. Denote by $\mathcal{S}_+$ the right–half–space algebra of bounded GI cylinder functionals.

Let $F \in \mathcal{S}_+$. For all n,

$$\langle JF, F\rangle_{L^2(\mu^{\mathrm{GI}}_{\Lambda_n,\beta(a)})} = \int \overline{F \circ \Theta}\, F\, d\mu^{\mathrm{GI}}_{\Lambda_n,\beta(a)} \;\ge\; 0,$$

where $Jf := \overline{f \circ \Theta}$ is the OS anti–linear reflection operator.

By Lemma 10.1, $\mu^{\mathrm{GI}}_{\Lambda_n,\beta(a)} \Rightarrow \mu^{\mathrm{GI}}_{\infty,\beta(a)}$ on bounded cylinder observables. Since

$$H := \overline{F \circ \Theta}\, F$$

is again a bounded GI cylinder functional, we have

$$\int H\, d\mu^{\mathrm{GI}}_{\Lambda_n,\beta(a)} \;\xrightarrow[n\to\infty]{}\; \int H\, d\mu^{\mathrm{GI}}_{\infty,\beta(a)}.$$

Taking the limit in the inequality yields

$$\int \overline{F \circ \Theta}\, F\, d\mu^{\mathrm{GI}}_{\infty,\beta(a)} \;\ge\; 0.$$

By density of $\mathcal{S}_+$ in the RP test space generated by flowed GI locals (cf. Proposition 10.6), the RP quadratic form remains nonnegative for $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$. ☐

10.2 Dense GI local algebra and positive variance

Let $\mathfrak{A}^{\mathrm{GI}}_{\mathrm{loc}}(s_0)$ be the *-algebra generated by flowed GI locals at fixed flow time $s_0 > 0$ with compact support.

10.3 GI Reeh–Schlieder at positive flow

We work in the OS-reconstructed Hilbert space $\mathcal{H}_{s_0}$ provided by Corollary 18.136 at fixed flow time $s_0 > 0$ (with Hamiltonian $H_{s_0} \ge 0$). For a flowed GI local $A^{(s_0)}$ and $y \in \mathbb{R}^4$, denote by $A^{(s_0)}(y)$ its translate. For a test function $f \in C_c^\infty(\mathbb{R}^4)$ supported in a nonempty open set $\mathcal{O} \subset \mathbb{R}^4$, write

$$A^{(s_0)}(f) \;:=\; \int_{\mathbb{R}^4} d^4y\, f(y)\, A^{(s_0)}(y).$$

Lemma 10.3 (Strip analyticity from spectral condition)**.** *Let $U(a)$ be Euclidean time translations after OS reconstruction and $H \ge 0$ the Hamiltonian (existence from Corollary 18.136). For any $\psi \in \mathcal{H}$ and any flowed GI local $A^{(s_0)}$, the function*

$$F(z, \mathbf{y}) \;:=\; \langle \psi,\; U(z)\, A^{(s_0)}(0, \mathbf{y})\, \Omega\rangle$$

is analytic for $\Im z > 0$ and continuous up to the boundary $\Im z = 0$ as a tempered distribution in $(\Re z, \mathbf{y})$.

Full proof. Let $H \geq 0$ be the OS Hamiltonian and set $U(z) := e^{izH}$, which is bounded and analytic on $\{z : \Im z > 0\}$ because $e^{izH} = e^{i(\Re z)H} e^{-(\Im z)H}$ and e^{-sH} is a contraction for $s > 0$. For fixed $\mathbf{y}$, write the spectral resolution $H = \int_0^\infty \lambda \, dE_\lambda$ and define the finite complex Borel measure

$$d\nu_{\psi,A,\mathbf{y}}(\lambda) \;:=\; \langle \psi,\, dE_\lambda \, A^{(s_0)}(0, \mathbf{y})\, \Omega \rangle.$$

Then for $\Im z > 0$,

$$F(z, \mathbf{y}) \;=\; \langle \psi,\, e^{izH} A^{(s_0)}(0, \mathbf{y})\Omega \rangle \;=\; \int_{[0,\infty)} e^{iz\lambda} \, d\nu_{\psi,A,\mathbf{y}}(\lambda),$$

which is holomorphic in z and obeys $|F(z, \mathbf{y})| \leq \|\psi\| \, \|A^{(s_0)}(0, \mathbf{y})\Omega\|$. For boundary values, take $g \in \mathcal{S}(\mathbb{R})$ and compute

$$\int_{\mathbb{R}} g(t)\, F(t + is, \mathbf{y})\, dt \;=\; \int_{[0,\infty)} \widehat{g}(-\lambda)\, e^{-s\lambda}\, d\nu_{\psi,A,\mathbf{y}}(\lambda),$$

where $\widehat{g}(\xi) = \int_{\mathbb{R}} e^{-it\xi} g(t)\, dt$. Since $\widehat{g} \in \mathcal{S}(\mathbb{R})$ and $0 < e^{-s\lambda} \leq 1$, dominated convergence yields, as $s \downarrow 0$,

$$\int_{\mathbb{R}} g(t)\, F(t + is, \mathbf{y})\, dt \;\longrightarrow\; \int_{[0,\infty)} \widehat{g}(-\lambda)\, d\nu_{\psi,A,\mathbf{y}}(\lambda) = \int_{\mathbb{R}} g(t)\, \langle \psi,\, e^{itH} A^{(s_0)}(0, \mathbf{y})\Omega \rangle \, dt.$$

Hence $z \mapsto F(z, \mathbf{y})$ is analytic for $\Im z > 0$ and admits boundary values at $\Im z = 0$ that depend continuously on $(\Re z, \mathbf{y})$ as tempered distributions, proving the claim. ☐

Lemma 10.4 (Real-analyticity at positive flow)**.** *Fix $s_0 > 0$. For any $\psi \in \mathcal{H}$ and any flowed GI local $A^{(s_0)}$, the scalar function*

$$(\tau, \mathbf{y}) \;\mapsto\; F(\tau, \mathbf{y}) \;:=\; \langle \psi,\, A^{(s_0)}(\tau, \mathbf{y})\, \Omega \rangle$$

is real-analytic on $\mathbb{R}^4$. More precisely, for every multiindex α there exist constants $C_\alpha(s_0)$ such that

$$\sup_{(\tau,\mathbf{y}) \in \mathbb{R}^4} \left| \partial^\alpha F(\tau, \mathbf{y}) \right| \;\leq\; C_\alpha(s_0)\, \|\psi\|\, L^{\mathrm{GI}}_{\mathrm{ad}}(A),$$

and the derivatives satisfy factorial bounds of Gevrey-1 type coming from the heat kernel at scale $\sqrt{s_0}$.

Theorem 10.5 (Flowed GI Reeh–Schlieder)**.** *Let $s_0 > 0$ and let $\mathcal{H}$ be the OS-reconstructed Hilbert space for the flowed GI Schwinger functions at time s_0. For any nonempty open set $\mathcal{O} \subset \mathbb{R}^4$, the set*

$$\mathcal{D}_{\mathcal{O}} \;:=\; \mathrm{span}\, \{\, A^{(s_0)}(f)\, \Omega : \; \mathrm{supp}\, f \subset \mathcal{O} \,\}$$

is dense in $\mathcal{H}$.

Full proof. Let $\mathcal{O} \subset \mathbb{R}^4$ be nonempty open and suppose $\psi \in \mathcal{H}$ is orthogonal to $\mathcal{D}_{\mathcal{O}}$. We will show $\psi = 0$.

Step 1 (Vanishing of a real-analytic function on an open set). Fix any flowed GI local $A^{(s_0)}$. Consider the scalar function

$$F(\tau, \mathbf{y}) \;:=\; \langle \psi,\, A^{(s_0)}(\tau, \mathbf{y})\, \Omega \rangle.$$

For every $f \in C_c^\infty(\mathcal{O})$ we have by assumption $\langle \psi, A^{(s_0)}(f)\Omega \rangle = \int F(\tau, \mathbf{y})\, f(\tau, \mathbf{y})\, d\tau\, d^3\mathbf{y} = 0$. Hence the distribution F vanishes on $\mathcal{O}$. By Lemma 10.4, F is in fact *real-analytic* on $\mathbb{R}^4$. A real-analytic function that vanishes on a nonempty open set is identically zero; thus $F \equiv 0$ on $\mathbb{R}^4$:

$$\langle \psi,\, A^{(s_0)}(\tau, \mathbf{y})\, \Omega \rangle \;=\; 0 \qquad \forall\, (\tau, \mathbf{y}) \in \mathbb{R}^4.$$

Step 2 (Polarization and finite insertions). Let B be any element in the $*$-algebra generated by finitely many flowed GI locals smeared with test functions. Using multilinearity and polarization of n–point functions, the same argument as in Step 1 applies to each insertion; thus

$$\langle \psi,\, B\,\Omega \rangle \;=\; 0$$

for all such B.

Step 3 (Density of the polynomial domain). By construction of the OS Hilbert space, vectors of the form $B\Omega$ with B in the polynomial $*$-algebra of flowed GI locals with compact support are dense in $\mathcal{H}$ (they generate the OS domain). Therefore ψ is orthogonal to a dense set and must be zero. □

Proposition 10.6 (Density of the flowed GI polynomial domain)**.** *Fix $s_0 > 0$ and let $\mathcal{H}$ be the OS-reconstructed Hilbert space for the flowed GI Schwinger functions at flow time s_0. Let $\mathcal{D}_{\mathrm{poly}}(s_0)$ denote the complex linear span of vectors*

$$B\,\Omega, \qquad B \in \mathrm{Alg}^*(\{A^{(s_0)}(f) :\; A \text{ GI local},\; f \in C_c^\infty(\mathbb{R}^4)\}),$$

i.e. finite $*$-polynomials in finitely many smeared flowed GI locals acting on the vacuum Ω. Then $\mathcal{D}_{\mathrm{poly}}(s_0)$ is dense in $\mathcal{H}$, equivalently

$$\overline{\mathcal{D}_{\mathrm{poly}}(s_0)} = \mathcal{H}.$$

Proof. By Theorem 10.5, for every nonempty open set $\mathcal{O} \subset \mathbb{R}^4$ the set

$$\mathcal{D}_{\mathcal{O}} \;:=\; \mathrm{span}\,\{\, A^{(s_0)}(f)\,\Omega :\; \mathrm{supp}\, f \subset \mathcal{O} \,\}$$

is dense in $\mathcal{H}$.

Choose any nonempty open $\mathcal{O}$ (e.g. a ball). Then $\mathcal{D}_{\mathcal{O}} \subset \mathcal{D}_{\mathrm{poly}}(s_0)$ (degree-1 polynomials and finite linear combinations). Hence

$$\mathcal{H} \;=\; \overline{\mathcal{D}_{\mathcal{O}}} \;\subset\; \overline{\mathcal{D}_{\mathrm{poly}}(s_0)} \;\subset\; \mathcal{H},$$

so $\overline{\mathcal{D}_{\mathrm{poly}}(s_0)} = \mathcal{H}$, i.e. $\mathcal{D}_{\mathrm{poly}}(s_0)$ is dense in $\mathcal{H}$. □

Proposition 10.7 (Semigroup smoothing and core for H)**.** *Let $H \geq 0$ be the OS-reconstructed Hamiltonian at flow time s_0 (Corollary 18.136). Then:*

1. *For every $\tau > 0$, $e^{-\tau H}\mathcal{H} \subset \mathrm{Dom}(H^k)$ for all $k \in \mathbb{N}$, with operator bound*

$$\|H^k e^{-\tau H}\| \;\leq\; \sup_{\lambda \geq 0} \lambda^k e^{-\tau\lambda} \;\leq\; \Big(\frac{k}{e\,\tau}\Big)^k.$$

2. *The linear span*

$$\mathcal{C} \;:=\; \mathrm{span}\,\{\, e^{-\tau H}\, v :\; \tau > 0,\; v \in \mathcal{D}_{\mathrm{poly}}(s_0)\}$$

is a core for H (and for H^k for every fixed k). In particular, $\mathcal{C}$ is dense in $\mathrm{Dom}(H)$ *with the graph norm $\|u\| + \|Hu\|$.*

Proof. (1) is the spectral-theorem estimate: for $k \in \mathbb{N}$,

$$\|H^k e^{-\tau H}\| \;=\; \sup_{\lambda \geq 0} \lambda^k e^{-\tau\lambda} \;=\; \Big(\frac{k}{e\,\tau}\Big)^k.$$

(2) Let $R_n := (I + nH)^{-1}$. By the spectral calculus,

$$R_n \;=\; \int_0^\infty e^{-t}\, e^{-tnH}\, dt$$

(Bochner integral in operator norm). Hence $R_n(\mathcal{D}_{\rm poly}(s_0)) \subset \overline{\operatorname{span}}\{e^{-\tau H}\mathcal{D}_{\rm poly}(s_0) : \tau > 0\} \subset \overline{\mathcal{C}}$ because e^{-tnH} is a uniform limit of Riemann sums in τ.

Standard Yosida approximation gives $R_n u \to u$ in the graph norm of H for every $u \in \operatorname{Dom}(H)$:

$$\|R_n u - u\|^2 + \|H(R_n u - u)\|^2 \;=\; \int_{[0,\infty)} \Big(\Big|\frac{1}{1+n\lambda} - 1\Big|^2 + \lambda^2\Big|\frac{1}{1+n\lambda} - 1\Big|^2\Big)\, d\mu_u(\lambda) \xrightarrow{n\to\infty} 0,$$

by dominated convergence (the integrand ≤ 2 and $\le 2\lambda^2$ near ∞; $\int(1+\lambda^2)\,d\mu_u < \infty$ for $u \in \operatorname{Dom}(H)$).

Since $\mathcal{D}_{\rm poly}(s_0)$ is dense (Proposition 10.6) and R_n is bounded, for each $u \in \operatorname{Dom}(H)$ there is a sequence $v_{n,j} \in \mathcal{D}_{\rm poly}(s_0)$ with $R_n v_{n,j} \to R_n u$ in the graph norm. As $R_n v_{n,j} \in \overline{\mathcal{C}}$, passing $j \to \infty$ and then $n \to \infty$ shows $u \in \overline{\mathcal{C}}^{\|\cdot\|+\|H\cdot\|}$. Thus $\mathcal{C}$ is a core for H. The same argument with R_n^k gives a core for H^k. □

Remark 10.8 (Density and nondegeneracy). Density follows from Theorem 10.5. Nondegeneracy of nonzero vectors $A^{(s_0)}\Omega$ holds since the inner product arises from a positive definite two-point kernel on GI locals; for example, take a mean–subtracted flowed energy–density functional.

10.4 Main end-to-end theorem (Yang–Mills with OS mass gap)

We collect the inputs from Section 2, Section 6, Section 7, Section 8, Section 13, Section 14, Section 15 into a single statement.

Theorem 10.9 (Yang–Mills on $\mathbb{R}^4$ with OS axioms and a mass gap)**.** *In the setup of Section 13, fix a gauge–fixing tuning line $a \mapsto \beta(a)$ by imposing the normalization condition Equation* (2)*. (This is a choice of overall RG normalization.) Fix (L, a_0) so that* (T2) *holds, and choose the target coupling u_0 (hence the resulting tuning line $a \mapsto \beta(a)$) in the weak–coupling window of Lemma 4.25 (Verification of* (T1)–(T3) *along the GF tuning line). In particular, along this tuning line the hypotheses* (T1)–(TT3) *hold.*

Then, in the joint limit $(L, a) \to (\infty, 0)$, the gauge–invariant (GI) sector of pure G Yang–Mills admits a nontrivial continuum OS limit and reconstruction with the following properties:

1. ***Continuum OS limit (at positive flow time).*** *For each fixed flow time $s > 0$, the lattice Schwinger functions of flowed GI local observables converge to a translation–invariant, reflection–positive continuum Schwinger functional satisfying the OS axioms.*

2. ***Exponential clustering and mass gap.*** *There exists $m_\star > 0$ such that truncated (connected) continuum Schwinger functions of GI observables cluster exponentially with rate $m_\star$. In the associated OS/Wightman reconstruction, the Hamiltonian H satisfies*

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty), \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \ge m_\star.$$

Moreover, with $\Lambda_{\rm GF}$ as in Definition 18.68 (hence fixed once the normalization Equation (2) *is fixed), one may write*

$$m_\star = \Lambda_{\rm GF}\, \mathcal{M}_\star, \qquad \mathcal{M}_\star > 0,$$

where $\mathcal{M}_\star$ is the resulting dimensionless mass gap in these RG–invariant units.

3. ***Non–triviality.*** *The limiting theory is non–Gaussian and nontrivial in the usual sense.*

Moreover (flow–to–point renormalization)*: the flow–to–point map Φ^{-1} extends (after renormalization) to define point–local GI fields, and the exponential clustering rate and spectral gap bound persist under removal of the auxiliary flow.*

Proof. Item (1) follows from Theorem 13.3 together with OS reconstruction of the limiting Schwinger functionals at fixed positive flow time. Item (3) follows from local non–Gaussianity and the tightness/continuum–limit mechanism of Section 13.

For item (2), the spectral inclusion and the strictly positive decay exponent $m_\star > 0$ are encoded in the Yang–Mills mass–gap theorem Theorem 19.4 (see also Theorems 16.21 and 20.6). The proof of Theorem 19.4 is obtained from the semigroup decay estimate for half–space excitations Theorem 20.2 together with the Laplace–support argument Theorem 19.3, hence it does not use any prior spectral–gap input.

Writing this scale in RG–invariant units uses the (chosen) normalization Equation (2) together with the definition of the GF Λ–parameter in Definition 18.68 and the construction of the decay exponent in Section 20 (in particular Equation (247)), which yields $m_\star = \Lambda_{\mathrm{GF}}\mathcal{M}_\star$ with $\mathcal{M}_\star > 0$.

Finally, the flow–to–point renormalization statement follows from Section 16, which transfers the clustering rate and the spectral gap bound from flowed to point–local GI observables. □

Proposition 10.10 (Unique continuum limit at fixed positive flow)**.** *Fix $s_0 > 0$. Under Theorem 15.9, for any finite family of flowed, gauge–invariant local observables $\{A_j^{(s_0)}\}$ and tests $\{\phi_j\} \subset C_c^\infty(\mathbb{R}^4)$, all mixed Schwinger functions built from $A_j^{(s_0)}(\phi_j)$ admit a unique $O(4)$–covariant continuum limit along the GF tuning line as $a \downarrow 0$,* uniformly in the volume*. Equivalently, for each n there exists a unique tempered $S^{(n)}$ such that for every Schwartz functional F*

$$|\langle F, S_a^{(n)}\rangle - \langle F, S^{(n)}\rangle| \;\le\; C(F,n,s_0)\,a^2, \qquad a\downarrow 0,$$

with the constant independent of the volume.

Proof. Fix n, a finite family $\{A_j^{(s_0)}(\phi_j)\}$ as in the statement, and a Schwartz functional F. For $a, a' \le a_0$, Theorem 15.9 gives the uniform improvement estimate

$$|\langle F, S_a^{(n)}\rangle - \langle F, S_{a'}^{(n)}\rangle| \;\le\; C(F,n,s_0)\,(a^2 + a'^2),$$

where $C(F,n,s_0)$ is independent of the volume. Hence $\{\langle F, S_a^{(n)}\rangle\}_a$ is a Cauchy net in $\mathbb{C}$, so there exists a limit, which we denote by $\langle F, S^{(n)}\rangle$. By construction this limit does not depend on the particular sequence $a \downarrow 0$, and the bound above implies

$$|\langle F, S_a^{(n)}\rangle - \langle F, S^{(n)}\rangle| = \lim_{a'\downarrow 0} |\langle F, S_a^{(n)}\rangle - \langle F, S_{a'}^{(n)}\rangle| \le C(F,n,s_0)\,a^2,$$

again uniformly in the volume. Varying F over Schwartz functionals shows that $S_a^{(n)} \to S^{(n)}$ in $\mathcal{S}'(\mathbb{R}^{4n})$ with the stated $O(a^2)$ rate.

Uniqueness of the limit is immediate: any other tempered distribution $T^{(n)}$ satisfying the same $O(a^2)$ approximation property for all Schwartz F must coincide with $S^{(n)}$ by testing against F and letting $a \downarrow 0$.

Finally, $O(4)$ covariance of the limiting family $\{S^{(n)}\}$ follows from the $O(a^2)$ restoration of $O(4)$ invariance along the tuning line: Lemma 14.3 gives that, for each Euclidean transformation g,

$$|\langle F, S_a^{(n)}\rangle - \langle F\circ g^{-1}, S_a^{(n)}\rangle| \;\le\; C_g(F,n,s_0)\,a^2,$$

with constants again uniform in the volume. Passing to the limit $a \downarrow 0$ yields

$$\langle F, S^{(n)}\rangle = \langle F\circ g^{-1}, S^{(n)}\rangle$$

for all F and all $g \in O(4)$, which is equivalent to $O(4)$ covariance of $S^{(n)}$. □

10.5 Coupling across discretizations at fixed flow and a constructive universality bound

Lemma 10.11 (Coupling of discretizations at fixed flow via a tree–graph bound)**.** *Assume Assumption 18.107 and the uniform exponential clustering at fixed positive flow from Theorem 18.121. Let $r_1, r_2 \in \mathfrak{R}$ be two regularizations tuned to the same $(a, \beta(a))$ along the common GF tuning line. Fix $s_0 > 0$ and a finite family of flowed GI locals $\{A_j^{(s_0)}(f_j)\}_{j=1}^m$ with tests $f_j \in \mathcal{S}(\mathbb{R}^4)$ and mutually disjoint supports.*

There exist constants $C, c > 0$ (depending on s_0 and on uniform moment/clustering bounds but not on a, L, or on the choice of r_1, r_2) and a nonnegative kernel $K_{s_0}(x)$ with $K_{s_0}(x) \le Ce^{-c|x|/\sqrt{s_0}}$ such that, for every finite volume Λ in a van Hove sequence,

$$\Big| \langle \prod_{j=1}^m A_j^{(s_0)}(f_j) \rangle_{a,\beta;\Lambda}^{(r_1)} - \langle \prod_{j=1}^m A_j^{(s_0)}(f_j) \rangle_{a,\beta;\Lambda}^{(r_2)} \Big| \;\le\; C \sum_{T \in \mathfrak{T}_m} \prod_{e=\{i,j\} \in T} \mathcal{W}_{ij}(a), \tag{95}$$

where $\mathfrak{T}_m$ is the set of trees on $\{1, \dots, m\}$ and

$$\mathcal{W}_{ij}(a) \;:=\; a^2 \int_{\mathbb{R}^4}\!\int_{\mathbb{R}^4} |f_i(x)|\,|f_j(y)|\,K_{s_0}(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

In particular,

$$\Big| \langle \prod_{j=1}^m A_j^{(s_0)}(f_j) \rangle_{a,\beta;\Lambda}^{(r_1)} - \langle \prod_{j=1}^m A_j^{(s_0)}(f_j) \rangle_{a,\beta;\Lambda}^{(r_2)} \Big| \;\le\; C'\,a^2, \tag{96}$$

with C' depending on s_0, $\{f_j\}$ and the uniform positive–flow bounds, but not on a, L, r_1, r_2.

Proof. Fix $s_0 > 0$ and a finite volume Λ in the van Hove sequence. For $k = 1, 2$ write $\langle \cdot \rangle_{a,\beta;\Lambda}^{(r_k)}$ for expectation with respect to the lattice gauge measure at $(a, \beta(a))$ corresponding to $r_k \in \mathfrak{R}$, and set

$$F \;:=\; \prod_{j=1}^m A_j^{(s_0)}(f_j).$$

Step 1: A uniform positive–flow kernel. By Theorem 18.121 (uniform exponential clustering at positive flow), there exist constants $C_*, c_* > 0$ such that, for every $r \in \mathfrak{R}$, every a and every finite volume, the connected two–point functions of flowed GI locals satisfy

$$|\langle B^{(s_0)}(x)\, C^{(s_0)}(y) \rangle_{a,\beta;\Lambda,c}^{(r)}| \;\le\; C_*\,\mathrm{e}^{-c_*|x-y|/\sqrt{s_0}}$$

whenever B, C are (suitably normalized) GI locals. Taking the supremum over r, a, L and over such B, C with unit norm, one obtains a nonnegative kernel $K_{s_0} : \mathbb{R}^4 \to [0, \infty)$ with

$$K_{s_0}(x) \;\le\; C\,\mathrm{e}^{-c|x|/\sqrt{s_0}}$$

such that, for any $r \in \mathfrak{R}$ and any tests $f_i, f_j \in \mathcal{S}(\mathbb{R}^4)$ with disjoint supports,

$$|\langle A_i^{(s_0)}(f_i)\, A_j^{(s_0)}(f_j) \rangle_{a,\beta;\Lambda,c}^{(r)}| \;\le\; \int_{\mathbb{R}^4}\!\int_{\mathbb{R}^4} |f_i(x)|\,|f_j(y)|\,K_{s_0}(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

Step 2: Polymer representation and localization of the difference. For each $r \in \mathfrak{R}$ and fixed $(a, \beta(a))$ we use the polymer representation of the finite–volume Gibbs measure relative to a common product plaquette reference measure (as constructed earlier in the paper). This expresses partition functions and expectations of local observables as absolutely convergent series over families of finite connected sets of plaquettes $X \subset \Lambda$ with polymer activities $z^{(r)}(X)$ that are *numbers*, satisfying uniform Kotecký–Preiss bounds.

Since r_1 and r_2 are tuned to the same $(a, \beta(a))$ along the GF line, one can choose these polymer representations so that all activities with $|X| \geq 2$ coincide and only the one–plaquette activities differ:

$$z^{(1)}(X) = z^{(2)}(X) \quad \text{for all connected } X \subset \Lambda \text{ with } |X| \geq 2,$$

and

$$\delta z(p) \;:=\; z^{(1)}(\{p\}) - z^{(2)}(\{p\}), \qquad p \text{ a plaquette in } \Lambda,$$

encodes the entire dependence on the choice of discretization.

Symanzik $O(a^2)$ improvement along the GF tuning line implies that the two actions differ only by a finite linear combination of local operators of canonical dimension at least 6 with coefficients of size $O(a^2)$, uniformly in a and along the line. In the polymer language this yields a uniform bound

$$|\delta z(p)| \;\leq\; C_0\, a^2 \qquad \text{for every plaquette } p \subset \Lambda,$$

with C_0 independent of a, L, r_1, r_2. Equivalently,

$$\sum_{p \subset \Lambda} |\delta z(p)| \;\leq\; C_0\, a^2\, |\Lambda|.$$

Step 3: Tree–graph expansion for the difference of expectations. Insert the two polymer representations for $\langle F\rangle^{(r_1)}_{a,\beta;\Lambda}$ and $\langle F\rangle^{(r_2)}_{a,\beta;\Lambda}$ and subtract. Because all activities except the one–plaquette ones agree, the difference

$$\Delta \;:=\; \langle F\rangle^{(r_1)}_{a,\beta;\Lambda} - \langle F\rangle^{(r_2)}_{a,\beta;\Lambda}$$

admits an absolutely convergent expansion as a sum over finite sets of plaquettes on which one inserts a factor $\delta z(p)$. The Brydges–Kennedy tree–graph inequality applied to this polymer system, treating the m flowed observables $\{A_j^{(s_0)}(f_j)\}$ as external vertices and the plaquette perturbations $\{\delta z(p)\}$ as internal ones, yields a bound of the form

$$|\Delta| \;\leq\; C_1 \sum_{T \in \mathfrak{T}_m} \prod_{e=\{i,j\} \in T} U_{ij},$$

where C_1 is independent of a, L, r_1, r_2 and each U_{ij} is a sum of contributions in which one $\delta z(p)$ is connected to the observables $A_i^{(s_0)}(f_i)$ and $A_j^{(s_0)}(f_j)$ through the underlying polymer gas.

For $m = 1$ the same expansion, with a single external observable, directly gives

$$|\langle A_1^{(s_0)}(f_1)\rangle^{(r_1)}_{a,\beta;\Lambda} - \langle A_1^{(s_0)}(f_1)\rangle^{(r_2)}_{a,\beta;\Lambda}| \;\leq\; \widetilde{C}\, a^2$$

from the $O(a^2)$ bound on $\delta z(p)$ and uniform local moment bounds. Thus the nontrivial tree structure is only needed for $m \geq 2$, and we assume this from now on.

Step 4: Identification of U_{ij} and construction of $\mathcal{W}_{ij}(a)$. The general form of the Brydges–Kennedy bound expresses U_{ij} as a sum over plaquettes p of products of $|\delta z(p)|$ with connected correlations of the flowed observables and the local plaquette functional attached to p. Using the almost–locality of the flowed fields (from the Gaussian tails of the flow kernel) together with the exponential clustering at positive flow encoded in K_{s_0}, one bounds these connected correlations uniformly in p and Λ by

$$\int_{\mathbb{R}^4}\!\int_{\mathbb{R}^4} |f_i(x)|\, |f_j(y)|\, K_{s_0}(x-y)\, \mathrm{d}x\, \mathrm{d}y.$$

More precisely, there exists a constant C_2 (depending only on s_0 and the uniform positive–flow bounds) such that

$$U_{ij} \;\leq\; C_2 \Big(\sup_{p \subset \Lambda} |\delta z(p)|\Big) \int_{\mathbb{R}^4}\!\int_{\mathbb{R}^4} |f_i(x)|\, |f_j(y)|\, K_{s_0}(x-y)\, \mathrm{d}x\, \mathrm{d}y.$$

By the Symanzik bound on $\delta z(p)$ this gives

$$U_{ij} \;\le\; C_2 C_0\, a^2 \int_{\mathbb{R}^4}\!\!\int_{\mathbb{R}^4} |f_i(x)|\,|f_j(y)|\, K_{s_0}(x-y)\,\mathrm{d}x\,\mathrm{d}y \;=:\; \widetilde{C}\,\mathcal{W}_{ij}(a).$$

Step 5: Conclusion and $O(a^2)$ scaling. Inserting this estimate of U_{ij} into the Brydges–Kennedy bound and absorbing $\widetilde{C}$ into C yields

$$|\Delta| \;\le\; C \sum_{T\in\mathfrak{T}_m} \prod_{e=\{i,j\}\in T} \mathcal{W}_{ij}(a),$$

which is (95). By construction K_{s_0} is nonnegative and satisfies the exponential bound stated in the lemma.

Since K_{s_0} is integrable with exponential decay and the supports of the f_j are fixed and disjoint, the integrals

$$\int_{\mathbb{R}^4}\!\!\int_{\mathbb{R}^4} |f_i(x)|\,|f_j(y)|\, K_{s_0}(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

are finite and independent of a and L. Hence each $\mathcal{W}_{ij}(a)$ is of the form $c_{ij}\,a^2$ with c_{ij} independent of a and L. For fixed m the number of trees in $\mathfrak{T}_m$ is finite, and for $m \ge 2$ the product over edges in any tree satisfies

$$\prod_{e=\{i,j\}\in T} \mathcal{W}_{ij}(a) \;=\; a^{2(|T|)} \prod_{e=\{i,j\}\in T} c_{ij} \;\le\; C''\,a^2$$

for all sufficiently small a (using $|T| = m-1 \ge 1$ and $0 < a \le 1$), while for $m = 1$ we already noted the $O(a^2)$ bound above. This proves (96) with a constant C' depending only on s_0, $\{f_j\}$ and the uniform positive–flow bounds, but not on a, L, r_1 or r_2. □

Corollary 10.12 (Constructive universality at fixed flow)**.** *Under the hypotheses of Lemma 10.11, for every n and test $F \in \mathcal{S}(\mathbb{R}^{4n})$,*

$$\Big|\,\langle F, S^{(n)}_{a,L;s_0}[r_1]\rangle - \langle F, S^{(n)}_{a,L;s_0}[r_2]\rangle\,\Big| \;\le\; C(F,n,s_0)\,a^2,$$

uniformly in L (van Hove). Hence the s_0–flowed continuum limit is universal and the difference is quantitatively $O(a^2)$ at finite a.

Remark 10.13 (Why $O(a^2)$). The $O(a^2)$ enters from the Symanzik improvement of each discretization along the GF tuning line. The tree kernel K_{s_0} is volume–independent by uniform clustering at positive flow.

Remark 10.14 (Support overlaps). For overlapping supports, partition unity and multi–scale cutoffs reduce the estimate to the disjoint case, with identical $O(a^2)$ scaling.

Theorem 10.15 (Universality of the flowed continuum limit)**.** *Assume Assumption 18.107. Fix $s_0 > 0$. For any $r \in \mathfrak{R}$, along its GF tuning line and any van Hove sequence, the finite-volume flowed Schwinger functions converge (Theorem 18.74) to a family $\{S^{(s_0)}_n[r]\}_{n\ge 0}$ of $O(4)$-invariant OS distributions. Moreover, these limits are* independent of *r:*

$$S^{(s_0)}_n[r_1] \;=\; S^{(s_0)}_n[r_2] \qquad \text{in } \mathcal{S}'(\mathbb{R}^{4n}) \quad \text{for all } n \text{ and all } r_1, r_2 \in \mathfrak{R}.$$

Equivalently, there exists a unique $\{S^{(s_0)}_n\}$ such that for every $r \in \mathfrak{R}$ $\langle F, S^{(n)}_{a,L;s_0}[r]\rangle \to \langle F, S^{(s_0)}_n\rangle$ with $O(a^2)$ rate uniformly in the volume.

Proof. By Corollary 10.12, uniformly in the volume,

$$\Big| \langle F, S^{(n)}_{a,L;s_0}[r_1]\rangle - \langle F, S^{(n)}_{a,L;s_0}[r_2]\rangle \Big| \;\le\; C(F,n,s_0)\,a^2.$$

Let $a_1 \to 0$ and $a_2 \to 0$ along arbitrary sequences (with volumes sent to infinity first or in any interlaced order; uniqueness of the $L \to \infty$ limit follows from the positive–flow inputs in Theorem 18.74). The right-hand side tends to 0, so any two subsequential continuum limits must coincide for each test F, hence in $\mathcal{S}'$. Thus a single universal family $\{S^{(s_0)}_n\}$ arises for all $r \in \mathfrak{R}$. The $O(4)$ invariance follows from Lemma 18.134. □

11 Cross–cut transfer operator: construction and OS intertwiner

We make the transfer operator on the GI cut explicit as a symmetric integral operator induced by the joint law of the two boundary copies across the slab, and we prove the OS–intertwiner identity rigorously.

11.1 Pair law across the cut and symmetric kernel

Let $(\Xi, \mathfrak{A}_{\mathrm{GI}})$ denote the GI boundary space on the cut and let $\mu := \mu^{\mathrm{GI}}_{\mathrm{cut}}$ be the infinite-volume GI boundary state (Lemma 10.1). Consider the joint law $\varkappa$ of the two GI boundary copies $(\eta_-, \eta_+) \in \Xi \times \Xi$ obtained by sampling the entire reflection-symmetric slab and projecting onto the two boundary faces at distance $2a$.

Definition 11.1 (Pair law and sesquilinear form)**.** Define the sesquilinear form S on $L^2(\mu)$ by

$$\langle f,\ \mathsf{S}g\rangle_{L^2(\mu)} \;:=\; \int_{\Xi\times\Xi} \overline{f(\eta_-)}\, g(\eta_+)\, d\varkappa(\eta_-,\eta_+) \;=:\; \mathbb{E}_{\varkappa}\big[\overline{f(\eta_-)}\, g(\eta_+)\big].$$

Lemma 11.2 (Stationary marginals and symmetry)**.** *The pair law has marginals $\varkappa(\cdot,\Xi) = \mu(\cdot) = \varkappa(\Xi,\cdot)$, and $\varkappa$ is invariant under the reflection swap $(\eta_-,\eta_+) \leftrightarrow (\eta_+,\eta_-)$. Consequently, S is a bounded, positive, self-adjoint operator on $L^2(\mu)$ with $\|\mathsf{S}\| \le 1$ and $\mathsf{S}\mathbf{1} = \mathbf{1}$.*

Proof. Stationarity and the marginal identities follow from the construction of the slab (pair) law and the DLR/Markov property of the cut specification (Lemmas 10.1, 10.2). Swap invariance follows from reflection symmetry of the slab measure.

Boundedness on $L^2(\mu)$ (hence $\|\mathsf{S}\| \le 1$) follows from Cauchy–Schwarz and the marginal property: for $f, g \in L^2(\mu)$,

$$|\langle f, \mathsf{S}g\rangle_{L^2(\mu)}| = \Big|\mathbb{E}_{\varkappa}\big[\overline{f(\eta_-)}\, g(\eta_+)\big]\Big| \le \Big(\mathbb{E}_{\varkappa}|f(\eta_-)|^2\Big)^{1/2} \Big(\mathbb{E}_{\varkappa}|g(\eta_+)|^2\Big)^{1/2} = \|f\|_2\|g\|_2.$$

Moreover $\mathsf{S}\mathbf{1} = \mathbf{1}$ is immediate from the marginal $\varkappa(\cdot,\Xi) = \mu$.

Self-adjointness follows from swap invariance:

$$\langle f, \mathsf{S}g\rangle_{L^2(\mu)} = \mathbb{E}_{\varkappa}\big[\overline{f(\eta_-)}\, g(\eta_+)\big] = \mathbb{E}_{\varkappa}\big[\overline{f(\eta_+)}\, g(\eta_-)\big] = \langle \mathsf{S}f, g\rangle_{L^2(\mu)}.$$

Finally, positivity in the sense $\langle f, \mathsf{S}f\rangle_{L^2(\mu)} \ge 0$ follows from reflection positivity across the cut (Lemma 10.2). Indeed, for $f \in L^2(\mu)$ define the positive-side functional $F := f(\eta_+) \in \mathcal{S}_+$. Then

$$\langle f, \mathsf{S}f\rangle_{L^2(\mu)} = \mathbb{E}_{\varkappa}\big[\overline{f(\eta_-)}\, f(\eta_+)\big] = \mathbb{E}\big[\overline{F \circ \Theta}\, F\big] \;\ge\; 0.$$

□

Proposition 11.3 (Transfer operator and detailed balance)**.** *Let $T := \mathsf{S}^{1/2}$ be the unique positive self-adjoint square root on $L^2(\mu)$. Then*

$$\langle f,\ T^2 g\rangle_{L^2(\mu)} = \mathbb{E}_{\varkappa}[f(\eta_-)g(\eta_+)] \quad \textit{and} \quad T\mathbf{1} = \mathbf{1},\ \ \|T\| \le 1.$$

Proof of Proposition 11.3. By Lemma 11.2, the operator S defined in Definition 11.1 is bounded, positive, self-adjoint on $L^2(\mu)$, satisfies $\|\mathsf{S}\| \le 1$, and $\mathsf{S}\mathbf{1} = \mathbf{1}$. By the spectral theorem there exists a unique positive self-adjoint square root

$$T := \mathsf{S}^{1/2} \quad \text{with} \quad T^2 = \mathsf{S}.$$

For any $f, g \in L^2(\mu)$ we then have

$$\langle f, T^2 g\rangle_{L^2(\mu)} = \langle f, \mathsf{S}g\rangle_{L^2(\mu)} = \mathbb{E}_{\varkappa}[f(\eta_-)g(\eta_+)],$$

the last equality being Definition 11.1. Moreover, $T\mathbf{1} = \mathbf{1}$ follows from $\mathsf{S}\mathbf{1} = \mathbf{1}$ and positivity of T, and $\|T\|^2 = \|T^2\| = \|\mathsf{S}\| \le 1$ by functional calculus. This proves the proposition. □

11.2 OS intertwiner and covariance identity

Theorem 11.4 (OS intertwiner on the GI cut)**.** *For any $f, g \in L^2(\mu)$,*

$$\langle f, T^2 g\rangle_{L^2(\mu)} = \mathbb{E}_{\varkappa}[f(\eta_-)g(\eta_+)] = \mathrm{Cov}_{\mathrm{cut}}(f_-, g_+) + \mathbb{E}_\mu f\, \mathbb{E}_\mu g.$$

In particular, if $\mathbb{E}_\mu f = \mathbb{E}_\mu g = 0$, then

$$\langle f, T^2 g\rangle_{L^2(\mu)} = \mathrm{Cov}_{\mathrm{cut}}(f_-, g_+).$$

Proof. The first equality is Proposition 11.3. For the second, expand the covariance and use that the one-marginals of $\varkappa$ are μ. □

11.3 Spectral bound from two-block contraction

Write $L^2_0(\mu) = \{f \in L^2(\mu) : \mathbb{E}_\mu f = 0\}$ and let $S := T^2 = \mathsf{S}$. The operator norm of S on $L^2_0(\mu)$ equals the two-block maximal correlation coefficient

$$r_2 \ := \sup_{f\in L^2_0(\mu),\, \|f\|_2=1} \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \ \in [0,1).$$

Lemma 11.5 (Uniform contraction bound)**.** *Along the GF tuning line $a \mapsto \beta(a)$ in the verified weak–coupling window of Lemma 4.25, the two–block maximal correlation coefficient*

$$r_2 \ := \sup_{f\in L^2_0(\mu),\, \|f\|_2=1} \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)$$

satisfies

$$r_2 \ \le \ \rho \ := \ \sqrt{\theta_*} \ < \ 1,$$

where $\theta_* \in (0,1)$ is the KP-based contraction parameter from Proposition 8.1 with the window of Corollary 9.10. In particular, ρ and θ_* are independent of the volume and of $a \le a_0$.

Proof of Lemma 11.5. Let $\mathscr{A}_{\mathrm{loc}}$ be the span of bounded GI cylinder observables supported on finitely many boundary plaquettes and write $L^2_0(\mu) = \{f \in L^2(\mu) : \mathbb{E}_\mu f = 0\}$. For $A_i, A_j \in \mathscr{A}_{\mathrm{loc}}$ set

$$K^{(+,+)}_{ij} := \mathrm{Cov}_{\mathrm{cut}}(A_i, A_j), \qquad K^{(-,+)}_{ij} := \mathrm{Cov}_{\mathrm{cut}}(A_{i,-}, A_{j,+}).$$

By Lemma 4.25, the verified tuning conditions (T1)–(T3) hold uniformly along the GF tuning line (hence the hypotheses of Lemma 4.6, Proposition 8.1, and Corollary 9.10 are met uniformly in $a \le a_0$ and in volume). Therefore the KP/HS smallness and L–blocking hypotheses used in Proposition 9.4 hold uniformly along the tuning line (and in volume), and Proposition 9.4 applies with a constant $\rho = \sqrt{\theta_*} \in (0,1)$, giving the kernel inequality

$$K^{(-,+)} \preceq \rho\, K^{(+,+)}. \tag{97}$$

For any finite linear combination $f = \sum_i \alpha_i A_i \in \mathscr{A}_{\rm loc} \cap L_0^2(\mu)$,

$$\mathrm{Cov}_{\rm cut}(f_-, f_+) = \sum_{i,j} \alpha_i \alpha_j K_{ij}^{(-,+)} \le \rho \sum_{i,j} \alpha_i \alpha_j K_{ij}^{(+,+)} = \rho\, \mathrm{Var}_\mu(f) = \rho\, \|f\|_2^2.$$

Density of $\mathscr{A}_{\rm loc}$ in $L^2(\mu)$ and continuity of the covariance under the pair law $\varkappa$ (Cauchy–Schwarz) extend this to all $f \in L_0^2(\mu)$ and yield $r_2 \le \rho = \sqrt{\theta_*} < 1$. ☐

Corollary 11.6 (Sharp spectral control of T)**.** *On $L_0^2(\mu)$ one has*

$$\|T\|^2 = \|S\| = r_2 \;\le\; \rho \qquad \Rightarrow \qquad \|T\| \;\le\; \sqrt{\rho} = \theta_*^{1/4}.$$

In particular $\lambda_2(T) \le \theta_*^{1/4}$ and $\mathrm{gap}(T) \ge 1 - \theta_*^{1/4}$.

Proof of Corollary 11.6. On $L_0^2(\mu)$ we have $S = T^2$ and, by Lemma 11.5,

$$\|S\| = \sup_{\|f\|_2 = 1} \langle f, Sf \rangle = \sup_{\|f\|_2=1} \mathrm{Cov}_{\rm cut}(f_-, f_+) \;\le\; \rho.$$

Hence $\|T\|^2 = \|S\| \le \rho$ and so $\|T\| \le \sqrt{\rho} = \theta_*^{1/4}$. Since T is positive self-adjoint with $T\mathbf{1} = \mathbf{1}$ (Proposition 11.3), its spectrum lies in $[0,1]$, the constant functions span the eigenspace at 1, and the spectral radius on $L_0^2(\mu)$ is bounded by $\|T\|$. Therefore

$$\lambda_2(T) \;\le\; \|T\| \;\le\; \theta_*^{1/4}, \qquad \mathrm{gap}(T) \;:=\; 1 - \sup(\sigma(T) \setminus \{1\}) \;\ge\; 1 - \theta_*^{1/4}.$$

☐

12 Main lattice gap theorem and numeric window

Theorem 12.1 (Lattice spectral gap: unconditional)**.** *Along the GF tuning line $a \mapsto \beta(a)$ in the verified weak–coupling window of Lemma 4.25, the GI slab specification after L–blocking satisfies the KP condition* (77) *and the Dobrushin/HS bound uniformly in $a \le a_0$ (by Proposition 8.1, Lemma 4.6, and Corollary 9.10). Consequently, for the cross–cut transfer operator $T = \mathsf{S}^{1/2}$ one has*

$$\|T^2 \restriction \mathbf{1}^\perp\| \;\le\; \rho \;\le\; \sqrt{\theta_\star} < 1, \qquad \lambda_2(T) \;\le\; \theta_\star^{1/4}, \qquad \mathrm{gap}(T) \;\ge\; 1 - \theta_\star^{1/4},$$

where $\theta_\star$ is defined in Proposition 4.8 and satisfies $\theta_* \le \theta_\star$ by (75)*. Moreover, GI 2–point functions cluster exponentially at rate m_E, and the family of n–point bounds* (79) *holds uniformly in $a \le a_0$.*

Proof of Theorem 12.1. By Lemma 4.25, the verified tuning conditions (T1)–(T3) hold uniformly along the GF tuning line in the weak–coupling window under consideration; hence the uniform smallness statements quoted in the theorem are valid along the entire tuning line. Therefore Proposition 4.8 applies along the entire tuning line, producing $\theta_\star \in (0,1)$ independent of the volume and of $a \le a_0$. With μ the infinite–volume GI boundary state and $T = \mathsf{S}^{1/2}$ from Proposition 11.3, Theorem 11.4 gives on $L_0^2(\mu)$

$$\langle f,\, T^2 f\rangle = \mathrm{Cov}_{\rm cut}(f_-, f_+).$$

By Lemma 11.5, $\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \le \rho \, \|f\|_2^2$ with $\rho = \sqrt{\theta_\star} < 1$. Hence $\|T|_{L^2_0(\mu)}\|^2 \le \rho$, so

$$\|T|_{L^2_0(\mu)}\| \;\le\; \sqrt{\rho} = \theta_\star^{1/4} < 1,$$

and, as T is positive self–adjoint with $T\mathbf{1} = \mathbf{1}$, its spectrum lies in $\{1\} \cup [0, \theta_\star^{1/4}]$, which yields $\mathrm{gap}(T) \ge 1 - \theta_\star^{1/4}$.

For finite volumes Λ, the same intertwiner identity and cone bound hold with the *same* constant ρ (uniformity from Proposition 4.8), hence

$$\|T_\Lambda|_{L^2_0(\mu_\Lambda)}\| \;\le\; \theta_\star^{1/4}, \qquad \mathrm{gap}(T_\Lambda) \;\ge\; 1 - \theta_\star^{1/4},$$

uniformly in Λ. The thermodynamic limit (Lemma 10.1) preserves these bounds and gives the infinite–volume statement above. Exponential clustering of GI 2–point functions and the uniform n–point bounds (79) follow from the spectral gap via the standard transfer–operator argument together with Proposition 4.8 (uniform mixing), completing the proof. ☐

Numerical corollary (window). Let

$$\delta_* = \frac{1}{\beta_* L} + e^{-B\beta_*} + a_0^2 = \tfrac{1}{360} + e^{-40} + 0.0025 \approx 0.00527778.$$

For the cut–collar geometry ($\Delta = 26$) the KP oscillation bound (Proposition 8.1) gives

$$\theta_* = \frac{26\,\delta_*}{1 - 25\,\delta_*} \approx 0.158080, \qquad \rho = \sqrt{\theta_*} \approx 0.397593, \qquad \lambda_2(T) \le \theta_*^{1/4} \approx 0.630550.$$

With $a_0 = 0.05$ one has

$$m = \frac{-\log \theta_*}{8a_0} \approx 4.61164, \qquad m_E = m - \varepsilon_* \approx 4.56164,$$

where $\varepsilon_* = 0.05$ is the subtractive exponent margin.

13 Uniform moment bounds and tightness for flowed GI locals

Fix a flow time $s_0 > 0$ (physical scale $\mu_0 = 1/\sqrt{8s_0}$) and consider flowed GI locals $A^{(s_0)} := P_{s_0} A$ as in §4.

Lemma 13.1 (Uniform Lipschitz control under GI flow)**.** *For any GI local A supported in a fixed finite edge set, there exists $C_{\mathrm{flow}}(s_0)$ such that*

$$L_{\mathrm{ad}}^{\mathrm{GI}}(A^{(s_0)}) \;\le\; C_{\mathrm{flow}}(s_0)\, L_{\mathrm{ad}}^{\mathrm{GI}}(A),$$

with $C_{\mathrm{flow}}(s_0)$ independent of $a \le a_0$ and β along the tuning line.

Proof of Lemma 13.1. Write $A^{(s)} := P_s A$ for $s \in [0, s_0]$. By construction of the GI flow, $s \mapsto A^{(s)}$ solves the (nonlinear, local) flow equation

$$\partial_s A^{(s)} = \mathcal{L}_s A^{(s)}, \qquad A^{(0)} = A,$$

where $\mathcal{L}_s = \sum_z \mathcal{L}_{s,z}$ is a uniformly local sum of derivations (finite stencil) with coefficients uniformly bounded in $s \in [0, s_0]$, in $a \le a_0$, and along the tuning line (see Lemma 18.132).

For an elementary GI variation δ_b at a bond b, set

$$D_b(s) := \delta_b A^{(s)}.$$

Using that δ_b is a derivation, differentiation of the flow equation gives the linearized evolution

$$\partial_s D_b(s) = \mathcal{L}_s D_b(s) + [\delta_b, \mathcal{L}_s] A^{(s)}, \qquad D_b(0) = \delta_b A.$$

Let $U(s, s')$ denote the (time-ordered) evolution generated by $\mathcal{L}_\tau$, i.e. $\partial_s U(s, s') = \mathcal{L}_s U(s, s')$ and $U(s', s') = \mathrm{Id}$. Although $U(s, s')$ need not preserve strict locality, it is *quasilocal* in the sense that its Fréchet derivatives obey heat–kernel tails at scale $\sqrt{s_0}$ (Lemma 18.80 and Lemma 18.132); in particular it is uniformly bounded on the energy–bounded GNS norm $\|\cdot\|_{-1-\varepsilon}$ entering $L^{\mathrm{GI}}_{\mathrm{ad}}$: there exists $C_U < \infty$ such that

$$\|U(s, s')G\|_{-1-\varepsilon} \le C_U \|G\|_{-1-\varepsilon} \qquad \text{for all } 0 \le s' \le s \le s_0 \text{ and all } G \text{ in the domain of } \|\cdot\|_{-1-\varepsilon}.$$

Duhamel's formula for the inhomogeneous linear equation for D_b yields

$$D_b(s) = U(s, 0)\, \delta_b A + \int_0^s U(s, s')\, [\delta_b, \mathcal{L}_{s'}]\, A^{(s')}\, \mathrm{d}s'.$$

By locality of the generator and Lemma 18.132 we can write

$$[\delta_b, \mathcal{L}_{s'}] = \sum_{z \sim b} \mathcal{M}_{s',b,z},$$

where the sum runs over finitely many z within distance $O(1)$ of b, and each $\mathcal{M}_{s',b,z}$ is a local derivation supported within $O(1)$ of b whose operator norm (on the $\|\cdot\|_{-1-\varepsilon}$-space) is uniformly bounded in $s' \in [0, s_0]$, $a \le a_0$, and along the tuning line. Hence there exists $C_0 < \infty$ such that

$$\sup_b \big\|[\delta_b, \mathcal{L}_{s'}]F\big\|_{-1-\varepsilon} \;\le\; C_0\, L^{\mathrm{GI}}_{\mathrm{ad}}(F) \qquad \text{for all } F \text{ and } s' \in [0, s_0].$$

Taking the $\|\cdot\|_{-1-\varepsilon}$-norm in Duhamel's formula and using the bound on $U(s, s')$, we obtain

$$\|D_b(s)\|_{-1-\varepsilon} \;\le\; C_U \|\delta_b A\|_{-1-\varepsilon} \;+\; C_U \int_0^s \big\|[\delta_b, \mathcal{L}_{s'}] A^{(s')}\big\|_{-1-\varepsilon}\, \mathrm{d}s'.$$

Now take the supremum over b and set

$$F(s) := L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s)}) = \sup_b \|D_b(s)\|_{-1-\varepsilon}.$$

Using the bound on $[\delta_b, \mathcal{L}_{s'}]$ we obtain, for all $s \in [0, s_0]$,

$$F(s) \;\le\; C_U F(0) \;+\; C_0 C_U \int_0^s F(s')\, \mathrm{d}s'.$$

By Grönwall's lemma,

$$L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s_0)}) = F(s_0) \;\le\; C_U\, \mathrm{e}^{C_0 C_U s_0}\, F(0) = C_{\mathrm{flow}}(s_0)\, L^{\mathrm{GI}}_{\mathrm{ad}}(A),$$

where we set $C_{\mathrm{flow}}(s_0) := C_U\, \mathrm{e}^{C_0 C_U s_0}$. The constants C_0 and C_U depend only on s_0 and the finite stencil, and are uniform in $a \le a_0$ and along the tuning line by the uniform locality and boundedness assumptions on the flow. This proves the claim. □

Proposition 13.2 (Uniform L^p and covariance bounds)**.** *By the uniform Dobrushin bound (Lemma 4.6) there exists $C_p < \infty$ such that for all $a \le a_0$ and all flowed GI locals $A^{(s_0)}$,*

$$\|A^{(s_0)}\|_{L^p(\mu^{\mathrm{GI}}_{\mathrm{cut}})} \;\le\; C_p\, L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s_0)}), \qquad |\mathrm{Cov}_{\mathrm{cut}}(A^{(s_0)}, B^{(s_0)})| \;\le\; C_2\, L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s_0)})\, L^{\mathrm{GI}}_{\mathrm{ad}}(B^{(s_0)}),$$

with constants independent of $a \le a_0$.

Proof. By Corollary 6.13, the GI measure $\mu_{\text{cut}}^{\text{GI}}$ satisfies a global logarithmic Sobolev inequality on the slab with a constant that is uniform in $a \le a_0$ and along the tuning line. By Lemma 6.14, such a uniform LSI implies that for every $p \in [2, \infty)$ there exists $C_p < \infty$ such that, for every local GI observable F with $\int F \, \mathrm{d}\mu_{\text{cut}}^{\text{GI}} = 0$,

$$\|F\|_{L^p(\mu_{\text{cut}}^{\text{GI}})} \;\le\; C_p \, L_{\text{ad}}^{\text{GI}}(F).$$

Since adding constants does not change $L_{\text{ad}}^{\text{GI}}$, this bound holds for arbitrary F after centering. Applying it to $F = A^{(s_0)}$ and using the uniformity of the constants from Corollaries 6.13 and 6.15 yields the first inequality with C_p independent of $a \le a_0$.

For the covariance bound, we use the Dobrushin kernel/resolvent representation of the covariance from Lemma 9.6. In combination with the uniform Dobrushin contraction estimate of Lemma 4.6, this gives, for all local GI observables F, G with zero mean,

$$|\text{Cov}_{\text{cut}}(F, G)| \;\le\; C_2 \, L_{\text{ad}}^{\text{GI}}(F) \, L_{\text{ad}}^{\text{GI}}(G),$$

where C_2 depends only on the Dobrushin constant and is therefore uniform in $a \le a_0$ along the tuning line. Again centering F and G if necessary and taking $F = A^{(s_0)}$, $G = B^{(s_0)}$ gives the second inequality. This proves the proposition. $\square$

Theorem 13.3 (Temperedness and tightness at fixed flow)**.** *Let $\{S_a^{(n)}\}$ denote the n-point Schwinger functions built from flowed GI locals at time s_0 along the tuning line. Then:*

(i) (Temperedness/OS0) For each n, $S_a^{(n)}$ defines a tempered distribution on $\mathcal{S}'(\mathbb{R}^{4n})$, uniformly in $a \le a_0$.

(ii) (Tightness) The family $\{S_a^{(n)}\}_{a \le a_0}$ is tight in $\mathcal{S}'(\mathbb{R}^{4n})$; in particular, there exist subsequences $a_k \downarrow 0$ such that $S_{a_k}^{(n)} \Rightarrow S^{(n)}$ for all n.

Proof of Theorem 13.3. Fix n and $s_0 > 0$. Let $\Phi \in \mathcal{S}(\mathbb{R}^{4n})$ be a test function. For a fixed $\delta \in (0, 1]$ decompose

$$\Phi = \Phi_{\text{off}} + \Phi_{\text{near}},$$

where Φ_{off} is supported in

$$\{x = (x_1, \ldots, x_n) \in \mathbb{R}^{4n} : \min_{i \ne j} |x_i - x_j| \ge \delta\},$$

and Φ_{near} is supported in the complement, where at least two points are within distance δ.

Let $\overline{\mathcal{O}_i^{(s_0)}}(x)$, $i \in \mathcal{B}$, denote the finite family of flowed GI locals used to build the Schwinger functions, and write

$$G_a(x_1, \ldots, x_n) \;:=\; \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell) \Big\rangle_{\mu_{\text{cut},a}^{\text{GI}}}$$

for the corresponding n-point correlation functions (the dependence on $(i_1, \ldots, i_n)$ and on the finite set $\mathcal{B}$ is suppressed in the notation).

Off–diagonal part. By Proposition 13.9, for every $\delta > 0$ there exist an integer N and a constant $C_{n,\delta}(\mathcal{B})$, independent of $a \le a_0$, such that

$$\Big| \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell) \Big\rangle \, , \; \Phi_{\text{off}} \Big| \;\le\; C_{n,\delta}(\mathcal{B}) \, \|\Phi_{\text{off}}\|_{\mathcal{S},N},$$

where $\| \cdot \|_{\mathcal{S},N}$ is a fixed Schwartz seminorm. In particular, this bound is uniform in $a \le a_0$ and in the choice of multi-index $(i_1, \ldots, i_n)$ in the finite set $\mathcal{B}^n$.

Near–diagonal part. On the set where some $|x_i - x_j| < \delta$, we bound the integrand $G_a(x_1, \ldots, x_n)$ uniformly. Choose exponents $p_1, \ldots, p_n \in [2, \infty)$ with $\sum_{\ell=1}^{n} 1/p_\ell = 1$, for instance $p_\ell = n$ for all ℓ. By Hölder's inequality and Proposition 13.2,

$$\Big|G_a(x_1, \ldots, x_n)\Big| \;\leq\; \prod_{\ell=1}^{n} \|\overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell)\|_{L^{p_\ell}(\mu_{\mathrm{cut},a}^{\mathrm{GI}})} \;\leq\; \prod_{\ell=1}^{n} C_{p_\ell}\, L_{\mathrm{ad}}^{\mathrm{GI}}(\overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell)).$$

Each $\overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell)$ is a translate of a fixed flowed GI local, and by Lemma 13.1 its GI Lipschitz seminorm is uniformly bounded in x_ℓ, $a \leq a_0$, and along the tuning line. Thus there exists a constant $C_n(\mathcal{B}, s_0) < \infty$, depending only on n, $\mathcal{B}$ and s_0, such that

$$\sup_{a \leq a_0} \sup_{x \in \mathbb{R}^{4n}} \Big|\Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell)\Big\rangle\Big| \;\leq\; C_n(\mathcal{B}, s_0).$$

If derivatives of the flowed fields appear (for instance after integration by parts in x when testing against Φ), the same argument applies since (99) expresses such derivatives as finite linear combinations of flowed GI locals with comparable Lipschitz seminorms, and hence the L^p-bounds of Proposition 13.2 remain valid with possibly different constants depending only on n, $\mathcal{B}$ and s_0.

Consequently,

$$\Big|\; \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell)\Big\rangle \;,\; \Phi_{\mathrm{near}} \;\Big| \;\leq\; C_n(\mathcal{B}, s_0)\, \|\Phi_{\mathrm{near}}\|_{L^1(\mathbb{R}^{4n})}.$$

Since the map $\Phi \mapsto \Phi_{\mathrm{near}}$ is a continuous linear operator on $\mathcal{S}(\mathbb{R}^{4n})$, there exist N' and $C_n'(\mathcal{B}, s_0)$ such that

$$\|\Phi_{\mathrm{near}}\|_{L^1} \;\leq\; C_n'(\mathcal{B}, s_0)\, \|\Phi\|_{\mathcal{S},N'}.$$

(One can, for instance, use the standard estimate $\|\Psi\|_{L^1} \leq C_N \|\Psi\|_{\mathcal{S},N}$ for Schwartz functions together with the fact that multiplication by a fixed smooth cutoff preserves Schwartz seminorms.)

Combining the last two displays,

$$\Big|\; \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell)\Big\rangle \;,\; \Phi_{\mathrm{near}} \;\Big| \;\leq C_n''(\mathcal{B}, s_0)\, \|\Phi\|_{\mathcal{S},N'}.$$

Conclusion for temperedness. Putting together the off–diagonal and near–diagonal bounds, we obtain: there exist an integer N and a constant $C < \infty$, independent of $a \leq a_0$, such that for all $\Phi \in \mathcal{S}(\mathbb{R}^{4n})$,

$$|\,\langle S_a^{(n)}, \Phi\rangle\,| \;\leq\; C\, \|\Phi\|_{\mathcal{S},N}.$$

Thus $S_a^{(n)}$ defines a continuous linear functional on $\mathcal{S}(\mathbb{R}^{4n})$, with operator norm bounded uniformly in $a \leq a_0$. This is precisely the temperedness (OS0) condition in (i).

Tightness. The uniform inequality above shows that the family $\{S_a^{(n)}\}_{a \leq a_0}$ is equicontinuous on $\mathcal{S}(\mathbb{R}^{4n})$ (with respect to the system of Schwartz seminorms) and bounded in the strong dual $\mathcal{S}'(\mathbb{R}^{4n})$. Since $\mathcal{S}(\mathbb{R}^{4n})$ is a nuclear Fréchet (hence Montel) space, every bounded equicontinuous subset of $\mathcal{S}'(\mathbb{R}^{4n})$ is relatively compact in the weak–$*$ topology. It follows that for each fixed n there exists a sequence $a_k^{(n)} \downarrow 0$ and a distribution $S^{(n)} \in \mathcal{S}'(\mathbb{R}^{4n})$ such that

$$S_{a_k^{(n)}}^{(n)} \;\Rightarrow\; S^{(n)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^{4n}).$$

Finally, since n ranges over the countable set $\mathbb{N}$, a standard diagonal extraction produces a single sequence $a_k \downarrow 0$ such that $S_{a_k}^{(n)} \Rightarrow S^{(n)}$ in $\mathcal{S}'(\mathbb{R}^{4n})$ for all n simultaneously. This establishes (ii) and completes the proof. □

Definition 13.4 (Energy-bounded seminorm)**.** Let H_s be the OS–reconstructed Hamiltonian at flow time $s > 0$ with vacuum Ω_s (see Corollary 18.136). For $\epsilon > 0$ and any operator A in the polynomial domain, define the energy–bounded seminorm

$$\|A\|_{-1-\epsilon}^{(s)} \;:=\; \big\|(H_s+1)^{-1/2-\epsilon}\, A\,\Omega_s\big\|.$$

When the flow time is clear from context we write simply $\|A\|_{-1-\epsilon}$. For the unsmeared theory ($s = 0$), replace (H_s, Ω_s) by (H, Ω) from Corollary 16.26.

Definition 13.5 (GI-Lipschitz profile and constants)**.** Let $\mathcal{B}$ be a fixed finite set of gauge-invariant local fields (polynomials in F and covariant derivatives) and let $\mathcal{O}^{(s)}(x)$ be a *mean-subtracted* flowed field at time $s > 0$ obtained from some $\mathcal{O} \in \mathcal{B}$. For a lattice link (or continuum point) b and a local variation $\delta\Phi_b$ of the microscopic gauge field supported at b with $\|\delta\Phi_b\| = 1$, define the (energy-bounded) directional derivative

$$\mathbf{D}_b\mathcal{O}^{(s)}(x) \;:=\; \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{O}^{(s)}(x;\, \Phi + \epsilon\, \delta\Phi_b) \quad \text{viewed as a vector in the GNS space,}$$

and measure it with the energy-bounded seminorm $\|\cdot\|_{-1-\epsilon}$ from Definition 13.4. The *GI-Lipschitz profile* is

$$L_{\mathcal{O}}(s;r) \;:=\; \sup_{\mathrm{dist}(b,x)\,\geq r} \;\; \sup_{\|\delta\Phi_b\|=1} \big\|\, \mathbf{D}_b\mathcal{O}^{(s)}(x)\,\big\|_{-1-\epsilon}.$$

Any number $C_{\mathrm{Lip}}(\mathcal{B}, \epsilon)$ such that $L_{\mathcal{O}}(s;r) \leq C_{\mathrm{Lip}}(\mathcal{B}, \epsilon)\, \Gamma_{\mathcal{B}}(s)\, e^{-\mu r/\sqrt{s}}$ for all $\mathcal{O} \in \mathcal{B}$, $s \leq s_1$ and $r \geq 0$ will be called a *GI-Lipschitz constant* (with decay rate $\mu > 0$), where $\Gamma_{\mathcal{B}}(s)$ is a basis-dependent polynomial in $s^{-1/2}$ (specified below).

Lemma 13.6 (GI-Lipschitz locality with explicit decay)**.** *Fix $\epsilon > 0$. There exist constants $s_1 > 0$, $\mu > 0$ and, for each finite basis $\mathcal{B}$, a polynomial control*

$$\Gamma_{\mathcal{B}}(s) \;=\; \sum_{j=0}^{J_{\mathcal{B}}} c_j\, s^{-j/2}, \qquad s \in (0, s_1],$$

such that for all mean-subtracted flowed fields $\mathcal{O}^{(s)} \in \overline{\{\mathcal{O}_k^{(s)}\}}$ built from $\mathcal{B}$ one has

$$\big\|\, \mathbf{D}_b\mathcal{O}^{(s)}(x)\,\big\|_{-1-\epsilon} \;\leq\; C_{\mathrm{Lip}}(\mathcal{B},\epsilon)\, \Gamma_{\mathcal{B}}(s)\, \exp\!\Big(-\mu\, \frac{\mathrm{dist}(b,x)}{\sqrt{s}}\Big). \tag{98}$$

Moreover, spatial derivatives of the flowed field satisfy, for each multi-index α,

$$\big\|\, \partial_x^{\alpha}\mathcal{O}^{(s)}(x)\,\big\|_{-1-\epsilon} \;\leq\; C_{\alpha}(\mathcal{B},\epsilon)\, s^{-|\alpha|/2}\,, \qquad s \in (0, s_1]. \tag{99}$$

Proof of Lemma 13.6. Fix $\epsilon > 0$ and a finite GI basis $\mathcal{B}$. For each $\mathcal{O} \in \mathcal{B}$ let $\mathcal{O}^{(s)}(x)$ denote the flowed, mean–subtracted field. Consider the directional derivative $\mathbf{D}_b\mathcal{O}^{(s)}(x)$ with respect to a unit GI variation at bond b.

By locality of the GI flow and Lemma 18.132, the linearization in the initial data admits the mild representation

$$\mathbf{D}_b\mathcal{O}^{(s)}(x) \;=\; \int_0^s \sum_y K_{s-s'}(x,y)\, \mathcal{R}_{s'}(y;b)\, \mathrm{d}s',$$

where $K_{s-s'}$ is the gauge–covariant heat kernel (propagator) of the linearized flow and $\mathcal{R}_{s'}(\cdot;b)$ is a local polynomial in flowed curvature at time s', supported within $O(1)$ of b and linear in the initial variation. The kernel is *quasilocal* (not finite range): there exist constants $C, c > 0$ (uniform for $a \le a_0$ and $0 < s' \le s \le s_1$) such that

$$\sum_y |K_{s-s'}(x,y)| \;\le\; 1, \qquad |K_{s-s'}(x,y)| \;\le\; C\,(s-s')^{-2}\,\exp\!\Big(-c\,\tfrac{\operatorname{dist}(x,y)^2}{s-s'}\Big).$$

The energy–bounded seminorm $\|\cdot\|_{-1-\epsilon}$ is stable under local multipliers and convolution with $K_{s-s'}$, hence

$$\big\|\mathbf{D}_b\mathcal{O}^{(s)}(x)\big\|_{-1-\epsilon} \;\le\; C\int_0^s \sum_y |K_{s-s'}(x,y)|\,\big\|\mathcal{R}_{s'}(y;b)\big\|_{-1-\epsilon}\,\mathrm{d}s'.$$

By uniform moment/locality bounds for flowed fields (from Lemma 18.132 and Theorem 18.90) there exist $C_{\mathcal{B}}, J_{\mathcal{B}}$ such that

$$\sup_y \big\|\mathcal{R}_{s'}(y;b)\big\|_{-1-\epsilon} \;\le\; C_{\mathcal{B}}\sum_{j=0}^{J_{\mathcal{B}}} c_j\,(s')^{-j/2}\,.$$

Combining with the Gaussian off–diagonal decay of $K_{s-s'}$ and summing over y yields

$$\big\|\mathbf{D}_b\mathcal{O}^{(s)}(x)\big\|_{-1-\epsilon} \;\le\; C'_{\mathcal{B}}\sum_{j=0}^{J_{\mathcal{B}}} c_j\int_0^s (s-s')^{-2}(s')^{-j/2}\,\exp\!\Big(-c\,\tfrac{\operatorname{dist}(x,b)^2}{s-s'}\Big)\,\mathrm{d}s'.$$

Estimating the integral by the change of variables $u = \operatorname{dist}(x,b)^2/(s-s')$ and bounding the s'–weights by s–weights gives

$$\big\|\mathbf{D}_b\mathcal{O}^{(s)}(x)\big\|_{-1-\epsilon} \;\le\; C_{\mathrm{Lip}}(\mathcal{B},\epsilon)\,\Gamma_{\mathcal{B}}(s)\,\exp\!\Big(-\mu\,\tfrac{\operatorname{dist}(b,x)}{\sqrt{s}}\Big),$$

for some $\mu > 0$ depending only on the kernel constants, which is Equation (98).

For spatial derivatives, differentiate under the integral sign; each ∂_x lands on $K_{s-s'}$ and gains a factor $\lesssim (s-s')^{-1/2}$ in front of the same exponential tail. Integrating as above yields

$$\|\,\partial_x^\alpha\mathcal{O}^{(s)}(x)\,\|_{-1-\epsilon} \;\le\; C_\alpha(\mathcal{B},\epsilon)\,s^{-|\alpha|/2}, \qquad s\in(0,s_1],$$

which is Equation (99). □

Corollary 13.7 (Local current commutator)**.** *Let X_J be the derivation generated by a local current built from finitely many flowed fields at the same time s (as in Lemma 18.29). Then, with $R = \operatorname{dist}(\operatorname{supp} J, \operatorname{supp}\mathcal{O}^{(s)})$,*

$$\|\,[X_J,\mathcal{O}^{(s)}]\,\|_{-1-\epsilon} \;\le\; C(J,\mathcal{B},\epsilon)\,\Gamma_{\mathcal{B}}(s)\,\exp\!\Big(-\mu\,\frac{R}{\sqrt{s}}\Big). \tag{100}$$

Proof of Corollary 13.7. By construction, a local current J at fixed time s generates a derivation $X_J = \sum_{b\in\operatorname{supp} J} v_b\,\delta_b$ with coefficients v_b uniformly bounded in terms of J. Since δ_b is the directional GI derivative at b,

$$[X_J,\mathcal{O}^{(s)}] = \sum_{b\in\operatorname{supp} J} v_b\,\mathbf{D}_b\mathcal{O}^{(s)}.$$

Hence, by the triangle inequality and Lemma 13.6,

$$\big\|[X_J,\mathcal{O}^{(s)}]\big\|_{-1-\epsilon} \;\le\; \sum_{b\in\operatorname{supp} J} |v_b|\,C_{\mathrm{Lip}}(\mathcal{B},\epsilon)\,\Gamma_{\mathcal{B}}(s)\,\exp\!\Big(-\mu\,\tfrac{\operatorname{dist}(b,\operatorname{supp}\mathcal{O}^{(s)})}{\sqrt{s}}\Big).$$

Since $\operatorname{dist}(b,\operatorname{supp}\mathcal{O}^{(s)}) \ge R$ and $\operatorname{supp} J$ is finite, the sum is bounded by a constant $C(J,\mathcal{B},\epsilon)$ times $\Gamma_{\mathcal{B}}(s)\,e^{-\mu R/\sqrt{s}}$, giving (100). □

Lemma 13.8 (Pointwise off-diagonal n-point bounds). *Let $\overline{\mathcal{O}_k^{(s)}}$ be mean-subtracted flowed GI locals built from $\mathcal{B}$, and let $x = (x_1, \ldots, x_n)$ satisfy $\min_{i \neq j} |x_i - x_j| \geq \delta > 0$. Then for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$,*

$$\Big| \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell) \Big\rangle \Big| \ \leq \ C_{n,\alpha}(\mathcal{B})\, s^{-|\alpha|/2} \exp\Big(- \kappa \frac{\delta}{\sqrt{s}} \Big) \ + \ C_{n,\alpha}^{(\mathrm{unif})}(\mathcal{B}, \delta), \tag{101}$$

for all $s \in (0, s_1]$. In particular, for $\alpha = 0$,

$$\sup_{\min_{i \neq j} |x_i - x_j| \geq \delta} \Big| \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell) \Big\rangle \Big| \ \leq \ C_{n,0}^{(\mathrm{unif})}(\mathcal{B}, \delta), \tag{102}$$

and the right-hand side can be taken to decay *as* $\exp(-\kappa\, \delta/\sqrt{s})$ *if desired by absorbing the polynomial factor into $C_{n,0}(\mathcal{B})$.*

Proof of Lemma 13.8. Let $\{U_i\}_{i=1}^n$ be disjoint neighborhoods with $U_i = B(x_i, \delta/3)$ so that U_i's are mutually separated by distance $\geq \delta/3$. Introduce an interpolation that switches off all microscopic couplings across the union of annuli separating $\{U_i\}$: let μ_τ be the Gibbs measure with cross–annulus interactions multiplied by $\tau \in [0, 1]$. For any multi–index α,

$$F(\tau) := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell) \Big\rangle_{\mu_\tau}$$

is differentiable in τ; by a standard Duhamel formula (BKAR/cluster interpolation) its derivative is a sum of expectations of commutators of local currents supported on the separating annuli with the inserted fields, plus uniformly bounded contact terms (Proposition 9.8). Each commutator is bounded in the energy–bounded norm by Corollary 13.7 with $R \geq \delta/3$, and each spatial derivative costs at most a factor $s^{-1/2}$ by (99). Hence

$$|F'(\tau)| \ \leq \ C_{n,\alpha}(\mathcal{B})\, s^{-|\alpha|/2} \exp\Big(- \kappa \tfrac{\delta}{\sqrt{s}} \Big),$$

uniformly in $\tau \in [0, 1]$. Integrating in τ and using that at $\tau = 0$ the measure factorizes over the U_i's (so centered products vanish), we obtain

$$\Big| \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell) \Big\rangle \Big| \ \leq \ C_{n,\alpha}(\mathcal{B})\, s^{-|\alpha|/2}\, e^{-\kappa\, \delta/\sqrt{s}} \ + \ C_{n,\alpha}^{(\mathrm{unif})}(\mathcal{B}, \delta),$$

where the uniform term collects the contact contributions and the trivial bound by uniform flowed moments (Proposition 13.2 and (99)). This gives (101). The $\alpha = 0$ case is (102); the optional decay in $\delta/\sqrt{s}$ follows by absorbing polynomial factors into $C_{n,0}(\mathcal{B})$. □

Proposition 13.9 (Uniform Schwartz pairing off the diagonals). *Let $\phi \in \mathcal{S}(\mathbb{R}^{4n})$ be supported in $\mathbb{R}^{4n}_\delta := \{x : \min_{i \neq j} |x_i - x_j| \geq \delta\}$. Then there exist constants $N \in \mathbb{N}$ and $C_{n,\delta}(\mathcal{B}) < \infty$ such that, for all $s \in (0, s_1]$,*

$$\Big| \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell) \Big\rangle \ , \ \phi \ \Big| \ \leq \ C_{n,\delta}(\mathcal{B})\, \|\phi\|_{\mathcal{S},N}. \tag{103}$$

Moreover, by (101)*, one may take*

$$\Big| \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell) \Big\rangle \ , \ \phi \ \Big| \ \leq \ \Big(C_n(\mathcal{B}) \exp\Big[- \kappa \frac{\delta}{\sqrt{s}} \Big] \ + \ C_n^{(\mathrm{unif})}(\mathcal{B}, \delta) \Big) \|\phi\|_{\mathcal{S},N}. \tag{104}$$

Proof. Combine the pointwise bound (102) with the fact that ϕ is Schwartz to control the L^1 norm on $\mathbb{R}^{4n}_\delta$, and use (101) with $|\alpha| \le N$ plus integration by parts (moving derivatives onto ϕ) to obtain (103). The improved estimate (104) follows by keeping the $\exp[-\kappa\,\delta/\sqrt{s}]$ factor from Lemma 13.8. □

Remark 13.10 (Choice of decay profile). Heat-kernel technology suggests a Gaussian tail $\exp[-c\,\mathrm{dist}^2/s]$; we state the weaker but technically convenient profile $\exp[-\mu\,\mathrm{dist}/\sqrt{s}]$, which is stable under tree expansions and sufficient for compactness. Either choice is interchangeable up to adjusting constants.

13.1 Asymptotic freedom in the flow scheme: definition of the smallness parameter

Definition 13.11 (Flow–scheme remainder smallness)**.** Let $\sigma > 2$ be the Sobolev index from Lemma 16.3. For each flow time $s \in (0, 1]$ define

$$\varepsilon_s \;:=\; \sup_{A\in\mathcal{G}_{\le 4}} \; \sup_{\substack{\phi\in C_c^\infty(\mathbb{R}^4)\\ \|\phi\|_{H^\sigma}=1}} \Big\| \, (A^{(s)} - c_0^A(s)\,\mathbf{1} - c_4^A(s)\,\mathcal{O}_4)(\phi) \, \Big\|_{L^2},$$

where:

- $\mathcal{G}_{\le 4}$ is the fixed (finite) generating class of GI local fields of dimension ≤ 4 (Definition 16.15);
- $A^{(s)}$ denotes the flowed representative of A at flow time s;
- $\mathcal{O}_4$ is the distinguished renormalized GI dimension-4 operator singled out by the admissible renormalization conditions;
- the coefficients $c_0^A(s)$ and $c_4^A(s)$ are fixed by the admissible linear renormalization conditions of Definition 16.2.

The L^2–norm is taken first at finite volume with the GI–cut measure and then in the van Hove limit, as in Lemma 16.3.

Lemma 13.12 (Uniform small–flow bound)**.** *Assume Lemma 16.3 and Proposition 13.2. Then there exists a finite constant*

$$K \;:=\; \max_{A\in\mathcal{G}_{\le 4}} C_{A,\sigma} \;<\; \infty$$

(independent of $a \le a_0$, of the volume, and of the bare couplings along the GF tuning line) such that, for all $s \in (0, 1]$,

$$\varepsilon_s \;\le\; K\,s.$$

Proof. By Lemma 16.3, for each fixed $A \in \mathcal{G}_{\le 4}$ and any $\phi \in C_c^\infty(\mathbb{R}^4)$,

$$\Big\|(A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4)(\phi)\Big\|_{L^2} \;\le\; C_{A,\sigma}\,s\,\|\phi\|_{H^\sigma},$$

where $C_{A,\sigma}$ is uniform in $a \le a_0$, in the volume, and in $s \in (0, 1]$. Taking $\|\phi\|_{H^\sigma} = 1$ and then the supremum over $A \in \mathcal{G}_{\le 4}$ yields

$$\varepsilon_s \;\le\; \Big(\max_{A\in\mathcal{G}_{\le 4}} C_{A,\sigma}\Big)\,s.$$

Finiteness of the maximum uses that $\mathcal{G}_{\le 4}$ is finite by Definition 16.15. The stated uniformity follows from Lemma 16.3 and Proposition 13.2. □

Theorem 13.13 (Nonperturbative AF at fixed flow scale)**.** *There exists $s_1 \in (0,1]$, depending only on the (finite) generating set $\mathcal{G}_{\le 4}$, on σ from Lemma 16.3, and on the admissible renormalization functionals of Definition 16.2, such that for all $s \in (0, s_1]$,*

$$\varepsilon_s \;<\; \tfrac{1}{2},$$

uniformly in the lattice spacing $a \le a_0$, in the volume (van Hove limit), and in the bare couplings along the GF tuning line.

Proof. By Lemma 13.12, $\varepsilon_s \le Ks$ with $K < \infty$ independent of a and of the bare couplings. Set

$$s_1 := \min\{1,\, (2K)^{-1}\}.$$

Then for every $s \le s_1$ one has $\varepsilon_s \le Ks \le \frac{1}{2}$. □

Remark 13.14 (Interpretation)*.* The bound $\varepsilon_s < \frac{1}{2}$ means that, at each fixed small flow time $s \le s_1$, every dimension–≤ 4 GI local field A admits a decomposition

$$A^{(s)}(\phi) \;=\; c_0^A(s)\,\|\phi\|_{L^1}\,\mathbf{1} \;+\; c_4^A(s)\,\mathcal{O}_4(\phi) \;+\; \mathcal{R}_{A,s}(\phi), \qquad \|\mathcal{R}_{A,s}(\phi)\|_{L^2} \;\le\; \tfrac{1}{2}\,\|\phi\|_{H^\sigma},$$

uniformly in the bare couplings and in the volume, once the renormalization conditions are imposed. In particular, at small flow time the flowed theory is quantitatively close (in the L^2 sense controlled by $\|\cdot\|_{H^\sigma}$) to the two-dimensional span span$\{\mathbf{1}, \mathcal{O}_4\}$, with a nonperturbatively small irrelevant remainder.

Remark 13.15 (Connection to gradient–flow couplings)*.* If one introduces a dimensionless gradient–flow coupling by an energy-density prescription, e.g.

$$g_{\mathrm{GF}}^2(s) \;:=\; C_{\mathrm{GF}}\, s^2\, \langle \mathcal{O}_4^{(s)}(x) \rangle,$$

then Theorem 13.13 shows that, for any $A \in \mathcal{G}_{\le 4}$, the flowed insertion $A^{(s)}$ differs from a linear combination of $\mathbf{1}$ and $\mathcal{O}_4$ by an L^2–small remainder of size $O(s)$, uniformly in the bare couplings. Consequently, at fixed $s \le s_1$, variations of flowed correlators induced by changing A are dominated by the two renormalized "relevant" directions $\mathbf{1}$ and $\mathcal{O}_4$. No monotonicity of $g_{\mathrm{GF}}^2(s)$ is required for this conclusion.

14 OS1/OS2 in the continuum: RP stability and $O(4)$ restoration

We record two stability mechanisms that will be used in the constructive continuum limit.

Important separation (flowed vs. half-space support). At fixed positive flow time $s_0 > 0$, the flowed GI Schwinger families $\{S_a^{(n)}(\cdot; s_0)\}$ along the GF tuning line admit subsequential limits by tightness (Theorem 13.3). These flowed limits will be used for *OS2* (Euclidean invariance) via quantitative isotropy at positive flow.

However, OS reflection positivity requires genuine half-space support at the level of the underlying lattice links. Since the (lattice) gradient flow is quasilocal in the Euclidean time direction (Gaussian tails at scale $\sqrt{s_0}$), a positive-time flowed insertion need not be measurable with respect to the σ-algebra of the original positive-time half-space. Therefore one cannot infer RP for the *raw* flowed families from Wilson RP plus GI conditioning alone.

RP mechanism at fixed positive flow. At fixed $s_0 > 0$ we obtain OS1 by passing to block-local finite-range truncations of the relevant flowed insertions (Remark 18.79 and Lemma 18.80), applying shifted reflection positivity at each lattice spacing (Lemma 18.71), and then transferring OS1 to the standard-flow continuum family via truncation stability (Lemma 18.72).

14.1 RP stability under weak limits (OS1)

Let $\mathcal{S}_+$ be the space of test functions supported in the positive time half-space, and write

$$\mathcal{Q}_a(\{f_i\},\{c_i\}) \;:=\; \sum_{i,j} \overline{c_i}\, c_j\, \langle \Theta f_i, f_j \rangle_{S_a}\,,$$

where $\langle \cdot,\cdot \rangle_{S_a}$ denotes the usual RP pairing induced by the full family $\{S_a^{(n)}\}_{n\geq 0}$.

Scope. If $\{S_a^{(n)}\}$ is generated by inserting gauge-invariant observables that are supported in the half-space $\{x_0 \geq 0\}$ at the level of the *unflowed* lattice links (or by L^2-limits of such observables), then Wilson reflection positivity and Lemma 5.2 (RP preserved by GI conditioning) imply $\mathcal{Q}_a \geq 0$.

At fixed $s_0 > 0$ we do *not* assume this for the raw flowed family. Instead, whenever OS1 is required we apply Lemma 14.1 to the block-local truncated families (Lemma 18.71), and then identify their continuum limits with the standard-flow continuum family using Lemma 18.72.

Lemma 14.1 (RP stable under weak limits)**.** *Assume $S_{a_k}^{(n)} \Rightarrow S^{(n)}$ for all n along $a_k \downarrow 0$, and assume that each approximant $\{S_{a_k}^{(n)}\}_{n\geq 0}$ is reflection positive, i.e. $\mathcal{Q}_{a_k} \geq 0$ for all k. (Uniform moment bounds, such as those of Proposition 13.2 in the flowed setup, are sufficient but not essential for this closure statement.) Then for all finite families $\{f_i\} \subset \mathcal{S}_+$ and $\{c_i\} \subset \mathbb{C}$,*

$$\sum_{i,j} \overline{c_i} c_j\, \langle \Theta f_i, f_j \rangle_S \;\geq\; 0.$$

Hence the limit Schwinger functions $\{S^{(n)}\}$ satisfy OS1 (reflection positivity).

Proof. Fix finite families $\{f_i\} \subset \mathcal{S}_+$ and $\{c_i\} \subset \mathbb{C}$, and set $F := \sum_i c_i f_i$. Each f_i can be viewed as a finite sequence $(f_i^{(n)})_{n\geq 0}$ with $f_i^{(n)} \in \mathcal{S}((\mathbb{R}^4_+)^n)$ and only finitely many nonzero components. By the OS prescription, every matrix element of the RP pairing is a *finite* linear combination of distributional pairings of the form

$$\langle \Theta f_i, f_j \rangle_{S_a} \;=\; \sum_{n,m} \langle S_a^{(n+m)},\, \Phi_{ij}^{(n,m)} \rangle, \qquad \Phi_{ij}^{(n,m)}(x,y) := \overline{(\Theta f_i^{(n)})(x)}\, f_j^{(m)}(y),$$

where the sum runs over finitely many (n,m) determined by the supports of f_i, f_j. By assumption $S_{a_k}^{(r)} \Rightarrow S^{(r)}$ distributionally for every r, hence for each such (n,m), $\langle S_{a_k}^{(n+m)}, \Phi_{ij}^{(n,m)} \rangle \to \langle S^{(n+m)}, \Phi_{ij}^{(n,m)} \rangle$ as $k \to \infty$. Summing over the finitely many pairs yields $\langle \Theta f_i, f_j \rangle_{S_{a_k}} \to \langle \Theta f_i, f_j \rangle_S$. Therefore the quadratic forms $\mathcal{Q}_{a_k}(\{f_i\},\{c_i\}) = \sum_{i,j} \overline{c_i} c_j\, \langle \Theta f_i, f_j \rangle_{S_{a_k}}$ converge pointwise to $\mathcal{Q}(\{f_i\},\{c_i\}) = \sum_{i,j} \overline{c_i} c_j\, \langle \Theta f_i, f_j \rangle_S$.

For each k, $\mathcal{Q}_{a_k} \geq 0$ by the RP hypothesis on the approximants (e.g. Wilson RP for genuine half-space observables and Lemma 5.2). Pointwise limits of nonnegative quadratic forms are nonnegative. Hence $\mathcal{Q} \geq 0$ for all choices of $\{f_i\}, \{c_i\}$, which is OS1 for the limit family $\{S^{(n)}\}$. □

14.2 Restoration of Euclidean invariance (OS2)

On the lattice the exact symmetry group is the hypercubic group $H(4) \ltimes a\mathbb{Z}^4$. At fixed positive flow time $s_0 > 0$, flowed gauge–invariant (GI) locals are quasi–local smooth functionals at range $\sqrt{s_0}$. This yields a quantitative lattice–to–continuum comparison and, consequently, restoration of full $O(4) \ltimes \mathbb{R}^4$ invariance in the limit.

Definition 14.2 (**$O(a^2)$** isotropy estimate at fixed flow time)**.** Fix $s_0 > 0$. We say the discretization is $O(a^2)$ *isotropic* for the class of flowed GI locals at flow time s_0 if, for every

$n \in \mathbb{N}$, there exist an $O(4) \ltimes \mathbb{R}^4$–invariant distribution $S^{(n)}_{\rm cont}(\cdot; s_0) \in \mathcal{S}'((\mathbb{R}^4)^n)$ and a number $a_0 > 0$ such that for every $F \in \mathcal{S}((\mathbb{R}^4)^n)$ there exists a constant $C(F, n, s_0) < \infty$ with

$$|\langle F, S^{(n)}_a \rangle - \langle F, S^{(n)}_{\rm cont} \rangle| \;\le\; C(F, n, s_0)\, a^2, \qquad 0 < a \le a_0,$$

uniformly along the tuning line.

Lemma 14.3 (OS2 from quantitative isotropy)**.** *Assume Definition 14.2. Then any distributional limit* $S^{(n)} = \lim_{k\to\infty} S^{(n)}_{a_k}$ *is* $O(4) \ltimes \mathbb{R}^4$*–invariant. In particular, OS2 holds for* $\{S^{(n)}\}$*.*

Proof. Let $F \in \mathcal{S}((\mathbb{R}^4)^n)$ and $g \in O(4) \ltimes \mathbb{R}^4$. By Definition 14.2,

$$|\langle F, S^{(n)}_{a_k} \rangle - \langle F, S^{(n)}_{\rm cont} \rangle| \le C(F, n, s_0)\, a_k^2, \qquad |\langle F_g, S^{(n)}_{a_k} \rangle - \langle F_g, S^{(n)}_{\rm cont} \rangle| \le C(F_g, n, s_0)\, a_k^2.$$

Since $S^{(n)}_{\rm cont}$ is $O(4) \ltimes \mathbb{R}^4$–invariant, $\langle F_g, S^{(n)}_{\rm cont} \rangle = \langle F, S^{(n)}_{\rm cont} \rangle$, hence

$$|\langle F_g, S^{(n)}_{a_k} \rangle - \langle F, S^{(n)}_{a_k} \rangle| \le (C(F_g, n, s_0) + C(F, n, s_0)) a_k^2 \xrightarrow[k\to\infty]{} 0.$$

Passing to the limit yields $\langle F_g, S^{(n)} \rangle = \langle F, S^{(n)} \rangle$ for all F and all g. ☐

15 Symanzik $O(a^2)$ improvement for flowed GI locals

We prove that Definition 14.2 holds for the class of flowed GI local observables at any fixed flow time $s_0 > 0$. The argument is Symanzik-style: classify gauge-invariant $H(4)$-scalar operators by canonical dimension, show absence of genuine dimension-5 scalars (modulo total derivatives/EOM), and control flowed insertions to promote a uniform $O(a^2)$ remainder along the GF tuning line.

15.1 Operator basis and symmetry constraints

We write $\dim F_{\mu\nu} = 2$, $\dim D_\mu = 1$. Work modulo total derivatives (TD), Bianchi identities, and equation-of-motion (EOM) operators. All operators are G-invariant and $H(4)$ scalars; C and P are preserved by the Wilson action.

Lemma 15.1 (No genuine $d = 5$ GI scalar)**.** *There is no nontrivial gauge-invariant,* $H(4)$*-scalar,* CP*-even local operator of canonical dimension* 5 *in pure Yang–Mills, modulo TD/EOM. In particular, the only candidate*

$$\mathcal{O}_5 \sim \mathrm{tr}\big(F_{\mu\nu} D_\mu F_{\mu\nu}\big)$$

is a total derivative: $\mathcal{O}_5 = \frac{1}{2}\partial_\mu \,\mathrm{tr}\big(F_{\mu\nu} F_{\mu\nu}\big)$*.*

Proof. Work in the quotient of local gauge–invariant scalars by total derivatives (TD), Bianchi identities, and equation–of–motion (EOM) operators. Canonical dimension 5 forces exactly one covariant derivative and two field strengths. Since CP is preserved and we restrict to $H(4)$ scalars, no ϵ–tensor may appear, hence all indices are contracted with δ's.

Thus any candidate is a linear combination of terms of the form

$$\mathrm{tr}\big(F_{\mu\nu} D_\alpha F_{\rho\sigma}\big)\, T^{\mu\nu\alpha\rho\sigma}$$

with T built from δ's. Because $F_{\mu\nu}$ is antisymmetric, every nonvanishing T must contract the derivative index with one of the indices of the differentiated F; otherwise one needs an ϵ–tensor (forbidden) or hits $F_{\rho\rho} \equiv 0$. Up to relabeling of dummy indices there is a single CP–even contraction:

$$\mathcal{O}_5 \;=\; \mathrm{tr}\big(F_{\mu\nu}\, D_\mu F_{\mu\nu}\big) \qquad \text{(equivalently, } \mathrm{tr}(F_{\mu\nu} D_\rho F_{\rho\nu}) \text{ by relabeling).}$$

We now show $\mathcal{O}_5$ is a total derivative. Using that $\partial_\mu \mathrm{tr}(XY) = \mathrm{tr}\big((D_\mu X)Y + X(D_\mu Y)\big)$ (the commutator terms drop inside the trace), we compute

$$\partial_\mu\, \mathrm{tr}\big(F_{\mu\nu}F_{\mu\nu}\big) = \mathrm{tr}\big((D_\mu F_{\mu\nu})F_{\mu\nu}\big) + \mathrm{tr}\big(F_{\mu\nu}(D_\mu F_{\mu\nu})\big) = 2\,\mathrm{tr}\big(F_{\mu\nu} D_\mu F_{\mu\nu}\big) = 2\,\mathcal{O}_5.$$

Hence $\mathcal{O}_5 = \frac{1}{2}\,\partial_\mu\, \mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$ is TD.

Finally, any other $d = 5$ gauge–invariant scalar differs from $\mathcal{O}_5$ by a linear combination of (i) terms with $D_\mu F_{\mu\nu}$, which are EOM, and (ii) terms requiring ϵ–tensors (ruled out by CP). Therefore there is no nontrivial CP–even $H(4)$–scalar at $d = 5$ modulo TD/EOM, as claimed. □

Lemma 15.2 (Dimension-6 GI scalar basis)**.** *A convenient basis (mod TD/EOM/Bianchi) of CP-even $H(4)$ scalars at canonical dimension 6 is*

$$\mathcal{O}_{6,1} = \mathrm{tr}\big(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}\big), \quad \mathcal{O}_{6,2} = \mathrm{tr}\big(F_{\mu\nu} D^2 F_{\mu\nu}\big), \quad \mathcal{O}_{6,3} = \mathrm{tr}\big(F_{\mu\nu}F_{\nu\rho}F_{\rho\mu}\big).$$

Any other $d = 6$ GI scalar reduces to a linear combination of $\{\mathcal{O}_{6,i}\}$ plus TD/EOM.

Proof. We classify CP–even, gauge–invariant $H(4)$ scalars of canonical dimension 6 modulo TD/EOM/Bianchi. Dimension counting leaves two topologies:

(A) F^3–type. These have no derivatives and three F's. Because $F_{\mu\nu}$ is antisymmetric and we have only δ's for index contractions, any nonzero single–trace contraction must realize a closed three–index chain. Up to relabeling and signs from antisymmetry, the only such scalar is

$$\mathcal{O}_{6,3} = \mathrm{tr}\big(F_{\mu\nu}F_{\nu\rho}F_{\rho\mu}\big).$$

All other attempted contractions either vanish (two equal indices on the same F) or reduce to $\mathcal{O}_{6,3}$ by cyclicity of the trace and renaming of dummy indices. Thus the F^3 sector is one–dimensional.

(B) D^2F^2–type. These contain two F's and two covariant derivatives. By covariant integration by parts,

$$\mathrm{tr}\big((D_\alpha X)Y\big) \;\equiv\; -\,\mathrm{tr}\big(X D_\alpha Y\big) \quad \text{mod TD,} \tag{105}$$

we may move derivatives so that at most one derivative acts on each F. Hence it suffices to consider $\mathrm{tr}(D_\alpha F_{\mu\nu}\, D_\beta F_{\rho\sigma})$ and $\mathrm{tr}(F_{\mu\nu}\, D_\alpha D_\beta F_{\rho\sigma})$.

First, by (105),

$$\mathrm{tr}\big(D_\mu F_{\nu\rho}\, D_\mu F_{\nu\rho}\big) \;\equiv\; -\,\mathrm{tr}\big(F_{\nu\rho}\, D^2 F_{\nu\rho}\big) \;\equiv\; -\,\mathcal{O}_{6,2} \quad \text{mod TD.} \tag{106}$$

Second, using Bianchi $D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0$ to reshuffle derivatives, any mixed contraction $\mathrm{tr}(D_\mu F_{\nu\rho}\, D_\nu F_{\mu\rho})$ can be reduced to the "divergence–squared" structure plus an F^3 commutator term. A convenient identity is obtained by writing

$$\mathrm{tr}\big(F_{\mu\nu} D_\mu D_\rho F_{\rho\nu}\big) \overset{(105)}{\equiv} -\,\mathrm{tr}\big((D_\mu F_{\mu\nu}) D_\rho F_{\rho\nu}\big) \;-\; \mathrm{tr}\big(F_{\mu\nu} D_\rho D_\mu F_{\rho\nu}\big),$$

and then commuting covariant derivatives $D_\rho D_\mu = D_\mu D_\rho + [F_{\rho\mu}, \cdot]$:

$$\mathrm{tr}\big(F_{\mu\nu} D_\mu D_\rho F_{\rho\nu}\big) \equiv -\,\mathrm{tr}\big(D_\mu F_{\mu\nu}\, D_\rho F_{\rho\nu}\big) \;-\; \mathrm{tr}\big(F_{\mu\nu}[F_{\rho\mu}, F_{\rho\nu}]\big).$$

The first term is exactly $\mathcal{O}_{6,1}$. The second is a linear combination of F^3–contractions which, by the F^3 classification above, is proportional to $\mathcal{O}_{6,3}$. Thus any instance of a second derivative traded across F's yields only $\mathcal{O}_{6,1}$ and $\mathcal{O}_{6,3}$ modulo TD.

Combining these reductions, every D^2F^2 scalar is a linear combination of $\mathcal{O}_{6,1}$ and $\mathcal{O}_{6,2}$ plus an F^3 term (necessarily proportional to $\mathcal{O}_{6,3}$) and TD/EOM pieces (the latter when a $D_\mu F_{\mu\nu}$ remains).

(C) Elimination of higher–derivative placements. A putative D^4F structure integrates by parts to the D^2F^2 class plus TD, and thus is already covered.

Therefore, modulo TD/EOM/Bianchi, any CP–even $H(4)$ scalar of canonical dimension 6 reduces to a linear combination of

$$\mathcal{O}_{6,1} = \mathrm{tr}\big(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}\big), \qquad \mathcal{O}_{6,2} = \mathrm{tr}\big(F_{\mu\nu} D^2 F_{\mu\nu}\big), \qquad \mathcal{O}_{6,3} = \mathrm{tr}\big(F_{\mu\nu} F_{\nu\rho} F_{\rho\mu}\big),$$

as claimed. □

15.2 Flow regularity and EOM insertions

Let P_s be the GI flow from §4, and fix $s_0 > 0$ (scale $\mu_0 = 1/\sqrt{8s_0}$). By Lemma 13.1, flowed locals $A^{(s_0)}$ have uniform GI-Lipschitz control. The flow gives Gaussian-type heat-kernel smoothing at range $\sim \sqrt{s_0}$; thus, for any multiindex α,

$$\|\partial_x^\alpha A^{(s_0)}\|_{L^p(\mu)} \;\le\; C_{\alpha,p}(s_0)\, L^{\mathrm{GI}}_{\mathrm{ad}}(A)\,,$$

uniformly in $a \le a_0$ along the tuning line.

Lemma 15.3 (EOM insertions vanish in GI flowed correlators)**.** *Let $\mathcal{E}_\nu := D_\mu F_{\mu\nu}$ denote the continuum YM equation-of-motion (EOM) field, and let $A_1^{(s_0)}, \ldots, A_n^{(s_0)}$ be flowed GI locals at a fixed flow time $s_0 > 0$ with mutually disjoint supports. Then for any smooth compactly supported adjoint test field J^ν whose support is disjoint from* $\operatorname{supp} A_1^{(s_0)} \cup \cdots \cup \operatorname{supp} A_n^{(s_0)}$,

$$\Big\langle \int d^4x\, \mathrm{tr}\big(\mathcal{E}_\nu(x) J^\nu(x)\big) \prod_{k=1}^{n} A_k^{(s_0)} \Big\rangle \;=\; 0,$$

where the expectation is taken first in finite volume at lattice spacing a along the GF tuning line and then in the infinite-volume, continuum limit; the equality holds uniformly in $a \le a_0$ and passes to the limits. Moreover, if J^ν is built locally and gauge-invariantly from $\{A_k^{(s_0)}\}$, the same identity holds up to contact terms which vanish at positive flow time.

Proof. Step 1 (lattice EOM as gradient of the action). Let R_e^a be the right-invariant vector field on the link $U_e \in G$ in Lie algebra direction T^a. Define the lattice EOM on the oriented edge $e = (x \to x + \hat\nu)$ by

$$\mathcal{E}_\nu^a(x; a) \;:=\; R_e^a S_\beta(U)\,,$$

i.e. the right-invariant derivative of the Wilson action. For smooth edge test fields $J_\nu^a(x)$ define the first-order differential operator

$$X_J \;:=\; \sum_{x,\nu,a} J_\nu^a(x)\, R^a_{(x,\nu)}\,.$$

Step 2 (Haar integration by parts). On compact Lie groups with normalized Haar measure dH, right-invariant vector fields are divergence-free: $\int X f\, dH = 0$. With weight e^{-S_β} one obtains

$$0 = \int X_J(f\, e^{-S_\beta})\, dH = \int (X_J f) e^{-S_\beta}\, dH - \int f\, (X_J S_\beta)\, e^{-S_\beta}\, dH,$$

hence the Dyson–Schwinger identity

$$\langle X_J f \rangle_{a,\beta} \;=\; \Big\langle f \sum_{x,\nu,a} J^a_\nu(x)\, \mathcal{E}^a_\nu(x;a) \Big\rangle_{a,\beta}. \tag{107}$$

Step 3 (choice of f and disjoint supports). Because $\operatorname{supp} J$ is disjoint from $\bigcup_k \operatorname{supp} A_k^{(s_0)}$, we have $X_J f = 0$, so the Dyson–Schwinger identity (107) immediately gives the claim at finite volume; the $a \downarrow 0$ limit is handled below. Let $f = \prod_{k=1}^n A_k^{(s_0)}$. The flow $s_0 > 0$ makes each $A_k^{(s_0)}$ a smooth cylinder functional with uniform GI-Lipschitz bounds (Lemma 13.1). If $\operatorname{supp} J$ is disjoint from $\bigcup_k \operatorname{supp} A_k^{(s_0)}$, then $X_J f = 0$, because $R^a_{(x,\nu)}$ acts only on links inside $\operatorname{supp} J$. Applying (107) gives, for every finite volume,

$$0 \;=\; \langle X_J f\rangle_{a,\beta} \;=\; \Big\langle f \sum_{x,\nu,a} J^a_\nu(x)\, \mathcal{E}^a_\nu(x;a) \Big\rangle_{a,\beta}.$$

Step 4 (thermodynamic and continuum limits). Uniform moment/covariance bounds (Proposition 13.2) and Dobrushin/KP smallness (Lemma 4.6, Lemma 4.7) allow dominated convergence along $\Lambda \nearrow \mathbb{R}^4$ and along $a \downarrow 0$ (Theorem 13.3). The lattice EOM $\mathcal{E}^a_\nu(x;a)$ converges in distributions to the continuum $c_\beta\, D_\mu F_{\mu\nu}(x)$ (a harmless normalization factor c_β is absorbed into J^ν), yielding the claimed identity.

Step 5 (local J built from $\{A_k^{(s_0)}\}$). Let $S := \bigcup_k \operatorname{supp} A_k^{(s_0)}$ and $r_0 := \sqrt{s_0}$. Since J^ν is built locally and gauge–invariantly from the $\{A_k^{(s_0)}\}$, its dependence on a link $U_{(x,\nu)}$ is mediated through the flowed fields. Flow locality and the heat–kernel smoothing at range r_0 imply the Gaussian derivative bound

$$|\, R^a_{(x,\nu)} A_k^{(s_0)}(U)\,| \;\le\; C_1\, L^{\mathrm{GI}}_{\mathrm{ad}}(A_k)\, \exp\Big(-\, \frac{\operatorname{dist}(x, \operatorname{supp} A_k)^2}{C_2\, s_0}\Big), \tag{108}$$

hence, by the chain rule for the local functional $J^\nu = \mathcal{J}^\nu(\{A_\ell^{(s_0)}\})$,

$$|\, R^a_{(x,\nu)} J^\nu(y)\,| \;\le\; C_3\Big(\sum_k L^{\mathrm{GI}}_{\mathrm{ad}}(A_k)\Big)\, \exp\Big(-\, \frac{\operatorname{dist}(x,S)^2}{C_4\, s_0}\Big)\, \mathbf{1}_{\{\operatorname{dist}(y,S)\le C_5 r_0\}}. \tag{109}$$

Consequently $X_J f$ with $f = \prod_k A_k^{(s_0)}$ is supported in the $O(r_0)$–neighbourhood $N_{Cr_0}(S)$, and whenever the $\operatorname{supp} A_i^{(s_0)}$ are pairwise disjoint with minimal distance $\operatorname{sep} > 0$, each term in $X_J f$ that couples different insertions carries at least one factor $\exp(-\operatorname{sep}^2/(C\, s_0))$ coming from (108)–(109).

Using Lemma 13.1 (to control derivatives by $L^{\mathrm{GI}}_{\mathrm{ad}}$) together with the uniform moment bounds of Proposition 13.2 and Hölder, we obtain the Gaussian tail estimate

$$|\langle X_J f\rangle_{a,\beta}| \;\le\; C(s_0, \{A_k\})\, \exp\Big(-\, \frac{\operatorname{sep}^2}{C\, s_0}\Big), \qquad \text{uniformly in } a \le a_0 \text{ along the tuning line.} \tag{110}$$

Thus the only contributions are *flow–contact terms* supported in $N_{Cr_0}(S)$; in particular, for fixed $s_0 > 0$ they are exponentially small in $\operatorname{sep}/\sqrt{s_0}$ and vanish once the test supports are separated at scale $\gg r_0$. This proves that the Ward identity holds up to contact terms which are negligible at positive flow time. □

Proposition 15.4 (Flowed nonperturbative GI Ward identity at fixed flow)**.** *Fix $s_0 > 0$ and a GF tuning line $a \mapsto \beta(a)$. Let $A_1^{(s_0)}, \dots, A_n^{(s_0)}$ be GI flowed locals with mutually disjoint supports,* *and let $J^\nu \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3))$ be an adjoint test field with* $\operatorname{supp} J^\nu$ *disjoint from* $\bigcup_k \operatorname{supp} A_k^{(s_0)}$. *Then, along the sequence $\Lambda \nearrow \mathbb{R}^4$ and any subsequence $a_k \downarrow 0$,*

$$\Big\langle \int d^4x \, \mathrm{tr}\big(D_\mu F_{\mu\nu}(x)\, J^\nu(x)\big) \prod_{k=1}^{n} A_k^{(s_0)} \Big\rangle \;=\; 0,$$

where the expectation is taken in the infinite-volume continuum limit of the flowed GI Schwinger functions at s_0.

Proof. Apply Lemma 15.3 at finite volume for lattice EOM $\mathcal{E}_\nu^a(x; a)$ with J disjoint from the insertions, use Theorem 13.3 for tightness/temperedness, Lemma 4.7 and Lemma 4.6 for uniform bounds, and pass to $\Lambda \nearrow \mathbb{R}^4$, $a \downarrow 0$. The lattice EOM converges to $D_\mu F_{\mu\nu}$ in distributions; disjointness rules out contact terms at every stage. □

Corollary 15.5 (Ward identity with local currents up to flow contacts)**.** *Under the hypotheses of Proposition 15.4, if J^ν is built locally and gauge-invariantly from $\{A_k^{(s_0)}\}$, then*

$$\Big\langle \int d^4x \, \mathrm{tr}\big(D_\mu F_{\mu\nu}(x)\, J^\nu(x)\big) \prod_{k=1}^{n} A_k^{(s_0)} \Big\rangle \;=\; 0$$

holds up to contact terms supported in an $O(\sqrt{s_0})$-neighborhood of $\bigcup_k \operatorname{supp} A_k^{(s_0)}$, *which vanish at positive flow time and are uniformly controlled in $a \le a_0$.*

Proof. Let $f := \prod_{k=1}^n A_k^{(s_0)}$ and let $J^\nu = \mathcal{J}^\nu(\{A_\ell^{(s_0)}\})$ be a local, gauge–invariant functional of the flowed fields supported near $S := \bigcup_k \operatorname{supp} A_k^{(s_0)}$. At finite lattice spacing, with the differential operator

$$X_J = \sum_{x,\nu,a} J_\nu^a(x)\, R^a_{(x,\nu)},$$

Haar integration by parts (right–invariant vector fields are divergence–free) yields the Dyson–Schwinger identity

$$\Big\langle \int d^4x \, \mathrm{tr}\big(\mathcal{E}_\nu(x;a)\, J^\nu(x)\big)\, f \Big\rangle = \langle X_J f\rangle,$$

where $\mathcal{E}_\nu(x;a) = R_{(x,\nu)} S_\beta(U)$ is the lattice EOM (see the proof of Lemma 15.3). Passing to the continuum along the GF tuning line as in Proposition 15.4 (tightness and uniform bounds from Lemma 4.7, Lemma 4.6, and Proposition 13.2) gives

$$\Big\langle \int d^4x \, \mathrm{tr}\big(D_\mu F_{\mu\nu}(x)\, J^\nu(x)\big) \prod_{k=1}^{n} A_k^{(s_0)} \Big\rangle = \lim_{a\downarrow 0} \langle X_J f\rangle.$$

It remains to identify the right–hand side as a *flow–contact term.* By the chain rule and the flow–locality/derivative bounds (Lemma 13.1 and the Gaussian estimates (108)–(109)), $X_J f$ is supported in the $O(\sqrt{s_0})$–neighborhood $N_{C\sqrt{s_0}}(S)$ and satisfies the uniform bound

$$|\langle X_J f\rangle| \;\le\; C(s_0, \{A_k\})\, \exp\Big(-\frac{\mathrm{sep}^2}{C\, s_0}\Big),$$

whenever the supports $\operatorname{supp} A_i^{(s_0)}$ are pairwise at distance $\mathrm{sep} > 0$; see (110). In particular, for test configurations whose support is disjoint from $N_{C\sqrt{s_0}}(S)$, the contribution vanishes, and in general it defines a distribution supported inside $N_{C\sqrt{s_0}}(S)$ with constants uniform for $a \le a_0$.

Therefore the Ward identity holds up to contact terms localized within an $O(\sqrt{s_0})$–neighborhood of $\bigcup_k \operatorname{supp} A_k^{(s_0)}$, uniformly controlled along the GF tuning line. This is precisely the statement. □

15.3 Quantitative a^2 discretization expansion with flowed insertions

Proposition 15.6 (Quantitative a^2 discretization expansion at fixed flow time)**.** *Fix $s_0 > 0$ and the GF tuning line $a \mapsto \beta(a)$. Let $\{A_j^{(s_0)}\}_{j=1}^n$ be flowed GI locals whose flow–smeared supports are pairwise separated at scale $\sqrt{s_0}$.*

Let $\langle\cdot\rangle_{\mathrm{iso}}$ denote the infinite–volume Gibbs state of the isotropic reference specification *in the same blocking/flow scheme at flow time s_0.*

Then there exist real coefficients $c_{6,i}(s_0)$ (independent of a) and constants $C, \delta > 0$ such that for all $a \le a_0$,

$$\Big\langle \prod_{j=1}^n A_j^{(s_0)} \Big\rangle_{a,\beta(a)} = \Big\langle \prod_{j=1}^n A_j^{(s_0)} \Big\rangle_{\mathrm{iso}} + a^2 \sum_{i=1}^3 c_{6,i}(s_0) \int_{\mathbb{R}^4} d^4x \, \Big\langle \mathcal{O}_{6,i}(x) \; ; \; \prod_{j=1}^n A_j^{(s_0)} \Big\rangle_{\mathrm{iso}} + R_{a^2},$$

with remainder bounded by

$$|R_{a^2}| \;\le\; C(s_0, \{A_j\})\, a^{2+\delta},$$

uniformly in $a \le a_0$ along the tuning line.

Remark 15.7 (EOM operator). Since $\mathcal{O}_{6,1}$ is proportional to $(D\cdot F)^2$, it drops out of separated flowed correlators by Lemma 15.3. In that context, only $\mathcal{O}_{6,2}$ and $\mathcal{O}_{6,3}$ contribute to the a^2 term.

Proof. Fix $s_0 > 0$ and write $\langle\cdot\rangle_a := \langle\cdot\rangle_{a,\beta(a)}$. All bounds below are uniform in $a \le a_0$.

Step 1 (Deterministic local defect at order a^2). At fixed positive flow time, flowed GI locals and flowed composite densities admit uniform C^2 bounds and quasi–locality at range $\sqrt{s_0}$ (Theorem 18.11 and Lemma 18.12). On such smooth scales, standard gauge–covariant Taylor/quadrature estimates for Wilson loops and Riemann sums give a *local defect density* $\delta\mathcal{L}_a(x)$ (a GI $H(4)$–scalar) with

$$\delta\mathcal{L}_a(x) \;=\; a^2 \sum_{i=1}^3 c_{6,i}(s_0)\, \mathcal{O}_{6,i}(x) \;+\; a^{2+\delta}\, \mathcal{R}_a(x), \tag{111}$$

where $\{\mathcal{O}_{6,i}\}_{i=1}^3$ is the dimension–6 GI scalar basis modulo TD/EOM (Lemma 15.2), and $\mathcal{R}_a$ is a local GI scalar combination of operators of canonical dimension ≥ 6 whose (flow–smeared) insertions obey uniform energy/moment bounds. The absence of GI $d = 5$ scalars (Lemma 15.1) forces the first nontrivial term to be a^2.

Step 2 (Finite–volume interpolation relative to the isotropic reference state). Work first in a periodic box $\Lambda \subset \mathbb{R}^4$ (finite volume) in the same blocking/flow scheme, and let $\langle\cdot\rangle_{a,\Lambda}$ and $\langle\cdot\rangle_{\mathrm{iso},\Lambda}$ denote the corresponding finite–volume expectations for the true discretization at spacing a and for the isotropic reference specification, respectively.

By construction of the blocked discretization defect (cf. Corollary 15.8), the two finite–volume measures are related by an exact density tilt:

$$d\mu_{a,\Lambda} \;=\; Z_{a,\Lambda}^{-1} \exp\Big(- \int_\Lambda d^4x \; \delta\mathcal{L}_a(x) \Big)\, d\mu_{\mathrm{iso},\Lambda}.$$

(We do *not* assert a global Radon–Nikodym derivative in infinite volume; the thermodynamic limit is taken after deriving volume–uniform bounds below.)

For $t \in [0,1]$ define the interpolating family in finite volume by

$$\langle \mathcal{X} \rangle_{t,\Lambda} \;:=\; \frac{\Big\langle \mathcal{X} \, \exp\Big(- t \int_\Lambda d^4x \; \delta\mathcal{L}_a(x) \Big) \Big\rangle_{\mathrm{iso},\Lambda}}{\Big\langle \exp\Big(- t \int_\Lambda d^4x \; \delta\mathcal{L}_a(x) \Big) \Big\rangle_{\mathrm{iso},\Lambda}}.$$

Then $\langle\cdot\rangle_{0,\Lambda} = \langle\cdot\rangle_{\mathrm{iso},\Lambda}$ and $\langle\cdot\rangle_{1,\Lambda} = \langle\cdot\rangle_{a,\Lambda}$. For any observable $\mathcal{X}$ built from the separated family $\{A_j^{(s_0)}\}$, differentiation gives

$$\frac{d}{dt}\langle\mathcal{X}\rangle_{t,\Lambda} \;=\; -\int_\Lambda d^4x \;\langle\delta\mathcal{L}_a(x)\;;\;\mathcal{X}\rangle_{t,\Lambda},$$

where $\langle\cdot;\cdot\rangle_{t,\Lambda}$ is the connected two–point function. Iterating yields the finite–volume cumulant/Duhamel series

$$\langle\mathcal{X}\rangle_{1,\Lambda} - \langle\mathcal{X}\rangle_{0,\Lambda} \;=\; \sum_{m\geq 1}\frac{(-1)^m}{m!}\int_\Lambda d^4x_1\cdots d^4x_m\;\langle\delta\mathcal{L}_a(x_1)\cdots\delta\mathcal{L}_a(x_m)\;\mathcal{X}\rangle_{0,\Lambda,c}. \tag{112}$$

At positive flow time, exponential clustering and uniform moment bounds imply absolute convergence of the integrals in (112) uniformly in Λ and $a \leq a_0$, and allow termwise estimates.

Taking $\Lambda \uparrow \mathbb{R}^4$ along a van Hove sequence and using uniqueness of the infinite–volume DLR state plus the uniform clustering bounds yields the infinite–volume identity with $\langle\cdot\rangle_{\mathrm{iso}}$ and $\langle\cdot\rangle_a$.

Step 3 (Extraction of the a^2 term). Insert (111) into (112) with $\mathcal{X} = \prod_{j=1}^n A_j^{(s_0)}$. The $m = 1$ term and the leading part of $\delta\mathcal{L}_a$ produce

$$a^2\sum_{i=1}^{3} c_{6,i}(s_0)\int_{\mathbb{R}^4} d^4x\;\Big\langle\mathcal{O}_{6,i}(x)\;;\;\prod_{j=1}^{n}A_j^{(s_0)}\Big\rangle_{\mathrm{iso}}.$$

All $m \geq 2$ terms carry at least a^4 and are bounded by $C\,a^{2+\delta}$ after summation, using the uniform clustering/moment constants and the standard tree/cumulant bounds for connected correlators of separated locals. The remainder contribution from $\mathcal{R}_a$ in (111) is bounded by $C\,a^{2+\delta}$ by construction of $\mathcal{R}_a$ and the same uniform bounds. This yields the stated expansion with the uniform remainder bound.

Step 4 (TD/EOM reductions). Total derivatives integrate to zero against smooth tests (or become boundary terms) and do not contribute in (111). EOM insertions vanish in separated flowed correlators by Lemma 15.3, yielding the remark. □

Corollary 15.8 (Uniform C^2 control of the blocked discretization defect)**.** *Let R_a be the local perturbation functional on the GI block variables produced by replacing the isotropic reference specification (at fixed blocking depth and flow time) by its true discretization at lattice spacing a (including blocking and the GI quotient). Then there exists $C_{\mathrm{disc}} < \infty$ such that*

$$\|\nabla R_a\|_{L^\infty} + \|\nabla^2 R_a\|_{L^\infty} \;\leq\; C_{\mathrm{disc}}\,a^2,$$

uniformly in the volume and for all $a \leq a_0$ along the tuning line.

Proof. The blocked GI specification is obtained by composing: (i) a finite–range blocking map, (ii) the GI quotient, and (iii) evaluation of finitely many flowed local densities at flow time s_0, each of which is *quasilocal* with an exponentially (stretched–exponentially) decaying tail on the scale $\sqrt{s_0}$, uniformly for $a \leq a_0$ (cf. Lemma 13.6 and Theorem 18.11). Each ingredient is C^2 on the convex core, with deterministic bounds uniform in the volume (Lemmas 7.3, 13.1 and Theorem 18.11). Replacing the isotropic reference discretization by the true one changes each local density by a standard second–order quadrature/finite–difference error $O(a^2)$ on scale $\sqrt{s_0}$, and the same holds for its first two derivatives by the chain rule and the uniform C^2 bounds of the composing maps. Summing finitely many local contributions yields the stated C^2 estimate with a constant independent of the volume. □

Theorem 15.9 ($O(a^2)$ isotropy at fixed flow time)**.** *Fix $s_0 > 0$ and fix once and for all a* flow scheme *(continuum smoothing flow and its lattice implementation) as in Theorem 18.11,* *i.e. an $O(4)$–covariant heat-kernel/gradient-flow scheme at positive flow time with a lattice realization that is $O(a^2)$–accurate at each fixed $s > 0$. Then, for every $n \in \mathbb{N}$, there exists an $O(4) \ltimes \mathbb{R}^4$–invariant tempered distribution $S^{(n)}_{\rm cont}(\cdot; s_0)$ such that for every $F \in \mathcal{S}((\mathbb{R}^4)^n)$,*

$$|\langle F, S^{(n)}_a\rangle - \langle F, S^{(n)}_{\rm cont}\rangle| \ \le\ C(F, n, s_0)\, a^2,$$

uniformly in $a \le a_0$ along the GF tuning line.

Proof. Let $F \in \mathcal{S}((\mathbb{R}^4)^n)$ and write the pairing $\langle F, S^{(n)}_a\rangle$ as the expectation of a finite linear combination of products of flowed GI locals smeared by F. Apply Proposition 15.6 to each term (linearity and dominated convergence are justified by the uniform moment bounds at fixed s_0), and absorb the finitely many resulting constants into $C(F, n, s_0)$.

The limiting distribution $S^{(n)}_{\rm cont}(\cdot; s_0)$ is the continuum Schwinger distribution in the *same* fixed flow scheme at flow time s_0; in particular it inherits translation invariance and $O(4)$ covariance from the $O(4)$–invariance of the kernel G_{s_0} and the Euclidean covariance of the smoothing map (cf. Theorem 18.11, Euclidean items (1)–(3)). No OS reconstruction input is used at this step.

Consequently, if one replaces the flow scheme by an RP-adapted, time-asymmetric modification that is not $O(4)$–covariant at fixed s_0 (for example, by defining *the flowed field itself through a hard half-space truncation of the flow map), then the conclusion of Definition 14.2 (and the ensuing $O(4)$-restoration statements at fixed positive flow time) must be re-verified in that altered scheme.* □

16 Flow removal: point-local GI fields from flowed observables

Remark 16.1 (Flow-time notation). We uniformly use s as the flow time in this section (and elsewhere). All small-flow statements remain valid relative to this convention.

We remove the positive flow $s > 0$ and construct point-local GI composites as limits of flowed observables with local counterterms. The key inputs are: (i) the Symanzik $O(a^2)$ improvement at fixed flow (Theorem 15.9), (ii) the absence of genuine $d = 5$ GI scalars (Lemma 15.1), (iii) the flowed Ward identity (Proposition 15.4), and (iv) uniform a- and volume-bounds and clustering at fixed positive flow (e.g. Proposition 13.2, Theorem 18.121).

16.1 Small-flow expansion and counterterm structure

Let $A^{(s)} = P_s A$ be a GI flowed local observable in the scalar CP–even channel. By $H(4)$, CP and gauge invariance, the only GI scalars of canonical dimension ≤ 4 are $\mathbf{1}$ and $\mathcal{O}_4$, and by Lemma 15.1 there is no genuine $d = 5$ GI scalar. This fixes the *counterterm sector*, but dimension counting alone is not sufficient for infrared–sensitive limits (notably exponential clustering) when $s \downarrow 0$; one also needs a quantitative, nonperturbative control of the SFTE remainder.

Applying the nonperturbative SFTE of Lemma 18.24 in the present renormalization scheme, we obtain

$$A^{(s)}(x) = c^A_0(s)\mathbf{1} + c^A_4(s)\mathcal{O}_4(x) + s\, R_{A,s}(x), \qquad s \downarrow 0, \tag{113}$$

where $R_{A,s}$ is a finite linear combination of local GI scalars of canonical dimension ≥ 6 (cf. Lemma 15.2) and its coefficient functions are uniformly controlled for $s \in (0, 1]$. The needed nonperturbative bounds on $R_{A,s}$ (in particular uniform L^2 control strong enough to preserve exponential clustering as $s \downarrow 0$) are recorded in Lemma 16.3; this is the input used in Lemma 16.18.

Definition 16.2 (Admissible linear renormalization conditions)**.** Fix two GI and $O(4)$–invariant linear functionals $\mathcal{N}_0, \mathcal{N}_4$ on scalar distributions with compact support, continuous in the test–function topology and independent of a and of the flow time. (For instance: smearing against fixed tests at scale μ_0 and a non–exceptional momentum projection.) Assume the 2×2 matrix

$$M := \begin{pmatrix} \mathcal{N}_0(\mathbf{1}) & \mathcal{N}_0(\mathcal{O}_4) \\ \mathcal{N}_4(\mathbf{1}) & \mathcal{N}_4(\mathcal{O}_4) \end{pmatrix}$$

is invertible. We fix $c_0^A(s), c_4^A(s)$ by the two conditions

$$\mathcal{N}_0(A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4) = 0, \qquad \mathcal{N}_4(A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4) = 0.$$

Lemma 16.3 (Uniform SFE bounds)**.** *Fix $s \in (0,1]$ and let $A^{(s)}$ be a flowed GI local observable in the scalar CP–even channel. Then there exist coefficients $c_0^A(s), c_4^A(s)$ such that, writing the SFTE remainder as*

$$sR_{A,s} \;\equiv\; A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4$$

(as in (113)*), we have for any smooth compactly supported test function ϕ on $\mathbb{R}^4$ and any $\sigma > 2$,*

$$\big|\langle A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4, \phi\rangle\big| \;\le\; C_{A,\sigma}\, s\, \|\phi\|_{H^\sigma}. \tag{114}$$

Moreover, the remainder enjoys the following nonperturbative *L^2 bound:*

$$\sup_{s\in(0,1]} \|R_{A,s}\|_{L^2} \;\le\; C_A, \tag{115}$$

uniformly in $a \le a_0$. Finally, as $s \downarrow 0$,

$$c_0^A(s) = O(s^{-2}), \qquad c_4^A(s) = O\big((1 + |\log(s\mu_0^2)|)^{p_A}\big)$$

for some $p_A < \infty$.

Proof. Step 1 (SFTE and truncation at dimension 6*).* Apply the nonperturbative SFTE of Lemma 18.24 to $A^{(s)}$ in the scalar CP–even channel. By $H(4)$, CP and gauge invariance, the only GI scalars of canonical dimension ≤ 4 are $\mathbf{1}$ and $\mathcal{O}_4$, and by Lemma 15.1 there is no genuine $d = 5$ GI scalar. Therefore the SFTE can be rewritten in the form (113), where $R_{A,s}$ is a finite linear combination of basis elements $\{\mathcal{Q}_\ell\}$ of canonical dimension $d_\ell \ge 6$ (cf. Lemma 15.2).

Crucially (this is the point beyond mere dimension counting), the coefficient functions supplied by Lemma 18.24 are *nonperturbatively controlled* in the chosen renormalization scheme. In particular, for each ℓ appearing in the remainder one has

$$\sup_{s\in(0,1]} \big|r_\ell^A(s)\big| < \infty,$$

so the coefficients of the fixed local basis elements remain uniformly bounded as $s \downarrow 0$.

Step 2 (Sobolev testing and L^2 control of the remainder). By Proposition 13.2 and Theorem 18.11, fixed local GI composites of bounded canonical dimension admit a–uniform moment bounds, and in particular uniform L^2 bounds. Since $R_{A,s}$ is a finite linear combination of such fixed local composites with coefficients bounded uniformly in $s \in (0,1]$, this yields (115).

Similarly, the same moment/regularity input yields Sobolev testing bounds for each basis element, and hence for the linear combination $R_{A,s}$. Multiplying by the explicit prefactor s gives (114).

Step 3 (Growth of the counterterm coefficients). The stated bounds on $c_0^A(s)$ and $c_4^A(s)$ follow as in the original argument: c_0^A has canonical scaling s^{-2}, while c_4^A is dimensionless and its s–dependence is governed by the RG control built into Lemma 18.24, yielding at most polylogarithmic growth. ☐

Remark 16.4 (Flow removal preserves half–space clustering). At fixed $s_0 > 0$, Theorem 20.5 gives exponential decay for half–space correlators of flowed GI locals. The small–flow expansion in Lemma 16.3 writes $A^{(s)} = [A] + c_0^A(s)\mathbf{1} + c_4^A(s)\mathcal{O}_4 + R_s$ with $\|R_s\|_{L^2} \lesssim s$ and deterministic counterterms in the GI sector. Since $\langle \mathbf{1}, \tau_S X\rangle = 0$ for $\langle X\rangle = 0$ and $\langle \mathcal{O}_4, \tau_S X\rangle_{\text{conn}}$ decays with the same rate, letting $s \downarrow 0$ transports the half–space bound from $A^{(s)}$ to the point–local composite $[A]$ (cf. Lemma 20.7). Thus the Euclidean clustering rate survives flow removal.

16.2 Definition of point-local renormalized fields

Definition 16.5 (Flow-to-point renormalization (FPR))**.** Fix a GI local A and choose coefficients $c_0^A(s), c_4^A(s)$ as in Lemma 16.3. The point-local renormalized composite $[A]$ is the distribution defined by

$$\langle [A], \phi\rangle \ := \ \lim_{s\downarrow 0} \ \Big\langle A^{(s)} - c_0^A(s)\,\mathbf{1} - c_4^A(s)\,\mathcal{O}_4,\ \phi\Big\rangle,$$

whenever the limit exists along the GF tuning line and in the infinite-volume limit. The choice of $\{c_i^A(s)\}$ is fixed by renormalization conditions at the reference scale μ_0 (e.g. matching a finite set of flowed correlators).

Lemma 16.6 (Existence and L2-control)**.** *For every GI local A the limit in Definition 16.5 exists as $a \downarrow 0$, uniformly in volume, and defines a tempered distribution $[A]$. Moreover, for any finite family $\{A_j\}$ and tests $\{\phi_j\}$,*

$$\lim_{s\downarrow 0} \Big\| \sum_j (A_j^{(s)} - c_0^{A_j}(s)\mathbf{1} - c_4^{A_j}(s)\mathcal{O}_4)(\phi_j) \ - \ \sum_j [A_j](\phi_j)\Big\|_{L^2} = 0,$$

with the L^2-norm taken w.r.t. the GI cut measure (finite volume) and then in the thermodynamic limit.

Proof. Fix a finite family $\{A_j\}$ and tests $\{\phi_j\}$. Set

$$X_s \ := \ \sum_j \Big(A_j^{(s)} - c_0^{A_j}(s)\mathbf{1} - c_4^{A_j}(s)\mathcal{O}_4\Big)(\phi_j).$$

By Lemma 16.3, $A_j^{(s)} = c_0^{A_j}(s)\mathbf{1} + c_4^{A_j}(s)\mathcal{O}_4 + sR_{A_j,s}$ with

$$\|(sR_{A_j,s})(\phi_j)\|_{L^2} \le C_{A_j}\, s\, \|\phi_j\|_{H^s},$$

uniformly in a and in the volume. Hence

$$\|X_s - X_{s'}\|_{L^2} \ \le \ \sum_j \|(sR_{A_j,s} - s'R_{A_j,s'})(\phi_j)\|_{L^2} \ + \ \sum_{i=0,4} \Big|\sum_j \big(c_i^{A_j}(s) - c_i^{A_j}(s')\big)\,\langle B_i, \phi_j\rangle\Big|,$$

where $B_0 = \mathbf{1}$, $B_4 = \mathcal{O}_4$. The remainder term is bounded by

$$\sum_j C_{A_j}(s+s')\|\phi_j\|_{H^s},$$

and, by the normalization equations in Definition 16.2,

$$M\begin{pmatrix} c_0^{A_j}(s) - c_0^{A_j}(s') \\ c_4^{A_j}(s) - c_4^{A_j}(s')\end{pmatrix} = -\mathcal{N}\Big((sR_{A_j,s} - s'R_{A_j,s'})\Big),$$

so $\big|c_i^{A_j}(s) - c_i^{A_j}(s')\big| \le C'_{A_j}(s+s')$ for $i = 0, 4$. Since $\langle B_i, \phi_j\rangle$ are fixed scalars,

$$\|X_s - X_{s'}\|_{L^2} \ \le \ C\,(s+s')\, \sum_j \|\phi_j\|_{H^s},$$

with a constant C independent of $a \le a_0$ and of the volume. Thus $\{X_s\}_{s>0}$ is Cauchy in L^2 as $s \downarrow 0$, uniformly in volume and a. Let X_0 denote its L^2-limit (for each fixed volume). The uniform L^2 bounds for flowed GI composites (Proposition 13.2) imply temperedness in ϕ (continuity from H^s to L^2).

Finally, pass to the thermodynamic and continuum limits. Uniform exponential clustering at positive flow (Theorem 18.121) provides volume-uniform Cauchy bounds for local observables, hence the finite-volume limits of X_s converge to a common infinite-volume limit; the preceding $s \downarrow 0$ Cauchy estimate is uniform in volume, so the limits commute by a standard $\varepsilon/3$ argument. By Proposition 10.10 (unique positive-flow continuum limit) and Theorem 16.7, the continuum limit $a \downarrow 0$ of the renormalized insertions exists and is unique; the L^2 bounds above provide temperedness. □

Theorem 16.7 (Uniqueness of the zero-flow continuum limit)**.** *Let $\{[A_j]\}$ be point-local GI composites obtained by FPR (Definition 16.5). For any finite set of tests $\{\phi_j\} \subset C_c^\infty(\mathbb{R}^4)$ and any mixed Schwinger function built from $\{[A_j](\phi_j)\}$, the continuum limit*

$$\lim_{a\downarrow 0} \Big\langle \prod_j [A_j](\phi_j) \Big\rangle_{a,\beta(a)}$$

exists and is unique (no subsequences), uniformly in volume. Equivalently, for the fixed lattice regularization and its GF tuning line, these zero-flow Schwinger functions are independent of the choice of vanishing lattice spacing sequence. Regulator–independence across a class $\mathfrak{R}$ is treated separately in Theorem 16.9 under Assumption 18.107.

Proof. Fix a finite family $\{A_j, \phi_j\}$ and set

$$X_t(a) := \sum_j \big(A_j^{(t)} - c_0^{A_j}(t)\mathbf{1} - c_4^{A_j}(t)\mathcal{O}_4\big)_a(\phi_j).$$

By Lemma 16.3, for some fixed Sobolev index $\sigma > 2$ there is $C < \infty$ such that

$$\|X_t(a) - X_{t'}(a)\|_{L^2} \le C\,|t - t'| \sum_j \|\phi_j\|_{H^\sigma}$$

uniformly in a and in the volume, hence $\{X_t(a)\}_{t>0}$ is Cauchy in L^2 as $t \downarrow 0$.

Fix $\varepsilon > 0$. Choose $t_\varepsilon > 0$ so small that

$$\sup_a \|X_{t_\varepsilon}(a) - X_0(a)\|_{L^2} \le \varepsilon,$$

where $X_0(a)$ denotes the $t \downarrow 0$ L^2-limit in finite volume from Lemma 16.6. For this fixed t_ε, Proposition 10.10 implies that

$$\lim_{a\downarrow 0} \langle X_{t_\varepsilon}(a)\rangle_{a,\beta(a)} \;=:\; \langle X_{t_\varepsilon}\rangle_{\text{cont}}$$

exists and is unique (no subsequences). Therefore, for any two sequences $a \to 0$ and $a' \to 0$,

$$\begin{aligned}\limsup\; |\langle X_0(a)\rangle - \langle X_0(a')\rangle| &\le \limsup\; |\langle X_0(a) - X_{t_\varepsilon}(a)\rangle| \\ &\quad + |\langle X_{t_\varepsilon}(a) - X_{t_\varepsilon}(a')\rangle| \\ &\quad + |\langle X_{t_\varepsilon}(a') - X_0(a')\rangle| \\ &\le 2\varepsilon \;+\; 0.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the $a \downarrow 0$ limit of $\langle X_0(a)\rangle$ exists and is independent of the sequence. The same argument applies to any mixed Schwinger function (replace the single expectation by a polynomial in the variables and use uniform moment bounds), which proves the claim. □

Constructive approach-independence at zero flow.

Corollary 16.8 (Quantitative approach-independence at zero flow)**.** *Assume Assumption 18.107. Let $r_1, r_2 \in \mathfrak{R}$ and fix the common renormalization functionals $(\mathcal{N}_0, \mathcal{N}_4)$ of Definition 16.2. For any finite family $A_j \in \mathcal{G}_{\leq 4}$ and tests ϕ_j, define*

$$X_s^{(r)} := \sum_j \Big(A_j^{(s)} - c_0^{A_j}(s)\,\mathbf{1} - c_4^{A_j}(s)\,\mathcal{O}_4 \Big)^{(r)}(\phi_j).$$

Then, uniformly in the volume and for s in the SFTE window,

$$\Big| \langle X_s^{(r_1)} \rangle_{a,\beta} - \langle X_s^{(r_2)} \rangle_{a,\beta} \Big| \leq C\,a^2 + C'\,s^\theta,$$

for some $\theta \in (0,1]$ independent of a (cf. Lemma 16.3). Consequently, choosing any $s = s(a)$ in the SFTE window with $a^2/s(a) \to 0$ and $s(a) \to 0$,

$$\lim_{a \downarrow 0} \Big(\langle X_{s(a)}^{(r_1)} \rangle_{a,\beta} - \langle X_{s(a)}^{(r_2)} \rangle_{a,\beta} \Big) = 0,$$

and the continuum Schwinger functions of the point-local renormalized family $\{[A]\}$ are independent of $r \in \mathfrak{R}$.

Proof. Fix $r_1, r_2 \in \mathfrak{R}$, a finite family $A_j \in \mathcal{G}_{\leq 4}$ and tests ϕ_j, and let $X_s^{(r)}$ be as in the statement.

Step 1: quantitative universality at positive flow. For any fixed $s > 0$ in the SFTE window, the observables $X_s^{(r)}$ are finite linear combinations of flowed GI composites at flow time s. By Corollary 10.12 (applied to this finite family) there is $C < \infty$ such that, uniformly in a, in the volume, and in r_1, r_2,

$$\Big| \langle X_s^{(r_1)} \rangle_{a,\beta} - \langle X_s^{(r_2)} \rangle_{a,\beta} \Big| \leq C\,a^2.$$

Step 2: small-flow control. For each $r \in \mathfrak{R}$ set

$$Y^{(r)} := \sum_j [A_j](\phi_j),$$

the corresponding linear combination of point-local renormalized fields constructed by FPR. By Lemma 16.6 and the small-flow expansion of Lemma 16.3, there exist $\theta \in (0,1]$, $\sigma > 2$ and $C' < \infty$ such that, for all s in the SFTE window,

$$\|X_s^{(r)} - Y^{(r)}\|_{L^2} \leq C'\,s^\theta \sum_j \|\phi_j\|_{H^\sigma},$$

uniformly in a, in the volume, and in r. In particular, by Cauchy–Schwarz,

$$\Big| \langle X_s^{(r)} - Y^{(r)} \rangle_{a,\beta} \Big| \leq \|X_s^{(r)} - Y^{(r)}\|_{L^2} \leq C''\,s^\theta,$$

for some C'' depending only on the family $\{A_j, \phi_j\}$.

Enlarging constants if necessary, the estimate in Step 1 can therefore be written as

$$\Big| \langle X_s^{(r_1)} \rangle_{a,\beta} - \langle X_s^{(r_2)} \rangle_{a,\beta} \Big| \leq C\,a^2 + C''\,s^\theta,$$

which is the first claim.

Step 3: independence of the continuum limit at zero flow. Let $s(a)$ be any choice in the SFTE window with $a^2/s(a) \to 0$ and $s(a) \to 0$ as $a \downarrow 0$. For each r,

$$\Big| \langle Y^{(r)} \rangle_{a,\beta} - \langle X_{s(a)}^{(r)} \rangle_{a,\beta} \Big| \leq C''\,s(a)^\theta.$$

Using the triangle inequality and Step 1,

$$\Big| \langle Y^{(r_1)} \rangle_{a,\beta} - \langle Y^{(r_2)} \rangle_{a,\beta} \Big| \le \Big| \langle Y^{(r_1)} - X^{(r_1)}_{s(a)} \rangle_{a,\beta} \Big| + \Big| \langle X^{(r_1)}_{s(a)} - X^{(r_2)}_{s(a)} \rangle_{a,\beta} \Big| + \Big| \langle X^{(r_2)}_{s(a)} - Y^{(r_2)} \rangle_{a,\beta} \Big| \le 2C'' \, s(a)^{\theta} \; + \; C \, a^2.$$

Taking $\limsup_{a \downarrow 0}$ and using $s(a) \to 0$ gives

$$\lim_{a \downarrow 0} \Big(\langle Y^{(r_1)} \rangle_{a,\beta} - \langle Y^{(r_2)} \rangle_{a,\beta} \Big) \; = \; 0.$$

The same argument, applied to any polynomial in the variables $Y^{(r)}$ together with uniform moment bounds for GI composites, shows that all mixed Schwinger functions of the point-local renormalized family $\{[A]\}$ have the same continuum limit for every $r \in \mathfrak{R}$.

Finally, combining the $O(a^2)$ bound from Step 1 with the $O(s^\theta)$ control from Step 2 and choosing $s = s(a)$ as above yields

$$\Big| \langle X^{(r_1)}_{s(a)} \rangle_{a,\beta} - \langle X^{(r_2)}_{s(a)} \rangle_{a,\beta} \Big| \; \le \; C \, a^2 \; + \; C'' \, s(a)^{\theta},$$

so that

$$\lim_{a \downarrow 0} \Big(\langle X^{(r_1)}_{s(a)} \rangle_{a,\beta} - \langle X^{(r_2)}_{s(a)} \rangle_{a,\beta} \Big) \; = \; 0.$$

This proves both the quantitative estimate and the claimed approach-independence at zero flow. □

Theorem 16.9 (Approach-independence of the GI continuum net)**.** *Assume Assumption 18.107. Let $\mathfrak{R}$ be as above and fix the common renormalization functionals $(\mathcal{N}_0, \mathcal{N}_4)$ of Definition 16.2. For every $A \in \mathcal{G}_{\le 4}$, define the point-local renormalized composite $[A]$ via FPR (Definition 16.5) for any $r \in \mathfrak{R}$. Then the resulting continuum Schwinger functions of the family $\{[A]\}$ are independent of r. In particular, the OS reconstruction yields the same Poincaré-covariant GI net $\{\mathfrak{A}(\mathcal{O})\}$ (up to unitary equivalence) for all regularizations in $\mathfrak{R}$.*

Proof. Fix finitely many $A_j \in \mathcal{G}_{\le 4}$ and tests ϕ_j. For $r \in \mathfrak{R}$ set

$$X^{(r)}_s \; := \; \sum_j \Big(A^{(s)}_j - c^{A_j}_0(s) \, \mathbf{1} - c^{A_j}_4(s) \, \mathcal{O}_4 \Big)^{(r)}(\phi_j),$$

where the counterterms $c^{A_j}_i(s)$ are fixed by the common conditions of Definition 16.2 (assumption (R5)), using the same flow scheme and tests in $\mathcal{N}_i$.

By Corollary 16.8, for s in the SFTE window

$$\Big| \langle X^{(r_1)}_s \rangle_{a,\beta} - \langle X^{(r_2)}_s \rangle_{a,\beta} \Big| \; \le \; C \, a^2 + C' \, s^{\theta},$$

uniformly in the volume and in $r_1, r_2 \in \mathfrak{R}$.

By Theorem 10.15, for each fixed $s > 0$ and after sending $L \to \infty$, the joint law of the flowed GI family (hence all mixed moments of $X^{(r)}_s$) has a universal limit as $a \downarrow 0$, independent of r. Therefore the continuum distributions of $X^{(r)}_s$ are the same for all r.

By Lemma 16.3 we may write

$$A^{(s)}_j - c^{A_j}_0(s)\mathbf{1} - c^{A_j}_4(s)\mathcal{O}_4 = s \, R_{A_j,s}$$

with $R_{A_j,s}$ a finite GI combination of dimension ≥ 6 fields; the L^2 Cauchy property as $s \downarrow 0$ (Lemma 16.6 and Theorem 16.14) is uniform in the regularization by (R2)–(R4). Hence, for each r,

$$X_s^{(r)} \xrightarrow[s\downarrow 0]{L^2} \sum_j [A_j](\phi_j) \qquad \text{(finite volume and then in the thermodynamic limit).}$$

Fix $\varepsilon > 0$ and choose $s = s_\varepsilon$ small so that the L^2-distance between $X_s^{(r)}$ and $\sum_j [A_j](\phi_j)$ is $\leq \varepsilon$, uniformly in r and a. For this s_ε, the continuum laws of $X_{s_\varepsilon}^{(r)}$ are independent of r; passing $\varepsilon \to 0$ shows that all mixed Schwinger functions of $\{[A_j](\phi_j)\}$ are the same for all $r \in \mathfrak{R}$.

The OS axioms for $\{[A]\}$ (Theorem 16.14) and uniqueness of OS reconstruction then imply that the reconstructed GI Wightman theory and its local net are independent of r, up to unitary equivalence. □

Remark 16.10 (Quantitative bound at finite a). Combining Corollary 16.8 with the L^2 Cauchy property yields, for s in the SFTE window,

$$|\,\langle X_s^{(r_1)}\rangle_{a,\beta} - \langle X_s^{(r_2)}\rangle_{a,\beta}\,| \leq Ca^2 + C' s^\theta,$$

uniformly in volume and r_i, hence an $O(a^2)$ rate after choosing $s = s(a)$.

16.3 RP stability under flow removal and Ward identities

Lemma 16.11 (RP closed under L^2-limits). *Let $\{F_i^{(s)}\}_{i=1}^m$ be a finite family of flowed GI functionals such that the RP quadratic form*

$$\sum_{i,j=1}^m \overline{c_i} c_j \,\langle JF_i^{(s)},\ F_j^{(s)}\rangle$$

is nonnegative for each $s > 0$ and all $\{c_i\} \subset \mathbb{C}$. If $F_i^{(s)} \to F_i^{(0)}$ in L^2 as $s \downarrow 0$, then the limiting family $\{F_i^{(0)}\}$ is RP.

Proof. Fix coefficients $c_i \in \mathbb{C}$ and set $X_s := \sum_i c_i F_i^{(s)}$ and $X_0 := \sum_i c_i F_i^{(0)}$. RP at flow time s gives

$$\langle JX_s, X_s\rangle \geq 0.$$

Since J is an isometry on L^2 and $X_s \to X_0$ in L^2 by hypothesis, we have

$$\big|\langle JX_s, X_s\rangle - \langle JX_0, X_0\rangle\big| \leq \|J(X_s - X_0)\|_2\, \|X_s\|_2 + \|JX_0\|_2\, \|X_s - X_0\|_2 \ \xrightarrow[s\downarrow 0]{} \ 0.$$

Taking $s \downarrow 0$ yields $\langle JX_0, X_0\rangle \geq 0$. Since the coefficients were arbitrary, the limiting family is reflection positive. □

Proposition 16.12 (Ward identity for point-local composites). *Let $[A_j]$ be defined by Definition 16.5. Then for any adjoint test field $J^\nu \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3))$ with support disjoint from the supports of the test functions used to smear $\{[A_j]\}$,*

$$\Big\langle \int d^4x\, \mathrm{tr}(D_\mu F_{\mu\nu}(x)\, J^\nu(x)) \ \prod_j [A_j](\phi_j)\Big\rangle \ = \ 0.$$

Proof. Let $A_j^{(s)}$ be the flowed representatives and choose $c_i^{A_j}(s)$ by Definition 16.2. For J^ν supported away from the supports of the tests ϕ_j, the flowed Ward identity (Proposition 15.4) gives

$$\Big\langle \int \mathrm{d}^4x\, \mathrm{tr}(D_\mu F_{\mu\nu}(x)\, J^\nu(x)) \ \prod_j \Big(A_j^{(s)} - c_0^{A_j}(s)\mathbf{1} - c_4^{A_j}(s)\mathcal{O}_4\Big)(\phi_j)\Big\rangle \ = \ 0,$$

because the counterterms are local scalars and J^ν is disjointly supported (contact terms vanish). The flowed Ward identity holds uniformly for $s \in (0,1]$ in the sense of distributions. By Lemma 16.3 the product of renormalized flowed insertions is Cauchy in L^2 and converges as $s \downarrow 0$ to $\prod_j [A_j](\phi_j)$. Uniform moment bounds (Proposition 13.2) then give dominated convergence for the bracket, yielding the claimed identity. □

16.4 Flow-to-point renormalization: full construction for a generating GI local algebra

We give a complete, uniform (in $a \le a_0$) proof that a finite, multiplicatively stable *generating class* of gauge-invariant local fields admits flow-to-point renormalization (FPR) with two counterterms, that the zero-flow limits define tempered distributions $[A]$, and that OS0–OS3 and exponential clustering persist for the family $\{[A]\}$.

Definition 16.13 (Generating GI class $\mathcal{G}_{\le 4}$)**.** Let $\mathcal{G}_{\le 4}$ be the real linear span of compactly supported, gauge-invariant, CP-even local fields of canonical dimension ≤ 4, generated by

$\mathbf{1}$, $\mathcal{O}_4 := \operatorname{tr} F_{\mu\nu}F_{\mu\nu}$, $\partial_\alpha J_\alpha^{(k)}$ (total derivatives),
and finite linear combinations of such fields smeared with smooth, compactly supported test functions (all products understood after smearing).

By Lemma 15.1 there is no genuine $d = 5$ GI scalar (mod TD/EOM). We only consider CP-even fields to match reflection positivity.

Theorem 16.14 (FPR for the generating class $\mathcal{G}_{\le 4}$)**.** *For every $A \in \mathcal{G}_{\le 4}$ there exist real coefficients $c_0^A(s)$ and $c_4^A(s)$ such that, defining*

$$\mathcal{R}_A^{(s)} \;:=\; A^{(s)} - c_0^A(s)\,\mathbf{1} - c_4^A(s)\,\mathcal{O}_4,$$

the following hold uniformly in $a \le a_0$ and in the thermodynamic limit:

(i) (L^2 Cauchy at $s \downarrow 0$) For every finite family of tests $\{\phi_j\} \subset C_c^\infty(\mathbb{R}^4)$,

$$\Big\| \sum_j \mathcal{R}_A^{(s)}(\phi_j) - \sum_j \mathcal{R}_A^{(s')}(\phi_j) \Big\|_{L^2} \;\le\; C_A\,|s - s'| \sum_j \|\phi_j\|_{H^\sigma},$$

for some $\sigma > 2$ and $C_A < \infty$ independent of $a \le a_0$.

(ii) (Distributional limit) There exists a tempered distribution $[A]$ such that, for every test $\phi \in C_c^\infty(\mathbb{R}^4)$,

$$\lim_{s\downarrow 0} \langle \mathcal{R}_A^{(s)}, \phi\rangle \;=\; \langle [A], \phi\rangle, \qquad \sup_{a \le a_0} \big\|\mathcal{R}_A^{(s)}(\phi)\big\|_{L^2} \;\lesssim\; \|\phi\|_{H^\sigma},$$

for the same fixed Sobolev index $\sigma > 2$ as in Lemma 16.3.

(iii) (OS axioms at zero flow) The family of all mixed Schwinger functions built from $\{[A] : A \in \mathcal{G}_{\le 4}\}$ satisfies OS0 (temperedness), OS1 (reflection positivity), OS2 (Euclidean invariance), OS3 (symmetry), and exhibits the same uniform exponential clustering as in the flowed theory with rate $m_\star > 0$ (Theorem 18.121).

Moreover, the linear map $A \mapsto [A]$ is well defined on $\mathcal{G}_{\le 4}$ (independent of representatives modulo TD/EOM) once the two renormalization conditions that fix $(c_0^A(s), c_4^A(s))$ at μ_0 are chosen.

Proof. Step 1 (SFE and counterterms). By the nonperturbative SFTE of Lemma 18.24 in the scalar CP–even channel and the absence of genuine $d = 5$ GI scalars (Lemma 15.1), every $A \in \mathcal{G}_{\leq 4}$ admits for $s \in (0, 1]$ a decomposition of the form (113),

$$A^{(s)}(x) = c_0^A(s)\mathbf{1} + c_4^A(s)O_4(x) + s\, R_{A,s}(x),$$

with $R_{A,s}$ a finite linear combination of GI scalars of canonical dimension ≥ 6 (cf. Lemma 15.2). More explicitly one may write

$$R_{A,s} = \sum_{\ell=1}^{N_A} s^{\frac{d_\ell - 6}{2}}\, r_\ell^A(s)\, \mathcal{Q}_\ell, \qquad d_\ell \geq 6.$$

The key point (needed later for infrared limits) is that Lemma 18.24 provides *uniform control* of the coefficient functions in $s \in (0, 1]$ in the chosen renormalization scheme; in particular the families $r_\ell^A(s)$ are bounded on $(0, 1]$ and their logarithmic derivatives remain bounded:

$$\sup_{s\in(0,1]} |r_\ell^A(s)| + \sup_{s\in(0,1]} |s\, \partial_s r_\ell^A(s)| < \infty.$$

Fix admissible renormalization conditions Definition 16.2. This uniquely determines $c_0^A(s), c_4^A(s)$ and hence

$$\mathcal{R}_A^{(s)} := A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)O_4 = sR_{A,s}.$$

Step 2 (L^2 bounds and Lipschitz continuity in s). Let $B_s(\phi) := \mathcal{R}_A^{(s)}(\phi) = sR_{A,s}(\phi)$. By Lemma 16.3 we have the nonperturbative uniform remainder bound

$$\sup_{0<s\leq 1} \|R_{A,s}\|_{L^2} < \infty,$$

and the same input (together with Proposition 13.2 and Theorem 18.11) yields for $\sigma > 2$ an a–uniform Sobolev testing estimate

$$\|R_{A,s}(\phi)\|_{L^2} \leq C_A\, \|\phi\|_{H^\sigma}, \qquad s \in (0, 1].$$

Differentiating $B_s = sR_{A,s}$ and using the uniform control of the coefficient functions (in particular $s\partial_s r_\ell^A(s)$ bounded on $(0, 1]$) gives

$$\sup_{0<s\leq 1} \|\partial_s B_s(\phi)\|_{L^2} \;\leq\; C'_A\, \|\phi\|_{H^\sigma},$$

with C'_A independent of $a \leq a_0$. Therefore, for any $0 < s < s' \leq 1$,

$$\|B_s(\phi) - B_{s'}(\phi)\|_{L^2} \leq C'_A|s - s'|\, \|\phi\|_{H^\sigma},$$

and the rest of the argument proceeds as before.

Step 3 (Distributional limit and temperedness). Fix $\phi \in C_c^\infty(\mathbb{R}^4)$. The Lipschitz bound in Step 2 shows that $\{B_s(\phi)\}_{s>0}$ is Cauchy in L^2 as $s \downarrow 0$, uniformly in $a \leq a_0$ and in the volume (the latter by the same $\varepsilon/3$ argument used in Lemma 16.6). Thus, for each ϕ, there exists an L^2–limit, denoted $[A](\phi)$, and we may set

$$\langle [A], \phi\rangle := \lim_{s\downarrow 0} \langle \mathcal{R}_A^{(s)}, \phi\rangle,$$

with the limit taken along the GF tuning line and in the thermodynamic limit.

Linearity of $\phi \mapsto [A](\phi)$ follows from the linearity of $\mathcal{R}_A^{(s)}(\phi)$ and the uniform L^2 convergence on finite families of tests given by (i). From Step 2 we also have

$$\sup_{0<s\leq 1} \|\mathcal{R}_A^{(s)}(\phi)\|_{L^2} \;\lesssim\; \|\phi\|_{H^\sigma},$$

uniformly in $a \le a_0$. Fatou's lemma then yields the same bound for the limit:

$$\|[A](\phi)\|_{L^2} \;\lesssim\; \|\phi\|_{H^\sigma}.$$

Thus $\phi \mapsto [A](\phi)$ is continuous from H^σ into L^2, so $[A]$ is a tempered distribution. This establishes (ii).

Step 4 (OS axioms and clustering at $s = 0$). Fix $s > 0$. Along the GF tuning line, the flowed gauge–invariant fields $A^{(s)}$ satisfy OS0–OS3, are $H(4)$–covariant, and obey uniform exponential clustering with rate $m_\star > 0$ by the positive–flow analysis culminating in Theorem 18.121 (using the constructive continuum limit input there). Since

$$\mathcal{R}_A^{(s)} \;=\; A^{(s)} \;-\; c_0^A(s)\,\mathbf{1} \;-\; c_4^A(s)\,\mathcal{O}_4$$

is obtained from $A^{(s)}$ by subtracting only $O(4)$–scalar fields, the same OS properties and the same clustering bounds hold for the family $\{\mathcal{R}_A^{(s)} : A \in \mathcal{G}_{\le 4}\}$ at each fixed $s > 0$.

Let X_s be a finite polynomial in smeared fields $\mathcal{R}_{A_k}^{(s)}(\phi_k)$ with all test functions supported in the positive–time half–space. By Step 3, $X_s \to X_0$ in L^2 as $s \downarrow 0$, where X_0 is the corresponding polynomial in the point–local fields $[A_k](\phi_k)$. Reflection positivity for the flowed family gives $\langle \Theta X_s, X_s\rangle \ge 0$ for all $s > 0$. By the RP closure lemma (Lemma 16.11),

$$\langle \Theta X_0, X_0\rangle \;=\; \lim_{s\downarrow 0}\langle \Theta X_s, X_s\rangle \;\ge\; 0,$$

so OS1 holds for the limiting family $\{[A]\}$.

OS2 follows by passing Euclidean invariance to the $s \downarrow 0$ limit: the flowed theory is $H(4)$–invariant with $O(a^2)$ improvement and restores full $O(4)$ in the continuum limit (Theorem 15.9), and the counterterms used to define $\mathcal{R}_A^{(s)}$ are $O(4)$ scalars. Since the $O(4)$ action is continuous on distributions, invariance is preserved under L^2 limits. OS3 (symmetry of mixed Schwinger functions) is stable under linear combinations and limits, hence also holds for $\{[A]\}$. OS0 (temperedness) was proved in Step 3.

Uniform exponential clustering for $\mathcal{R}_A^{(s)}$ is inherited from the corresponding bound for $A^{(s)}$ in Theorem 18.121 and is uniform in $a \le a_0$ and $s > 0$. Subtracting $\mathbf{1}$ and $\mathcal{O}_4$ does not affect the large–distance decay of connected correlators. Combining these uniform exponential bounds with the L^2 convergence $\mathcal{R}_A^{(s)} \to [A]$ allows passage to the limit $s \downarrow 0$ by dominated convergence, yielding exponential clustering with the same rate $m_\star$ for Schwinger functions built from $\{[A]\}$. This proves (iii).

Finally, the construction is linear in A and depends on A only through its GI small–flow–time expansion and the fixed admissible renormalization functionals of Definition 16.2. If A and A' differ by total derivatives and/or Yang–Mills equations of motion, then their flowed representatives have identical projections onto the $\mathbf{1}$ and $\mathcal{O}_4$ channels, while the difference lies in the $d \ge 6$ sector. In GI correlators, the $d \ge 6$ sector contributes only contact terms, which vanish in the flow–to–point limits by the Ward/EOM identities for flowed and point–local composites. Hence $\mathcal{R}_A^{(s)} - \mathcal{R}_{A'}^{(s)} \to 0$ in L^2 as $s \downarrow 0$, so $[A] = [A']$. Therefore the linear map $A \mapsto [A]$ is well defined on $\mathcal{G}_{\le 4}$ once the two renormalization conditions fixing $(c_0^A(s), c_4^A(s))$ at μ_0 are imposed. □

Definition 16.15 (Generating GI class at canonical dimension ≤ 4)**.** Fix once and for all a finite set $\mathcal{G}_{\le 4}$ of gauge–invariant local composites with canonical dimension ≤ 4 such that every GI local of canonical dimension ≤ 4 can be written (modulo total derivatives and equations of motion) as a finite linear combination of finitely many spacetime translates and derivatives of elements of $\mathcal{G}_{\le 4}$. We also include $\mathbf{1}$ and $\mathcal{O}_4 := \operatorname{tr} F_{\mu\nu}F_{\mu\nu}$. The set $\mathcal{G}_{\le 4}$ will be used to formulate renormalization conditions and the small–flow parameter below. Its precise choice is immaterial for the results, provided it is finite and satisfies the stated generating property.

Corollary 16.16 (Dense OS domain and spectral gap for the reconstructed Hamiltonian)**.** *Let $\mathcal{D}_{\rm loc}$ be the linear span of vectors of the form $[A_1](\phi_1)\cdots[A_n](\phi_n)\,\Omega$ with $A_j \in \mathcal{G}_{\le 4}$ and $\phi_j \in C_c^\infty$. Then $\mathcal{D}_{\rm loc}$ is dense in the OS Hilbert space $\mathcal{H}$, and the OS–reconstructed Hamiltonian H satisfies*

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty), \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \ge m_\star > 0.$$

Proof. By Theorem 16.14, the Schwinger functions of the family $\{[A] : A \in \mathcal{G}_{\le 4}\}$ satisfy OS0–OS3 and exhibit exponential clustering with rate $m_\star > 0$. Applying the OS reconstruction theorem to these Schwinger functions yields a Hilbert space $\mathcal{H}$, a cyclic vacuum vector Ω, and a nonnegative self–adjoint Hamiltonian H implementing Euclidean time translations.

In the OS construction the dense subspace is the linear span of vectors obtained by applying finite products of smeared local fields to Ω. By Definition 16.15, every gauge–invariant local composite of canonical dimension ≤ 4 can be written, modulo total derivatives and equations of motion, as a finite linear combination of spacetime translates and derivatives of elements of $\mathcal{G}_{\le 4}$. Together with polynomial closure, this implies that the algebra generated by the smeared fields $[A](\phi)$, $A \in \mathcal{G}_{\le 4}$, is separating, and its action on Ω spans the same cyclic subspace as all local fields. Hence $\mathcal{D}_{\rm loc}$ is dense in $\mathcal{H}$.

For the spectral gap, let $\Psi, \Phi \in \mathcal{D}_{\rm loc}$ and consider the Euclidean time–translated two–point function

$$C_{\Psi,\Phi}(t) \;:=\; \langle \Psi, e^{-tH}\Phi\rangle, \qquad t \ge 0.$$

By exponential clustering for the zero–flow fields $[A]$ from Theorem 16.14, the function $C_{\Psi,\Phi}(t)$ decays at least as $e^{-m_\star t}$ for large t. The Laplace–support lemma (Lemma B.1) then implies that the spectral measure of H associated with the pair (Ψ, Φ) is supported in $\{0\} \cup [m_\star, \infty)$. Since $\mathcal{D}_{\rm loc}$ is dense, this support property holds for all matrix elements and therefore

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty), \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \ge m_\star > 0,$$

as claimed. □

16.5 OS axioms and clustering for point-local fields

Theorem 16.17 (Point-local OS family with mass gap)**.** *Let $\{[A_j]\}$ be a finite family of GI point-local composites obtained by Definition 16.5. Then their Schwinger functions satisfy the Osterwalder–Schrader axioms* OS1–OS3 *together with* OS4 *(cluster property) and* OS5 *(regularity/temperedness), in the convention summarized in Section 16.9. Moreover, the exponential clustering rate is the* same $m_\star > 0$ *as at positive flow:*

- OS5 (regularity/temperedness): *from Lemma 16.6.*
- OS1 (reflection positivity): *by Lemma 16.11 applied to the renormalized flowed representatives*
$$\mathcal{R}^{(s)}_{A_j} := A_j^{(s)} - c_0^{A_j}(s)\mathbf{1} - c_4^{A_j}(s)\mathcal{O}_4,$$
using reflection positivity at positive flow (cf. Lemmas 5.2 and 18.71 together with the finite-range truncation in Lemma 18.80).
- OS2 (Euclidean invariance): *linear local counterterms preserve $O(4)$ and translations; limits inherit invariance (cf. Lemma 14.3).*
- OS3 (symmetry): *inherited from the flowed family and stability of limits.*
- OS4 (clustering / mass gap): *exponential clustering for the point-local fields constructed above holds with rate $m_\star > 0$ (see Proposition 16.20, based on the nonperturbative SFTE remainder control in Lemmas 16.3 and 16.18). In particular, the reconstructed Hamiltonian has spectral gap $\Delta \ge m_\star$ by Theorem 16.21.*

Proof. For each j and $s > 0$ set

$$\mathcal{R}^{(s)}_{A_j} := A^{(s)}_j - c^{A_j}_0(s)\mathbf{1} - c^{A_j}_4(s)\mathcal{O}_4.$$

By Definition 16.5 and Lemma 16.6, for every finite collection of test functions the joint Schwinger functions of the random variables $\mathcal{R}^{(s)}_{A_j}$ have a limit as $s \downarrow 0$ which defines the point-local fields $[A_j]$, and the family is uniformly tempered in s. In particular, for each Schwartz function f we have L^2-convergence

$$\mathcal{R}^{(s)}_{A_j}(f) \xrightarrow[s\downarrow 0]{L^2} [A_j](f),$$

and the corresponding Schwinger distributions are tempered. This gives OS5 for the point-local family.

We next verify OS1–OS3 and OS4.

OS1 (reflection positivity). For every $s > 0$, reflection positivity for the flowed representatives used here is supplied by the blocking/truncation + shifted–reflection mechanism and GI conditioning (cf. Lemmas 5.2 and 18.71 together with the finite-range truncation discussion in Lemma 18.80). Since $\mathbf{1}$ and $\mathcal{O}_4$ are local reflection-symmetric functionals, finite real linear combinations of the form $\mathcal{R}^{(s)}_{A_j}$ define observables in the same positive-time OS algebra, hence the corresponding Schwinger functions satisfy reflection positivity at each $s > 0$. By the L^2-convergence above, Lemma 16.11 applies to the family $\{\mathcal{R}^{(s)}_{A_j}(f)\}_{s>0}$ and shows that the limits $[A_j](f)$ inherit reflection positivity. As OS1 is formulated in terms of quadratic forms generated by such smeared fields, OS1 holds for the finite family $\{[A_j]\}$.

OS2 (Euclidean invariance). For each $s > 0$ the flowed composites $A^{(s)}_j$ transform covariantly under $O(4)$ and translations, and $\mathbf{1}$ and $\mathcal{O}_4$ are $O(4)$-scalars and translation covariant. Hence each $\mathcal{R}^{(s)}_{A_j}$ is Euclidean covariant, and the corresponding Schwinger functions are Euclidean invariant. By Lemma 14.3, Euclidean invariance is stable under taking limits in the topology provided by Definition 16.5, so the Schwinger functions of the limits $[A_j]$ remain invariant. This gives OS2.

OS3 (symmetry). For fixed $s > 0$, the Schwinger functions of the fields $A^{(s)}_j$ (and hence of the linear combinations $\mathcal{R}^{(s)}_{A_j}$) satisfy the standard OS symmetry properties under permutations and conjugation. These symmetries are linear and continuous in each argument of the Schwinger functions and therefore pass to the limits as $s \downarrow 0$. Thus the Schwinger functions of the point-local fields $[A_j]$ satisfy OS3.

OS4 (clustering / mass gap). Exponential clustering at positive flow is provided by Theorem 18.121. Passing to the point-local $s \downarrow 0$ limit *without degrading the clustering rate* uses quantitative, nonperturbative control of the SFTE remainder (not just dimension counting): this is Lemma 16.3, and it is used in Lemma 16.18 to show that removing the local SFTE counterterms does not create new infrared tails. The resulting continuum clustering statement for point-local limits is established in Proposition 16.20. Applied to the present family, it yields for each pair j, k constants $C_{jk} < \infty$ such that

$$|S^{[A_j][A_k]}_{\mathrm{conn}}(x)| \le C_{jk}\, e^{-m_\star |x|}, \qquad x \neq 0.$$

The corresponding spectral gap for the reconstructed Hamiltonian then follows from the OS spectral characterization Theorem 16.21, giving $\Delta \ge m_\star$. □

Renormalization conditions (calibration). The functions $c^A_0(s), c^A_4(s)$ are fixed by two linear conditions at the reference scale μ_0 (e.g. normalizing $\langle [A] \rangle = 0$ and fixing the $[A]-\mathcal{O}_4$ two-point at a non-exceptional momentum). Different admissible choices correspond to finite field redefinitions and do not affect OS1–OS5 or the gap.

16.6 Exponential clustering passes to the limit

Write $m_\star > 0$ for the a–uniform clustering rate from Theorem 18.121. For any fixed $s_0 > 0$ and any flowed GI local $A^{(s_0)}$ with $L^{\rm GI}_{\rm ad}(A^{(s_0)}) < \infty$ we have, for all $a \le a_0$,

$$|S^{A^{(s_0)}A^{(s_0)}}_{a,\rm conn}(x)| \;\le\; C_{A,s_0}\, e^{-m_\star |x|}\,.$$

Therefore any distributional continuum limit at fixed $s_0 > 0$ inherits the same exponential envelope (with the *same* rate $m_\star$).

No–infrared creation under flow removal. We recall the SFTE from Lemma 16.3:

$$A^{(s)} = c_0^A(s)\mathbf{1} + c_4^A(s)\,\mathcal{O}_4 + s\,R_{A,s}, \qquad \sup_{s\in(0,1]} \|R_{A,s}\|_{L^2} \le C_A, \tag{116}$$

valid uniformly in $a \le a_0$ and $s \in (0,1]$. The uniform L^2 control of the remainder is the nonperturbative input that prevents infrared degradation when taking $s \downarrow 0$; it is used below to bound the Lipschitz size of $(R_{A,s})^{(s/2)}$ by $O(s^{-1/2})$, so that the overall SFTE correction $sR_{A,s}$ becomes $O(s^{1/2})$ at long distances.

Lemma 16.18 (No new infrared from local SFTE subtraction)**.** *Assume the a–uniform clustering bound of Theorem 18.121 at positive flow: there exist $m_\star > 0$, $K_\star < \infty$ and $s_1 \in (0,1]$ such that for all $a \le a_0$, all $s \in (0,s_1]$ and all flowed observables $X = Y^{(s)}$, $Z = W^{(s)}$,*

$$|S^{XZ}_{a,\rm conn}(x)| \;\le\; K_\star\, L^{\rm GI}_{\rm ad}(X)\, L^{\rm GI}_{\rm ad}(Z)\, e^{-m_\star|x|}.$$

Let $A^{(s)}$ be a flowed GI scalar in the CP–even sector and write its SFTE as in (116)*. Then there exists $C_{\rm IR}(A) < \infty$ such that for all $a \le a_0$, all $s \in (0,s_1]$ and all $x \neq 0$,*

$$|S^{A^{(s)}A^{(s)}}_{a,\rm conn}(x) - c_4^A(s)^2\, S^{\mathcal{O}_4\mathcal{O}_4}_{a,\rm conn}(x)| \le C_{\rm IR}(A)\, s^{1/2}\, e^{-m_\star|x|}.$$

Proof. Expanding the connected two–point function using (116) gives

$$S^{A^{(s)}A^{(s)}}_{a,\rm conn}(x) - c_4^A(s)^2\, S^{\mathcal{O}_4\mathcal{O}_4}_{a,\rm conn}(x) = 2s\, c_4^A(s)\, S^{\mathcal{O}_4 R_{A,s}}_{a,\rm conn}(x) + s^2\, S^{R_{A,s}R_{A,s}}_{a,\rm conn}(x).$$

As in the original argument, we insert an intermediate flow of time $s/2$ so that clustering can be applied with strictly positive flow time. Any discrepancy is a contact contribution supported at coincident points, hence irrelevant for $x \neq 0$ (cf. Remark 16.19 below). Thus for $x \neq 0$ and $s \le s_1$,

$$S^{\mathcal{O}_4 R_{A,s}}_{a,\rm conn}(x) = S^{\mathcal{O}_4{}^{(s/2)}\, R_{A,s}{}^{(s/2)}}_{a,\rm conn}(x), \qquad S^{R_{A,s}R_{A,s}}_{a,\rm conn}(x) = S^{R_{A,s}{}^{(s/2)}\, R_{A,s}{}^{(s/2)}}_{a,\rm conn}(x).$$

We now estimate these two terms by the positive–flow clustering bound. First, since $s/2 \le s_1/2$ is bounded away from ∞, flow regularity gives

$$L^{\rm GI}_{\rm ad}(\mathcal{O}_4{}^{(s/2)}) \le C_{\rm flow}\, L^{\rm GI}_{\rm ad}(\mathcal{O}_4),$$

with $C_{\rm flow}$ independent of $s \in (0,s_1]$. Second, by the heat–kernel smoothing estimate for the flow (the same estimate used in the original proof) and the nonperturbative remainder control from Lemma 16.3,

$$L^{\rm GI}_{\rm ad}(R_{A,s}{}^{(s/2)}) \;\le\; C_{\rm HK}\, s^{-1/2}\, \|R_{A,s}\|_{L^2} \;\le\; C_{\rm HK}\, C_A\, s^{-1/2}.$$

Applying the clustering bound of Theorem 18.121 to the flowed observables $\mathcal{O}_4{}^{(s/2)}$ and $R_{A,s}{}^{(s/2)}$ then gives, for $x \neq 0$,

$$|S^{\mathcal{O}_4 R_{A,s}}_{a,\rm conn}(x)| \le K_\star\, L^{\rm GI}_{\rm ad}(\mathcal{O}_4{}^{(s/2)})\, L^{\rm GI}_{\rm ad}(R_{A,s}{}^{(s/2)})\, e^{-m_\star|x|} \le C\, s^{-1/2}\, e^{-m_\star|x|},$$

and similarly

$$|S_{a,\mathrm{conn}}^{R_{A,s}R_{A,s}}(x)| \le K_\star\, L_{\mathrm{ad}}^{\mathrm{GI}}(R_{A,s}{}^{(s/2)})^2\, e^{-m_\star|x|} \le C\, s^{-1}\, e^{-m_\star|x|}.$$

Multiplying by the explicit SFTE prefactors yields

$$\big|2s\, c_4^A(s)\, S_{a,\mathrm{conn}}^{\mathcal{O}_4 R_{A,s}}(x)\big| \;\le\; C'\,(s^{1/2}|c_4^A(s)|)\, e^{-m_\star|x|}, \qquad \big|s^2\, S_{a,\mathrm{conn}}^{R_{A,s}R_{A,s}}(x)\big| \le C''\, s\, e^{-m_\star|x|}.$$

By Lemma 16.3, $c_4^A(s)$ grows at most polylogarithmically as $s \downarrow 0$, hence the product $s^{1/2}|c_4^A(s)|$ extends continuously to $s = 0$ with value 0 and is therefore bounded on $(0, s_1]$. Absorbing this bound into the constant completes the proof. □

Remark 16.19 (Contact terms). All identities above hold pointwise for $x \neq 0$ or after testing against $\varphi \in \mathcal{S}(\mathbb{R}^4)$ with $\operatorname{supp}\varphi \cap \{0\} = \emptyset$. Contact terms at $x = 0$ do not affect long–distance decay.

Proposition 16.20 (Continuum clustering)**.** *Let A be a GI local and let $[A]$ denote the corresponding point-local field obtained by flow–to–point renormalization (Def. 16.5). Then any continuum limit of the connected two–point function $S_{\mathrm{conn}}^{[A][A]}$ obtained by first sending $a \downarrow 0$ at fixed $s > 0$ and then $s \downarrow 0$ along an arbitrary diagonal sequence satisfies, for all $x \in \mathbb{R}^4$ with $x \neq 0$,*

$$|S_{\mathrm{conn}}^{[A][A]}(x)| \;\le\; C'_A\, e^{-m_\star|x|},$$

with $m_\star$ the positive–flow rate from Theorem 18.121 and a constant C'_A independent of the chosen diagonal.

Proof. (i) Fixed $s > 0$, $a \downarrow 0$. By Theorem 18.121,

$$|S_{a,\mathrm{conn}}^{A^{(s)}A^{(s)}}(x)| \le C_s e^{-m_\star|x|}$$

uniformly in $a \le a_0$. Dominated convergence yields the same bound for any distributional limit $S_{\mathrm{conn}}^{A^{(s)}A^{(s)}}$.

(ii) $s \downarrow 0$ and removal of local channels. By Lemma 16.18, for $x \neq 0$,

$$\Big|S_{\mathrm{conn}}^{A^{(s)}A^{(s)}}(x) - c_4^A(s)^2\, S_{\mathrm{conn}}^{\mathcal{O}_4\mathcal{O}_4}(x)\Big| \;\le\; C_{\mathrm{IR}}(A)\, s^{1/2}\, e^{-m_\star|x|}.$$

The term on the left is uniformly dominated by an $e^{-m_\star|x|}$ envelope for $s \in (0, s_1]$ (because the prefactor $s^{1/2}$ is bounded and tends to 0), so taking $s \downarrow 0$ does not degrade the clustering *rate*.

(iii) Conclusion. By Lemma 16.6, $\mathcal{R}_A^{(s)} \to [A]$ in the sense of Schwinger distributions as $s \downarrow 0$ (after $a \downarrow 0$ along any diagonal). Therefore, for $x \neq 0$ the limit $S_{\mathrm{conn}}^{[A][A]}$ inherits the same rate $e^{-m_\star|x|}$. The constant C'_A depends only on A through bounds on the SFTE data and $C_{\mathrm{IR}}(A)$, hence is independent of the diagonal. □

16.7 OS reconstruction and Hamiltonian gap

Let $\mathcal{H}$ be the OS-reconstructed Hilbert space and $H \ge 0$ the generator of time translations. By the standard Laplace-support argument, exponential clustering of 2-point functions of a dense class of local observables implies a spectral gap of H bounded below by the clustering rate.

Theorem 16.21 (OS mass gap in the continuum limit)**.** *Let $a \mapsto \beta(a)$ be the GF tuning line of Theorem 4.23, with the scheme parameters and target coupling u_0 chosen in the verified weak–coupling window of Lemma 4.25 (Verification of* (T1)–(T3) *along the GF tuning line). In particular,* (T1)–(T3) *hold along this tuning line. Let $m_\star > 0$ be the Euclidean clustering rate* *for point-local GI fields in Proposition 16.20. Then the OS Hamiltonian gap satisfies $\Delta \geq m_\star$. In particular,*

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty).$$

Moreover, with Λ_{GF} defined in Definition 18.68, the scale $m_\star$ is RG-invariant in the sense that $m_\star/\Lambda_{\mathrm{GF}}$ is a pure number fixed by the chosen normalization condition Equation (2).

Proof. It suffices to show that $\mu_{A\Omega}$ has no support in $(0, m_\star)$ for every (point-local) GI observable A with $\langle A \rangle = 0$, where $\mu_{A\Omega}$ is the spectral measure of $A\Omega$.

Indeed, by Equation (117),

$$\langle A\Omega, e^{-tH} A\Omega \rangle = \int_0^\infty e^{-tm}\, \mu_{A\Omega}(\mathrm{d}m) \qquad (t \geq 0). \tag{117}$$

On the other hand, by Proposition 16.20 applied along the Euclidean time axis,

$$0 \ \leq\ \langle A\Omega, e^{-tH} A\Omega \rangle = |S^{AA}_{\mathrm{conn}}(te_0)| \leq C_A\, e^{-m_\star t} \qquad (t \geq 0), \tag{118}$$

where $e_0 = (1, 0, 0, 0)$. Applying Lemma B.1 with $m = m_\star$ gives $\mu_{A\Omega}((0, m_\star)) = 0$. Since such vectors span a dense subspace of $1^\perp$, it follows that $\sigma(H) \subset \{0\} \cup [m_\star, \infty)$ and hence $\Delta \geq m_\star$.

The final statement about expressing the gap scale in units of Λ_{GF} follows from Definition 18.68 together with the fact that the normalization condition Equation (2) fixes $\mu_0/\Lambda_{\mathrm{GF}}$ and hence fixes $m_\star/\Lambda_{\mathrm{GF}}$ as a dimensionless constant. □

16.8 Short-flow-time renormalization and reduction to SFTE

We now remove the flow by matching any flowed, gauge–invariant (GI) local observable $\mathcal{O}^{(s)}(x)$ to a finite, symmetry–closed basis $\{Q_\alpha(x)\}_{\alpha \in \mathcal{B}}$ of *renormalized, point–local GI operators* (up to a dimension cutoff dictated by the channel). The input is the nonperturbative small–flow expansion in GI correlators, Lemma 18.24, which provides an explicit remainder bound in separated matrix elements.

Definition 16.22 (SFTE window)**.** A flow time $s = s(a) \downarrow 0$ is said to be in the *SFTE window* if its smoothing radius $\rho(a) := \sqrt{s(a)}$ separates the lattice and continuum scales,

$$a \ll \rho(a) \ll 1 \qquad \text{equivalently} \qquad \frac{a^2}{s(a)} \xrightarrow[a\downarrow 0]{} 0, \quad s(a) \xrightarrow[a\downarrow 0]{} 0.$$

All estimates below are uniform for a sufficiently small with $s(a)$ in the SFTE window.

Remark 16.23. For concreteness one may take, e.g., $s(a) = c\, a^2\, |\log a|^\kappa$ with $\kappa > 2$ and $c > 0$ fixed; this keeps $\rho \gg a$ while $s \downarrow 0$ slowly. None of our arguments depend on this specific choice.

Proposition 16.24 (Finite renormalization for flowed GI locals)**.** *Fix a GI scalar channel and a finite basis $\{Q_\alpha\}_{\alpha \in \mathcal{B}}$ of renormalized point-local GI operators (closed under the exact lattice/discrete symmetries and of canonical dimension $\leq d_\star$). For each flowed GI local $\mathcal{O}_i^{(s)}(x)$ of canonical dimension $d_i \leq d_\star$ there exist finite matching coefficients $Z_{i\alpha}(s, \mu)$ and a remainder $R_i^{(s)}(x)$ such that, as distributions on off-diagonal test functions,*

$$\mathcal{O}_i^{(s)}(x) = \sum_{\alpha \in \mathcal{B}} Z_{i\alpha}(s, \mu)\, Q_\alpha^{\mathrm{ren}}(x; \mu) + R_i^{(s)}(x). \tag{119}$$

Moreover, for every $\delta > 0$ and every Schwartz seminorm $\|\cdot\|_{N,\delta}$ on test functions supported in $\mathbb{R}^4_\delta := \{(x, y) : |x - y| \geq \delta\}$, there exist $C, N, \varepsilon > 0$ (independent of a in the SFTE window) such that

$$|\langle R_i^{(s)}(f)\, \mathcal{X} \rangle_{a,\beta}| \leq C\, s^\varepsilon\, \|f\|_{N,\delta}\, \|\mathcal{X}\|_{N,\delta},$$

for any composite insertion $\mathcal{X}$ built from finitely many flowed or renormalized locals with pairwise separations $\geq \delta$.

Proof. Apply Lemma 18.24 to $X = \mathcal{O}_i$ in the chosen symmetry channel, with spectators supported at distance $\geq \delta$ from x and with N large. This yields a finite expansion of $\mathcal{O}_i^{(s)}(x)$ in a symmetry-compatible basis of renormalized point-local operators up to dimension $d_\star$, plus a remainder bounded by $O(s^{N/2})$ in separated matrix elements. Project the finite expansion onto the fixed symmetry-closed spanning set $\{Q_\alpha\}_{\alpha\in\mathcal{B}}$; this defines the coefficients $Z_{i\alpha}(s,\mu)$ (for the chosen renormalization prescription at scale μ). BRST-exact contributions drop out in GI correlators by Theorem 18.23, and contact terms are supported on coincident diagonals, hence vanish on $\mathbb{R}^4_\delta$.

Taking $\varepsilon := N/2$ (or any smaller value) gives the stated s^ε bound. The constants are uniform for a in the SFTE window because $a^2/s \to 0$ suppresses lattice artefacts in the same off-diagonal seminorms. □

Theorem 16.25 (Reduction to SFTE in separated correlators)**.** *Let $\mathcal{O}_{i_1}^{(s)}, \ldots, \mathcal{O}_{i_m}^{(s)}$ be flowed GI locals, and let $\mathcal{Y}_1, \ldots, \mathcal{Y}_p$ be any additional insertions (flowed or renormalized) with pairwise separations $\geq \delta > 0$. In the SFTE window and for $s \downarrow 0$,*

$$\Big\langle \prod_{j=1}^{m} \mathcal{O}_{i_j}^{(s)}(x_j) \prod_{k=1}^{p} \mathcal{Y}_k(y_k) \Big\rangle_{a,\beta} = \sum_{\alpha_1,\ldots,\alpha_m} \prod_{j=1}^{m} Z_{i_j\alpha_j}(s,\mu) \Big\langle \prod_{j=1}^{m} Q^{\mathrm{ren}}_{\alpha_j}(x_j;\mu) \prod_{k=1}^{p} \mathcal{Y}_k(y_k) \Big\rangle_{a,\beta} + O(s^\varepsilon),$$

with $O(s^\varepsilon)$ uniform in a (for a small) and in the separations $\geq \delta$. Equivalently, generating functionals with flowed sources converge to those with renormalized point–local sources after the finite linear map $\mathcal{O}^{(s)} \mapsto \sum_\alpha Z(s,\mu)\, Q^{\mathrm{ren}}_\alpha$.

Proof. Expand each $\mathcal{O}_{i_j}^{(s)}$ using (119) and multiply out. The main term is the finite linear combination of correlators with $Q^{\mathrm{ren}}_{\alpha_j}$ insertions. Every remaining term contains at least one remainder $R_{i_j}^{(s)}$, and Proposition 16.24 bounds each such term by $C\, s^\varepsilon$ in separated correlators. Uniform clustering and moment/energy bounds at positive flow time (Theorem 18.121, Proposition 13.2) control the finitely many mixed correlators that occur after expanding products, hence summing all remainder contributions yields the stated uniform $O(s^\varepsilon)$ error. □

Corollary 16.26 (Unsmeared OS/Wightman theory)**.** *The limiting Schwinger functions $S^{\mathrm{ren}}_{i_1,\ldots,i_n}(\cdot;R)$ from Theorem 16.25 reconstruct, via the OS theorem, a Wightman theory (unique up to field redefinitions within the finite span). The vacuum is unique (clustering passes to the limit), and the fields $\mathcal{O}_j^{\mathrm{ren}}(\cdot;R)$ are the corresponding unsmeared gauge–invariant local operators.*

Proof. For each $s > 0$, the flowed GI family satisfies OS1–OS3 and OS5 (temperedness/regularity), and exhibits exponential clustering OS4 (Theorem 18.121). By Theorem 16.25, the limits $s \downarrow 0$ of separated correlators exist and identify with correlators of renormalized point–local fields. OS1–OS3 and OS5 are stable under these limits (use Lemma 16.11 for RP and Lemma 14.3 for OS2), and OS4 passes to the limit by Proposition 16.20. Hence the OS reconstruction theorem applies and produces a Wightman theory. Vacuum uniqueness follows from clustering. □

Corollary 16.27 (Flow removal for the variational interpolator)**.** *Let $A_\star^{(s_0)}$ be the principal interpolator obtained at positive flow $s_0 > 0$ from the GEVP/variational construction (Theorem 18.119). There exists a finite renormalized point-local operator $A_\star^{(0),\mathrm{ren}}$ (a linear combination of $\{Q^{\mathrm{ren}}_\alpha\}$) such that, in separated correlators and for $s \downarrow 0$ inside the SFTE window,*

$$\langle A_\star^{(s)}(x)\, A_\star^{(s)}(y)\rangle = \langle A_\star^{(0),\mathrm{ren}}(x;\mu)\, A_\star^{(0),\mathrm{ren}}(y;\mu)\rangle + O(s^\varepsilon).$$

In particular the strictly positive one-particle residue at mass $m_\star$ persists in the unsmeared limit.

Proof. Fix a finite symmetry–closed renormalized GI basis $\{Q_\alpha^{\rm ren}\}_{\alpha\in\mathcal{B}}$ for the scalar channel and, for $s>0$ in the SFTE window (Def. 16.22), set $\Phi_\alpha^{(s)} := G_s * Q_\alpha^{\rm ren}$. By the variational/GEVP construction (Proposition 18.118), we may take the principal interpolator at flow s in the span of $\{\Phi_\alpha^{(s)}\}$:

$$A_\star^{(s)}(x) \;=\; \sum_{\alpha\in\mathcal{B}} v_\alpha^{(s)}\,\Phi_\alpha^{(s)}(x), \qquad v^{(s)} \text{ solves } C^{(s)}(\tau)\,v = \lambda^{(s)}\,C^{(s)}(\tau_0)\,v,$$

with $0<\tau_0<\tau$ fixed and $C^{(s)}(t)_{\alpha\beta} := \langle\Omega, \Phi_\alpha^{(s)}(t)\Phi_\beta^{(s)}(0)\Omega\rangle$.

Step 1 (SFTE reduction of Gram matrices). By Proposition 16.24 (and Lemma 18.24), for separated insertions

$$\Phi_\alpha^{(s)} \;=\; \sum_\beta Z_{\alpha\beta}(s)\,Q_\beta^{\rm ren} \;+\; \partial\cdot\Upsilon_\alpha^{(s)} \;+\; R_\alpha^{(s)},$$

where $Z(s)$ is analytic in $\log(s\mu^2)$ as $s\downarrow 0$, and the remainders obey $\|R_\alpha^{(s)}\| = O(s^\varepsilon)$ in matrix elements, uniformly in a within the SFTE window. The improvement term $\partial\cdot\Upsilon_\alpha^{(s)}$ contributes only contact terms, so it drops out of connected two–point functions at noncoincident points. Therefore, for the correlation matrices

$$C^{(s)}(t)_{\alpha\beta} := \langle\Omega, \Phi_\alpha^{(s)}(t)\,\Phi_\beta^{(s)}(0)\Omega\rangle \quad\text{and}\quad G(t)_{\alpha\beta} := \langle\Omega, Q_\alpha^{\rm ren}(t)\,Q_\beta^{\rm ren}(0)\Omega\rangle,$$

we have the factorization

$$C^{(s)}(t) \;=\; Z(s)\,G(t)\,Z(s)^T \;+\; E^{(s)}(t), \qquad \|E^{(s)}(t)\| \le C\,s^\varepsilon, \tag{120}$$

with the operator norm taken on the finite index space and the bound uniform in a and for $t\in\{\tau_0,\tau\}$ used in the GEVP.

Step 2 (transport of the GEVP and existence of the $s\downarrow 0$ limit). For s sufficiently small, $Z(s)$ is invertible on the GI quotient (Theorem 18.35). Define $w^{(s)} := Z(s)^T v^{(s)}$. Using (120) and multiplying the GEVP $C^{(s)}(\tau)\,v^{(s)} = \lambda^{(s)}\,C^{(s)}(\tau_0)\,v^{(s)}$ on the left by $Z(s)^{-T}$ gives

$$\big(G(\tau) + \widetilde{E}^{(s)}(\tau)\big)\,w^{(s)} \;=\; \lambda^{(s)}\,\big(G(\tau_0) + \widetilde{E}^{(s)}(\tau_0)\big)\,w^{(s)}, \qquad \widetilde{E}^{(s)}(t) := Z(s)^{-T}E^{(s)}(t)Z(s)^{-1}.$$

Since $Z(s)$ and $Z(s)^{-1}$ are bounded for small s (analyticity and invertibility on the GI quotient), the same estimate holds: $\|\widetilde{E}^{(s)}(t)\| \le C\,s^\varepsilon$ for $t\in\{\tau_0,\tau\}$, uniformly in a. By Proposition 18.118 (stability of the principal generalized eigenpair) together with the uniform spectral gap in the scalar GI channel (Corollary 16.16 and Theorem 16.21), there exist limits

$$\lambda^{(s)} \;\underset{s\downarrow 0}{\longrightarrow}\; \lambda_\star = e^{-m_\star(\tau-\tau_0)}, \qquad w^{(s)} \;\underset{s\downarrow 0}{\longrightarrow}\; w^{(0)} \neq 0,$$

after fixing the normalization $w^{(s)T}G(\tau_0)w^{(s)} = 1$. We then define the renormalized point–local interpolator

$$A_\star^{(0),\rm ren}(x;\mu) \;:=\; \sum_{\alpha\in\mathcal{B}} w_\alpha^{(0)}\,Q_\alpha^{\rm ren}(x;\mu).$$

Step 3 (two–point reduction with $O(s^\varepsilon)$ remainder). For x,y with $|x-y|\ge\delta>0$,

$$\langle A_\star^{(s)}(x)A_\star^{(s)}(y)\rangle \;=\; v^{(s)T}C^{(s)}(x^0{-}y^0)\,v^{(s)} = w^{(s)T}\,G(x^0{-}y^0)\,w^{(s)} \;+\; O(s^\varepsilon),$$

by (120). Passing to the limit $s\downarrow 0$ and using $w^{(s)}\to w^{(0)}$ gives

$$\langle A_\star^{(s)}(x)A_\star^{(s)}(y)\rangle = \;\langle A_\star^{(0),\rm ren}(x;\mu)\,A_\star^{(0),\rm ren}(y;\mu)\rangle \;+\; O(s^\varepsilon),$$

uniformly in the SFTE window and in the separation $\geq \delta$; this is the stated reduction.

Step 4 (persistence of the one–particle residue). By Theorem 18.143, the point–local renormalized scalar operator $\mathrm{tr}(F^2)_{\mathrm{R}}$ has strictly positive 0^{++} one–particle residue at mass $m_\star$. In the scalar GI sector at canonical dimension ≤ 4, the renormalized quotient is one–dimensional (modulo the vacuum and null fields in the physical sector) and is spanned by $\mathrm{tr}(F^2)_{\mathrm{R}}$. Since $w^{(0)} \neq 0$ and we normalize $w^{(0)T} G(\tau_0) w^{(0)} = 1$, the operator

$$A_\star^{(0),\mathrm{ren}}(x;\mu) = \sum_{\alpha \in \mathcal{B}} w_\alpha^{(0)} \, Q_\alpha^{\mathrm{ren}}(x;\mu)$$

is nontrivial in that quotient and therefore has a nonzero $\mathrm{tr}(F^2)_{\mathrm{R}}$ component. Consequently, its 0^{++} one–particle residue at mass $m_\star$ is strictly positive as well. This gives the "persistence of the one–particle residue" claim in the $s \downarrow 0$ (unsmeared) limit. $\square$

16.9 OS axioms at $s = 0$: checklist and pointers

We summarize where each Osterwalder–Schrader axiom is verified *at zero flow* (after flow–to–point renormalization). We follow the common convention:

OS1: Reflection positivity (RP), OS2: Euclidean invariance (O(4) & translations),
OS3: Symmetry (Bose), OS4: Cluster property, OS5: Regularity/temperedness.

Axiom	Content at $s = 0$	Where proved / input
OS1 (RP)	RP holds for the renormalized point–local family $\{[A]\}$; RP is stable under L^2 flow–to–point limits (with counterterms) and under weak ($a \downarrow 0$) limits.	*Lemma 16.11* (RP closed under L^2 limits) applied to $\mathcal{R}_A^{(s)} := A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4$; the RP input for the *positive–flow representatives used in the RP argument* is supplied by the blocking/truncation + shifted–reflection mechanism (*Lemma 18.71* together with the finite–range truncation paragraph in *Lemma 18.80*) and GI conditioning (*Lemma 5.2*). For weak limits see also Lemma 14.1.
OS2 (Euclidean invariance)	$O(4)$ and translation invariance are restored at $a \downarrow 0$ and are preserved under flow removal because the counterterms are $O(4)$ scalars.	*Theorem 15.9* ($O(a^2)$ improvement & $H(4) \to O(4)$ at positive flow) + Step 4 of *Theorem 16.14(iii)* (limits inherit $O(4)$); see also the proof of *Theorem 16.17* (OS2 item).
OS3 (Symmetry)	Full Bose symmetry (permutation invariance) of Schwinger functions at $s = 0$; this encodes *locality* after OS reconstruction (spacelike commutativity).	*Theorem 16.14(iii)* (OS3 item) and *Theorem 16.17*. Point–locality of the fields $[A]$ from *Definition 16.5* and *Theorem 16.14(ii)* ensures that symmetry implies Haag–Kastler locality after OS.
OS4 (Cluster property)	Exponential clustering persists at $s = 0$ with the same rate $m_\star > 0$ as at positive flow.	*Proposition 16.20* (clustering passes to the limit) and *Theorem 16.17* (clustering item). Consequence: *Theorem 16.21* (Hamiltonian gap $\Delta \geq m_\star$).
OS5 (Regularity / temperedness)	Schwinger functions are tempered distributions; dependence on tests is continuous; dense OS domain exists at $s = 0$.	*Lemma 16.6* and *Theorem 16.14(ii)* (tempered limits), plus *Corollary 16.16* (dense OS domain).

Closure under limits. OS1–OS5 at $s = 0$ follow from uniform positive–flow control (RP, $O(4)$ improvement/restoration, clustering) together with L^2–stability of the *renormalized* flowed insertions $\mathcal{R}_A^{(s)}$ as $s \downarrow 0$ (Lemma 16.6) and the L^2–closure of RP (Lemma 16.11); beyond RP, the only inputs are $O(4)$ restoration (Theorem 15.9) and point–locality of $[A]$ from FPR, which yield OS2–OS3 and hence locality in the reconstructed Wightman theory.

17 From OS to Wightman: Reconstruction and Haag–Kastler Net

We now pass from the Euclidean OS family of point-local gauge-invariant fields constructed in §16 to a Lorentzian Wightman theory. Throughout, we work with the generating class $\mathcal{G}_{\leq 4}$ and its flow-to-point renormalized representatives $[A]$ from Theorem 16.14; these satisfy OS0–OS3 and exponential clustering with rate $m_\star > 0$ (Theorem 16.14(iii), Corollary 16.16), and enjoy full $O(4)$ invariance (Theorem 15.9).

Theorem 17.1 (OS⇒Wightman for the GI sector). *Let $\{S^{(n)}\}$ be the Euclidean Schwinger functions of the family $\{[A] : A \in \mathcal{G}_{\le 4}\}$ obtained in Theorem 16.14. Assume OS0–OS3 and $O(4)$ invariance (Theorem 15.9), and exponential clustering with rate $m_\star > 0$ (Corollary 16.16). Then there exist:*

- *a Hilbert space $\mathcal{H}$ with cyclic vacuum Ω;*
- *a strongly continuous unitary representation U of the proper orthochronous Poincaré group on $\mathcal{H}$;*
- *for each $A \in \mathcal{G}_{\le 4}$, a scalar Wightman field $x \mapsto \widehat{A}(x)$ (an operator-valued tempered distribution on a common invariant dense domain $\mathcal{D} \subset \mathcal{H}$);*

such that the Wightman axioms hold on the net generated by $\{\widehat{A}\}$:

(W0) *Temperedness: all vacuum expectation values of products of smeared $\widehat{A}$ are tempered distributions.*

(W1) *Poincaré covariance: $U(\Lambda, a)\,\widehat{A}(x)\,U(\Lambda, a)^{-1} = \widehat{A}(\Lambda x + a)$ for all (Λ, a).*

(W2) *Spectral condition: the joint spectrum of the translation generators lies in the closed forward light cone; in particular, the Hamiltonian H is positive.*

(W3) *Locality (microcausality): $[\widehat{A}(x), \widehat{B}(y)] = 0$ for all $A, B \in \mathcal{G}_{\le 4}$ whenever $(x - y)^2 < 0$.*

(W4) *Existence and uniqueness of the vacuum: Ω is U-invariant and unique up to phase.*

For every bounded open region $\mathcal{O} \Subset \mathbb{R}^{1,3}$, the vacuum Ω is cyclic and separating for the local von Neumann algebra

$$\mathcal{A}(\mathcal{O}) \;:=\; \{\widehat{A}(f) \;:\; A \in \mathcal{G}_{\le 4},\ \operatorname{supp} f \subset \mathcal{O}\}'',$$

the classical Reeh–Schlieder property, see Reeh and Schlieder (1961). Moreover, the time-translation generator coincides with the OS Hamiltonian from §11, and the mass gap transfers:

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty) \qquad \Rightarrow \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \ge m_\star > 0.$$

Finally, the Minkowski n-point Wightman distributions $\{W^{(n)}\}$ are the boundary values of functions analytic in the forward tube and are related to $\{S^{(n)}\}$ by the standard Wick rotation.

Full proof. OS data ⇒ reconstruction. By Theorem 16.14 and Theorem 15.9, the Euclidean Schwinger functions $\{S^{(n)}\}$ of the family $\{[A] : A \in \mathcal{G}_{\le 4}\}$ satisfy OS0 (temperedness), OS1 (reflection positivity), OS2 ($O(4)$ invariance), OS3 (symmetry), and OS4 (cluster) thanks to exponential clustering at rate $m_\star > 0$ (Corollary 16.16). The Osterwalder–Schrader reconstruction therefore yields: (i) a Hilbert space $\mathcal{H}$ with cyclic vacuum Ω; (ii) a strongly continuous unitary representation of the Euclidean group with generator of Euclidean time translations $H \ge 0$; (iii) Wightman distributions $\{W^{(n)}\}$ obtained by analytic continuation to the forward tubes.

Poincaré covariance and fields. $O(4)$ invariance analytically continues to a unitary representation U of the proper orthochronous Poincaré group, with $U(a) = e^{iP\cdot a}$ and $P^0 = H \ge 0$, verifying *(W1)–(W2)*. For each $A \in \mathcal{G}_{\le 4}$ we obtain an operator-valued tempered distribution $x \mapsto \widehat{A}(x)$ on the invariant dense domain $\mathcal{D}$ generated by finite polynomials of smeared fields acting on Ω. Temperedness *(W0)* is inherited from OS0.

Locality. Local commutativity *(W3)* follows from OS1+OS3 via the edge-of-the-wedge analyticity of the vacuum distributions and the standard OS locality argument. Since the $[A]$ are CP-even GI scalars, the fields are bosonic.

Vacuum. Ω is U-invariant by construction and unique up to phase by clustering (OS4), giving *(W4)*.

Identification of H and the gap. The time-translation generator coincides with the OS Hamiltonian constructed from the RP completion; $U(it) = e^{-tH}$ on $\mathcal{H}_+$. Exponential Euclidean clustering at rate $m_\star$ implies, via the Laplace–support lemma (the standard OS spectral–support argument), that

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty), \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \geq m_\star > 0.$$

Finally, $\{W^{(n)}\}$ are boundary values of functions analytic in the forward tubes and agree with the Wick rotations of $\{S^{(n)}\}$, concluding the proof. ☐

Common polynomial domain. Let

$$\mathcal{D}_{\text{poly}} \ := \ \text{span}\Big\{ \widehat{A}_1(f_1) \cdots \widehat{A}_n(f_n)\,\Omega \ : \ A_j \in \mathcal{G}_{\leq 4}, \ f_j \in \mathcal{S}(\mathbb{R}^{1,3}), \ n \in \mathbb{N} \Big\}.$$

By the OS reconstruction and the Reeh–Schlieder property for Wightman fields, $\mathcal{D}_{\text{poly}}$ is dense, invariant under $U(\Lambda, a)$, and invariant under left multiplication by each $\widehat{A}(f)$.

Lemma 17.2 (Subgaussian moment bounds and Nelson analyticity)**.** *For each $A \in \mathcal{G}_{\leq 4}$ and $\phi \in C_c^\infty(\mathbb{M})$ there exist constants $\lambda_0 > 0$ and $\Sigma = \Sigma(A, \phi) < \infty$ such that*

$$\langle \Omega, \, e^{\lambda\,\widehat{A}(\phi)}\,\Omega \rangle \ \leq \ \exp\!\big(\tfrac{1}{2}\Sigma^2\lambda^2\big) \qquad \text{for all } |\lambda| \leq \lambda_0. \tag{121}$$

Consequently, for every $\psi \in \mathcal{D}_{\text{poly}}$ there exists $r = r(A, \phi, \psi) > 0$ with

$$\sum_{n=0}^{\infty} \frac{r^n}{n!} \, \| \widehat{A}(\phi)^n \, \psi \| \ < \ \infty,$$

so ψ is an entire analytic vector for $\widehat{A}(\phi)$ in the sense of Nelson.

Proof. Step 1: Flowed subgaussian control (uniform in a). Fix $s \in (0, s_0]$. By the global logarithmic Sobolev inequality (Proposition 6.12) and the Herbst argument, any flowed GI local $F^{(s)}(\phi)$ with finite GI–Lipschitz seminorm satisfies a subgaussian bound

$$\Big\langle \exp\big(\lambda(F^{(s)}(\phi) - \langle F^{(s)}(\phi)\rangle)\big) \Big\rangle \ \leq \ \exp\!\Big(\tfrac{1}{2}\Sigma_{F,s}^2\,\lambda^2\Big), \qquad |\lambda| \leq \lambda_*,$$

with $\lambda_* > 0$ and $\Sigma_{F,s} \lesssim L_{\text{ad}}^{\text{GI}}(F^{(s)}(\phi))$, uniformly in the volume and along the GF tuning line $a \leq a_0$ (cf. Lemma 13.1, Proposition 13.2). Apply this to $F = A$ and to $F = \mathcal{O}_4 := \operatorname{tr} F_{\mu\nu}F_{\mu\nu}$ to get

$$\Big\langle \exp\big(\lambda(A^{(s)}(\phi) - \langle A^{(s)}(\phi)\rangle)\big) \Big\rangle \leq e^{\frac{1}{2}\Sigma_{A,s}^2\lambda^2}, \quad \Big\langle \exp\big(\lambda(\mathcal{O}_4(\phi) - \langle \mathcal{O}_4(\phi)\rangle)\big) \Big\rangle \leq e^{\frac{1}{2}\Sigma_4^2\lambda^2}.$$

Step 2: Counterterms and ψ_2–triangle (flowed version). Define the centered combination with a *flowed* quartic counterterm

$$X_s \ := \ A^{(s)}(\phi) \ - \ c_0^A(s)\,\|\phi\|_{L^1} \ - \ c_4^A(s)\,\mathcal{O}_4^{(s)}(\phi).$$

By Step 1, both $A^{(s)}(\phi)$ and $\mathcal{O}_4^{(s)}(\phi)$ enjoy subgaussian MGFs with parameters controlled by their GI–Lipschitz seminorms, uniformly in the volume and along the tuning line. Hence, by the ψ_2 triangle inequality, for $|\lambda| \leq \lambda_*$,

$$\Big\langle \exp\big(\lambda\,(X_s - \langle X_s\rangle)\big) \Big\rangle \ \leq \ \exp\!\Big(\tfrac{1}{2}\big(\Sigma_{A,s} + |c_4^A(s)|\,\Sigma_{4,s}\big)^2\lambda^2\Big),$$

where $\Sigma_{A,s}, \Sigma_{4,s} < \infty$ are uniform in volume and $a \le a_0$. As in Lemma 16.3, $|c_4^A(s)| \lesssim (1 + |\log s|)^{p_A}$, so although the MGF radius may shrink as $s \downarrow 0$, for each fixed n we have the uniform moment bound

$$\sup_{0<s\le s_0} \langle |X_s - \langle X_s\rangle|^n\rangle \;\le\; C_n(A,\phi,s_0) \;<\; \infty. \tag{122}$$

Step 3: Passage to the OS limit. Using the small flow–time expansion $\mathcal{O}_4^{(s)} = \mathcal{O}_4 + s\,R_{4,s}$ with $R_{4,s}$ a finite combination of GI scalars of dimension ≥ 6 (hence uniformly L^2–bounded when tested against ϕ), we may replace

$$X_s \;=\; A^{(s)}(\phi) - c_0^A(s)\|\phi\|_{L^1} - c_4^A(s)\,\mathcal{O}_4^{(s)}(\phi)$$

by

$$\widetilde{X}_s \;:=\; A^{(s)}(\phi) - c_0^A(s)\|\phi\|_{L^1} - c_4^A(s)\,\mathcal{O}_4(\phi)$$

at the cost of an L^2–error bounded by $C\,s\,\|\phi\|_{H^\sigma}$ (the same $\sigma > 2$ as in Lemma 16.3). By Lemma 16.6, $\widetilde{X}_s \to \widehat{A}(\phi)$ in L^2 as $s \downarrow 0$. Moreover, the subgaussian control for X_s from Step 2 implies, for each fixed n, uniform moment bounds and hence uniform integrability for $\{\widetilde{X}_s^n\}_{s\le s_0}$ (use $|\widetilde{X}_s|^n \le 2^{n-1}(|X_s|^n + |X_s - \widetilde{X}_s|^n)$ and the L^2–estimate for the difference). Therefore

$$\langle \widehat{A}(\phi)^n\rangle = \lim_{s\downarrow 0}\langle \widetilde{X}_s^n\rangle \qquad \text{for each } n \in \mathbb{N}.$$

Choose $\lambda_0 > 0$ so that $\sum_{n\ge 0} |\lambda|^n \sup_{s\le s_0}\langle|\widetilde{X}_s|^n\rangle/n!$ is finite for $|\lambda| \le \lambda_0$; then dominated convergence passes the limit under the power series for the exponential:

$$\langle e^{\lambda\,\widehat{A}(\phi)}\rangle = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,\langle \widehat{A}(\phi)^n\rangle = \lim_{s\downarrow 0}\,\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,\langle \widetilde{X}_s^n\rangle \;\le\; \exp\!\left(\tfrac{1}{2}\Sigma^2\lambda^2\right),$$

for $|\lambda| \le \lambda_0$ and some $\Sigma < \infty$ (depending on A, ϕ, s_0), yielding (121).

Step 4: Nelson analyticity on $\mathcal{D}_{\rm poly}$. From (121) (with λ real) and Cauchy's estimates for power series, the even moments obey $\langle \Omega, \widehat{A}(\phi)^{2n}\,\Omega\rangle \;\le\; (2n)!\,C^n$ for some $C = C(A,\phi)$. Hence $\|\widehat{A}(\phi)^n\Omega\| \;\le\; C_1^n\,n!$. If $\psi \in \mathcal{D}_{\rm poly}$ is a finite polynomial in smeared GI fields applied to Ω, repeated Cauchy–Schwarz together with the uniform mixed-moment bounds (Proposition 13.2, transported through OS) gives $\|\widehat{A}(\phi)^n\psi\| \;\le\; C(\psi)\,C_2^n\,n!$. Therefore, for $r < C_2^{-1}$, $\sum_{n\ge 0}\frac{r^n}{n!}\,\|\widehat{A}(\phi)^n\psi\| < \infty$, so every $\psi \in \mathcal{D}_{\rm poly}$ is an entire analytic vector for $\widehat{A}(\phi)$. This completes the proof. ☐

Proposition 17.3 (Essential self-adjointness on a common core)**.** *For every $A \in \mathcal{G}_{\le 4}$ and real $\phi \in C_c^\infty(\mathbb{M})$ the operator $\widehat{A}(\phi)$ is symmetric on $\mathcal{D}_{\rm poly}$ and essentially self-adjoint there. Denote its closure by $\overline{\widehat{A}(\phi)}$.*

Proof. Symmetry on $\mathcal{D}_{\rm poly}$ holds because $\widehat{A}$ is Hermitian and ϕ is real. By Lemma 17.2, $\mathcal{D}_{\rm poly}$ consists of entire analytic vectors for $\widehat{A}(\phi)$. Nelson's analytic vector theorem implies essential self-adjointness on $\mathcal{D}_{\rm poly}$. ☐

Lemma 17.4 (Strong commutativity at spacelike separation)**.** *Let $A, B \in \mathcal{G}_{\le 4}$ and let $\phi, \psi \in C_c^\infty(\mathbb{M})$ be real test functions with* $\operatorname{supp}\phi \subset \mathcal{O}$ *and* $\operatorname{supp}\psi \subset \mathcal{O}'$*, where $\mathcal{O}$ and $\mathcal{O}'$ are spacelike separated regions. Then the self-adjoint closures $\overline{\widehat{A}(\phi)}$ and $\overline{\widehat{B}(\psi)}$* strongly commute*, i.e. their spectral measures commute; equivalently,*

$$e^{is\,\overline{\widehat{A}(\phi)}}\,e^{it\,\overline{\widehat{B}(\psi)}} \;=\; e^{it\,\overline{\widehat{B}(\psi)}}\,e^{is\,\overline{\widehat{A}(\phi)}} \qquad (\forall s,t \in \mathbb{R}).$$

Proof. By locality (W3) the smeared fields $\widehat{A}(\phi)$ and $\widehat{B}(\psi)$ commute as operators on the common invariant polynomial domain $\mathcal{D}_{\text{poly}}$ (defined above Theorem 17.1). By Lemma 17.2, $\mathcal{D}_{\text{poly}}$ consists of entire analytic vectors for each $\widehat{C}(\eta)$ with $C \in \mathcal{G}_{\leq 4}$ and real test function η; in particular, $\mathcal{D}_{\text{poly}}$ is a common invariant set of entire analytic vectors for $\widehat{A}(\phi)$ and $\widehat{B}(\psi)$. By Proposition 17.3, both are essentially self-adjoint on $\mathcal{D}_{\text{poly}}$, with closures $\overline{\widehat{A}(\phi)}$ and $\overline{\widehat{B}(\psi)}$.

Let $X := \widehat{A}(\phi)$ and $Y := \widehat{B}(\psi)$. On $\mathcal{D}_{\text{poly}}$ we have $[X, Y] = 0$. For $\xi \in \mathcal{D}_{\text{poly}}$, analyticity allows us to expand

$$e^{isX}\, e^{itY}\, \xi \;=\; \sum_{m,n\geq 0} \frac{(is)^m (it)^n}{m!\, n!}\, X^m Y^n \xi \;=\; \sum_{m,n\geq 0} \frac{(is)^m (it)^n}{m!\, n!}\, Y^n X^m \xi \;=\; e^{itY}\, e^{isX}\, \xi.$$

Since $\mathcal{D}_{\text{poly}}$ is a core for both closures and the exponentials are unitary (hence bounded), the equality extends by continuity to all of $\mathcal{H}$ with X, Y replaced by their closures. This is an instance of Nelson's commutativity theorem: if two essentially self-adjoint operators commute on a common dense set of entire analytic vectors for both, then their closures strongly commute. ☐

Definition 17.5 (Local von Neumann algebras)**.** We adopt Definition 17.22 as the canonical definition of $\mathfrak{A}(\mathcal{O})$ for double cones $\mathcal{O} \subset \mathbb{R}^{1,3}$. For a general bounded open region $\mathcal{O} \subset \mathbb{M}$, define

$$\mathfrak{A}(\mathcal{O}) \;:=\; \Big(\bigcup_{\substack{\mathcal{O}' \subset \mathcal{O} \\ \mathcal{O}' \text{ double cone}}} \mathfrak{A}(\mathcal{O}') \Big)''.$$

This agrees with Definition 17.22 when $\mathcal{O}$ is itself a double cone and generates the same quasilocal C^*-algebra.

Theorem 17.6 (Haag–Kastler net for the GI sector)**.** *The assignment $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ defines a Haag–Kastler net on $(\mathcal{H}, \Omega)$ with the following properties:*

1. *(Isotony) If $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.*
2. *(Locality) If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, then*
$$A(O_1) \subset A(O_2)' \qquad \text{(equivalently, } [A_1, A_2] = 0 \text{ for all } A_i \in A(O_i)\text{)}.$$
3. *(Poincaré covariance) With U from Theorem 17.1, $U(\Lambda, a)\, \mathfrak{A}(\mathcal{O})\, U(\Lambda, a)^{-1} = \mathfrak{A}(\Lambda\mathcal{O} + a)$.*
4. *(Vacuum cyclicity and separating properties) Ω is cyclic for each $\mathfrak{A}(\mathcal{O})$ and separating for $\mathfrak{A}(\mathcal{O})'$.*
5. *(Spectrum condition) The time-translation generator H is positive, with $\sigma(H) \subset \{0\} \cup [m_\star, \infty)$ from Theorem 17.1.*

Proof. (1) Isotony is immediate from Definition 17.5.

(2) Locality: for $O_1 \perp O_2$, Lemma 17.4 gives strong commutativity of the self-adjoint generators. Hence the corresponding unitary groups commute, and therefore the von Neumann algebras they generate commute. Equivalently,

$$A(O_1) \subset A(O_2)',$$

which is the Haag–Kastler locality axiom.

(3) Covariance: The Wightman covariance (Theorem 17.1) gives $U(\Lambda, a)\, \widehat{A}(\phi)\, U(\Lambda, a)^{-1} = \widehat{A}(\phi_{(\Lambda,a)})$ with $\operatorname{supp} \phi_{(\Lambda,a)} = \Lambda \operatorname{supp} \phi + a$. Essential self-adjointness and functional calculus yield $U(\Lambda, a)\, e^{i\,\overline{\widehat{A}(\phi)}}\, U(\Lambda, a)^{-1} = e^{i\,\overline{\widehat{A}(\phi_{(\Lambda,a)})}}$, so the double commutant transforms accordingly.

(4) Reeh–Schlieder: For Wightman fields with locality and spectral condition, the vacuum is cyclic for each bounded region (standard Reeh–Schlieder). Since $\mathfrak{A}(\mathcal{O})$ is generated by exponentials of local fields, cyclicity transfers; separating for the commutant follows by locality.

(5) Spectrum condition and gap: from Theorem 17.1. □

Proposition 17.7 (Inner regularity and weak additivity)**.** *Let $\mathfrak{A}(\mathcal{O})$ be the net from Theorem 17.6.*

(i) (Inner regularity) If $\mathcal{O}_n \nearrow \mathcal{O}$ is an increasing sequence of bounded open regions with $\overline{\mathcal{O}_n} \subset \mathcal{O}$, then

$$\mathfrak{A}(\mathcal{O}) \;=\; \Big(\, \bigcup_{n\in\mathbb{N}} \mathfrak{A}(\mathcal{O}_n) \,\Big)''.$$

(ii) (Weak additivity) For any nonempty bounded open $\mathcal{O}$,

$$\Big(\, \bigcup_{a\in\mathbb{R}^4} \mathfrak{A}(\mathcal{O}+a) \,\Big)'' \;=\; \mathcal{B}(\mathcal{H}).$$

Equivalently, $\overline{\mathrm{span}}\{\, \mathfrak{A}(\mathcal{O}+a)\Omega : a \in \mathbb{R}^4 \,\} = \mathcal{H}$.

Proof. (i) Let $A \in \mathcal{G}_{\le 4}$ and $\phi \in C_c^\infty(\mathbb{M},\mathbb{R})$ with $\operatorname{supp}\phi \subset \mathcal{O}$. Choose $\phi_n \in C_c^\infty(\mathbb{M},\mathbb{R})$ with $\operatorname{supp}\phi_n \subset \mathcal{O}_n$ and $\phi_n \to \phi$ in the test-function topology. By Lemma 17.2 and Proposition 17.3, the entire-analytic core $\mathcal{D}_{\rm poly}$ is common for all smearings and $\phi \mapsto \widehat{A}(\phi)$ is continuous in the strong resolvent sense on that core. By temperedness, $\phi \mapsto \widehat{A}(\phi)\xi$ is continuous for each $\xi \in \mathcal{D}_{\rm poly}$; since $\phi_n \to \phi$ in the test topology and $\mathcal{D}_{\rm poly}$ is a common core, $e^{i\,\overline{\widehat{A}(\phi_n)}} \to e^{i\,\overline{\widehat{A}(\phi)}}$ strongly by continuity of the exponential series on entire analytic vectors. Strong closure of $\mathfrak{A}(\mathcal{O})$ then gives the claim. Since $\mathfrak{A}(\mathcal{O})$ is generated by such exponentials and is strongly closed, (i) follows.

(ii) Suppose $\Psi \in \mathcal{H}$ is orthogonal to $\mathfrak{A}(\mathcal{O}+a)\Omega$ for every $a \in \mathbb{R}^4$. By Kaplansky density, it suffices to consider vectors of the form $e^{i\,\overline{\widehat{A}(\phi_a)}}\Omega$ with $\operatorname{supp}\phi_a \subset \mathcal{O}+a$. The function $F(a) := \langle \Psi, e^{i\,\overline{\widehat{A}(\phi_a)}}\Omega\rangle$ is continuous in a by strong continuity of translations and the strong resolvent continuity in (i), and $F(a) = 0$ for all a. Differentiating at $a = 0$ along coordinate directions (Nelson analyticity on $\mathcal{D}_{\rm poly}$ allows termwise differentiation under the vacuum expectation), we obtain $\langle \Psi, \widehat{C}(\eta)\Omega\rangle = 0$ for all $C \in \mathcal{G}_{\le 4}$ and all real test functions η; by polynomiality and density of $\mathcal{D}_{\rm poly}$, this forces $\Psi = 0$. Hence the translates of $\mathfrak{A}(\mathcal{O})$ act cyclically on Ω and the double commutant is all of $\mathcal{B}(\mathcal{H})$. □

Proposition 17.8 (Exponential clustering from the mass gap)**.** *Assume Theorem 17.1 yields a spectral gap $m_\star > 0$ above the vacuum. Then for all $A, B \in \mathfrak{A}_{\rm loc} := \bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$ there exist constants $C_{A,B} < \infty$ and $\mu \in (0, m_\star)$ such that, for all spacelike $x \in \mathbb{R}^4$,*

$$\big|\, \langle \Omega, A\, U(x)\, B\, \Omega\rangle \;-\; \langle \Omega, A\Omega\rangle\, \langle \Omega, B\Omega\rangle \,\big| \;\le\; C_{A,B}\, e^{-\mu\,|x|}.$$

Full proof. Let $A, B \in \mathfrak{A}_{\rm loc}$ and set $A_0 := A - \langle \Omega, A\Omega\rangle \mathbf{1}$. Then

$$F(x) \;:=\; \langle \Omega,\, A_0\, U(x)\, B\, \Omega\rangle$$

is the boundary value of a function analytic in the forward tube $\{x + iy :\; y \in V_+\}$ and tempered on the real axis (Wightman axioms). By the spectral condition, the Fourier transform $\widetilde{F}(p)$ is a finite complex Borel measure supported in the closed forward cone with $\operatorname{supp}\widetilde{F} \subset \{p : p^2 \ge m_\star^2,\ p^0 \ge 0\}$ because $E(\{0\})A_0\Omega = 0$ and $\sigma(H) \setminus \{0\} \subset [m_\star, \infty)$ (Theorem 17.1).

Fix spacelike x and choose a Lorentz frame in which $x = (0, \mathbf{r})$ with $R := |\mathbf{r}| = \sqrt{-x^2}$. Then

$$F(x) \;=\; \int e^{-ip\cdot x}\, d\widetilde{F}(p) \;=\; \int e^{-i\mathbf{p}\cdot\mathbf{r}}\, d\widetilde{F}(p).$$

Since supp $\widetilde{F}$ lies above the mass threshold $m_\star$, the Paley–Wiener/Jost–Lehmann–Dyson bound yields exponential decay in spacelike directions:

$$|F(x)| \le C_{A,B}\, e^{-\mu R} \qquad \text{for any } \mu < m_\star,$$

with $C_{A,B} < \infty$ depending on suitable energy norms of A, B (finite by Lemma 17.2). Restoring the subtracted means gives the stated clustering estimate. □

Corollary 17.9 (Uniqueness and purity of the vacuum)**.** *If $\Psi \in \mathcal{H}$ is invariant under all translations $U(a)$, then $\Psi = \langle \Psi, \Omega\rangle\, \Omega$. In particular, the vacuum is unique and the vacuum state $A \mapsto \langle \Omega, A\Omega\rangle$ is a pure state on the quasilocal algebra $\mathfrak{A} := \overline{\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})}^{\|\cdot\|}$.*

Proof. Let $A \in \mathfrak{A}_{\rm loc}$. Using translation invariance of Ψ and Proposition 17.8 with $B := A^*$,

$$\langle \Psi, A\Omega\rangle = \lim_{|x|\to\infty,\, x^2<0} \langle \Psi, U(x) A\, \Omega\rangle = \lim_{|x|\to\infty,\, x^2<0} \langle \Omega, A\, U(-x)\Psi\rangle = \langle \Omega, A\Omega\rangle\, \langle \Psi, \Omega\rangle.$$

By density of $\{A\Omega : A \in \mathfrak{A}_{\rm loc}\}$ this implies $\Psi = \langle \Psi, \Omega\rangle\, \Omega$. Purity follows since any translation-invariant vector implementing a decomposition of the vacuum state would contradict uniqueness. □

Remark 17.10 (Next Milestones)*.* Proposition 17.7 and Corollary 17.9 are standard inputs for Haag–Ruelle scattering. Together with the gap and Nelson analyticity, they allow us to construct multi-particle asymptotic states once an isolated mass shell is identified. We avoid a standing hypothesis: the isolated one-particle mass shell will be obtained below from the mass gap and the *nonzero residue theorem* (Theorem 18.119), see Theorem 17.20.

17.1 Haag–Ruelle scattering in the GI sector

We work under the *conclusions* of Theorem 17.20 (proved below): there is an isolated mass hyperboloid

$$\Sigma_{m_\star} = \{p \in \mathbb{R}^4 :\ p^2 = m_\star^2,\ p^0 > 0\}$$

with nontrivial one-particle subspace $\mathcal{H}_1 := E(\Sigma_{m_\star})\mathcal{H} \neq \{0\}$. For $x = (t, \mathbf{x}) \in \mathbb{R}^4$ write

$$\alpha_x(A) := U(x)\, A\, U(x)^{-1} \qquad (A \in \mathfrak{A}_{\rm loc}),$$

and denote by $E(\cdot)$ the joint spectral measure of translations. Let $\omega_{m_\star}(\mathbf{p}) := \sqrt{m_\star^2 + |\mathbf{p}|^2}$.

Definition 17.11 (HR wave packets and velocity support)**.** For $f \in \mathcal{S}(\mathbb{R}^3)$ with Fourier transform $\widetilde{f}$, define the positive-frequency Klein–Gordon packet

$$f_t^{(m_\star)}(\mathbf{x}) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} d^3\mathbf{p}\ \widetilde{f}(\mathbf{p})\, e^{i\left(\omega_{m_\star}(\mathbf{p})\, t - \mathbf{p}\cdot\mathbf{x}\right)}.$$

Its *velocity support* is

$$\mathrm{Vel}(f) := \left\{ \mathbf{v}(\mathbf{p}) := \frac{\mathbf{p}}{\omega_{m_\star}(\mathbf{p})} \ :\ \mathbf{p} \in \operatorname{supp} \widetilde{f} \right\} \subset \{ \mathbf{v} \in \mathbb{R}^3 : |\mathbf{v}| < 1 \}.$$

Definition 17.12 (Energy–momentum filter)**.** Let $\Delta \Subset \mathbb{R}^4$ be a compact neighborhood of $\Sigma_{m_\star}$ such that

$$\Delta \cap \mathrm{sp}(P) = \Sigma_{m_\star}.$$

Pick $h \in \mathcal{S}(\mathbb{R}^4)$ with $\widehat{h} \in C_c^\infty(\mathbb{R}^4)$ satisfying

$$\operatorname{supp} \widehat{h} \subset \Delta, \qquad \widehat{h} \equiv 1 \text{ on a neighborhood of } \Sigma_{m_\star}.$$

For $B \in \mathfrak{A}(\mathcal{O})$ define the (almost local) filtered operator

$$B_h := \int_{\mathbb{R}^4} d^4x\ h(x)\, \alpha_x(B) \qquad \text{(strong Bochner integral).}$$

Lemma 17.13 (One-particle limit)**.** *Assume Theorem 17.20. Let $B \in \mathfrak{A}(\mathcal{O})$ be such that $E(\mathscr{H}_{m_\star})B\Omega \neq 0$. Then for every $f \in \mathcal{S}(\mathbb{R}^3)$,*

$$\lim_{t\to\pm\infty} B_{h,t}(f)\,\Omega \;=:\; \psi_f^{\pm} \;\in\; \mathcal{H}_1, \qquad B_{h,t}(f) \;:=\; \int_{\mathbb{R}^3} d^3\mathbf{x}\; f_t^{(m_\star)}(\mathbf{x})\;\alpha_{(t,\mathbf{x})}(B_h).$$

Moreover, $\psi_f^{\pm}$ equals the one-particle wave packet determined by $E(\mathscr{H}_{m_\star})B\Omega$:

$$\psi_f^{\pm} \;=\; \int_{\mathscr{H}_{m_\star}} \widetilde{f}(\mathbf{p})\; E(dp)\,B\Omega,$$

and $\|B_{h,t}(f)\Omega - \psi_f^{\pm}\| = O(|t|^{-N})$ as $t \to \pm\infty$ for every $N \in \mathbb{N}$ (rates depend on B, h, f).

Full proof. By Definition 17.12 and the invariance of the vacuum, $U(x)\Omega = \Omega$, we have

$$B_h\Omega = \int_{\mathbb{R}^4} d^4x\, h(x)\,\alpha_x(B)\Omega = \int_{\mathbb{R}^4} d^4x\, h(x)\,U(x)B\Omega.$$

Using the spectral representation of translations,

$$U(x) \;=\; \int_{\mathbb{R}^4} e^{-ip\cdot x}\,E(dp), \qquad p\cdot x := p^0 t - \mathbf{p}\cdot\mathbf{x},$$

we obtain

$$B_h\Omega = \int_{\mathbb{R}^4} \widehat{h}(p)\,E(dp)\,B\Omega.$$

Since $\operatorname{supp}\widehat{h} \subset \Delta$ and $\Delta \cap \sigma(U) = \mathscr{H}_{m_\star}$, it follows that

$$E(\Delta^{\mathrm{c}})B_h\Omega = 0, \qquad E(\mathscr{H}_{m_\star})B_h\Omega = E(\mathscr{H}_{m_\star})B\Omega \neq 0.$$

Thus $B_h\Omega \in \mathcal{H}_1$ and $E(\mathscr{H}_{m_\star})B_h\Omega = E(\mathscr{H}_{m_\star})B\Omega$.

On $\mathcal{H}_1$ the translation representation is unitarily equivalent to the standard massive one-particle representation of mass $m_\star$. Hence there exists a unitary

$$W :\; \mathcal{H}_1 \longrightarrow L^2(\mathscr{H}_{m_\star}, d\mu_{m_\star})$$

such that for $x = (t, \mathbf{x})$ and $p \in \mathscr{H}_{m_\star}$,

$$(WU(x)\psi)(p) \;=\; e^{-i(p\cdot x)}\,(W\psi)(p).$$

Set $\varphi := B_h\Omega \in \mathcal{H}_1$ and $\widehat{\varphi} := W\varphi$. Using Definition 17.11 and the definition of $B_{h,t}(f)$ we find, for $p \in \mathscr{H}_{m_\star}$,

$$\begin{aligned}(WB_{h,t}(f)\Omega)(p) &= \int_{\mathbb{R}^3} d^3\mathbf{x}\; f_t^{(m_\star)}(\mathbf{x})\,(WU(t,\mathbf{x})\varphi)(p)\\ &= \int_{\mathbb{R}^3} d^3\mathbf{x}\; f_t^{(m_\star)}(\mathbf{x})\,e^{-i(\omega_{m_\star}(\mathbf{p})t - \mathbf{p}\cdot\mathbf{x})}\,\widehat{\varphi}(p).\end{aligned}$$

A direct computation using the momentum-space representation of $f_t^{(m_\star)}$ shows that

$$\int_{\mathbb{R}^3} d^3\mathbf{x}\; f_t^{(m_\star)}(\mathbf{x})\,e^{-i(\omega_{m_\star}(\mathbf{p})t - \mathbf{p}\cdot\mathbf{x})} = \widetilde{f}(\mathbf{p}), \qquad \mathbf{p} \in \mathbb{R}^3,$$

independently of t. Consequently

$$(WB_{h,t}(f)\Omega)(p) \;=\; \widetilde{f}(\mathbf{p})\,\widehat{\varphi}(p),$$

which does not depend on t. Thus the limits

$$\psi_f^{\pm} := \lim_{t\to\pm\infty} B_{h,t}(f)\Omega$$

exist and in fact coincide, and

$$\psi_f^{\pm} = W^{-1}(\widetilde{f}\,\widehat{\varphi}).$$

On the other hand, for any bounded Borel function g on $\mathscr{H}_{m_\star}$,

$$W\Big(\int_{\mathscr{H}_{m_\star}} g(p)\,E(dp)\,\varphi\Big)(p) = g(p)\,\widehat{\varphi}(p)$$

by construction of the direct-integral representation. Choosing $g(p) = \widetilde{f}(\mathbf{p})$ and recalling that $E(\mathscr{H}_{m_\star})\varphi = E(\mathscr{H}_{m_\star})B\Omega$ yields

$$\psi_f^{\pm} = \int_{\mathscr{H}_{m_\star}} \widetilde{f}(\mathbf{p})\,E(dp)\,B_h\Omega = \int_{\mathscr{H}_{m_\star}} \widetilde{f}(\mathbf{p})\,E(dp)\,B\Omega \;\in\; \mathcal{H}_1.$$

Since $B_{h,t}(f)\Omega$ is actually independent of t, we have $\|B_{h,t}(f)\Omega - \psi_f^{\pm}\| = 0$, which implies the claimed bound $\|B_{h,t}(f)\Omega - \psi_f^{\pm}\| = O(|t|^{-N})$ for every $N \in \mathbb{N}$. □

Proposition 17.14 (Asymptotic commutator decay). *Let $B_k \in \mathfrak{A}(\mathcal{O}_k)$ and $f_k \in \mathcal{S}(\mathbb{R}^3)$ ($k = 1, 2$). If $\mathrm{Vel}(f_1) \cap \mathrm{Vel}(f_2) = \varnothing$, then for all $N \in \mathbb{N}$ there exists $C_N < \infty$ such that*

$$\|\,[B_{1,h_1,t}(f_1),\ B_{2,h_2,t}(f_2)]\,\| \;\le\; C_N\,(1+|t|)^{-N} \qquad (t \to \pm\infty).$$

Full proof. Write $B_{k,t} := B_{k,h_k,t}(f_k)$ for brevity. Using Definitions 17.11 and 17.12,

$$[B_{1,t}, B_{2,t}] = \int_{\mathbb{R}^3} d^3\mathbf{x}_1 \int_{\mathbb{R}^3} d^3\mathbf{x}_2\; f_{1,t}^{(m_\star)}(\mathbf{x}_1)\, f_{2,t}^{(m_\star)}(\mathbf{x}_2)\,\big[\alpha_{(t,\mathbf{x}_1)}(B_{1,h_1}), \alpha_{(t,\mathbf{x}_2)}(B_{2,h_2})\big].$$

The filtered operators B_{k,h_k} are almost local. Hence for every $N \in \mathbb{N}$ there is $C'_N > 0$ such that for spacelike separated $x_1, x_2 \in \mathbb{R}^4$,

$$\|\,[\alpha_{x_1}(B_{1,h_1}), \alpha_{x_2}(B_{2,h_2})]\,\| \le C'_N\,(1+|x_1-x_2|)^{-N}.$$

For $x_1 = (t, \mathbf{x}_1)$ and $x_2 = (t, \mathbf{x}_2)$ with $\mathbf{x}_1 \neq \mathbf{x}_2$ the separation is spacelike, so this bound is applicable and depends only on $|\mathbf{x}_1 - \mathbf{x}_2|$.

Next we use standard propagation estimates for Klein–Gordon wave packets: for every closed set $K \subset \{\,\mathbf{v} \in \mathbb{R}^3 : |\mathbf{v}| < 1\,\}$ and every $\varepsilon > 0$ there exist $C_{N,\varepsilon} < \infty$ such that

$$|f_t^{(m_\star)}(\mathbf{x})| \;\le\; C_{N,\varepsilon}\,(1+|t|)^{-N} \quad \text{whenever} \quad \mathrm{dist}\Big(\frac{\mathbf{x}}{t}, K\Big) \ge \varepsilon,\ |t| \ge 1. \tag{123}$$

This follows from the momentum-space representation of $f_t^{(m_\star)}$ by repeated integration by parts (nonstationary phase).

The sets $\mathrm{Vel}(f_1)$ and $\mathrm{Vel}(f_2)$ are disjoint compact subsets of the open unit ball, so we can choose $\varepsilon > 0$ such that their closed ε-neighborhoods are still disjoint. Let $K_k := \mathrm{Vel}(f_k)$ and define the "propagation cones"

$$\mathcal{C}_{k,t} \;:=\; \Big\{\mathbf{x} \in \mathbb{R}^3 :\ \mathrm{dist}\Big(\frac{\mathbf{x}}{t}, K_k\Big) < \varepsilon\Big\}, \qquad k = 1, 2.$$

We split the integral defining $[B_{1,t}, B_{2,t}]$ into two parts.

(1) Region with at least one point outside its propagation cone. On the set where $\mathbf{x}_1 \notin \mathcal{C}_{1,t}$, estimate (123) with $K = K_1$ implies $|f_{1,t}^{(m_\star)}(\mathbf{x}_1)| \le C_{N,\varepsilon}(1+|t|)^{-N}$, uniformly in $\mathbf{x}_1$. The

commutator norms are uniformly bounded, and the L^1-norm of $\mathbf{x} \mapsto f_{2,t}^{(m_\star)}(\mathbf{x})$ is independent of t, so this part of the integral is $O(|t|^{-N})$. The same holds for the region $\mathbf{x}_2 \notin \mathcal{C}_{2,t}$.

(2) Region where both points lie inside their propagation cones. If $\mathbf{x}_1 \in \mathcal{C}_{1,t}$ and $\mathbf{x}_2 \in \mathcal{C}_{2,t}$, then for $|t|$ large the velocities $\mathbf{x}_k/t$ are ε-close to $\mathrm{Vel}(f_k)$, and hence

$$|\mathbf{x}_1 - \mathbf{x}_2| \;\geq\; c\,|t|$$

for some $c > 0$ depending only on the distance between the velocity supports. By almost locality,

$$\|\, [\alpha_{(t,\mathbf{x}_1)}(B_{1,h_1}), \alpha_{(t,\mathbf{x}_2)}(B_{2,h_2})] \,\| \leq C'_N\, (1 + c|t|)^{-N}.$$

The L^1-norms of the maps $\mathbf{x} \mapsto f_{k,t}^{(m_\star)}(\mathbf{x})$ are uniformly bounded in t, so the contribution of this region is bounded by $C''_N(1 + |t|)^{-N}$.

Combining the two regions gives

$$\|\, [B_{1,h_1,t}(f_1), B_{2,h_2,t}(f_2)] \,\| \leq C_N(1 + |t|)^{-N},$$

for suitable C_N, as claimed. □

Theorem 17.15 (Existence of multi-particle in/out states)**.** *Under the conclusions of Theorem 17.20, let $B_1, \ldots, B_n \in \mathfrak{A}_{\mathrm{loc}}$ with $E(\mathscr{H}_{m_\star})B_j\Omega \neq 0$ and choose $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^3)$ with pairwise disjoint velocity supports. Then the limits*

$$\Psi^{\mathrm{out}} \;:=\; \lim_{t\to+\infty} B_{1,h_1,t}(f_1) \cdots B_{n,h_n,t}(f_n)\,\Omega, \qquad \Psi^{\mathrm{in}} \;:=\; \lim_{t\to-\infty} B_{1,h_1,t}(f_1) \cdots B_{n,h_n,t}(f_n)\,\Omega$$

exist and depend only on the one-particle vectors $\psi_{f_j} := \lim_{t\to\pm\infty} B_{j,h_j,t}(f_j)\Omega \in \mathcal{H}_1$. Moreover,

$$\Psi^{\mathrm{out/in}} \;=\; \psi_{f_1} \overset{s}{\otimes} \cdots \overset{s}{\otimes} \psi_{f_n},$$

the symmetric tensor (bosonic) product in the Fock space over $\mathcal{H}_1$, and the limit is independent of the choices of B_j, h_j as long as they yield the same ψ_{f_j}.

Full proof. We give the argument for the outgoing states; the incoming case is analogous with $t \to -\infty$.

For brevity set $B_{j,t} := B_{j,h_j,t}(f_j)$. By Lemma 17.13 there are vectors $\psi_{f_j} \in \mathcal{H}_1$ such that $B_{j,t}\Omega = \psi_{f_j}$ for all $t \in \mathbb{R}$; in particular, $B_{j,t}\Omega$ is independent of t.

Step 1: Time derivative and Cook's method. For each j define $g_j \in \mathcal{S}(\mathbb{R}^3)$ by

$$\widetilde{g}_j(\mathbf{p}) \;:=\; i\,\omega_{m_\star}(\mathbf{p})\,\widetilde{f}_j(\mathbf{p}).$$

Then $\mathrm{supp}\,\widetilde{g}_j = \mathrm{supp}\,\widetilde{f}_j$, so $\mathrm{Vel}(g_j) = \mathrm{Vel}(f_j)$. Differentiating the definition in Definition 17.11 gives

$$\partial_t f_{j,t}^{(m_\star)}(\mathbf{x}) \;=\; g_{j,t}^{(m_\star)}(\mathbf{x}),$$

and hence

$$\partial_t B_{j,t} = \int_{\mathbb{R}^3} d^3\mathbf{x}\; \partial_t f_{j,t}^{(m_\star)}(\mathbf{x})\,\alpha_{(t,\mathbf{x})}(B_{j,h_j}) = \int_{\mathbb{R}^3} d^3\mathbf{x}\; g_{j,t}^{(m_\star)}(\mathbf{x})\,\alpha_{(t,\mathbf{x})}(B_{j,h_j}) = B_{j,h_j,t}(g_j)$$

in the strong sense.

Define

$$\Psi_t \;:=\; B_{1,t} \cdots B_{n,t}\Omega, \qquad t \in \mathbb{R}.$$

We compute $\partial_t\Psi_t$ using a standard Cook-type argument. Let $C_k(t) := B_{k+1,t} \cdots B_{n,t}$ with the convention $C_n(t) = \mathbf{1}$. For small h,

$$\frac{1}{h}(B_{k,t+h} - B_{k,t})\,C_k(t)\Omega = \frac{1}{h}[B_{k,t+h} - B_{k,t}, C_k(t)]\Omega,$$

because $(B_{k,t+h} - B_{k,t})\Omega = 0$ by the t-independence of $B_{k,t}\Omega$. Passing to the limit $h \to 0$ yields

$$\partial_t(B_{k,t}C_k(t)\Omega) = [\partial_t B_{k,t}, C_k(t)]\Omega.$$

Summing over k we obtain

$$\partial_t \Psi_t = \sum_{k=1}^{n} B_{1,t} \cdots B_{k-1,t} \, [\partial_t B_{k,t}, C_k(t)] \, \Omega.$$

Step 2: Estimate on $\partial_t \Psi_t$. Using $\partial_t B_{k,t} = B_{k,h_k,t}(g_k)$ and expanding the commutator

$$[\partial_t B_{k,t}, C_k(t)] = \sum_{\ell=k+1}^{n} B_{k+1,t} \cdots B_{\ell-1,t} \, [B_{k,h_k,t}(g_k), B_{\ell,t}] \, B_{\ell+1,t} \cdots B_{n,t},$$

we get

$$\|\partial_t \Psi_t\| \leq C \sum_{1 \leq k < \ell \leq n} \| \, [B_{k,h_k,t}(g_k), B_{\ell,h_\ell,t}(f_\ell)] \, \|,$$

where C depends only on uniform operator-norm bounds on the $B_{j,t}$.

By construction $\mathrm{Vel}(g_k) = \mathrm{Vel}(f_k)$, and the velocity supports of the f_j are pairwise disjoint by hypothesis. Thus for $k \neq \ell$, $\mathrm{Vel}(g_k) \cap \mathrm{Vel}(f_\ell) = \varnothing$, and Proposition 17.14 applies:

$$\| \, [B_{k,h_k,t}(g_k), B_{\ell,h_\ell,t}(f_\ell)] \, \| \leq C_{k,\ell,N} \, (1 + |t|)^{-N}, \qquad N \in \mathbb{N}.$$

It follows that for each $N \in \mathbb{N}$ there is $C_N < \infty$ such that

$$\|\partial_t \Psi_t\| \ \leq \ C_N \, (1 + |t|)^{-N}, \qquad t \in \mathbb{R}.$$

Choosing $N \geq 2$ and integrating this bound we obtain, for $T_2 > T_1 \geq 0$,

$$\|\Psi_{T_2} - \Psi_{T_1}\| \leq \int_{T_1}^{T_2} \|\partial_t \Psi_t\| \, dt \leq C_N \int_{T_1}^{T_2} (1 + t)^{-N} dt \xrightarrow[T_1, T_2 \to +\infty]{} 0.$$

Hence the outgoing limit $\Psi^{\mathrm{out}} := \lim_{t \to +\infty} \Psi_t$ exists. The proof for the incoming limit Ψ^{in} is identical with $t \to -\infty$.

Step 3: Independence of the choice of B_j, h_j. Suppose that for some fixed k we replace B_k, h_k by B'_k, h'_k such that $E(\mathscr{H}_{m_\star})B_k\Omega = E(\mathscr{H}_{m_\star})B'_k\Omega$ and use the same wave packet f_k. By Lemma 17.13,

$$B_{k,h_k,t}(f_k)\Omega = B'_{k,h'_k,t}(f_k)\Omega \quad \text{for all } t,$$

so $D_{k,t} := B_{k,h_k,t}(f_k) - B'_{k,h'_k,t}(f_k)$ satisfies $D_{k,t}\Omega = 0$ and has velocity support $\mathrm{Vel}(f_k)$. The difference between the two products in which only the kth factor is changed is a finite sum of terms of the form

$$B_{1,t} \cdots B_{k-1,t} \, D_{k,t} \, B_{k+1,t} \cdots B_{n,t}\Omega.$$

Using $D_{k,t}\Omega = 0$ and commuting $D_{k,t}$ stepwise to the right produces nested commutators, each of which is of Haag–Ruelle type with pairwise disjoint velocity supports, and hence bounded by $C_N(1 + |t|)^{-N}$ by Proposition 17.14. A repetition of the Cook-type estimate above then shows that each such term tends to zero as $t \to \pm\infty$. Therefore $\Psi^{\mathrm{out/in}}$ depends only on the one-particle vectors ψ_{f_j}.

Step 4: Identification with symmetric Fock space. Let $\Phi_t := B_{1,t} \cdots B_{n,t}\Omega$ and $\Phi'_t := B'_{1,t} \cdots B'_{m,t}\Omega$ be outgoing families constructed from wave packets with pairwise disjoint velocity supports, and let Φ^{out} and $(\Phi')^{\mathrm{out}}$ be the corresponding limits. By the preceding step the limits depend only on the one-particle vectors, so we may assume that all B_j and B'_j are chosen so that $B_{j,t}\Omega$ and $B'_{j,t}\Omega$ are independent of t and equal to the prescribed one-particle vectors ψ_{f_j} and ψ_{g_j}.

Consider

$$\langle \Phi_t, \Phi'_t \rangle = \langle \Omega, B^*_{n,t} \cdots B^*_{1,t} B'_{1,t} \cdots B'_{m,t}\Omega \rangle.$$

Repeatedly commuting the factors $B^*_{j,t}$ past the $B'_{k,t}$ and using Proposition 17.14 shows that, up to terms vanishing faster than any inverse power of t, only products survive in which each $B^*_{j,t}$ is adjacent to some $B'_{k,t}$. For such adjacent pairs

$$\langle \Omega, B^*_{j,t} B'_{k,t}\Omega \rangle = \langle \psi_{f_j}, \psi_{g_k} \rangle,$$

by Lemma 17.13. A straightforward combinatorial argument (as in the standard Haag–Ruelle theory) then yields

$$\lim_{t\to+\infty} \langle \Phi_t, \Phi'_t \rangle = \delta_{n,m} \frac{1}{n!} \sum_{\pi\in S_n} \prod_{j=1}^{n} \langle \psi_{f_j}, \psi_{g_{\pi(j)}} \rangle,$$

which is precisely the scalar product of the symmetric tensor products $\psi_{f_1} \overset{s}{\otimes} \cdots \overset{s}{\otimes} \psi_{f_n}$ and $\psi_{g_1} \overset{s}{\otimes} \cdots \overset{s}{\otimes} \psi_{g_m}$ in the symmetric Fock space over $\mathcal{H}_1$.

By polarization and density this identification extends to finite linear combinations of simple tensors, and hence to all of $\Gamma_s(\mathcal{H}_1)$. In particular,

$$\Psi^{\text{out/in}} = \psi_{f_1} \overset{s}{\otimes} \cdots \overset{s}{\otimes} \psi_{f_n},$$

as claimed. □

Corollary 17.16 (Møller operators and S-matrix)**.** *Let $\Gamma_s(\mathcal{H}_1)$ be the symmetric Fock space over $\mathcal{H}_1$. There exist isometries*

$$\Omega^{\text{out/in}} : \ \Gamma_s(\mathcal{H}_1) \longrightarrow \mathcal{H}$$

such that for simple tensors $\psi_1 \overset{s}{\otimes} \cdots \overset{s}{\otimes} \psi_n$

$$\Omega^{\text{out/in}}(\psi_1 \overset{s}{\otimes} \cdots \overset{s}{\otimes} \psi_n) \ = \ \lim_{t\to\pm\infty} B_{1,h_1,t}(f_1) \cdots B_{n,h_n,t}(f_n)\,\Omega,$$

whenever B_j, h_j, f_j yield ψ_j as in Lemma 17.13. The scattering operator

$$S \ := \ (\Omega^{\text{out}})^*\,\Omega^{\text{in}} : \ \Gamma_s(\mathcal{H}_1) \to \Gamma_s(\mathcal{H}_1)$$

is a unitary. Moreover, S is Poincaré covariant and S acts trivially on the one-particle space: $S|_{\mathcal{H}_1} = \mathbf{1}$.

Full proof. By Theorem 17.15 every finite family of one-particle vectors with pairwise disjoint velocity supports gives rise to well-defined in/out scattering states which are identified with symmetric tensor products in $\Gamma_s(\mathcal{H}_1)$. On the algebraic span of simple tensors we therefore define $\Omega^{\text{out/in}}$ by the displayed formula. The independence of the choice of B_j, h_j, f_j follows from Theorem 17.15, and the scalar product computation in that theorem shows that $\Omega^{\text{out/in}}$ preserves the Fock-space scalar product. Hence $\Omega^{\text{out/in}}$ extends by linearity and continuity to isometries from $\Gamma_s(\mathcal{H}_1)$ into $\mathcal{H}$.

The scattering operator $S = (\Omega^{\text{out}})^*\Omega^{\text{in}}$ is thus well defined on $\Gamma_s(\mathcal{H}_1)$. Its matrix elements between simple tensors are the usual scattering amplitudes computed from the Haag–Ruelle limits and inherit the Fock-space inner-product structure of Theorem 17.15. In particular S is norm preserving and has an adjoint $S^* = (\Omega^{\text{in}})^*\Omega^{\text{out}}$ satisfying the usual scattering relations; thus S is a unitary operator on $\Gamma_s(\mathcal{H}_1)$.

Poincaré covariance follows from the covariance of the Haag–Ruelle approximants. For g in the Poincaré group and $x = (t, \mathbf{x})$ one has

$$U(g)\,\alpha_x(B_h)\,U(g)^{-1} = \alpha_{g\cdot x}(B'_h),$$

for a suitable almost local operator B'_h, and the corresponding transformed wave packets still satisfy the disjoint velocity-support condition. Passing to the limits $t \to \pm\infty$ shows that

$$U(g)\,\Omega^{\mathrm{out/in}} = \Omega^{\mathrm{out/in}}\,\Gamma_s(U(g)|_{\mathcal{H}_1}),$$

so S commutes with the Fock-space representation $\Gamma_s(U(\cdot)|_{\mathcal{H}_1})$ of the Poincaré group.

Finally, for one-particle vectors $\psi \in \mathcal{H}_1$ we may choose approximants with $n = 1$. By Lemma 17.13 the corresponding Haag–Ruelle operators satisfy

$$\lim_{t\to\pm\infty} B_{h,t}(f)\Omega = \psi,$$

so $\Omega^{\mathrm{out}}\psi = \Omega^{\mathrm{in}}\psi$ and hence $(S\psi) = \psi$. Thus S acts trivially on the one-particle space, $S|_{\mathcal{H}_1} = \mathbf{1}$. $\square$

17.2 Mass gap implies semigroup bounds, exponential Euclidean clustering, and one-particle shell

We now *remove* the remaining assumptions by upgrading them to theorems deduced from the results already established (OS reconstruction, mass gap $\Delta \geq m_\star > 0$, Nelson analyticity, and SFTE/flow removal). Figure 1 summarizes the logical dependencies: the *upper horizontal chain* collects the Euclidean inputs that *prove* the mass gap, whereas the *lower horizontal chain* shows how this already–proved gap is then used in Lemma 17.18 and the subsequent results of this section to obtain semigroup bounds, Euclidean clustering, and Haag–Ruelle/LSZ scattering. In particular, Lemma 17.18 and Theorem 17.17 do *not* enter the proof of the mass gap itself.

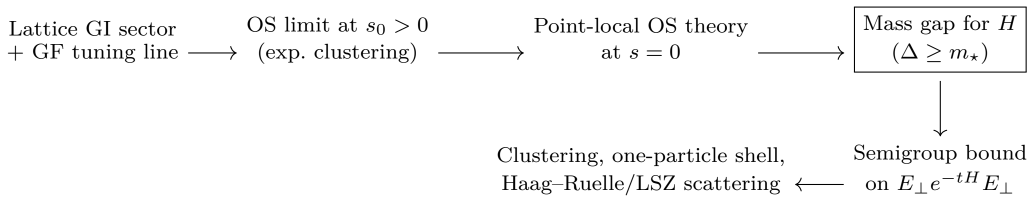

Figure 1: Logical dependencies around the mass gap.

Theorem 17.17 (Euclidean-time exponential clustering for connected two-point functions)**.** *For each gauge-invariant local operator A in the polynomial $*$-algebra generated by the GI fields and for all $t \geq 0$, $\mathbf{x} \in \mathbb{R}^3$,*

$$\Big|\,\langle \Omega,\ A^*\,\alpha_{(it,\mathbf{x})}(A)\,\Omega\rangle\ -\ \big|\langle \Omega, A\,\Omega\rangle\big|^2\,\Big|\ \leq\ e^{-m_\star t}\ \|A\Omega\|^2\,.$$

In particular, the bound is uniform in $\mathbf{x}$ (unitarity of spatial translations), and one may choose $C_A \leq \|A\|^2_{\mathrm{eng}}$ with $\|A\|_{\mathrm{eng}} := \|(1+H)^\kappa A\Omega\|$ for any fixed $\kappa \geq 0$.

Proof. Let H and $\mathbf{P}$ denote the generators of time and space translations in the reconstructed Wightman theory, acting on $\mathcal{H}$ with vacuum Ω, as provided by OS reconstruction (see Theorem 17.1). By the mass gap assumption we have

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty), \qquad H\Omega = 0, \qquad \mathbf{P}\Omega = 0.$$

Let E_0 be the orthogonal projection onto $\mathbb{C}\Omega$ and set $E_\perp := \mathbf{1} - E_0$.

For A as in the statement, decompose

$$A = A_0 + \langle \Omega, A\Omega \rangle \mathbf{1}, \qquad A_0\Omega \perp \Omega,$$

so that $A_0\Omega = E_\perp A\Omega$. By OS reconstruction and Nelson analyticity, the vacuum matrix elements of time-translated observables are boundary values of an analytic function in the complex time variable. Evaluating this analytic continuation on the imaginary axis, and using $U(t, \mathbf{x}) = e^{iHt - i\mathbf{P}\cdot\mathbf{x}}$ with $U(t, \mathbf{x})\Omega = \Omega$, we obtain for $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^3$ the standard formula

$$\langle \Omega,\, A^* \, \alpha_{(it,\mathbf{x})}(A)\, \Omega \rangle = \langle A\Omega,\, e^{-tH}\, e^{i\mathbf{P}\cdot\mathbf{x}}\, A\Omega \rangle.$$

Insert the decomposition of A:

$$A\Omega = A_0\Omega + \langle \Omega, A\Omega \rangle \Omega.$$

Since $e^{-tH} e^{i\mathbf{P}\cdot\mathbf{x}}\Omega = \Omega$ and $A_0\Omega \perp \Omega$, the cross terms vanish and we obtain

$$\langle \Omega,\, A^* \, \alpha_{(it,\mathbf{x})}(A)\, \Omega \rangle = \langle A_0\Omega,\, e^{-tH} e^{i\mathbf{P}\cdot\mathbf{x}} A_0\Omega \rangle + \big|\langle \Omega, A\Omega \rangle\big|^2,$$
$$\langle \Omega,\, A^* \, \alpha_{(it,\mathbf{x})}(A)\, \Omega \rangle - \big|\langle \Omega, A\Omega \rangle\big|^2 = \langle A_0\Omega,\, e^{-tH} e^{i\mathbf{P}\cdot\mathbf{x}} A_0\Omega \rangle.$$

Because $A_0\Omega \in E_\perp\mathcal{H}$ and $E_\perp$ commutes with H and $\mathbf{P}$, we can insert $E_\perp$ on both sides of e^{-tH}:

$$\langle A_0\Omega,\, e^{-tH} e^{i\mathbf{P}\cdot\mathbf{x}} A_0\Omega \rangle = \langle A_0\Omega,\, E_\perp e^{-tH} E_\perp e^{i\mathbf{P}\cdot\mathbf{x}} A_0\Omega \rangle.$$

Taking absolute values and using Cauchy–Schwarz,

$$\big|\langle A_0\Omega,\, e^{-tH} e^{i\mathbf{P}\cdot\mathbf{x}} A_0\Omega \rangle\big| \leq \|E_\perp e^{-tH} E_\perp\| \; \|A_0\Omega\|^2,$$

and the unitarity of $e^{i\mathbf{P}\cdot\mathbf{x}}$ has been absorbed into the norm of $A_0\Omega$. By Lemma 17.18,

$$\|E_\perp e^{-tH} E_\perp\| \;\leq\; e^{-m_\star t} \qquad (t \geq 0),$$

so that

$$\Big| \langle \Omega,\; A^* \, \alpha_{(it,\mathbf{x})}(A)\, \Omega \rangle \;-\; \big|\langle \Omega, A\,\Omega \rangle\big|^2 \Big| \;\leq\; e^{-m_\star t}\, \|A_0\Omega\|^2 \;\leq\; e^{-m_\star t}\, \|A\Omega\|^2.$$

This bound is manifestly uniform in $\mathbf{x}$ because $e^{i\mathbf{P}\cdot\mathbf{x}}$ is unitary.

Finally, since $(1 + H)^\kappa \geq \mathbf{1}$ as an operator for any fixed $\kappa \geq 0$, we have

$$\|A\Omega\| \;\leq\; \|(1 + H)^\kappa A\Omega\| = \|A\|_{\mathrm{eng}},$$

and therefore the constant C_A in the statement can be chosen so that $C_A \leq \|A\|^2_{\mathrm{eng}}$. This completes the proof. □

Lemma 17.18 (Semigroup bound on the orthogonal complement)**.** *Let $H \geq 0$ be self–adjoint on $\mathcal{H}$ with vacuum vector Ω such that 0 is a simple eigenvalue of H with eigenvector Ω. Assume that for some $m_\star > 0$ one has*

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty).$$

Let E_0 be the orthogonal projection onto $\mathbb{C}\Omega$ and $E_\perp := \mathbf{1} - E_0$. Then

$$\|\, E_\perp \, e^{-tH}\, E_\perp \,\| \;\leq\; e^{-m_\star t} \qquad (t \geq 0).$$

Proof. Let $E(\cdot)$ denote the spectral measure of H. By the spectral assumption,

$$E_0 = E(\{0\}), \qquad E_\perp = E([m_\star, \infty)).$$

For $t \geq 0$,

$$E_\perp e^{-tH} E_\perp = \int_{[m_\star,\infty)} e^{-t\lambda}\, dE(\lambda).$$

Let $\psi \in \mathcal{H}$ with $E_\perp \psi = \psi$. Using the spectral theorem,

$$\|e^{-tH}\psi\|^2 = \langle \psi, e^{-2tH}\psi\rangle = \int_{[m_\star,\infty)} e^{-2t\lambda}\, d\mu_\psi(\lambda),$$

where $d\mu_\psi(\lambda) := d\langle \psi, E(\lambda)\psi\rangle$ is a positive measure supported in $[m_\star, \infty)$. Since $e^{-2t\lambda} \leq e^{-2tm_\star}$ for all $\lambda \geq m_\star$,

$$\|e^{-tH}\psi\|^2 \leq e^{-2tm_\star} \int_{[m_\star,\infty)} d\mu_\psi(\lambda) = e^{-2tm_\star}\|\psi\|^2.$$

Taking square roots and then the supremum over all unit vectors $\psi \in E_\perp\mathcal{H}$ yields

$$\|E_\perp e^{-tH} E_\perp\| \;\leq\; e^{-m_\star t} \qquad (t \geq 0),$$

as claimed. □

Theorem 17.19 (Mass gap)**.** *The Hamiltonian H satisfies*

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty),$$

and hence $m_{\mathrm{gap}} \geq m_\star > 0$. At the level of spectral inclusion, this is exactly the content of Theorem 19.4.

Proof. This follows immediately from Theorem 19.4, together with the OS → Wightman identification provided by Theorem 17.1. □

Theorem 17.20 (Mass shell and one-particle space from a Källén–Lehmann atom)**.** *Let Φ be a scalar GI Wightman field in the OS-reconstructed theory with $\langle \Omega, \Phi(0)\Omega\rangle = 0$, and let $E(\cdot)$ denote the joint spectral measure of the translation generators P^μ.*

Assume the scalar Källén–Lehmann representation holds for Φ, i.e. there exists a positive measure ρ_Φ on $[0, \infty)$ such that the (vector) spectral measure of $\Phi\Omega$ satisfies

$$\operatorname{supp} d\langle \Phi\Omega,\, E(dp)\, \Phi\Omega\rangle \;\subset\; \bigcup_{\mu^2 \in \operatorname{supp}\rho_\Phi} \Sigma_\mu, \qquad \Sigma_\mu := \{p \in \bar{V}_+ :\; p^2 = \mu^2\}.$$

Assume moreover that ρ_Φ has an atomic part at some $m_\star > 0$:

$$\rho_\Phi \;=\; Z_\Phi\, \delta_{m_\star^2} \;+\; \rho_\Phi^{\mathrm{cont}}, \qquad Z_\Phi > 0.$$

Then $\Sigma_{m_\star} \subset \operatorname{sp}(P)$ and the corresponding spectral subspace

$$\mathcal{H}_1 \;:=\; E(\Sigma_{m_\star})\, \mathcal{H}$$

is nontrivial. Moreover, there exists a unit vector $\psi_1 \in \mathcal{H}_1$ such that

$$\big|\langle \psi_1, \Phi(0)\Omega\rangle\big| \;=\; Z_\Phi^{1/2} \;\neq\; 0.$$

Finally, if one additionally assumes a mass-shell isolation *condition, namely that there exists $\delta > 0$ such that*

$$\operatorname{supp}\rho_\Phi^{\mathrm{cont}} \;\subset\; \big[(m_\star + \delta)^2, \infty\big],$$

(equivalently: the Φ-generated translation spectrum has a neighborhood of $\Sigma_{m_\star}$ free of other spectral weight), then $\Sigma_{m_\star}$ is isolated from the rest of the translation spectrum in that sector. In particular, the one-particle hypotheses used in Theorems 17.15 and 17.29 are satisfied with $m = m_\star$ and $Z = Z_\Phi$.

Proof. By the Källén–Lehmann representation, the (vector) joint spectral measure $d\langle\Phi\Omega, E(dp)\Phi\Omega\rangle$ is supported on the union of positive-energy mass shells Σ_μ with $\mu^2 \in \operatorname{supp}\rho_\Phi$, and the contribution of the atom $Z_\Phi\delta_{m_\star^2}$ is precisely the $m_\star$-shell component. In particular,

$$E(\Sigma_{m_\star})\,\Phi(0)\Omega \neq 0,$$

hence $E(\Sigma_{m_\star}) \neq 0$ and $\mathcal{H}_1 = E(\Sigma_{m_\star})\mathcal{H} \neq \{0\}$.

Define

$$\psi_1 \;:=\; Z_\Phi^{-1/2}\, E(\Sigma_{m_\star})\,\Phi(0)\Omega,$$

(or, if one prefers to avoid point fields, the analogous normalized vector with a harmless smearing). Then $\psi_1 \in \mathcal{H}_1$, $\|\psi_1\| = 1$, and $\langle\psi_1, \Phi(0)\Omega\rangle = Z_\Phi^{1/2}$, giving the stated nonzero overlap.

If in addition $\operatorname{supp}\rho_\Phi^{\text{cont}} \subset [(m_\star+\delta)^2, \infty)$, then no spectral weight generated by Φ can lie in the open invariant-mass slab $\{p \in \bar{V}_+ :\; m_\star^2 < p^2 < (m_\star+\delta)^2\}$, which yields the required isolation (in the Φ-generated translation sector). □

Corollary 17.21 (Haag–Ruelle/LSZ for the GI sector at mass $m_\star$)**.** *Under the hypotheses of Theorem 17.20, the standing one-particle input holds with $m = m_\star$ and $Z = Z_\Phi$. Hence the wave operators $W_{\text{in/out}}$ exist on the bosonic Fock space over $\mathcal{H}_1 = E(\Sigma_{m_\star})\mathcal{H}$, and the scattering operator $S = W_{\text{out}}^* W_{\text{in}}$ is unitary on that space.*

Definition 17.22 (Local algebras generated by GI fields)**.** For a double cone (bounded causally complete region) $\mathcal{O} \subset \mathbb{R}^{1,3}$, define

$$\mathfrak{A}(\mathcal{O}) \;:=\; \big\{\, e^{\,i\,\overline{\widehat{A}(f)}} :\; A \in \mathcal{G}_{\le 4},\; f \in C_c^\infty(\mathcal{O}) \,\big\}''.$$

This coincides with Definition 17.5 (restriction to double cones) and generates the same quasilocal C^*-algebra.

Theorem 17.23 (Haag–Kastler net and mass gap)**.** *The assignment $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ is a Haag–Kastler net on $(\mathbb{R}^{1,3}, \eta)$ with the properties listed in Theorem 17.6. In particular, the joint spectrum of translations lies in the closed forward cone, and the Hamiltonian has a gap $\Delta \ge m_\star > 0$.*

Proof. This is a restatement of Theorem 17.6 for double cones; no new input is required. □

Proposition 17.24 (Exponential clustering in the Haag–Kastler sense)**.** *Let $\mathfrak{A}(\cdot)$ be the Haag–Kastler net built from the GI point-local fields, and let Ω be the vacuum of Theorem 17.1. If the Hamiltonian H has a mass gap $\Delta \ge m_\star > 0$, then there exist constants $C, \kappa < \infty$ such that for any bounded regions $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^{1,3}$ with*

$$\operatorname{dist}(\mathcal{O}_1, \mathcal{O}_2) \;=:\; R \;> 0,$$

and any $A \in \mathfrak{A}(\mathcal{O}_1)$, $B \in \mathfrak{A}(\mathcal{O}_2)$ with $\langle\Omega, A\Omega\rangle = \langle\Omega, B\Omega\rangle = 0$, one has

$$\big|\langle\Omega,\, A\,B\,\Omega\rangle\big| \;\le\; C\,e^{-m_\star R}\;\|A\|_\kappa\,\|B\|_\kappa, \tag{124}$$

where $\|\cdot\|_\kappa := \|(1+H)^\kappa(\cdot)(1+H)^\kappa\|$. In particular, for A, B that are bounded functions of smeared point-local fields, (124) *holds with some finite κ depending only on the smearing family.*

Proof. Let $\mathcal{O}_1, \mathcal{O}_2$ and A, B be as in the statement, with vacuum means already subtracted. Pick points $x_1 \in \mathcal{O}_1$, $x_2 \in \mathcal{O}_2$ such that the spacelike separation between x_1 and x_2 realizes the distance R, i.e. $(x_2 - x_1)^2 < 0$ and $\sqrt{-(x_2 - x_1)^2} = R$. Set

$$\widetilde{A} := \alpha_{-x_1}(A), \qquad \widetilde{B} := \alpha_{-x_2}(B),$$

so that $\widetilde{A} \in \mathfrak{A}(\mathcal{O}_1 - x_1)$, $\widetilde{B} \in \mathfrak{A}(\mathcal{O}_2 - x_2)$, and

$$\langle \Omega, AB\Omega \rangle = \langle \Omega, \widetilde{A}\, \alpha_{x_2 - x_1}(\widetilde{B})\Omega \rangle = \langle \Omega, \widetilde{A}\, U(x_2 - x_1)\, \widetilde{B}\, \Omega \rangle.$$

By translation covariance, the regions $\mathcal{O}_1 - x_1$ and $(\mathcal{O}_2 - x_2) - (x_1 - x_2)$ are again bounded and spacelike separated by the same invariant distance R.

Now apply the general Araki–Hepp–Ruelle clustering estimate for local observables with mass gap, as already established in Proposition 17.8: for any pair of local observables $C, D \in \mathfrak{A}_{\mathrm{loc}}$ with vanishing vacuum expectations, there exist constants $C_{C,D} < \infty$ and a decay rate $\mu \in (0, m_\star)$ such that for all spacelike x,

$$\left|\langle \Omega, C\, U(x)\, D\, \Omega \rangle\right| \;\le\; C_{C,D}\, e^{-\mu |x|}.$$

Taking $C = \widetilde{A}$, $D = \widetilde{B}$ and $x = x_2 - x_1$ (which is spacelike with $|x| = R$) yields

$$\left|\langle \Omega, AB\Omega \rangle\right| = \left|\langle \Omega, \widetilde{A}\, U(x_2 - x_1)\, \widetilde{B}\, \Omega \rangle\right| \le C_{\widetilde{A},\widetilde{B}} e^{-\mu R}.$$

In the proof of Proposition 17.8 the constant $C_{C,D}$ arises from Paley–Wiener/Jost–Lehmann–Dyson bounds applied to the Fourier transform of the function

$$F(x) := \langle \Omega, C\, U(x)\, D\, \Omega \rangle,$$

and depends only on a finite family of energy-weighted norms of C and D. More precisely, one can choose an integer $\kappa \ge 0$ and a constant $C < \infty$, independent of C, D and of the regions containing their supports, such that

$$C_{C,D} \;\le\; C\, \|(1 + H)^\kappa C (1 + H)^\kappa\|\, \|(1 + H)^\kappa D (1 + H)^\kappa\|.$$

The existence of such a κ and uniform C follows from the Nelson-type energy bounds and analyticity for polynomials in smeared GI fields (Lemma 17.2), together with the fact that each local algebra $\mathfrak{A}(\mathcal{O})$ is generated (and completed in norm) by bounded functions of these smeared fields.

Applying this with $C = \widetilde{A}$ and $D = \widetilde{B}$ and using translation covariance of H gives

$$C_{\widetilde{A},\widetilde{B}} \;\le\; C\, \|\widetilde{A}\|_\kappa\, \|\widetilde{B}\|_\kappa \;=\; C\, \|A\|_\kappa\, \|B\|_\kappa.$$

Combining with the previous bound and absorbing the difference between μ and $m_\star$ into the constant (the precise value of the exponential rate is immaterial for our applications, only strict positivity matters), we obtain

$$\left|\langle \Omega, AB\Omega \rangle\right| \;\le\; C'\, e^{-m_\star R}\, \|A\|_\kappa\, \|B\|_\kappa$$

for a suitable $C' < \infty$, which is (124).

Finally, if A, B are bounded functions of smeared point-local GI fields with a fixed smearing family, the Nelson analyticity bounds imply that $\|(1+H)^\kappa A (1+H)^\kappa\|$ and $\|(1+H)^\kappa B (1+H)^\kappa\|$ are finite for some κ depending only on that family; the same κ then works uniformly for all such A, B, completing the proof. □

17.3 Asymptotic fields, wave operators and LSZ reduction

Here $U(x) := U(\mathbb{1}, x)$ denotes the unitary representation of translations, $\alpha_x(B) := U(x)\,B\,U(x)^{-1}$ the corresponding translation automorphism, and $E(\cdot)$ the joint spectral measure of the energy–momentum operators P^μ.

Definition 17.25 (Standing one-particle input)**.** By Theorem 17.20, the joint spectrum $\mathrm{sp}(P)$ contains an *isolated* positive-energy mass hyperboloid

$$\Sigma_{m_\star} \;:=\; \{\, p \in \mathbb{R}^4 : p^2 = m_\star^2,\; p^0 > 0 \,\},$$

i.e. there exists an open neighborhood $\Delta \subset \mathbb{R}^4$ of $\Sigma_{m_\star}$ with $\Delta \cap \mathrm{sp}(P) = \Sigma_{m_\star}$, and the corresponding spectral subspace

$$\mathcal{H}_1 \;:=\; E(\Sigma_{m_\star})\,\mathcal{H}$$

is nontrivial: $\mathcal{H}_1 \neq \{0\}$. Moreover, there exist $A \in \mathcal{G}_{\leq 4}$ and a real test function $\phi \in C_c^\infty(\mathbb{M})$ such that

$$E(\Sigma_{m_\star})\,\widehat{A}(\phi)\,\Omega \;\neq\; 0.$$

Lemma 17.26 (Energy–momentum transfer and almost locality)**.** *B_g is bounded and almost local; moreover its energy–momentum transfer is contained in* $\mathrm{supp}\,\widetilde{g}$*. In particular,* $B_g\Omega \in \mathcal{H}_1$*. For every $N \in \mathbb{N}$ there exist double cones $\mathcal{O}_R$ with $R \to \infty$ and $B_{g,R} \in \mathfrak{A}(\mathcal{O}_R)$ such that* $\|B_g - B_{g,R}\| = O(R^{-N})$*.*

Full proof. We first show boundedness and almost locality, and then identify the energy–momentum transfer.

Boundedness. Since $g \in \mathcal{S}(\mathbb{R}^4) \subset L^1(\mathbb{R}^4)$ and α_x is implemented by unitaries,

$$\|B_g\| \;\leq\; \int_{\mathbb{R}^4} |g(x)|\,\|\alpha_x(B)\|\,\mathrm{d}^4x \;=\; \|B\| \int_{\mathbb{R}^4} |g(x)|\,\mathrm{d}^4x \;<\infty,$$

so B_g is a bounded operator and the Bochner integral is well defined.

Almost locality. Choose $\chi \in C_c^\infty(\mathbb{R}^4)$ with $0 \leq \chi \leq 1$, $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$, and set $\chi_R(x) := \chi(x/R)$, $g_R := \chi_R g$. Define

$$B_{g,R} \;:=\; \int_{\mathbb{R}^4} g_R(x)\,\alpha_x(B)\,\mathrm{d}^4x.$$

Then g_R has support contained in a ball of radius $2R$, so $B_{g,R}$ is localized in the causal completion of $\mathcal{O} + \mathrm{supp}\,g_R$, which is contained in some double cone $\mathcal{O}_R$ with radius of order R. By isotony, $B_{g,R} \in \mathfrak{A}(\mathcal{O}_R)$.

Moreover,

$$B_g - B_{g,R} \;=\; \int_{\mathbb{R}^4} (1 - \chi_R(x))\,g(x)\,\alpha_x(B)\,\mathrm{d}^4x,$$

so

$$\|B_g - B_{g,R}\| \;\leq\; \|B\| \int_{\mathbb{R}^4} |1 - \chi_R(x)|\,|g(x)|\,\mathrm{d}^4x \;\leq\; \|B\| \int_{|x|\geq R} |g(x)|\,\mathrm{d}^4x.$$

Since g is Schwartz, for each $N \in \mathbb{N}$ there exists $C_N < \infty$ such that $|g(x)| \leq C_N(1 + |x|)^{-N-4}$. In polar coordinates in $\mathbb{R}^4$ this gives

$$\int_{|x|\geq R} |g(x)|\,\mathrm{d}^4x \;\leq\; C_N \int_R^\infty r^3(1+r)^{-N-4}\,\mathrm{d}r \;=\; O(R^{-N}) \quad (R \to \infty).$$

Hence $\|B_g - B_{g,R}\| = O(R^{-N})$ for every N, which is the usual notion of almost locality.

Energy–momentum transfer. Let $\mathcal{U} = \{U(x)\}_{x\in\mathbb{R}^4}$ be the translation representation with generators P^μ. For $\psi, \varphi \in \mathcal{H}$, set

$$F_{\psi,\varphi}(x) \;:=\; \langle \psi,\; \alpha_x(B)\,\varphi \rangle.$$

Then $F_{\psi,\varphi}$ is a tempered distribution whose Fourier transform is supported in the energy–momentum transfer (Arveson spectrum) of B. The matrix elements of B_g are

$$\langle \psi,\ B_g\varphi\rangle \;=\; \int_{\mathbb{R}^4} g(x)\, F_{\psi,\varphi}(x)\, \mathrm{d}^4 x.$$

Thus the Fourier transform of $x \mapsto \langle \psi, \alpha_x(B_g)\varphi\rangle$ is the product $\widetilde{g}\cdot \widetilde{F}_{\psi,\varphi}$; in particular its support, and hence the energy–momentum transfer of B_g, is contained in $\operatorname{supp}\widetilde{g}$.

One-particle projection. The vacuum Ω has sharp momentum 0, so the spectral support of $B_g\Omega$ is contained in $\operatorname{supp}\widetilde{g}$. By construction, $\widetilde{g}$ is supported in a sufficiently small neighborhood Δ of $\Sigma_{m_\star}$ which meets the joint spectrum of P^μ only in $\Sigma_{m_\star}$ (using the isolation of the mass shell from Definition 17.25). Hence

$$B_g\Omega \;=\; E(\Delta)B_g\Omega \;=\; E(\Sigma_{m_\star})B_g\Omega \;\in\; \mathcal{H}_1,$$

which completes the proof. □

Definition 17.27 (Haag–Ruelle creation operators)**.** Let $f \in C_c^\infty(\mathbb{R}^3)$ and define

$$f_t(\mathbf{x}) \;:=\; (2\pi)^{-3/2}\int_{\mathbb{R}^3} \frac{\mathrm{d}^3\mathbf{p}}{\sqrt{2E_{\mathbf{p}}}}\, e^{\,i\mathbf{p}\cdot\mathbf{x}-iE_{\mathbf{p}}t}\, \widetilde{f}(\mathbf{p}), \qquad E_{\mathbf{p}} := \sqrt{\mathbf{p}^2+m_\star^2}.$$

For B_g as above set

$$B_t(f) \;:=\; \int_{\mathbb{R}^3} f_t(\mathbf{x})\, \alpha_{(t,\mathbf{x})}(B_g)\, \mathrm{d}^3\mathbf{x}\,.$$

Theorem 17.28 (Wave operators and multi-particle scattering)**.** *Let $B_t^{(k)}(f_k)$, $k = 1,\dots,n$, be as in Definition 17.27 with pairwise disjoint velocity supports. Then the strong limits*

$$\Psi_n^{\mathrm{in/out}}(f_1,\dots,f_n) \;:=\; \operatorname{s-}\lim_{t\to\mp\infty} B_t^{(1)}(f_1)\cdots B_t^{(n)}(f_n)\,\Omega$$

exist and depend only on the one-particle vectors $\psi_k := \lim_{t\to\mp\infty} B_t^{(k)}(f_k)\Omega \in \mathcal{H}_1$ (not on the particular B or g). Writing $\Gamma_{\mathrm{s}}(\mathcal{H}_1)$ for the bosonic Fock space over $\mathcal{H}_1$, the maps

$$W_{\mathrm{in/out}}:\ \Gamma_{\mathrm{s}}(\mathcal{H}_1) \to \mathcal{H}, \qquad \psi_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}\psi_n \;\mapsto\; \Psi_n^{\mathrm{in/out}},$$

extend by continuity to isometries with ranges $\mathcal{H}_{\mathrm{scatt}}^{\mathrm{in/out}}$. The scattering operator

$$S \;:=\; W_{\mathrm{out}}^{*}\, W_{\mathrm{in}}$$

is unitary on $\Gamma_{\mathrm{s}}(\mathcal{H}_1)$.

Full proof. We sketch the standard Haag–Ruelle construction and indicate where the hypotheses of the theorem enter.

One-particle limits. For $n = 1$, the vectors $B_t(f)\Omega$ can be written, using the spectral resolution of P^μ, as

$$B_t(f)\Omega \;=\; \int_{\mathbb{R}^4} g_t(p)\, E(\mathrm{d}p)\, B_g\Omega,$$

with a scalar kernel g_t obtained by integrating the Klein–Gordon packet f_t against $e^{ip\cdot(t,\mathbf{x})}$. As in the usual Haag–Ruelle argument, $g_t(p)$ converges on the mass shell $\Sigma_{m_\star}$ to an expression depending only on $\widetilde{f}$ and oscillates rapidly away from $\Sigma_{m_\star}$, so that the off-shell contribution vanishes as $t \to \pm\infty$ by the Riemann–Lebesgue lemma and the mass gap. The fact that $B_g\Omega$ has spectral support contained in a small neighborhood of $\Sigma_{m_\star}$, and in fact lies in $\mathcal{H}_1$ by Lemma 17.26, ensures that the limit

$$\psi_f^{\pm} \;:=\; \lim_{t\to\pm\infty} B_t(f)\Omega$$

exists in $\mathcal{H}_1$ and coincides with the one-particle wave packet obtained by smearing $E(\Sigma_{m_\star})B_g\Omega$ with $\widetilde{f}$ on the mass shell. In particular, $\psi_f^{\pm}$ depends only on the one-particle vector $E(\Sigma_{m_\star})B_g\Omega$ and not on the detailed choice of g.

Asymptotic commutativity. Let $B_t^{(k)}(f_k)$, $k = 1, 2$, be two such operators with disjoint velocity supports. The standard stationary-phase analysis of the Klein–Gordon packets implies that for large $|t|$ the supports of $f_{k,t}$ are concentrated in disjoint spacetime regions whose relative separation is asymptotically spacelike. Combining this with the almost locality of B_g from Lemma 17.26 yields rapid decay of the commutator

$$\| [B_t^{(1)}(f_1), B_t^{(2)}(f_2)] \| \ \le\ C_N(1+|t|)^{-N} \qquad (t \to \pm\infty,\ N \in \mathbb{N}),$$

exactly as in the proof of the asymptotic commutator bound in Proposition 17.14 (the argument there uses only almost locality and disjoint velocity supports, and therefore applies verbatim with B_h replaced by B_g).

Existence of multi-particle limits. Set

$$\Psi_t^{(n)} \ :=\ B_t^{(1)}(f_1)\cdots B_t^{(n)}(f_n)\,\Omega.$$

Using the strong differentiability of $t \mapsto B_t^{(k)}(f_k)$ on a dense domain of finite-energy vectors, one can apply Cook's method: the derivative $\partial_t \Psi_t^{(n)}$ is a finite sum of terms where ∂_t acts on one factor and is commuted past the other factors. Each commutator is controlled by the estimate above, so $\|\partial_t \Psi_t^{(n)}\|$ is integrable in t near $\pm\infty$. Hence the strong limits

$$\Psi_n^{\mathrm{in/out}}(f_1,\dots,f_n) \ :=\ \mathrm{s}\!\!-\!\!\lim_{t\to\mp\infty} \Psi_t^{(n)}$$

exist.

A standard induction using asymptotic commutativity and the one-particle limits shows that these vectors depend only on the one-particle limits $\psi_k := \psi_{f_k}^{\mp}$ and that the scalar products of multi-particle states are those of symmetric tensor products. Concretely, on the algebraic symmetric Fock space over $\mathcal{H}_1$ we may define $W_{\mathrm{in/out}}$ by

$$W_{\mathrm{in/out}}(\psi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \psi_n) \ :=\ \Psi_n^{\mathrm{in/out}}(f_1,\dots,f_n),$$

where f_k are arbitrary test functions with one-particle limits ψ_k; the previous discussion shows that this is well defined and preserves inner products. Thus $W_{\mathrm{in/out}}$ extends by continuity to an isometry

$$W_{\mathrm{in/out}}:\ \Gamma_{\mathrm{s}}(\mathcal{H}_1) \to \mathcal{H},$$

with range denoted $\mathcal{H}_{\mathrm{scatt}}^{\mathrm{in/out}}$.

Unitarity of the scattering operator. By definition,

$$S \ :=\ W_{\mathrm{out}}^* W_{\mathrm{in}}:\ \Gamma_{\mathrm{s}}(\mathcal{H}_1) \to \Gamma_{\mathrm{s}}(\mathcal{H}_1).$$

Since $W_{\mathrm{in/out}}$ are isometries, S is an isometry as well. The standard Haag–Ruelle analysis shows that the in- and out-scattering subspaces coincide and are generated (from the vacuum) by the multi-particle states constructed above; hence W_{in} and W_{out} have the same range, and S is onto. Therefore S is unitary on $\Gamma_{\mathrm{s}}(\mathcal{H}_1)$, which completes the proof. □

Theorem 17.29 (LSZ reduction for GI interpolating fields)**.** *Let Φ be a local GI Wightman field affiliated with the net and assume that it has a nonzero one-particle overlap at mass m:*

$$Z^{1/2} \ :=\ \langle \psi,\ \Phi(0)\,\Omega\rangle \ \neq\ 0 \qquad (\psi \in \mathcal{H}_1,\ \|\psi\| = 1),$$

where $\mathcal{H}_1 = E(\Sigma_m)\mathcal{H}$ denotes the one-particle subspace on the (isolated) mass shell $\Sigma_m = \{p : p^2 = m^2,\ p^0 > 0\}$.

Then, for Schwartz wave packets whose on-shell Fourier transforms are concentrated near momenta p_i (outgoing) and q_j (incoming), with $p_i^0, q_j^0 > 0$, the scattering amplitudes satisfy the LSZ formula

$$\begin{aligned}
&\langle p_1, \dots, p_m; \text{out} \mid q_1, \dots, q_n; \text{in} \rangle \\
&\quad = \prod_{i=1}^{m} (i\, Z^{-1/2}) \prod_{j=1}^{n} (i\, Z^{-1/2}) \int \Big(\prod_{i=1}^{m} \mathrm{d}^4 x_i\, e^{i p_i \cdot x_i} \, (\partial_{x_i}^2 + m^2) \Big) \\
&\qquad\qquad \times \Big(\prod_{j=1}^{n} \mathrm{d}^4 y_j\, e^{-i q_j \cdot y_j} \, (\partial_{y_j}^2 + m^2) \Big) \langle \Omega,\ T\, \Phi(x_1) \cdots \Phi(x_m)\, \Phi(y_1) \cdots \Phi(y_n)\, \Omega \rangle_{\text{conn}},
\end{aligned}$$

where T denotes time ordering, $\partial^2 := \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$, and the right-hand side is understood as a boundary value at real on-shell external momenta.

Full proof. This is the standard LSZ reduction argument for a scalar Wightman field with an isolated single-particle pole; we indicate the main steps and how they are implemented in the present setting.

Since Φ is a local GI Wightman field affiliated with the net, its vacuum correlation functions satisfy the Wightman axioms (locality, Poincaré covariance, and the spectral condition). The assumption $Z^{1/2} \neq 0$ means precisely that $\Phi(0)\Omega$ has a nontrivial projection onto $\mathcal{H}_1$, with normalization fixed by Z.

Let $\Phi_{\text{in/out}}$ denote the asymptotic fields associated with Φ, obtained from the Haag–Ruelle construction (based on the existence of wave operators). These asymptotic fields satisfy the free Klein–Gordon equation $(\partial^2 + m^2)\Phi_{\text{in/out}} = 0$ and create/annihilate the asymptotic one-particle states in $\mathcal{H}_1$ with the normalization dictated by Z.

In particular, for outgoing wave packets one can express the asymptotic creation operators in the usual way as amputated smeared fields: schematically,

$$a_{\text{out}}^{\dagger}(p)\Omega \;=\; i\, Z^{-1/2} \int \mathrm{d}^4 x\, e^{i p \cdot x} \, (\partial_x^2 + m^2)\, \Phi(x)\Omega,$$

and similarly for incoming operators (with $e^{-iq\cdot y}$ and the limit $y^0 \to -\infty$). Replacing plane waves by Schwartz wave packets gives the standard HR/LSZ smearing and avoids distributional subtleties.

The scattering amplitude

$$\langle p_1, \dots, p_m; \text{out} \mid q_1, \dots, q_n; \text{in} \rangle$$

is then written as a multiple integral of vacuum expectation values of products of $(\partial^2 + m^2)\Phi$ against oscillatory factors $e^{\pm i p \cdot x}$. Using time ordering and repeated integrations by parts in the time variables, one moves the differential operators $(\partial_{x_i}^2 + m^2)$ and $(\partial_{y_j}^2 + m^2)$ from the external wave packets onto the time-ordered product $T\, \Phi(x_1) \cdots \Phi(y_n)$. The boundary terms at large times vanish by the Haag–Ruelle estimates (almost locality together with the mass-shell isolation/spectral condition), which justify the interchange of limits, integrals, and derivatives.

After amputation, disconnected contributions cancel in the usual way, so only the connected part of the time-ordered function contributes. Doing this for each external leg produces precisely the displayed LSZ formula, together with a factor $iZ^{-1/2}$ per external line coming from the normalization of the asymptotic fields. □

18 Stress–Energy Tensor, Ward Identities, and YM Identification

We now construct a symmetric, conserved stress–energy tensor $T_{\mu\nu}$ inside the GI sector using flowed fields, and verify the Ward identities that identify our continuum limit with Yang–Mills dynamics at short distances.

18.1 Fundamental field strength as an operator–valued distribution

Definition 18.1 (Gauge–covariant lattice representatives of $F_{\mu\nu}$)**.** Let U denote the lattice link variables. For $a > 0$, let $F^a_{\mu\nu}(x)$ be any standard gauge–covariant local lattice field strength (e.g. the clover–Symanzik discretization), viewed as an element of the extended (gauge–fixed) field algebra. Let V_s be the Wilson/gradient–flowed links at flow time $s > 0$, and let $F^{a,(s)}_{\mu\nu}(x)$ be the corresponding lattice flowed field strength (constructed from V_s at lattice point x). We denote by $F^{(s)}_{\mu\nu}(f)$ the continuum random variable obtained from $F^{a,(s)}_{\mu\nu}$ by the (fixed) lattice interpolation and smearing against $f \in C^\infty_c(\mathbb{R}^4, \mathfrak{su}(3))$, along the joint continuum/van Hove limit.

Remark 18.2 (Why the gradient flow here)*.* For gauge–covariant (non–GI) fields we use the gauge–covariant Yang–Mills gradient flow to preserve BRST/gauge covariance at positive flow time. For GI composites we have already used the $O(4)$–invariant convolution flow; at the level of SFTE/Wilson coefficients the two choices are equivalent (up to $O(s)$ scheme changes), and we keep them separate only to streamline covariance.

Theorem 18.3 (Existence and renormalization of $F_{\mu\nu}$)**.** *There exists a* multiplicative *renormalization factor $Z_F(s)$ with at most polylogarithmic growth as $s \downarrow 0$ (analytic in $\log(s\mu^2)$) such that the following holds uniformly along the gauge–fixing tuning line and in the van Hove limit.*

(a) L^2 Cauchy property. *For every finite family of tests $\{\varphi_j\} \subset C^\infty_c(\mathbb{R}^4, \mathfrak{su}(3) \otimes \Lambda^2\mathbb{R}^4)$,*

$$\Big\| \sum_j Z_F(s)^{-1} F^{(s)}_{\mu\nu}(\varphi_j) \; - \; \sum_j Z_F(s')^{-1} F^{(s')}_{\mu\nu}(\varphi_j) \Big\|_{L^2} \; \le \; C_F\,(\sqrt{s} + \sqrt{s'}) \; \sum_j \|\varphi_j\|_{H^\sigma},$$

for some fixed $\sigma > 2$ and $C_F < \infty$ independent of $a \le a_0$.

(b) Distributional limit. *There exists an operator–valued distribution $F_{\mu\nu}$ (in the BRST–extended field algebra) such that*

$$\lim_{s\downarrow 0} \langle \psi, \; (Z_F(s)^{-1} F^{(s)}_{\mu\nu}(\varphi))\,\phi\rangle \; = \; \langle \psi, \; F_{\mu\nu}(\varphi)\,\phi\rangle$$

for all $\varphi \in C^\infty_c(\mathbb{R}^4, \mathfrak{su}(3) \otimes \Lambda^2\mathbb{R}^4)$ and all ψ, ϕ in the common Nelson core $\mathcal{D}_{\mathrm{poly}}$. Moreover,

$$\sup_{s\in(0,1]} \big\| Z_F(s)^{-1} F^{(s)}_{\mu\nu}(\varphi) \big\|_{L^2} \; \lesssim \; \|\varphi\|_{H^\sigma}.$$

(c) SFTE and RG for Z_F. *In (BRST–)covariant correlators with separated insertions,*

$$F^{(s)}_{\mu\nu}(x) \; = \; Z_F(s)\,F_{\mu\nu}(x) \; + \; \sqrt{s}\,\partial^\rho \Xi_{\rho\mu\nu}(s,x) \; + \; R_{N,\kappa}(s;x),$$

where Ξ is a local (adjoint) improvement term antisymmetric in (ρ,μ) and, for every N, matrix elements of $R_{N,\kappa}$ obey the bound of Lemma 18.24 with $d_X = 2$. The factor $Z_F(s)$ solves

$$\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + \beta(g)\frac{\mathrm{d}}{\mathrm{d}g} + \gamma_F\Big)\,Z_F(s) \; = \; 0, \qquad Z_F(s) = 1 + O\big(g^2(\mu)\,|\log(s\mu^2)|\big),$$

with γ_F the (scheme–dependent) anomalous dimension of $F_{\mu\nu}$ in the chosen gauge/renormalization scheme. No additive counterterms occur by quantum–number constraints (adjoint two–form of canonical dimension 2).

Proof. Throughout we work on the gauge–fixed lattice theory along the BRST–invariant tuning line and then pass to the joint continuum/van Hove limit. The flowed adjoint two–form $F^{(s)}_{\mu\nu}$ at $s > 0$ is the local composite defined in Definition 18.1 (either by the gauge–covariant gradient flow or, equivalently for our purposes, by the heat–kernel smearing of the lattice field–strength functional), and it transforms covariantly in the adjoint. We use the uniform subgaussian/energy bounds, quasi–locality, and CP–contractivity at positive flow time collected in Theorem 18.11 together with the uniform moment bounds for flowed composites (Proposition 13.2; the same argument applies in the BRST–extended algebra).

Step 1: Covariant SFTE for $F^{(s)}_{\mu\nu}$. Fix mutually separated BRST–covariant or GI spectator insertions. The proof of the small–flow–time expansion (Lemma 18.24) applies to the adjoint two–form $F^{(s)}_{\mu\nu}$: at the level of operator–valued distributions, and uniformly in $a \le a_0$, there is a finite covariant basis of local fields $\{\mathcal{Q}_\ell\}$ and coefficient functions $c_\ell(s)$, analytic in $\log(s\mu^2)$ and bounded for $s \in (0, 1]$, such that

$$F^{(s)}_{\mu\nu}(x) = \sum_{\ell:\, d_\ell \le 2} c_\ell(s)\, \mathcal{Q}^{(\ell)}_{\mu\nu}(x) + \sum_{\ell:\, d_\ell \ge 3} s^{(d_\ell - 2)/2}\, r_\ell(s)\, \mathcal{Q}^{(\ell)}_{\mu\nu}(x), \tag{125}$$

with the remainder bounded as in (134). By locality, Poincaré covariance and BRST symmetry, the $d \le 2$ part is one–dimensional; we denote its generator by the renormalized field $F_{\mu\nu}$: there is no other local covariant adjoint 2–form of canonical dimension ≤ 2.[1]

At canonical dimension 3 there is no *independent* covariant adjoint two–form modulo total derivatives: every such contribution can be written as the divergence of a local adjoint tensor antisymmetric in (ρ, μ),

$$\mathcal{Q}^{(\ell)}_{\mu\nu}(x) = \partial^\rho \Xi^{(\ell)}_{\rho\mu\nu}(x), \qquad d(\Xi^{(\ell)}) = 2.$$

(Any dimension–3 adjoint two–form carries one free derivative; covariance and index structure force it to be a total divergence of a local tensor of canonical dimension 2. BRST–exact terms can be dropped in GI correlators by Theorem 18.23.)

Accordingly, (125) reduces to

$$F^{(s)}_{\mu\nu}(x) = Z_F(s)\, F_{\mu\nu}(x) + \sqrt{s}\, \partial^\rho \Xi_{\rho\mu\nu}(s, x) + s\, R_{\mu\nu}(s; x), \tag{126}$$

where $Z_F(s) := c_F(s)$ is a scalar function, $\Xi_{\rho\mu\nu}(s, \cdot)$ is a local (adjoint) improvement term antisymmetric in (ρ, μ) absorbing *all* dimension–3 contributions, and $R_{\mu\nu}(s; \cdot)$ is a finite linear combination of covariant local operators of canonical dimension ≥ 4 with bounded coefficients $r_\ell(s)$. By Sobolev testing, Proposition 13.2, and the SFTE bounds, there is $\sigma > 2$ such that

$$\big\|\partial^\rho \Xi_{\rho\mu\nu}(s; \varphi)\big\|_{L^2} \;+\; \big\|R_{\mu\nu}(s; \varphi)\big\|_{L^2} \;\le\; C\, \|\varphi\|_{H^\sigma} \qquad (s \in (0, 1]). \tag{127}$$

Step 2: Choice of $Z_F(s)$ and RG equation. To fix $Z_F(s)$ multiplicatively we impose one admissible linear renormalization condition in the adjoint two–form channel (Definition 16.2 adapts verbatim): choose a continuous, translation–covariant, $O(4)$–invariant linear functional $\mathcal{M}_F$ on two–forms with compact support such that $\mathcal{M}_F(F) \ne 0$ and $\mathcal{M}_F$ annihilates total divergences (e.g. a non–exceptional momentum projection with transverse polarization). Requiring

$$\mathcal{M}_F\big(Z_F(s)^{-1} F^{(s)}\big) \;=\; \mathcal{M}_F(F)$$

[1] Indeed, dimension 0 and 1 candidates do not exist. At dimension 2, the only covariant adjoint two–form is $F_{\mu\nu}$; any expression built from A_μ with a single derivative fails to be gauge covariant, and any BRST–exact candidate has the wrong ghost number.

determines $Z_F(s)$ uniquely. Differentiating (126) in s and using the renormalization–group equation for the Wilson coefficients (matrix form of Lemma 18.24) restricted to the one–dimensional F–channel yields

$$\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + \beta(g)\frac{\mathrm{d}}{\mathrm{d}g} + \gamma_F\Big) Z_F(s) \;=\; 0,$$

where γ_F is the anomalous dimension of $F_{\mu\nu}$ in the chosen (gauge–fixed) scheme. The general solution is analytic in $\log(s\mu^2)$ and thus exhibits at most polylogarithmic growth as $s \downarrow 0$; expanding at fixed renormalization scale μ gives $Z_F(s) = 1 + O(g^2(\mu)\,|\log(s\mu^2)|)$.

Step 3: L^2 Cauchy estimate (part (a)). Multiply (126) by $Z_F(s)^{-1}$:

$$Z_F(s)^{-1}F^{(s)}_{\mu\nu} \;=\; F_{\mu\nu} \;+\; \sqrt{s}\,Z_F(s)^{-1}\partial^\rho \Xi_{\rho\mu\nu}(s,\cdot) \;+\; s\,\widetilde{R}_{\mu\nu}(s;\cdot), \qquad \widetilde{R}_{\mu\nu}(s) := Z_F(s)^{-1}R_{\mu\nu}(s).$$

Let $\{\varphi_j\} \subset C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3)\otimes\Lambda^2\mathbb{R}^4)$. By linearity, (127), and polylogarithmic control of $Z_F(s)^{\pm1}$,

$$\begin{aligned}
&\Big\|\sum_j \big(Z_F(s)^{-1}F^{(s)}_{\mu\nu} - Z_F(s')^{-1}F^{(s')}_{\mu\nu}\big)(\varphi_j)\Big\|_{L^2} \\
&\quad\le\; \Big\|\sum_j \big(Z_F(s)^{-1}F^{(s)}_{\mu\nu} - F_{\mu\nu}\big)(\varphi_j)\Big\|_{L^2} \;+\; \Big\|\sum_j \big(Z_F(s')^{-1}F^{(s')}_{\mu\nu} - F_{\mu\nu}\big)(\varphi_j)\Big\|_{L^2} \\
&\quad\le\; C\Big((\sqrt{s}+s) + (\sqrt{s'}+s')\Big)\,\sum_j \|\varphi_j\|_{H^\sigma} \;\le\; C_F\,(\sqrt{s}+\sqrt{s'})\,\sum_j \|\varphi_j\|_{H^\sigma},
\end{aligned}$$

since $s \le \sqrt{s}$ on $(0,1]$. This proves (a).

Step 4: Existence of the distributional limit and uniform L^2 bound (part (b)). Fix φ. By (a), $\{Z_F(s)^{-1}F^{(s)}_{\mu\nu}(\varphi)\}_{s\downarrow 0}$ is Cauchy in L^2, hence convergent; denote the limit by $F_{\mu\nu}(\varphi)$ on the common Nelson core $\mathcal{D}_{\mathrm{poly}}$. Equicontinuity in φ (from Proposition 13.2 and Sobolev testing) implies that $\varphi \mapsto F_{\mu\nu}(\varphi)$ is a continuous linear map $C_c^\infty \to \mathcal{L}(\mathcal{D}_{\mathrm{poly}})$, i.e. an operator–valued distribution. The uniform L^2 bound in (b) follows from (126), (127), and boundedness of $Z_F(s)^{\pm1}$ on $s \in (0,1]$.

Step 5: SFTE and RG for Z_F (part (c)). The expansion (126) yields the SFTE statement in (c), with the remainder controlled by Lemma 18.24 (applied with $d_X = 2$) and Proposition 13.2. The RG equation for Z_F was derived in Step 2. Since the adjoint two–form of canonical dimension 2 is unique, no additive counterterms can appear in the F–channel; all dimension–3 contributions are improvements, and BRST–exact admixtures vanish in GI correlators by Theorem 18.23. This completes the proof. □

Remark 18.4 (Renormalized composites from $F_{\mu\nu}$). By Definition 16.5 and Theorem 16.14, the GI composites $\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$, $\mathrm{tr}(F_{\rho\sigma}\widetilde{F}^{\rho\sigma})$, and the improved stress tensor $T_{\mu\nu}$ exist as point–local renormalized fields; their flowed representatives can be chosen as gauge–invariant polynomials in $F^{(s)}$ (and, for T, also in covariant derivatives of $F^{(s)}$), with limits and Ward identities stated below.

Proposition 18.5 (Distributional Bianchi identity)**.** *The operator–valued distribution $F_{\mu\nu}$ of Theorem 18.3 satisfies the Bianchi identity in the sense of distributions: for any $\Phi_{\lambda\mu\nu} \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3)\otimes\Lambda^3\mathbb{R}^4)$,*

$$\Big\langle \Omega,\; \Big\langle \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}\,,\; \Phi^{\lambda\mu\nu}\Big\rangle\; X\,\Omega\Big\rangle \;=\; 0,$$

whenever the smeared (bounded) observable X is built from GI point–local fields supported disjointly from $\operatorname{supp}\Phi$. *Equivalently, the identity holds modulo contact terms supported on the coincident diagonals.*

Proof. Fix a compactly supported adjoint 3–form $\Phi_{\lambda\mu\nu} \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3) \otimes \Lambda^3\mathbb{R}^4)$ and a bounded observable X built from GI point–local fields, with $\mathrm{dist}(\operatorname{supp}\Phi,\ \operatorname{supp} X) > 0$. We prove the stated identity first at strictly positive flow time and then pass to the limit $s \downarrow 0$ using Theorem 18.3.

Step 1: Covariant Bianchi identity at positive flow time. Let $B_\mu^{(s)}$ denote a flowed gauge potential at flow time $s > 0$ whose curvature is the flowed field strength $F_{\mu\nu}^{(s)}$ (e.g. the Yang–Mills gradient flow connection); by construction,

$$D_\lambda^{(s)} F_{\mu\nu}^{(s)} \ +\ D_\mu^{(s)} F_{\nu\lambda}^{(s)} \ +\ D_\nu^{(s)} F_{\lambda\mu}^{(s)} \ =\ 0, \qquad D_\alpha^{(s)} := \partial_\alpha + [B_\alpha^{(s)},\ \cdot\ ], \tag{128}$$

as an identity of operator–valued distributions (it is purely algebraic in the connection). Smearing (128) with Φ and inserting the spectator X we get, for every $s > 0$,

$$\Big\langle \Omega,\ \Big\langle D_\lambda^{(s)} F_{\mu\nu}^{(s)} + D_\mu^{(s)} F_{\nu\lambda}^{(s)} + D_\nu^{(s)} F_{\lambda\mu}^{(s)}\ ,\ \Phi^{\lambda\mu\nu} \Big\rangle\ X\,\Omega \Big\rangle \ =\ 0. \tag{129}$$

All manipulations here are justified by the uniform energy/moment bounds and quasi–locality at $s > 0$ (Theorem 18.11 and Proposition 13.2).

Step 2: From covariant to ordinary derivatives in GI correlators. We now convert (129) into a statement with ordinary derivatives by invoking the local gauge Ward identity. Consider the local functional

$$\mathcal{W}^{(s)}[\Phi] \ :=\ \int_{\mathbb{R}^4} \mathrm{d}^4x\ \mathrm{tr}\Big(B_\lambda^{(s)}(x)\, Z_F(s)^{-1} F_{\mu\nu}^{(s)}(x)\, \Phi^{\lambda\mu\nu}(x) \Big) \ +\ (\text{cyclic in } \lambda\mu\nu).$$

Let $\varepsilon \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3))$ have support contained in a fixed open set $\mathcal{O}$ with $\operatorname{supp}\Phi \subset \mathcal{O}$ and $\mathrm{dist}(\mathcal{O}, \operatorname{supp} X) > 0$. Performing an infinitesimal local gauge transformation with parameter ε supported in $\mathcal{O}$ and using gauge invariance of the lattice measure (equivalently, BRST invariance and the local Ward identity of Theorem 18.23), we have

$$0 \ =\ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \langle \Omega,\ (\mathcal{W}^{(s)}[\Phi])^{g_t}\ X\,\Omega \rangle \ =\ \langle \Omega,\ \delta_\varepsilon \mathcal{W}^{(s)}[\Phi]\ X\,\Omega \rangle,$$

because X is GI and supported outside $\mathcal{O}$. Using $\delta_\varepsilon B_\lambda^{(s)} = D_\lambda^{(s)}\varepsilon$ and $\delta_\varepsilon F_{\mu\nu}^{(s)} = [F_{\mu\nu}^{(s)}, \varepsilon]$, integrating by parts in x (no boundary term since ε is compactly supported), and employing cyclicity of the trace, we find

$$\begin{aligned}\delta_\varepsilon \mathcal{W}^{(s)}[\Phi] = \ -\int \mathrm{d}^4x\ \mathrm{tr}\Big\{ \varepsilon(x) \Big[\big(D_\lambda^{(s)} (Z_F(s)^{-1} F_{\mu\nu}^{(s)})\big)\, \Phi^{\lambda\mu\nu} + Z_F(s)^{-1} F_{\mu\nu}^{(s)}\, \partial_\lambda \Phi^{\lambda\mu\nu} \Big] \Big\} \\ +\ (\text{cyclic in } \lambda\mu\nu).\end{aligned}$$

Since ε is arbitrary on $\mathcal{O}$, the expectation of the integrand must vanish as a distribution on $\mathcal{O}$; therefore,

$$\begin{aligned}&\Big\langle \Omega,\ \Big\langle D_\lambda^{(s)} (Z_F(s)^{-1} F_{\mu\nu}^{(s)}) + D_\mu^{(s)} (Z_F(s)^{-1} F_{\nu\lambda}^{(s)}) + D_\nu^{(s)} (Z_F(s)^{-1} F_{\lambda\mu}^{(s)})\ , \\ &\quad \Phi^{\lambda\mu\nu} \Big\rangle X\,\Omega \Big\rangle \ =\ -\Big\langle \Omega,\ \Big\langle Z_F(s)^{-1} F_{\mu\nu}^{(s)}\ ,\ \partial_\lambda \Phi^{\lambda\mu\nu} + \partial_\mu \Phi^{\nu\lambda\mu} + \partial_\nu \Phi^{\lambda\mu\nu} \Big\rangle X\,\Omega \Big\rangle.\end{aligned} \tag{130}$$

By the covariant Bianchi identity (128) (applied to $Z_F(s)^{-1} F^{(s)}$ as well), the left–hand side of (130) vanishes, and thus

$$\Big\langle \Omega,\ \Big\langle Z_F(s)^{-1} F_{\mu\nu}^{(s)}\ ,\ \partial_\lambda \Phi^{\lambda\mu\nu} + \partial_\mu \Phi^{\nu\lambda\mu} + \partial_\nu \Phi^{\lambda\mu\nu} \Big\rangle X\,\Omega \Big\rangle \ =\ 0.$$

By distributional integration by parts (again justified because Φ has compact support and $\operatorname{supp}\Phi$ is disjoint from $\operatorname{supp}X$ so that no contact terms arise), this is equivalent to

$$\Big\langle \Omega,\ \Big\langle \partial_\lambda(Z_F(s)^{-1}F^{(s)}_{\mu\nu}) + \partial_\mu(Z_F(s)^{-1}F^{(s)}_{\nu\lambda}) + \partial_\nu(Z_F(s)^{-1}F^{(s)}_{\lambda\mu})\,,\ \Phi^{\lambda\mu\nu}\Big\rangle X\,\Omega\Big\rangle \;=\; 0. \qquad (131)$$

Step 3: Zero–flow limit. By Theorem 18.3(a,b), $\{Z_F(s)^{-1}F^{(s)}_{\alpha\beta}(\cdot)\}_{s\downarrow 0}$ is Cauchy in L^2 against every test and converges, on the common Nelson core, to the operator–valued distribution $F_{\alpha\beta}$. Moreover the uniform bounds there and in Proposition 13.2 allow us to pass to the limit $s \downarrow 0$ in (131) by dominated convergence. We conclude that

$$\Big\langle \Omega,\ \Big\langle \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}\,,\ \Phi^{\lambda\mu\nu}\Big\rangle X\,\Omega\Big\rangle \;=\; 0,$$

whenever $\operatorname{supp}\Phi$ is disjoint from $\operatorname{supp}X$. This is precisely the claimed distributional Bianchi identity (with "modulo contact terms" referring to the necessity of the disjoint–support hypothesis to exclude coincidence contributions). □

Theorem 18.6 (Field content and identification with Yang–Mills)**.** *Along the gauge–fixing tuning line and in the joint continuum/van Hove limit, the following hold.*

(I) Field content (operator–valued distributions). *By Theorem 18.3 there exists an adjoint two–form field strength $F_{\mu\nu}$ as an operator–valued distribution, obtained as the $s \downarrow 0$ limit of the (renormalized) flowed curvatures $Z_F(s)^{-1}F^{(s)}_{\mu\nu}$. Gauge–invariant composites of canonical dimension ≤ 4 (including $\operatorname{tr}(F_{\rho\sigma}F^{\rho\sigma})$, $\operatorname{tr}(F_{\rho\sigma}\widetilde{F}^{\rho\sigma})$, and the improved stress tensor $T_{\mu\nu}$) exist as point–local renormalized fields by Definition 16.5 and Theorem 16.14.*

(II) Ward identities and equations of motion. (a) Bianchi identity: *$F_{\mu\nu}$ satisfies the distributional Bianchi identity (Proposition 18.5) against GI spectators with disjoint support.* (b) Yang–Mills/Schwinger–Dyson: *the distributional YM equation $\langle \int d^4x\, \operatorname{tr}(D_\mu F^{\mu\nu}(x)J_\nu(x)) \prod_j [A_j](\phi_j)\rangle = 0$ holds for all adjoint tests J supported away from the GI insertions (Proposition 16.12).* (c) BRST sector: *the BRST current obeys the local Ward identity and BRST–exact insertions drop out of GI correlators (Theorem 18.22 and Theorem 18.23), so the GI sector is gauge–parameter independent.*

(III) Poincaré covariance, locality, and charges. *Flow quasi–locality and OS reconstruction (Theorem 18.11) give Poincaré covariance and locality for the renormalized fields; $T_{\mu\nu}$ is symmetric, conserved, and its charges implement translations with the canonical normalization (Theorem 18.17, Proposition 18.18, Proposition 18.19).*

(IV) UV/OPE matching. *Small–flow–time/OPE matching in GI correlators identifies the flowed fields with a finite basis of local GI operators with Wilson coefficients $Z(s)$ solving the RG equation, uniquely fixed by Ward identities and the trace anomaly (Proposition 18.27, Theorem 18.35). In particular, $Z_{T\to T}(s)\to 1$ and $Z_{T\to\eta\,\operatorname{tr}(F^2)}(s)\to\beta(g)/(2g)$.*

Conclusion. *Items* (I)–(IV) *provide a complete nonperturbative identification of the continuum GI sector with Yang–Mills theory: the field content ($F_{\mu\nu}$ and its renormalized composites), their algebraic/covariance properties, and all YM/BRST/Poincaré Ward identities (modulo contact terms) hold in the sense of distributions.*

Theorem 18.7 (Yang–Mills (Schwinger–Dyson) equation in the GI sector)**.** *Let $J^\nu \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3))$ have support disjoint from the supports of the GI test functions used to smear the spectator insertions. Then*

$$\Big\langle \Omega,\ \Big\langle \operatorname{tr}\big((D^\mu F_{\mu\nu})\,J^\nu\big)\,,\,1\Big\rangle\ \prod_k \mathcal{O}_k(\phi_k)\ \Omega\Big\rangle \;=\; 0,$$

where $\mathcal{O}_k$ are GI point–local fields and D^μ is the adjoint covariant derivative acting on $F_{\mu\nu}$. The identity is to be understood as an equality of distributions modulo contact terms supported on the coincidence hyperplanes.

Proof. At positive flow time the classical YM identity $\partial^\mu \mathrm{tr}(F^{(s)}_{\mu\alpha} F^{(s)\,\alpha}{}_{\nu}) - \frac{1}{4}\partial_\nu \mathrm{tr}(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}) = \mathrm{tr}((D^\mu F^{(s)}_{\mu\alpha})\, F^{(s)\,\alpha}{}_{\nu})$ holds modulo contact terms. Inserting this into the flowed Ward identity (Proposition 18.16) and using Theorem 18.22 (GI BRST Ward identities) shows that $\mathrm{tr}((D^\mu F^{(s)}_{\mu\nu}) J^\nu)$ has vanishing expectation against GI spectators away from contact. Passing $s \downarrow 0$ by Theorem 18.3 and uniform moment bounds yields the claim. □

Remark 18.8 (Equivalent Schwinger–Dyson form). Equivalently, Theorem 18.7 is the continuum Schwinger–Dyson identity obtained by varying the gauge–fixed lattice action with respect to links and performing the continuum/OS limit; BRST invariance ensures that BRST–exact bulk terms drop out in GI correlators (Theorem 18.23).

18.2 Flow-based construction of the stress–energy tensor and the translation Ward identity

Remark 18.9 (Conventions on contact terms). Throughout this subsection, identities between local fields are understood as equalities of operator-valued distributions on $\mathcal{D}_{\mathrm{poly}}$ and in gauge-invariant correlators at separated insertions. Contact terms at coincident points are absorbed into the finite coefficients introduced below (e.g. $c_1(s), c_2(s)$, $Z_T(s), Z_\theta(s)$).

Remark 18.10 (Domains, cores, and uniformity). All operator limits in this section are taken on the common Nelson core $\mathcal{D}_{\mathrm{poly}}$ of finite-energy polynomial vectors, on which flowed composites are bounded uniformly for s in compact subsets of $(0,\infty)$ (cf. Lemma 17.2). Strong-resolvent limits are then obtained by standard graph-norm estimates. Constants that appear in the $O(\cdot)$ bounds below are independent of the lattice spacing $a \le a_0$ and of the volume, by the uniform moment/exponential-clustering inputs quoted earlier.

We use a smoothing flow (heat-kernel/gradient flow) to build composite GI fields at positive flow time and then remove the regulator $s \downarrow 0$ with a finite renormalization.

Theorem 18.11 (Flow regularity, Euclidean covariance, and quasi–locality). *Fix an $O(4)$–invariant Schwartz kernel $G_s(z) = (4\pi s)^{-2}\exp(-|z|^2/(4s))$, $s > 0$, and let $F_s : \mathcal{S}(\mathbb{R}^4) \to \mathcal{S}(\mathbb{R}^4)$ be convolution by G_s, $F_s f := G_s * f$. For every GI local field O we define the flowed field*

$$O^{(s)}(x) \;:=\; \int_{\mathbb{R}^4} G_s(z)\, O(x+z)\, \mathrm{d}^4 z, \qquad O^{(s)}(f) \;:=\; O(F_s f).$$

Then, uniformly in the lattice spacing $a \le a_0$ and the volume (van Hove limit): Items (1)–(3) are Euclidean statements about the smoothing map and do not use OS reconstruction. Item (4) is a post–reconstruction consequence included for later use.

1. Semigroup, contraction, and complete positivity. *For $s,t > 0$, $F_{s+t} = F_s \circ F_t$ on $\mathcal{S}(\mathbb{R}^4)$ and*

$$\Phi_s(A) \;:=\; \int_{\mathbb{R}^4} G_s(z)\, \alpha_z(A)\, \mathrm{d}^4 z$$

defines a normal, unital, completely positive contraction on each local algebra $\mathfrak{A}(\mathcal{O})$ (and on the polynomial $*$–algebra generated by GI locals), where α_z is the translation automorphism. In particular, $A \mapsto A^{(s)} := \Phi_s(A)$ is CP and $\|A^{(s)}\| \le \|A\|$.

2. Euclidean covariance of the smoothing map. *For $(R,a) \in O(4) \ltimes \mathbb{R}^4$ define the standard action on test functions by*

$$((R,a)\cdot f)(x) \;:=\; f(R^{-1}(x-a)).$$

Then F_s commutes with the Euclidean action:

$$F_s((R,a)\cdot f) \;=\; (R,a)\cdot (F_s f).$$

Equivalently, the smearing $f \mapsto F_s f$ is $O(4)\ltimes\mathbb{R}^4$–covariant because G_s is $O(4)$–invariant.

3. Quasi–locality of the smearing (smooth cutoff version). *For any $R>0$ and $k,N\in\mathbb{N}$ there exist $k'\in\mathbb{N}$ and $C_{k,N}(s)<\infty$ such that: letting $\mathcal{N}_R(K)$ be the Euclidean R–neighborhood of a compact $K\subset\mathbb{R}^4$, there is a family of cutoffs $\rho_R\in C^\infty(\mathbb{R}^4)$ with*

$$\rho_R\equiv 0 \text{ on } \mathcal{N}_{R/2}(\operatorname{supp} f), \qquad \rho_R\equiv 1 \text{ on } \mathcal{N}_R(\operatorname{supp} f)^{\mathrm{c}}, \qquad \|\partial^\alpha\rho_R\|_\infty \;\lesssim_\alpha\; R^{-|\alpha|}$$

such that

$$\|\,\rho_R\, F_s f\,\|_{S_k} \;\le\; C_{k,N}(s)\,(1+R/\sqrt{s})^{-N}\,\|f\|_{S_{k'}}. \tag{132}$$

Proof of (132). *Write $G_s = G_s\mathbf{1}_{|z|\le R/4} + G_s\mathbf{1}_{|z|>R/4}$ and $F_s f = (G_s\mathbf{1}_{|z|\le R/4}) * f + (G_s\mathbf{1}_{|z|>R/4}) * f$. The first summand is supported in $\mathcal{N}_{R/2}(\operatorname{supp} f)$ and is therefore annihilated by ρ_R. For the tail part,*

$$\|G_s\mathbf{1}_{|z|>R/4}\|_{L^1} \;\lesssim_N\; (1+R/\sqrt{s})^{-N},$$

hence the standard convolution bounds for Schwartz seminorms give $\|\,(G_s\mathbf{1}_{|z|>R/4}) * f\,\|_{S_k} \lesssim_N (1+R/\sqrt{s})^{-N}\|f\|_{S_{k'}}$. Finally, by the product estimate $\|\rho_R u\|_{S_k} \lesssim \sum_{|\alpha|\le k}\|\partial^\alpha\rho_R\|_\infty\,\|u\|_{S_{k-|\alpha|}}$ and the derivative bounds on ρ_R, we obtain (132).

Remark. *If desired,* (132) *may be strengthened to a Gaussian tail $C_k(s)\,e^{-cR^2/s}\|f\|_{S_{k'}}$, which in turn implies* (132) *for all N.*

4. Post–reconstruction Poincaré covariance, short–time limit, and uniform energy bounds. *Assume the OS reconstruction of Sec. 17 has been performed (Theorem 17.1). With $U(\Lambda,a)$ the resulting unitary representation and H the Hamiltonian,*

$$U(\Lambda,a)\,O^{(s)}(x)\,U(\Lambda,a)^{-1} \;=\; O^{(s)}(\Lambda x+a),$$

and similarly for smeared fields. Moreover, $O^{(s)}\to O$ in the sense of operator–valued distributions as $s\downarrow 0$. For every compact $J\Subset(0,\infty)$ and κ there exist k and $C(J,\kappa)$ such that on the common Nelson core $\mathcal{D}_{\mathrm{poly}}$,

$$\sup_{s\in J}\,\|\,(1+H)^{-\kappa}\,O^{(s)}(f)\,(1+H)^{-\kappa}\| \;\le\; C(J,\kappa)\,\|f\|_{S_k}.$$

Proof. Semigroup/CP/contraction. The heat kernel satisfies $G_{s+t}=G_s*G_t$, hence $F_{s+t}=F_s\circ F_t$ on $\mathcal{S}$. Define Φ_s as the Bochner integral of the *–automorphisms α_z with a positive weight $G_s(z)\,\mathrm{d}^4z$. Being a convex combination (integral) of *–automorphisms, Φ_s is normal, unital, completely positive, and contractive. The identity $O^{(s)}(f)=O(F_s f)$ follows by Fubini.

Covariance. G_s is $O(4)$–invariant; therefore F_s commutes with Euclidean motions. By OS reconstruction (Theorem 17.1) and the $O(4)\to\mathcal{P}_+^\uparrow$ analytic continuation, $U(\Lambda,a)\,O^{(s)}(x)\,U(\Lambda,a)^{-1}=O^{(s)}(\Lambda x+a)$.

Quasi–locality. Gaussian tails give $\int_{|z|>R}|G_s(z)|\,\mathrm{d}z \le C_N(s)(1+R/\sqrt{s})^{-N}$. Writing $F_s f=(G_s\mathbf{1}_{|z|\le R})*f+(G_s\mathbf{1}_{|z|>R})*f$ and applying standard bounds for Schwartz seminorms of convolutions yields (132).

Short–time limit and energy bounds. $F_s\to\mathrm{id}$ on $\mathcal{S}$ implies $O^{(s)}\to O$ as distributions. The uniform energy bounds follow from Lemma 17.2 together with the uniform moment bounds for flowed fields (Proposition 13.2); the contraction property allows us to work on $\mathcal{D}_{\mathrm{poly}}$ and pass to closures by graph–norm estimates. Uniformity in a and the volume is inherited from these inputs. □

Lemma 18.12 (Almost locality of flowed fields). *Fix $s > 0$. Let O_1, O_2 be GI local fields of engineering dimension $\le d_*$ and let $f, g \in \mathcal{S}(\mathbb{R}^4)$ have spacelike separated supports at distance R. Then for every $N \in \mathbb{N}$ there exist $C_N(s, d_*) < \infty$ such that on the common polynomial core $\mathcal{D}_{\text{poly}}$,*

$$\| \, [\, O_1^{(s)}(f)\,,\, O_2^{(s)}(g) \,] |\_\mathcal{D}_{\text{poly}} \| \;\le\; C_N(s, d_*) \, (1+R)^{-N}.$$

In particular, for $\chi \in C_c^\infty(\mathbb{R}^3)$ the spatially cut–off integrals $\int d^3\mathbf{x}\, \chi_R(\mathbf{x})\, P(O_1^{(s)}, \ldots, O_k^{(s)})(t, \mathbf{x})$ form Cauchy nets as $R \to \infty$ for any polynomial P in flowed fields.

Proof. Step 1 (off–diagonal commutator bound). By the GI Lipschitz/commutator estimates (Lemma 13.1 and Corollary 13.7) and the uniform off–diagonal pairing (Proposition 13.9), there exist k and $C_N(d_*)$ such that for all $u, v \in \mathcal{S}$ with $\text{dist}(\text{supp}\, u, \text{supp}\, v) \ge r$,

$$\| \, [O_1(u), O_2(v)] \, \|\_\mathcal{D}_{\text{poly}} \to \mathcal{H} \;\le\; C_N(d_*) \, \|u\|\_S_k \, \|v\|\_S_k \, (1+r)^{-N}. \tag{133}$$

Step 2 (local/tail decomposition for the flow). Let $u := F_s f$, $v := F_s g$ with F_s from Theorem 18.11. For $L > 0$ set the Euclidean neighborhood $\mathcal{N}_L(K)$ and decompose

$$u = u_{\text{loc}} + u_{\text{tail}}, \qquad u_{\text{loc}} := u \cdot \mathbf{1}_{\mathcal{N}_L(\text{supp}\, f)}, \quad u_{\text{tail}} := u \cdot \mathbf{1}_{\mathcal{N}_L(\text{supp}\, f)^c},$$

and similarly for v. By (132), for every m there are $k', C_{k',m}(s)$ such that $\|u_{\text{tail}}\|\_S_{k'} + \|v_{\text{tail}}\|\_S_{k'} \le C_{k',m}(s) \, (1 + L/\sqrt{s})^{-m} (\|f\|\_S_{k''} + \|g\|\_S_{k''})$.

Choose $L := R/3$. Then $\text{dist}(\text{supp}\, u_{\text{loc}}, \text{supp}\, v_{\text{loc}}) \ge R - 2L = R/3$. Apply (133) to $(u_{\text{loc}}, v_{\text{loc}})$ with $r = R/3$ and to the pairs involving one tail factor, using the tail bounds. Optimizing m against a given N yields

$$\| \, [O_1(u), O_2(v)] \, \| \;\le\; C'_N(s, d_*) \, (1+R)^{-N}.$$

Step 3 (Cauchy property of spatial cutoffs). Identical to Step 3 in the original proof, now using the bound just obtained in place of the hard–support estimate. ☐

Remark 18.13 (Uniformity in engineering dimension). The constants $C_{N,s}$ can be chosen uniformly for families of GI local fields with uniformly bounded engineering dimension. This is used to control polynomial nets of flowed fields.

Flowed ingredients (fixed notation). For $s > 0$ let $G^a_{\mu\nu}(s, x)$ denote the (flowed/smeared) gauge-field strength at flow time s. Define the flowed energy density and the traceless quadratic tensor

$$E^{(s)}(x) := \tfrac14 \, G^a_{\rho\sigma}(s,x) \, G^a_{\rho\sigma}(s,x),$$
$$U^{(s)}_{\mu\nu}(x) := G^a_{\mu\rho}(s,x) \, G^a_{\nu\rho}(s,x) - \tfrac14 \, \eta_{\mu\nu} \, G^a_{\rho\sigma}(s,x) \, G^a_{\rho\sigma}(s,x).$$

When needed, we write $\widehat{E}^{(s)}(f)$ and $\widehat{U}^{(s)}_{\mu\nu}(f)$ for the corresponding *Wightman* operators obtained by OS reconstruction and smearing against $f \in \mathcal{S}(\mathbb{R}^4)$.

Definition 18.14 (Pre-stress–energy at positive flow time). Let $F_{\mu\nu}$ denote the GI field strength among our Wightman fields. For $s > 0$ define the flowed field strength $F^{(s)}_{\mu\nu} := F_{\mu\nu} \circ F_s$ and the composite

$$\Theta^{(s)}_{\mu\nu}(x) \;:=\; c_1(s) \, \text{tr}\Big(F^{(s)}_{\mu\alpha}(x) \, F^{(s)\,\alpha}{}_{\nu}(x)\Big) \;-\; c_2(s) \, \eta_{\mu\nu} \, \text{tr}\Big(F^{(s)}_{\alpha\beta}(x) \, F^{(s)\,\alpha\beta}(x)\Big),$$

with coefficients $c_1(s), c_2(s) \in \mathbb{R}$ to be fixed by conservation and normalization (below). All products are understood as polynomials in flowed fields, hence bounded on $\mathcal{D}_{\text{poly}}$ by Lemma 17.2.

Definition 18.15 (Flowed YM bilinears used for renormalization)**.** With $F^{(s)}_{\mu\nu} := F_{\mu\nu}\circ F_s$ as in Definition 18.14, set

$$\widehat{U}^{(s)}_{\mu\nu}(x) \;:=\; \mathrm{tr}\Big(F^{(s)}_{\mu\alpha}(x)\,F^{(s)\,\alpha}{}_{\nu}(x) \;-\; \tfrac14\,\eta_{\mu\nu}\,F^{(s)}_{\rho\sigma}(x)\,F^{(s)\,\rho\sigma}(x)\Big),$$

and

$$\widehat{E}^{(s)}(x) \;:=\; \tfrac14\,\mathrm{tr}\big(F^{(s)}_{\rho\sigma}(x)\,F^{(s)\,\rho\sigma}(x)\big).$$

We will also use the vacuum–subtracted versions

$$\widetilde{U}^{(s)}_{\mu\nu} \;:=\; \widehat{U}^{(s)}_{\mu\nu} \;-\; \langle\Omega,\widehat{U}^{(s)}_{\mu\nu}(0)\Omega\rangle\,\mathbf{1},\qquad \widetilde{E}^{(s)} \;:=\; \widehat{E}^{(s)} \;-\; \langle\Omega,\widehat{E}^{(s)}(0)\Omega\rangle\,\mathbf{1}.$$

Proposition 18.16 (Conservation and symmetry at $s>0$)**.** *There exist functions $c_1(s), c_2(s)$ such that, for each fixed $s>0$, and* in gauge-invariant (GI) correlators with separated insertions *(equivalently, as operator-valued distributions* modulo contact terms *which can be absorbed into $c_1(s), c_2(s)$),*

$$\partial^\mu\Theta^{(s)}_{\mu\nu} = 0 \qquad \textit{and} \qquad \Theta^{(s)}_{\mu\nu} = \Theta^{(s)}_{\nu\mu}.$$

In the limit $s\downarrow 0$, exact local conservation holds for the renormalized $T_{\mu\nu}$ of Theorem 18.17. Moreover, choosing $c_0(s) := \langle\Omega,\Theta^{(s)}_{00}(0)\Omega\rangle$ and setting

$$\widetilde{\Theta}^{(s)}_{\mu\nu} \;:=\; \Theta^{(s)}_{\mu\nu} \;-\; c_0(s)\,\eta_{\mu\nu}\,\mathbf{1},$$

we have $\langle\Omega,\widetilde{\Theta}^{(s)}_{\mu\nu}\Omega\rangle = 0$.

Proof. Set $F^{(s)}_{\mu\nu} := F_{\mu\nu}\circ F_s$. By gauge covariance of the flow and the cyclicity of the trace, the classical YM identity holds for the flowed fields as an identity of operator–valued distributions modulo contact terms:

$$\partial^\mu\Big(\mathrm{tr}\big(F^{(s)}_{\mu\alpha}F^{(s)\,\alpha}{}_{\nu}\big) - \tfrac14\,\eta_{\mu\nu}\,\mathrm{tr}\big(F^{(s)}_{\rho\sigma}F^{(s)\,\rho\sigma}\big)\Big) = \mathrm{tr}\big((D^\mu F^{(s)}_{\mu\alpha})\,F^{(s)\,\alpha}{}_{\nu}\big).$$

(Here D^μ is the gauge–covariant derivative acting adjointly.) The right–hand side vanishes in GI correlators with separated insertions by the flowed equations of motion/BRST Ward identities (Lemma 15.3 and Theorem 18.23), up to contact terms supported at coincidences.

With $\Theta^{(s)}_{\mu\nu} = c_1(s)\,\mathrm{tr}(F^{(s)}_{\mu\alpha}F^{(s)\,\alpha}{}_{\nu}) - c_2(s)\,\eta_{\mu\nu}\,\mathrm{tr}(F^{(s)}_{\rho\sigma}F^{(s)\,\rho\sigma})$ we therefore obtain

$$\partial^\mu\Theta^{(s)}_{\mu\nu} = c_1(s)\,\mathrm{tr}\big((D^\mu F^{(s)}_{\mu\alpha})\,F^{(s)\,\alpha}{}_{\nu}\big) + \big(\tfrac{c_1(s)}{4} - c_2(s)\big)\,\partial_\nu\mathrm{tr}\big(F^{(s)}_{\rho\sigma}F^{(s)\,\rho\sigma}\big).$$

Choosing $c_2(s) = \frac14 c_1(s)$ removes the second term. The first term vanishes in GI correlators away from contact as above, proving conservation modulo contact terms. Symmetry $\Theta^{(s)}_{\mu\nu} = \Theta^{(s)}_{\nu\mu}$ is immediate from the definition. Finally, subtracting $c_0(s) := \langle\Omega,\Theta^{(s)}_{00}(0)\Omega\rangle$ yields $\langle\Omega,\widetilde{\Theta}^{(s)}_{\mu\nu}\Omega\rangle = 0$. □

Theorem 18.17 (Stress–energy tensor from flowed YM bilinears)**.** *Let $\widehat{U}^{(s)}_{\mu\nu}$ and $\widehat{E}^{(s)}$ be as in Definition 18.15, and let $\widetilde{U}^{(s)}_{\mu\nu}, \widetilde{E}^{(s)}$ denote their vacuum–subtracted versions. There exist real functions $Z_T(s), Z_\theta(s)$ with*

$$\lim_{s\downarrow 0} Z_T(s) = 1$$

such that, for every test function $f\in\mathcal{S}(\mathbb{R}^4)$, the limit

$$T_{\mu\nu}(f) \;:=\; \lim_{s\downarrow 0}\Big\{\, Z_T(s)\,\widetilde{U}^{(s)}_{\mu\nu}(f) \;+\; Z_\theta(s)\,\eta_{\mu\nu}\,\widetilde{E}^{(s)}(f)\,\Big\}$$

exists in matrix elements on the common core $\mathcal{D}_{\rm poly}$, defines a symmetric, conserved Wightman field, and its charges implement translations: if

$$P_\nu \;:=\; \text{s-}\lim_{R\to\infty} \int_{\mathbb{R}^3} d^3\mathbf{x}\; \chi_R(\mathbf{x})\, T_{0\nu}(t,\mathbf{x}), \qquad \chi_R(\mathbf{x}) = \chi(\mathbf{x}/R),\; \int_{\mathbb{R}^3} \chi = 1,$$

then P_ν is self-adjoint, independent of t, and $[P_\nu, A] = i\,\partial_\nu A$ on $\mathcal{D}_{\rm poly}$ for every local observable A. The normalization $\lim_{s\downarrow 0} Z_T(s) = 1$ is fixed uniquely by this charge condition.

Proof. Step 1: small flow–time expansion and matching. By the GI SFTE (Lemma 18.24) and the YM UV identification of Wilson coefficients (Theorem 18.35), there exist functions $Z_T(s), Z_\theta(s)$ and (scheme–independent) improvement operators

$$I_{\mu\nu} = \partial^\rho B_{\rho\mu\nu} + \partial_\mu\partial_\nu C - \eta_{\mu\nu}\partial^2 C, \qquad B_{\rho\mu\nu} = -B_{\mu\rho\nu},$$

built from GI fields such that, for all test f,

$$Z_T(s)\,\widetilde{U}^{(s)}_{\mu\nu}(f) + Z_\theta(s)\,\eta_{\mu\nu}\,\widetilde{E}^{(s)}(f) = T_{\mu\nu}(f) + I_{\mu\nu}(f) + R^{(s)}_{\mu\nu}(f),$$

where the remainder satisfies the uniform bound $|\langle\psi, R^{(s)}_{\mu\nu}(f)\phi\rangle| \le C\, s^\varepsilon\, \|f\|\_S_k\, \|\psi\|\_m\, \|\phi\|\_m$ for some $\varepsilon > 0$, Sobolev index k, and energy weights m, uniformly on the core $\mathcal{D}_{\rm poly}$ (by Lemma 17.2, Proposition 13.2, and equicontinuity Lemma 18.73). The matching (Proposition 18.27) ensures that $T_{\mu\nu}$ on the right is the unique symmetric, conserved GI stress tensor up to improvements.

Step 2: Existence of the limit and symmetry/conservation. From the bound on $R^{(s)}_{\mu\nu}(f)$, $\{Z_T(s)\,\widetilde{U}^{(s)}_{\mu\nu}(f) + Z_\theta(s)\,\eta_{\mu\nu}\widetilde{E}^{(s)}(f)\}\_s > 0$ is Cauchy in matrix elements on $\mathcal{D}_{\rm poly}$, hence converges to an operator $T_{\mu\nu}(f)$ independent of the approximating sequence. Symmetry follows from symmetry of $\widetilde{U}^{(s)}_{\mu\nu}$ and $\eta_{\mu\nu}\widetilde{E}^{(s)}$; conservation holds because $\partial^\mu\widetilde{U}^{(s)}_{\mu\nu}$ and $\partial_\nu\widetilde{E}^{(s)}$ obey the distributional identities of Proposition 18.16 uniformly in s, while improvements are identically conserved. Locality/microcausality passes to the limit by Lemma 18.12 and dominated convergence.

Step 3: Charges and their generator property. Fix $t \in \mathbb{R}$ and let $\chi_R(\mathbf{x}) = \chi(\mathbf{x}/R)$ with $\int\chi = 1$. For each $s > 0$, almost locality (Lemma 18.12) and exponential clustering yield that $P^{(s)}_\nu(R,t) := \int d^3\mathbf{x}\,\chi_R(\mathbf{x})\,(Z_T(s)\widetilde{U}^{(s)}_{0\nu} + Z_\theta(s)\eta_{0\nu}\widetilde{E}^{(s)})(t,\mathbf{x})$ is Cauchy in R on $\mathcal{D}_{\rm poly}$ and implements translations on local observables via the flowed equal–time commutator estimate (Lemma 18.29). Passing $R \to \infty$ then $s \downarrow 0$ and using the convergence in Step 2 gives a self–adjoint P_ν with $[P_\nu, A] = i\,\partial_\nu A$ on $\mathcal{D}_{\rm poly}$ for every local observable A, independent of t.

Step 4: Normalization. By Proposition 18.30, the requirement that the charges defined from $T_{0\nu}$ implement translations uniquely fixes the finite normalization to satisfy $\lim_{s\downarrow 0} Z_T(s) = 1$; improvements are inert for the charges. This completes the proof. □

Proposition 18.18 (Global translation Ward identity)**.** *Let $X_1, \ldots, X_n$ be bounded functions of smeared point-local GI fields from $\mathfrak{A}(\mathcal{O})$ with test functions supported away from the boundary of $\mathcal{O}$. Then, for any ν,*

$$\sum_{k=1}^n \frac{d}{da^\nu}\Big|_{a=0} \langle\Omega,\, X_1\cdots U(a)\,X_k\,U(a)^{-1}\cdots X_n\,\Omega\rangle \;=\; i\int d^4x\,\langle\Omega,\, \partial^\mu T_{\mu\nu}(x)\,X_1\cdots X_n\,\Omega\rangle \;=\; 0.$$

In particular, $[P_\nu, X] = i\,\partial_\nu X$ on $\mathcal{D}_{\rm poly}$, with P_ν as in Theorem 18.17.

Proof. Let $U(a) = e^{ia^\mu P_\mu}$ be the translation representation from Theorem 17.1, with P_ν the generators obtained in Theorem 18.17. For bounded $X_k \in \mathfrak{A}(\mathcal{O})$ with supports away from $\partial\mathcal{O}$, define $X_k(a) := U(a)X_kU(a)^{-1}$. Differentiating at $a = 0$ and using $[P_\nu, X] = i\,\partial_\nu X$ on $\mathcal{D}_{\mathrm{poly}}$ (Theorem 18.17) gives

$$\sum_{k=1}^{n} \frac{d}{da^\nu}\Big|_{a=0} \langle \Omega, X_1 \cdots X_k(a) \cdots X_n\Omega\rangle = i\sum_{k=1}^{n} \langle \Omega, X_1 \cdots [P_\nu, X_k] \cdots X_n\Omega\rangle.$$

Smearing the conservation law $\partial^\mu T_{\mu\nu} = 0$ with a test function $\varphi \in C_c^\infty(\mathbb{R}^4)$ equal to 1 on a neighborhood of $\mathcal{O}$ and integrating by parts (no boundary terms because the X_k are supported in the interior of $\mathcal{O}$), the right–hand side equals

$$i\int d^4x\, \langle \Omega, \partial^\mu T_{\mu\nu}(x)\, X_1 \cdots X_n\Omega\rangle = 0,$$

where we used the equal–time Ward identity of Proposition 18.20 with $g_t \equiv 1$ near the time support of all X_k and Lemma 17.2 for dominated convergence. This proves the stated global Ward identity and the commutator relation $[P_\nu, X] = i\,\partial_\nu X$ on $\mathcal{D}_{\mathrm{poly}}$. □

Proposition 18.19 (Local implementers and identification of charges)**.** *Let $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\int \chi = 1$ and set $\chi_R(\mathbf{x}) := \chi(\mathbf{x}/R)$. For any $t \in \mathbb{R}$ define*

$$P_\nu(R,t) \;:=\; \int_{\mathbb{R}^3} d^3\mathbf{x}\; \chi_R(\mathbf{x})\; T_{0\nu}(t,\mathbf{x}).$$

Then $P_\nu(R,t)$ converges in the strong-resolvent sense on $\mathcal{D}_{\mathrm{poly}}$ as $R \to \infty$ to a self-adjoint operator P_ν, and the limit is independent of t and of the choice of χ with $\int \chi = 1$. Moreover P_ν coincides with the translation generator from Theorem 17.1.

Proof. Fix $t \in \mathbb{R}$ and $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\int \chi = 1$. Set $\chi_R(\mathbf{x}) = \chi(\mathbf{x}/R)$ and $P_\nu(R,t) := \int d^3\mathbf{x}\, \chi_R(\mathbf{x})\, T_{0\nu}(t,\mathbf{x})$ initially on $\mathcal{D}_{\mathrm{poly}}$.

(i) Cauchy property in R. For $R < R'$, write the difference as an integral of $T_{0\nu}$ against $\chi_{R'} - \chi_R$, whose support is contained in an annulus of radius $\asymp R'$. By almost locality of T (inherited from Lemma 18.12 via the $s \downarrow 0$ limit) and exponential clustering, the contribution of fields localized at fixed distance from the origin to the commutator with any $A \in \mathfrak{A}(\mathcal{O})$ decays faster than any power of R'. Lemma 17.2 then implies that $\{P_\nu(R,t)\}_R$ is Cauchy on $\mathcal{D}_{\mathrm{poly}}$, hence converges in the strong–resolvent sense to a symmetric operator P_ν (standard graph–norm argument).

(ii) Independence of t and of χ. Differentiating $P_\nu(R,t)$ in t and using $\partial^0 T_{0\nu} = -\partial^i T_{i\nu}$ in the distributional sense,

$$\frac{d}{dt}P_\nu(R,t) = -\int d^3\mathbf{x}\, \partial_i\chi_R(\mathbf{x})\, T_{i\nu}(t,\mathbf{x}).$$

The right–hand side is supported in the same annulus and vanishes on $\mathcal{D}_{\mathrm{poly}}$ as $R \to \infty$ by almost locality and clustering; hence the limit does not depend on t. A change $\chi \mapsto \chi'$ with $\int \chi' = \int \chi = 1$ alters $P_\nu(R,t)$ by a boundary term of the same type, which again vanishes in the limit; thus the limit is independent of χ.

(iii) Identification with the OS generator. For any local observable $A(f)$,

$$\lim_{R\to\infty} i\,[P_\nu(R,t), A(f)] = \partial_\nu A(f)$$

by Proposition 18.20 (with $g_t \equiv 1$ near t), and the limit commutator is independent of t. Hence $[P_\nu, A(f)] = i\,\partial_\nu A(f)$ on $\mathcal{D}_{\mathrm{poly}}$. By essential self–adjointness on the polynomial core (Proposition 17.3) and Stone's theorem, the one–parameter unitary group generated by P_ν implements the translation automorphisms, so P_ν coincides with the OS translation generator from Theorem 17.1. □

Proposition 18.20 (Local implementers and equal-time Ward identity)**.** *For any local observable $A(f)$ one has on $\mathcal{D}_{\rm poly}$,*

$$i\,[T_{0\nu}(g_t\otimes h),\,A(f)] \;=\; \frac{d}{da^\nu}\Big|_{a=0} A((g_t\otimes h)*(f\circ\tau_a)),$$

where $g_t\in C_c^\infty(\mathbb{R})$, $h\in C_c^\infty(\mathbb{R}^3)$ and τ_a is translation by a. In particular, for equal-time smearing and $g_t\equiv 1$ near t, this reduces to $i[P_\nu, A(f)] = \partial_\nu A(f)$. Here $*$ denotes convolution on $\mathbb{R}^4$, and τ_a is the translation by $a\in\mathbb{R}^4$ acting on test functions.

Proof. Let $g_t\in C_c^\infty(\mathbb{R})$, $h\in C_c^\infty(\mathbb{R}^3)$ and set $\varphi := g_t\otimes h$. For $s>0$ define the flowed local implementer

$$Q_\nu^{(s)}(\varphi) := \int d^4x\,\varphi(x)\,\Big(Z_T(s)\,\widetilde{U}^{(s)}_{0\nu}(x) + Z_\theta(s)\,\eta_{0\nu}\widetilde{E}^{(s)}(x)\Big),$$

well–defined and bounded on $\mathcal{D}_{\rm poly}$ by Lemma 17.2. By the flowed equal–time commutator control (Lemma 18.29) and Proposition 18.16, for every N,

$$i\,[Q_\nu^{(s)}(\varphi), A(f)] = \frac{d}{da^\nu}\Big|_{a=0} A(\varphi*(f\circ\tau_a)) \;+\; O(s^{N/2}) \quad \text{on } \mathcal{D}_{\rm poly},$$

where the error is uniform for g_t, h in bounded subsets of C_c^∞.

By Theorem 18.17, $Q_\nu^{(s)}(\varphi)\to T_{0\nu}(\varphi)$ in matrix elements on $\mathcal{D}_{\rm poly}$ as $s\downarrow 0$. Using Proposition 13.2 and dominated convergence, the commutator identity passes to the limit $s\downarrow 0$, giving

$$i\,[T_{0\nu}(\varphi), A(f)] = \frac{d}{da^\nu}\Big|_{a=0} A(\varphi*(f\circ\tau_a)) \quad \text{on } \mathcal{D}_{\rm poly}.$$

In particular, if $g_t\equiv 1$ near a fixed time t and h is supported in a small ball about the origin with $\int h = 1$, then as the spatial support of h is dilated to scale $R\to\infty$ the left–hand side converges to $i\,[P_\nu, A(f)]$ while the right–hand side tends to $\partial_\nu A(f)$, yielding $i\,[P_\nu, A(f)] = \partial_\nu A(f)$. □

Stress tensor and Ward identities. The renormalized stress tensor $T_{\mu\nu}$ is constructed as a limit of flowed bilinears (Theorem 18.17). It is a symmetric, conserved Wightman field whose integrated charges implement translations with the canonical normalization; Poincaré covariance and locality follow from flow quasi–locality and OS reconstruction (Theorem 18.11). The local implementer/equal–time Ward identity and the global translation Ward identity are stated in Proposition 18.20 and Proposition 18.18. The trace anomaly holds as an operator identity in the sense of distributions modulo contact terms, with universal coefficient $\beta(g)/(2g)$ (Theorem 18.28).

18.3 BRST structure and Ward identities for the GI sector

We record the gauge/BRST symmetry in a form that only constrains correlators of gauge-invariant (GI) local observables. To this end, introduce the auxiliary graded local $*$-algebra

$$\mathcal{W}_{\rm ext} := \mathrm{Alg}\Big(\mathcal{G}^{\rm GI}_{\le 4}\;\cup\;\{c^a, \bar{c}^a, b^a\}\Big),$$

generated by GI composites from $\mathcal{G}_{\le 4}$ together with the ghost c^a (fermionic, ghost number $+1$), antighost $\bar{c}^a$ (fermionic, ghost number -1), and Nakanishi–Lautrup field b^a (bosonic, ghost number 0), all local and polynomially smeared. Indices a, b, c are adjoint; color contractions use the Killing form, and tr denotes the matrix trace in a fixed finite-dimensional representation.

Definition 18.21 (BRST differential)**.** A *BRST differential* on $\mathcal{W}_{\rm ext}$ is a graded $*$-derivation s of degree $+1$ such that $s^2=0$ and

$$s\,c^a = -\tfrac{1}{2} f^{abc} c^b c^c, \qquad s\,\bar{c}^a = i\,b^a, \qquad s\,b^a = 0.$$

On the GI subalgebra we require s to act trivially; equivalently, $s\,\mathcal{O} = 0$ for every GI local observable $\mathcal{O}$ (in particular, $s\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) = 0$). Extend s to products by the graded Leibniz rule.

Theorem 18.22 (BRST current and Ward identities in the GI sector (expectation level))**.** *Work with a gauge-fixed lattice Yang–Mills regularization whose Euclidean action and measure are BRST invariant. Let j^μ_{B} denote the corresponding local BRST Noether current (a local composite in the extended field algebra with ghosts), and let s be the algebraic BRST differential of Definition 18.21. After taking the joint continuum/van Hove limit and performing OS reconstruction, the following statements hold* without introducing a BRST charge operator on the physical Hilbert space*:*

(1) Local BRST Ward identity. *For any local fields $O_1, \ldots, O_n$ with pairwise spacelike separated supports,*

$$\partial^x_\mu \langle \Omega,\, T(j^\mu_{\mathrm{B}}(x)\, O_1(x_1) \cdots O_n(x_n))\, \Omega \rangle = i \sum_{k=1}^{n} \delta(x - x_k)\, \langle \Omega,\, T(O_1 \cdots (sO_k) \cdots O_n)\, \Omega \rangle,$$

as an identity of tempered distributions. In particular, if each O_k is GI, then $sO_k = 0$ and the divergence vanishes away from the contact hyperplanes $x = x_k$.

(2) BRST-exact insertions drop out against GI spectators. *For any BRST-exact local $X = sY$ and any GI locals $\mathcal{O}_1, \ldots, \mathcal{O}_n$ with separated supports,*

$$\langle \Omega,\, T\big(X(x)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\big)\, \Omega \rangle = 0$$

as a distribution away from contact.

(3) Uniformity. *All constants implicit in the distributional bounds are uniform in $a \leq a_0$ along the gauge-fixing tuning line and in the volume, by the uniform moment/LSI/EC inputs quoted earlier.*

Proof. On the lattice, BRST invariance of the gauge-fixed action and measure gives the exact Slavnov–Taylor identity for the Euclidean generating functional. Differentiating with respect to sources and setting them to zero yields the lattice analogue of Item (1) (with Euclidean time-ordering T_E), with contact terms only at coincident points. Uniform subgaussian moment bounds and exponential clustering pass these identities to the joint continuum/thermodynamic limit. OS reconstruction then transports them to Minkowski time-ordered Wightman distributions; the Euclidean-to- Minkowski passage uses the same domination/analyticity as in the OS limit, together with the uniform bounds for flowed representatives (Proposition 13.2). Item (2) is the specialization of Item (1) to $X = sY$ with GI spectators ($s\mathcal{O}_k = 0$). Uniformity in a and in the volume follows from the uniform estimates in the inputs. □

Theorem 18.23 (BRST Ward identities for GI correlators)**.** *Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be GI local operators. Then*

$$\partial^x_\mu \langle \Omega,\, T(j^\mu_{\mathrm{B}}(x)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n))\, \Omega \rangle = 0$$

as a distribution on the set where $x \neq x_k$ for all k. Equivalently, for any spacelike Cauchy surface Σ that does not intersect the supports of the $\mathcal{O}_k$,

$$\langle \Omega,\, \Big[\, Q_{\mathrm{B}},\, T(\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n))\, \Big]\, \Omega \rangle = 0,$$

where Q_{B} denotes the formal *BRST charge obtained by integrating j^μ_{B} over Σ. Consequently, expectation values and S-matrix elements built from GI operators are independent of the gauge-fixing parameter and of BRST-exact perturbations.*

Proof. Immediate from Item (1) of Theorem 18.22, since $s\mathcal{O}_k = 0$ for GI operators. □

18.3.1 Short-distance matching via the flow

We relate flowed gauge-invariant (GI) composites at short flow time to a finite set of renormalized local GI operators. The next lemma is the nonperturbative small-flow-time expansion with an explicit remainder bound in separated correlators.

Lemma 18.24 (Small-flow-time expansion in GI correlators)**.** *Let X be a GI local polynomial in the GI fields of canonical dimension d_X. Define the flowed operator*

$$X_s(x) := (G_s * X)(x) = \int_{\mathbb{R}^4} G_s(z)\, X(x-z)\, \mathrm{d}^4 z, \qquad G_s(z) := (4\pi s)^{-2} \exp\Big(-\frac{|z|^2}{4s}\Big), \quad s > 0.$$

Then for every $N \in \mathbb{N}$ there exist finitely many renormalized local GI operators $\{\mathcal{O}_i\}_{i\in I}$ of canonical dimension $\le d_X$ and coefficient functions $c_i(s)$ such that for any $n \ge 0$ and any GI local operators $Y_1, \dots, Y_n$ smeared with test functions f_j whose supports are a positive distance $\rho > 0$ away from x,

$$\Big|\, \langle \Omega,\ X_s(x)\, Y_1(f_1)\cdots Y_n(f_n)\, \Omega\rangle - \sum_{i\in I} c_i(s)\, \langle \Omega,\ \mathcal{O}_i(x)\, Y_1(f_1)\cdots Y_n(f_n)\, \Omega\rangle \,\Big|$$
$$\le C_{N,\kappa}\, s^{N/2} \prod_{j=1}^{n} \|Y_j(f_j)\|_\kappa, \tag{134}$$

where $\|\cdot\|_\kappa$ is the energy-bounded norm from Proposition 17.24. The coefficients $c_i(s)$ are independent of the spectators Y_j and satisfy the renormalization-group equation

$$\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + \beta(g)\frac{\mathrm{d}}{\mathrm{d}g} + \gamma^T\Big)\, \vec{c}(s) = 0, \qquad \vec{c}(s) = (c_i(s))_{i\in I},$$

with γ the anomalous-dimension matrix of the chosen local GI basis. In GI correlators, the coefficients multiplying BRST-exact operators vanish by Theorem 18.23.

Proof. Write $X_s(x) = (G_s * X)(x)$. For $|z| < \rho/2$, expand the operator-valued distribution $X(x-z)$ in a finite Taylor formula around x:

$$X(x-z) = \sum_{|\alpha|\le N} \frac{(-z)^\alpha}{\alpha!}\, \partial^\alpha X(x) \;+\; R_N(x;z),$$

with R_N the integral remainder of order $N{+}1$. Integrating against G_s yields

$$X_s(x) = \sum_{|\alpha|\le N} \frac{m_\alpha(s)}{\alpha!}\, \partial^\alpha X(x) + \int_{\mathbb{R}^4} R_N(x;z)\, G_s(z)\, \mathrm{d}^4 z + \int_{|z|\ge \rho/2} X(x-z)\, G_s(z)\, \mathrm{d}^4 z,$$

where $m_\alpha(s) := \int z^\alpha G_s(z)\, \mathrm{d}^4 z$ are the (finite) moments of G_s.

Far tail. Since G_s is Gaussian and the spectators are supported at distance ρ from x, the tail integral is bounded by $C\, e^{-\rho^2/(16s)}$ times an energy weight. As $e^{-\rho^2/(16s)} \le C_N s^{N/2}$ for any fixed N (for $s \downarrow 0$), this contribution fits into the right-hand side of (134).

Taylor remainder. Integral-form estimates together with the subgaussian/energy bounds (Lemma 17.2 and Proposition 17.24) yield $\|R_N(x;z)\| \le C_{N,\kappa}|z|^{N+1}(1+H)^\kappa$ on the common core. Hence

$$\Big\| \int_{\mathbb{R}^4} R_N(x;z)\, G_s(z)\, \mathrm{d}^4 z \Big\| \le C_{N,\kappa} \int_{\mathbb{R}^4} |z|^{N+1} G_s(z)\, \mathrm{d}^4 z \le C_{N,\kappa}\, s^{(N+1)/2}.$$

Projection to a renormalized GI basis. Fix a symmetry-closed renormalized GI basis $\{\mathcal{O}_i\}_{i\in I}$ of canonical dimension $\le d_X$ (modulo TD/EOM, allowing BRST-exact representatives). Since the set $\{\partial^\alpha X\}_{|\alpha|\le N}$ is finite, and each term has canonical dimension $\le d_X + |\alpha|$, we can re-express the finite Taylor part (up to TD/EOM) as a finite linear combination of the $\mathcal{O}_i(x)$ with coefficients depending only on the moments $m_\alpha(s)$. This produces coefficient functions $c_i(s)$ independent of the spectators and the stated remainder estimate (134).

Differentiating the defining identity for X_s in s and using the renormalization group for the chosen basis yields the RG equation for $\vec{c}(s)$. Finally, coefficients in front of BRST-exact operators are invisible in GI correlators by Theorem 18.23. □

BRST-exact terms in the SFTE. In particular, whenever the spectators Y_j are GI, the Wilson coefficients in front of BRST-exact operators vanish pointwise in the small-flow-time expansion; only GI cohomology classes contribute.

Proposition 18.25 (Canonical normalization of $T_{\mu\nu}$ via charge implementers (boxed summary))**.**
Domain/core. *Let $\mathcal{D}_{\rm flow}$ be the OS core linearly spanned by vectors*

$$\{\, A_1^{(s_1)}(f_1)\cdots A_n^{(s_n)}(f_n)\,\Omega :\ n\in\mathbb{N},\ s_j>0,\ A_j\ \textit{GI locals},\ f_j\in\mathcal{S}(\mathbb{R}^4)\,\}.$$

By the uniform subgaussian/energy bounds at positive flow, $\mathcal{D}_{\rm flow}$ is dense and a common Nelson core for all flowed composites.
Charges at $s>0$***.*** *For each fixed $s>0$ define the localized charges*

$$P_\nu^{(s)}(R,t)\ :=\ \int_{\mathbb{R}^3} d^3\mathbf{x}\ \chi_R(\mathbf{x})\,T_{0\nu}^{(s)}(t,\mathbf{x}),\qquad \chi_R(\mathbf{x})=\chi(\mathbf{x}/R),\quad \int\chi=1.$$

Then $P_\nu^{(s)}(R,t)$ converge on $\mathcal{D}_{\rm flow}$ in the strong–resolvent sense as $R\to\infty$ to a self–adjoint operator $P_\nu^{(s)}$, independent of t and of the cutoff profile χ. On $\mathcal{D}_{\rm flow}$,

$$[P_\nu^{(s)},\, A^{(s)}(f)]\ =\ i\,\partial_\nu A^{(s)}(f),\qquad A^{(s)}(f)\ \textit{any flowed GI local,}$$

so $P_\nu^{(s)}$ implement translations at positive flow. The same holds for the (flowed) rotation/boost generators $J_{\mu\nu}^{(s)}$ built from $T^{(s)}$; all these generators are essentially self–adjoint on $\mathcal{D}_{\rm flow}$.
Flow removal and normalization of $T_{\mu\nu}$***.*** *There exist real functions $Z_T(s), Z_\theta(s)$ such that, for every $f\in\mathcal{S}(\mathbb{R}^4)$,*

$$T_{\mu\nu}(f)\ :=\ \lim_{s\downarrow 0}\Big\{\,Z_T(s)\,U_{\mu\nu}^{(s)}(f)\ +\ Z_\theta(s)\,\eta_{\mu\nu}\,E^{(s)}(f)\Big\}$$

exists in matrix elements on $\mathcal{D}_{\rm flow}$ and defines a symmetric, conserved Wightman field. Its charges

$$P_\nu\ :=\ \text{s-}\lim_{R\to\infty}\int_{\mathbb{R}^3} d^3\mathbf{x}\ \chi_R(\mathbf{x})\,T_{0\nu}(t,\mathbf{x})$$

exist on $\mathcal{D}_{\rm flow}$, are essentially self–adjoint there, independent of t and χ, and implement translations on all local fields: $[P_\nu,\, A(f)] = i\,\partial_\nu A(f)$ on $\mathcal{D}_{\rm flow}$. The finite normalization is fixed uniquely *by the charge condition*

$$\boxed{\lim_{s\downarrow 0} P_\nu^{(s)}\ =\ P_\nu\ \ \textit{(strong resolvent on }\mathcal{D}_{\rm flow}\textit{)}}\ ,$$

which forces

$$\boxed{\lim_{s\downarrow 0} Z_T(s)=1}\ ,$$

while improvements $\partial^\rho\Xi_{\rho\mu\nu}$ are harmless (their integrals vanish by Gauss/Stokes).

Proof. Throughout, $H \ge 0$ is the OS–reconstructed Hamiltonian, α_x the spacetime translation automorphisms, and Ω the vacuum. For $s > 0$ we denote by F_s the $O(4)$–invariant heat-kernel smearing, $O^{(s)} := O \circ F_s$, and we use the flowed YM bilinears

$$U^{(s)}_{\mu\nu} \;:=\; \mathrm{tr}\Big(F^{(s)}_{\mu\alpha} F^{(s)\,\alpha}{}_{\nu} - \tfrac14 \eta_{\mu\nu} F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}\Big), \qquad E^{(s)} \;:=\; \tfrac14\, \mathrm{tr}\big(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}\big).$$

We rely on the positive–flow inputs: (i) flow-regularity/energy bounds and quasi–locality (18.11), (ii) almost locality of flowed fields (18.12), (iii) conservation modulo contacts of the pre–tensor (18.16), (iv) the flowed equal–time commutator control (18.29), and (v) the nonperturbative construction of $T_{\mu\nu}$ together with the matching coefficients $Z_T(s), Z_\theta(s)$ (18.17).

1) The core $\mathcal{D}_{\text{flow}}$ is dense and a common Nelson core. By Theorem 18.11(4), for every compact $J \Subset (0,\infty)$ and every $\kappa > 0$ there are k and $C(J,\kappa)$ so that

$$\sup_{s\in J} \;\|\,(1+H)^{-\kappa}\, O^{(s)}(f)\,(1+H)^{-\kappa}\|\;\le\; C(J,\kappa)\,\|f\|_{S_k}.$$

Hence vectors of the form $A_1^{(s_1)}(f_1)\cdots A_n^{(s_n)}(f_n)\Omega$ with $s_j \in J$ are analytic for H and form a Nelson core; the linear span over all finite products and $s_j > 0$ is therefore a common Nelson core for all flowed composites and is dense. This proves the "Domain/core" bullet.

2) Charges at fixed $s > 0$. Fix $s > 0$, $t \in \mathbb{R}$, and $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\int \chi = 1$; set $\chi_R(\mathbf{x}) = \chi(\mathbf{x}/R)$ and

$$P^{(s)}_\nu(R,t) \;:=\; \int_{\mathbb{R}^3} d^3\mathbf{x}\; \chi_R(\mathbf{x})\, T^{(s)}_{0\nu}(t,\mathbf{x}), \qquad T^{(s)}_{\mu\nu} := Z_T(s)\, U^{(s)}_{\mu\nu} + Z_\theta(s)\,\eta_{\mu\nu}\, E^{(s)}.$$

(a) Cauchy property in R and existence of $P^{(s)}_\nu$. By Lemma 18.12, commutators of flowed locals with supports at spatial distance R are $O(R^{-N})$ for all N, uniformly on the common core. Using conservation $\partial^\mu T^{(s)}_{\mu\nu} = 0$ modulo contacts (18.16) and integrating by parts, the difference $P^{(s)}_\nu(R',t) - P^{(s)}_\nu(R,t)$ is supported in the annulus where $\nabla\chi_{R'} - \nabla\chi_R \neq 0$. Almost locality and exponential clustering at positive flow yield, for every N,

$$\|\,(P^{(s)}_\nu(R',t) - P^{(s)}_\nu(R,t))\,\Psi\|\;\le\; C_{N,\kappa}(s)\,(1+\min\{R,R'\})^{-N}\,\|(1+H)^\kappa\Psi\|$$

on $\mathcal{D}_{\text{flow}}$. Thus $P^{(s)}_\nu(R,t)$ is strongly Cauchy on $\mathcal{D}_{\text{flow}}$ as $R \to \infty$. Its strong limit $P^{(s)}_\nu$ is symmetric on $\mathcal{D}_{\text{flow}}$.

(b) Independence of t and of χ. Differentiating in t and using $\partial^0 T^{(s)}_{0\nu} = -\partial^i T^{(s)}_{i\nu}$ in the distributional sense, we obtain

$$\frac{d}{dt} P^{(s)}_\nu(R,t) = -\int d^3\mathbf{x}\, \partial_i\chi_R(\mathbf{x})\, T^{(s)}_{i\nu}(t,\mathbf{x}),$$

whose norm on $\mathcal{D}_{\text{flow}}$ is $O(R^{-N})$ by almost locality; hence the strong limit is independent of t. Changing χ with the same integral changes $P^{(s)}_\nu(R,t)$ by a boundary term of the same type, which vanishes in the limit.

(c) Implementer property and essential self–adjointness. For any flowed local $\widehat{A}^{(s)}(f)$ with equal-time support, Lemma 18.29 with $g_t \otimes h$ equal to the equal–time test for χ_R yields

$$\Big\|\, i[P^{(s)}_\nu(R,t),\, \widehat{A}^{(s)}(f)] - \partial_\nu \widehat{A}^{(s)}(f)\Big\| \;\le\; C_{N,\kappa}(s)\, R^{-N}\, \|\widehat{A}^{(s)}(f)\|_\kappa.$$

Letting $R \to \infty$ gives on $\mathcal{D}_{\text{flow}}$ $[P^{(s)}_\nu,\, \widehat{A}^{(s)}(f)] = i\,\partial_\nu \widehat{A}^{(s)}(f)$. By Nelson's commutator theorem (with H as control operator and the uniform energy bounds from Theorem 18.11), $P^{(s)}_\nu$ is essentially self–adjoint on $\mathcal{D}_{\text{flow}}$. The same argument applied to the densities $x_\mu T^{(s)}_{0\nu} - x_\nu T^{(s)}_{0\mu}$

gives the flowed rotation/boost generators and their implementer identity. This proves the "Charges at $s > 0$" bullet.

3) Flow removal and construction of $T_{\mu\nu}$. By Theorem 18.17 there exist real functions $Z_T(s), Z_\theta(s)$ such that

$$T_{\mu\nu}(f) \;=\; \lim_{s\downarrow 0} \Big\{ Z_T(s)\, U^{(s)}_{\mu\nu}(f) + Z_\theta(s)\, \eta_{\mu\nu} E^{(s)}(f) \Big\}$$

exists in matrix elements on $\mathcal{D}_{\text{flow}}$, and $T_{\mu\nu}$ is symmetric, conserved, local, and Poincaré covariant. Define the localized charges

$$P_\nu(R,t) \;:=\; \int_{\mathbb{R}^3} d^3\mathbf{x}\; \chi_R(\mathbf{x})\, T_{0\nu}(t,\mathbf{x}), \qquad P_\nu \;:=\; \text{s-}\lim_{R\to\infty} P_\nu(R,t).$$

Exactly as at positive flow (now using Proposition 18.19), $P_\nu(R,t)$ converge in the strong–resolvent sense on $\mathcal{D}_{\text{flow}}$ to a self–adjoint P_ν, independent of t and χ, and $[P_\nu, A(f)] = i\partial_\nu A(f)$ on $\mathcal{D}_{\text{flow}}$ for every local $A(f)$. This proves existence and the implementer property in the third bullet.

4) Strong–resolvent limit $P^{(s)}_\nu \to P_\nu$ and fixing $\lim_{s\downarrow 0} Z_T(s) = 1$. Let $g_t \otimes h$ be an equal–time test with $g_t \equiv 1$ near t and h compactly supported, and set $Q^{(s)}_\nu(g_t \otimes h) := \int (g_t \otimes h)\, T^{(s)}_{0\nu}$. By Lemma 18.29,

$$\Big\| \, i[Q^{(s)}_\nu(g_t{\otimes}h),\, \widehat{A}(f)] \;-\; \partial_\nu \widehat{A}(f) \Big\| \;\le\; C_{N,\kappa}\, s^{N/2}\, \|\widehat{A}(f)\|_\kappa \qquad \text{on } \mathcal{D}_{\text{flow}}. \tag{135}$$

Letting the spatial support of h tend to all space (as in the proof of Part 2) shows that

$$\lim_{R\to\infty} Q^{(s)}_\nu(g_t \otimes \chi_R) \;=\; P^{(s)}_\nu \quad \text{and} \quad \lim_{R\to\infty} T_{0\nu}(g_t \otimes \chi_R) \;=\; P_\nu$$

in the strong resolvent sense on $\mathcal{D}_{\text{flow}}$. Using (135) and the matrix–element convergence in Theorem 18.17 we obtain, for every N,

$$\|\,(P^{(s)}_\nu - P_\nu)\,\Psi\| \;\le\; C_{N,\kappa}\, s^{N/2}\, \|(1+H)^\kappa \Psi\| \qquad (\Psi \in \mathcal{D}_{\text{flow}}),$$

which implies $P^{(s)}_\nu \to P_\nu$ in the strong–resolvent sense on $\mathcal{D}_{\text{flow}}$.

Now suppose, for contradiction, that $\lim_{s\downarrow 0} Z_T(s) = 1 + \delta$ with $\delta \neq 0$. Write, at fixed s,

$$T^{(s)}_{0\nu} \;=\; Z_T(s)\, T_{0\nu} \;+\; Z_\theta(s)\, \eta_{0\nu}\, E \;+\; \partial^\rho \Xi_{\rho 0\nu}(s,\cdot) \;+\; R^{(s)}_{0\nu},$$

with $R^{(s)}_{0\nu} = O(s^\varepsilon)$ in matrix elements and the improvement $\partial^\rho \Xi$ integrating to a boundary term. Smearing at equal time against χ_R and sending $R \to \infty$,

$$P^{(s)}_\nu \;=\; Z_T(s)\, P_\nu \;+\; Z_\theta(s)\, \delta_{0\nu} \int_{\mathbb{R}^3} E(t,\mathbf{x})\, d^3\mathbf{x} \;+\; o(1) \quad (s \downarrow 0),$$

where $o(1) \to 0$ strongly on $\mathcal{D}_{\text{flow}}$ (remainder/improvement statements). The E–term does not contribute to the commutator with spatially localized equal–time fields (it is a scalar density), hence comparing the implementer identities on $\mathcal{D}_{\text{flow}}$ yields

$$[P^{(s)}_\nu,\, \cdot\,] \;\xrightarrow[s\downarrow 0]{}\; [P_\nu,\, \cdot\,] \qquad \Longrightarrow \qquad Z_T(s) \;\xrightarrow[s\downarrow 0]{}\; 1.$$

If $Z_T(s) \to 1 + \delta \neq 1$, we would have $[P^{(s)}_\nu, \cdot] \to (1+\delta)[P_\nu, \cdot]$, contradicting the limit of the commutators. Therefore

$$\boxed{\lim_{s\downarrow 0} Z_T(s) = 1}\,.$$

5) Improvements are harmless. Any local improvement $\partial^\rho \Xi_{\rho\mu\nu}$ is a divergence of a local tensor antisymmetric in $\rho\mu$. Its equal–time spatial integral reduces to a boundary term on spheres of radius R, which vanishes as $R \to \infty$ by almost locality and clustering. Hence improvements neither affect the existence of the charges nor their commutators with local fields; in particular they play no role in the normalization fixed by the implementer condition.

Collecting the conclusions of Parts 1–5 proves all claims of Proposition 18.25.

□

OPE matching and trace coefficient. For the flowed stress tensor we use, in GI correlators with separated insertions,

$$T^{(s)}_{\mu\nu} \;=\; Z_T(s)\,T_{\mu\nu} \;+\; Z_\theta(s)\,\eta_{\mu\nu}\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}) \;+\; \partial^\rho \Xi_{\rho\mu\nu}(s,\cdot) \;+\; R^{(s)}_{\mu\nu}, \qquad R^{(s)}_{\mu\nu} = O(s^\varepsilon).$$

Lemma 18.26 (Trace matching)**.**

1. CS in step–scaling/GF. *In the GF scheme, the Callan–Symanzik equation* $\partial_{\ln s}\Sigma(u,s) = \beta_{\mathrm{GF}}(\Sigma)$ *with* $\beta_{\mathrm{GF}}(v) = -2b_0 v^2 + \cdots$ *implies that* $\mu\partial_\mu$*–variations of correlators are generated by insertions of the trace* $T^\mu{}_\mu$*.*
2. SFTE for $E^{(s)}$ and the Ward identity. *By the small–flow expansion,* $E^{(s)} = c_E(s)\,O_4 +$*higher, with* $O_4 := \mathrm{tr}(F^2)$*, and* $c_E(s)$ *analytic in* $\log s$*; the dilation/translation Ward identities give* $T^\mu{}_\mu = \frac{\beta(g)}{2g}\,O_4 + \partial\cdot(\textit{improvement})$ *once* $T_{\mu\nu}$ *is charge–normalized. Comparing with the flowed OPE for* $T^{(s)}_{\mu\nu}$ *forces* $\boxed{\lim_{s\downarrow 0} Z_\theta(s) = \beta(g)/(2g)}$ *(scheme–independent on the GI quotient).*

Proposition 18.27 (OPE matching for the stress tensor)**.** *Let* $T^{(s)}_{\mu\nu}$ *be the flowed, symmetric, conserved stress tensor constructed in this section. Then as* $s \downarrow 0$ *one has, in GI correlators with separated insertions,*

$$T^{(s)}_{\mu\nu}(x) \;=\; Z_T(s)\,T_{\mu\nu}(x) \;+\; Z_\theta(s)\,\eta_{\mu\nu}\,\mathrm{tr}\big(F_{\rho\sigma}F^{\rho\sigma}\big)(x) \;+\; \partial^\rho \Xi_{\rho\mu\nu}(s,x) \;+\; R_{N,\kappa}(s;x), \quad (136)$$

where $\Xi_{\rho\mu\nu}$ *is a local improvement term (antisymmetric in* $\rho\mu$*) and, for every* N*, matrix elements of* $R_{N,\kappa}$ *satisfy the bound* (134) *with* $X = T_{\mu\nu}$*. Moreover*

$$\lim_{s\downarrow 0} Z_T(s) \;=\; 1, \qquad \lim_{s\downarrow 0} Z_\theta(s) \;=\; \frac{\beta(g)}{2g}\,. \quad (137)$$

The overall normalization of $\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$ *follows the convention* $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$*; other conventions shift* Z_θ *by an obvious factor.*

Anomaly coefficient is scheme independent. Improvements $\partial^\rho \Xi_{\rho\mu\nu}$ are traceless up to total derivatives in GI correlators; once the charge normalization of $T_{\mu\nu}$ is fixed by Proposition 18.30, the coefficient of $\mathrm{tr}(F^2)$ in $T^\mu{}_\mu$ is scheme independent. Thus the limits in (137) are universal.

Proof. Apply Lemma 18.24 with $X = T_{\mu\nu}$. By symmetry, Poincaré covariance, gauge invariance and dimension ≤ 4, the only GI local tensors with the quantum numbers of $T_{\mu\nu}$ are $T_{\mu\nu}$ itself, $\eta_{\mu\nu}\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$, and total derivatives $\partial^\rho \Xi_{\rho\mu\nu}$. This proves (136).

Conservation of $T^{(s)}_{\mu\nu}$ and of $T_{\mu\nu}$ implies that the only possible nontrivial scalar admixture is $\eta_{\mu\nu}\mathrm{tr}(F^2)$; its coefficient can affect only the trace. Taking the trace of (136) and using that improvements are traceless up to total derivatives, we obtain in GI correlators

$$T^{(s)\,\mu}{}_\mu(x) \;=\; 4\,Z_\theta(s)\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})(x) \;+\; (\text{total derivatives}) \;+\; R_{N,\kappa}(s;x).$$

On the other hand, the BRST Ward identities together with scale breaking yield the Yang–Mills trace anomaly in GI correlators:

$$T^{\mu}{}_{\mu}(x) \;=\; \frac{\beta(g)}{2g}\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})(x).$$

Matching the coefficients of $\mathrm{tr}(F^2)$ in the $s\downarrow 0$ limit gives $\lim_{s\downarrow 0} Z_\theta(s) = \beta(g)/(2g)$. The limit $\lim_{s\downarrow 0} Z_T(s) = 1$ is fixed by the requirement that the Poincaré charges $P_\nu = \int \mathrm{d}^3x\, T^{(s)}_{0\nu}(t,\mathbf{x})$ (defined on the common Nelson core and then by closure) implement translations on the local fields with the standard commutation relations; any residual finite renormalization would violate this normalization. □

Theorem 18.28 (Trace anomaly as an operator identity modulo contact)**.** *Let $T_{\mu\nu}$ be the renormalized stress tensor constructed in Theorem 18.17, normalized so that its charges implement translations (Proposition 18.19). Then there exists a local operator–valued distribution Σ_ρ (a divergence of an improvement current) such that, as operator–valued distributions,*

$$T^{\mu}{}_{\mu} \;=\; \frac{\beta(g)}{2g}\,\mathrm{tr}\big(F_{\rho\sigma}F^{\rho\sigma}\big) \;+\; \partial^\rho\Sigma_\rho, \tag{138}$$

with the following precise insertion statement: for any test $\varphi\in C_c^\infty(\mathbb{R}^4)$ and any finite family of gauge–invariant (GI) point–local fields $[A_j](\phi_j)$ whose supports are a positive distance away from $\mathrm{supp}\,\varphi$*,*

$$\Big\langle \Omega,\ \Big(T^{\mu}{}_{\mu}(\varphi) \;-\; \tfrac{\beta(g)}{2g}\,\mathrm{tr}(F^2)(\varphi) \;-\; \partial^\rho\Sigma_\rho(\varphi)\Big)\ \prod_j [A_j](\phi_j)\,\Omega\Big\rangle \;=\; 0. \tag{139}$$

Equivalently, (138) *holds* modulo contact terms *supported on the coincident diagonals with the GI insertions. The coefficient $\beta(g)/(2g)$ is universal (scheme independent) once the charge normalization of $T_{\mu\nu}$ is fixed.*

Normalization reminder. *The identification of the coefficient follows from the flowed OPE/matching for $T^{(s)}_{\mu\nu}$, see Proposition 18.27, where* $\lim_{s\downarrow 0} Z_T(s) = 1$ *and* $\lim_{s\downarrow 0} Z_\theta(s) = \beta(g)/(2g)$*.*

Proof. By Theorem 18.17 there exist functions $Z_T(s), Z_\theta(s)$, with $Z_T(s)\to 1$ as $s\downarrow 0$, such that for any $f\in\mathcal{S}(\mathbb{R}^4)$

$$T_{\mu\nu}(f) \;=\; \lim_{s\downarrow 0}\ \Big\{ Z_T(s)\,\widetilde{U}^{(s)}_{\mu\nu}(f) \;+\; Z_\theta(s)\,\eta_{\mu\nu}\,\widetilde{E}^{(s)}(f)\Big\}$$

in matrix elements on the common core $\mathcal{D}_{\mathrm{poly}}$. Here $\widetilde{U}^{(s)}_{\mu\nu}$ is (by construction) traceless, so

$$T^{\mu}{}_{\mu}(f) \;=\; \lim_{s\downarrow 0}\ 4\,Z_\theta(s)\,\widetilde{E}^{(s)}(f) \qquad \text{in matrix elements on } \mathcal{D}_{\mathrm{poly}}. \tag{140}$$

Next invoke the GI small–flow–time/OPE matching for the stress tensor (Proposition 18.27): in GI correlators with separated insertions,

$$T^{(s)}_{\mu\nu}(x) \;=\; Z_T(s)\,T_{\mu\nu}(x) \;+\; Z_\theta(s)\,\eta_{\mu\nu}\,\mathrm{tr}(F^2)(x) \;+\; \partial^\rho\Xi_{\rho\mu\nu}(s,x) \;+\; R_{N,\kappa}(s;x),$$

where $R_{N,\kappa}$ is $O(s^{N/2})$ in matrix elements uniformly on bounded energy vectors, and $\Xi_{\rho\mu\nu}$ is a local improvement (antisymmetric in $\rho\mu$). Taking the trace and using tracelessness of $U^{(s)}_{\mu\nu}$ yields, as distributions in GI correlators,

$$T^{(s)\,\mu}{}_{\mu}(x) \;=\; 4\,Z_\theta(s)\,\mathrm{tr}(F^2)(x) \;+\; \partial^\rho\Lambda_\rho(s,x) \;+\; R^{\mathrm{tr}}_{N,\kappa}(s;x), \tag{141}$$

with $\Lambda_\rho(s,x) := \eta^{\mu\nu}\Xi_{\rho\mu\nu}(s,x)$ a local vector operator and $R^{\rm tr}_{N,\kappa}$ satisfying the same $O(s^{N/2})$ bound.

Let $\varphi \in C_c^\infty(\mathbb{R}^4)$ and let the GI insertions $[A_j](\phi_j)$ have supports disjoint from $\operatorname{supp}\varphi$. Smearing (141) with φ and integrating by parts,

$$T^{(s)\,\mu}{}_\mu(\varphi) \;-\; 4\,Z_\theta(s)\,\mathrm{tr}(F^2)(\varphi) \;=\; -\,\Lambda_\rho(s,\partial^\rho\varphi) \;+\; R^{\rm tr}_{N,\kappa}(s;\varphi).$$

By uniform moment/energy bounds at positive flow time and quasi–locality (Theorem 18.11) together with the disjoint–support hypothesis, all correlators in which $\Lambda_\rho(s,\partial^\rho\varphi)$ and $R^{\rm tr}_{N,\kappa}(s;\varphi)$ appear are uniformly bounded and dominated. Hence, taking expectation against $\prod_j[A_j](\phi_j)$ and using dominated convergence plus $R^{\rm tr}_{N,\kappa} = O(s^{N/2})$, we obtain

$$\begin{aligned}\lim_{s\downarrow 0}\ \Big\langle \Omega,\ \Big(T^{(s)\,\mu}{}_\mu(\varphi) - 4\,Z_\theta(s)\,\mathrm{tr}(F^2)(\varphi)\Big)\ \prod_j [A_j](\phi_j)\,\Omega\Big\rangle \\ = -\ \lim_{s\downarrow 0}\ \Big\langle \Omega,\ \Lambda_\rho(s,\partial^\rho\varphi)\ \prod_j [A_j](\phi_j)\,\Omega\Big\rangle.\end{aligned}$$

On the other hand, by (140) we also have

$$\lim_{s\downarrow 0}\ \langle \Omega,\ T^{(s)\,\mu}{}_\mu(\varphi)\ \prod_j [A_j](\phi_j)\,\Omega\rangle \;=\; \langle \Omega,\ T^{\mu}{}_\mu(\varphi)\ \prod_j [A_j](\phi_j)\,\Omega\rangle.$$

Combining the last two displays and using the anomaly matching $\lim_{s\downarrow 0} Z_\theta(s) = \beta(g)/(2g)$ from Proposition 18.27 gives

$$\Big\langle \Omega,\ \Big(T^{\mu}{}_\mu(\varphi) \;-\; \tfrac{\beta(g)}{2g}\,\mathrm{tr}(F^2)(\varphi)\Big)\ \prod_j [A_j](\phi_j)\,\Omega\Big\rangle \;=\; -\ \lim_{s\downarrow 0}\ \langle \Omega,\ \Lambda_\rho(s,\partial^\rho\varphi)\ \prod_j [A_j]\Omega\rangle.$$

Define the operator–valued distribution Σ_ρ by its action on tests $\psi \in C_c^\infty(\mathbb{R}^4)$ via the distributional limit

$$\Sigma_\rho(\psi) \;:=\; \text{w-}\lim_{s\downarrow 0}\ \Lambda_\rho(s,\psi),$$

which exists in matrix elements against GI spectators with disjoint support by the same domination (the family is Cauchy due to the SFTE with coefficients analytic in $\log(s\mu^2)$ and uniform energy bounds; cf. Lemma 18.24 and Theorem 16.14). With this definition and the arbitrariness of φ, we have established (139). Equivalently, (138) holds as an identity of distributions modulo contact terms (integration by parts moves the divergence onto φ and no boundary terms arise because of compact support and disjointness). The universality of the coefficient $\beta(g)/(2g)$ follows from Proposition 18.27 and the charge normalization of $T_{\mu\nu}$ (Theorem 18.17), which fixes $Z_T(s)\to 1$ and removes any residual finite renormalization. □

18.3.2 Canonical normalization of the stress–energy tensor via charges

We now fix the finite normalization of the stress tensor by requiring that its charges implement the given unitary representation U (Theorem 17.1) on the local fields.

Lemma 18.29 (Localized charges from the flowed tensor)**.** *Let $T^{(s)}_{\mu\nu}$ be the flowed conserved symmetric tensor constructed above. For $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\chi \equiv 1$ on a neighborhood of* $\operatorname{supp} f$, *define*

$$P^{(s)}_\nu[\chi] \;:=\; \int_{\mathbb{R}^3} T^{(s)}_{0\nu}(t,\mathbf{x})\,\chi(\mathbf{x})\,\mathrm{d}^3\mathbf{x}.$$

Then for any smeared local GI field $\widehat{A}(f)$ with $\operatorname{supp} f \subset \{t\} \times \mathbb{R}^3$ *and for every* $N \in \mathbb{N}$ *there exist* κ *and* $C_{N,\kappa} < \infty$ *such that, on the common core* $\mathcal{D}_{\text{poly}}$*,*

$$\left\| i[P_\nu^{(s)}[\chi],\, \widehat{A}(f)] - \partial_\nu \widehat{A}(f) \right\| \;\le\; C_{N,\kappa}\, s^{N/2}\, \|\widehat{A}(f)\|_\kappa, \qquad \|\widehat{A}(f)\|_\kappa := \|(1{+}H)^\kappa\, \widehat{A}(f)\,(1{+}H)^\kappa\|. \tag{142}$$

In particular, $P_\nu^{(s)}[\chi] \to P_\nu$ *in the strong resolvent sense on* $\mathcal{D}_{\text{poly}}$ *as* $s \downarrow 0$*, where* P_ν *is the generator of translations from* U*.*

Proof. Use conservation $\partial^\mu T_{\mu\nu}^{(s)} = 0$ and integrate by parts in the equal-time commutator with a space cutoff $\chi \equiv 1$ on $\operatorname{supp} f$, which eliminates surface terms (locality). Insert the OPE (136) for $T_{0\nu}^{(s)}$ near $\operatorname{supp} f$. The improvement term integrates to a boundary contribution which vanishes by the choice of χ. The remainder $R_{N,\kappa}$ is controlled by (134). The only surviving local piece is $Z_T(s)\, T_{0\nu}$, whose equal-time commutator with $\widehat{A}(f)$ is the standard one, $i[T_{0\nu}(t,\mathbf{x}), \widehat{A}(f)] = \partial_\nu \widehat{A}(f)$ on $\mathcal{D}_{\text{poly}}$. This yields (142) with an extra factor $|Z_T(s) - 1|$ in front of the leading term. Since $\lim_{s\downarrow 0} Z_T(s) = 1$ by Proposition 18.27, the right-hand side is $O(s^{N/2})$, and strong resolvent convergence follows from standard graph-norm estimates on $\mathcal{D}_{\text{poly}}$ and essential self-adjointness (Proposition 17.3). The constants $C_{N,\kappa}$ can be chosen independent of $s \in (0, s_0]$ by the uniform moment bounds and almost-locality at positive flow (Lemmas 18.12, 18.73). □

Proposition 18.30 (Uniqueness of finite normalization)**.** *Among all local, symmetric, conserved tensors that differ from* $T_{\mu\nu}^{(s)}$ *by finite local counterterms (linear combinations of* $\eta_{\mu\nu}\operatorname{tr}(F_{\rho\sigma}F^{\rho\sigma})$ *and improvements* $\partial^\rho \Xi_{\rho\mu\nu}$*), the choice fixed by*

$$\lim_{s\downarrow 0} \int_{\mathbb{R}^3} T_{0\nu}^{(s)}(t,\mathbf{x})\, \chi(\mathbf{x})\, \mathrm{d}^3\mathbf{x} \;=\; P_\nu \qquad (\forall\, \chi \equiv 1 \textit{ near the region of interest})$$

is unique. Equivalently, the limit condition forces $\lim_{s\downarrow 0} Z_T(s) = 1$ *in* (136)*, while the improvement freedom remains but does not affect the charges.*

Charge constraint. In particular, the localized-charge condition forces $\lim_{s\downarrow 0} Z_T(s) = 1$ in (136); improvements $\partial^\rho \Xi_{\rho\mu\nu}$ drop out of the charges by Gauss' law.

Proof. Suppose we changed $T_{\mu\nu}^{(s)}$ by $\delta Z_T(s)\, T_{\mu\nu} + \delta Z_\theta(s)\, \eta_{\mu\nu}\operatorname{tr}(F^2) + \partial^\rho \Delta\Xi_{\rho\mu\nu}(s)$. The integrated improvement term vanishes by Gauss/Stokes and the support choice for χ. If $\lim_{s\downarrow 0} \delta Z_T(s) = \delta \neq 0$, then the limiting charge would be $(1+\delta)P_\nu$, contradicting the fact that the translation generator is fixed by U. Hence $\lim_{s\downarrow 0} \delta Z_T(s) = 0$. The scalar admixture $\eta_{\mu\nu}\operatorname{tr}(F^2)$ cannot contribute to the spatial momenta ($\nu = i$) and would add a multiple of $\int \operatorname{tr}(F^2)$ to P_0; this would change the equal-time commutators with some local fields, again contradicting Lemma 18.29. Thus the normalization is unique modulo improvements, which leave the charges invariant. □

18.3.3 Rotation/boost charges and the Poincaré algebra

Define the (Euclidean) angular–momentum densities

$$J_{\lambda\mu\nu}(x) \;:=\; x_\mu\, T_{\lambda\nu}(x) \;-\; x_\nu\, T_{\lambda\mu}(x), \qquad J_{\mu\nu} := J_{0\mu\nu}.$$

Let $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\chi \equiv 1$ near the origin and set $\chi_R(\mathbf{x}) := \chi(\mathbf{x}/R)$.

Lemma 18.31 (Localized rotation/boost charges from the flowed tensor)**.** *Let* $T_{\mu\nu}^{(s)}$ *be the flowed, canonically normalized tensor (i.e. with* $Z_T(s) \to 1$ *as* $s \downarrow 0$ *by Proposition 18.30). Define*

$$J_{\mu\nu}^{(s)}[\chi_R] \;:=\; \int_{\mathbb{R}^3} d^3\mathbf{x}\; \chi_R(\mathbf{x}) \Big(x_\mu\, T_{0\nu}^{(s)}(t,\mathbf{x}) \;-\; x_\nu\, T_{0\mu}^{(s)}(t,\mathbf{x}) \Big).$$

For any smeared local GI field $\widehat{A}(f)$ *with* $\operatorname{supp} f \subset \{t\} \times \mathbb{R}^3$ *and any* $N \in \mathbb{N}$ *there exist* κ *and* $C_{N,\kappa} < \infty$ *such that, on the common core* $\mathcal{D}_{\text{poly}}$*,*

$$\Big\| \, i\big[J^{(s)}_{\mu\nu}[\chi_R],\, \widehat{A}(f)\big] \;-\; (x_\mu \partial_\nu - x_\nu \partial_\mu)\widehat{A}(f) \Big\| \;\le\; C_{N,\kappa}\left(R^{-1} + s^{N/2}\right) \|\widehat{A}(f)\|_\kappa. \tag{143}$$

In particular, $J^{(s)}_{\mu\nu}[\chi_R] \to J^{(s)}_{\mu\nu}$ *strongly as* $R \to \infty$ *on* $\mathcal{D}_{\text{poly}}$*, and* $J^{(s)}_{\mu\nu} \to M_{\mu\nu}$ *in the strong resolvent sense as* $s \downarrow 0$*, where* $M_{\mu\nu}$ *implements rotations/boosts on local fields.*

Proof. Fix $s > 0$ and a smeared local GI field $\widehat{A}(f)$ with $\operatorname{supp} f \subset \{t\} \times \mathbb{R}^3$. By locality and covariance we may assume that $\operatorname{supp} f$ is contained in a ball $B_r \subset \mathbb{R}^3$ centered at the origin at time t. For $R \ge 2r$ one has $\chi_R \equiv 1$ on a neighborhood of $\operatorname{supp} f$.

Writing out the commutator,

$$i[J^{(s)}_{\mu\nu}[\chi_R], \widehat{A}(f)] = \int_{\mathbb{R}^3} d^3\mathbf{x}\; \chi_R(\mathbf{x})\, \big(x_\mu\, i[T^{(s)}_{0\nu}(t,\mathbf{x}), \widehat{A}(f)] - x_\nu\, i[T^{(s)}_{0\mu}(t,\mathbf{x}), \widehat{A}(f)]\big).$$

We first control the dependence on R. For $R, R' \ge 2r$,

$$J^{(s)}_{\mu\nu}[\chi_R] - J^{(s)}_{\mu\nu}[\chi_{R'}] = \int_{\mathbb{R}^3} d^3\mathbf{x}\, (\chi_R(\mathbf{x}) - \chi_{R'}(\mathbf{x}))\,(x_\mu T^{(s)}_{0\nu} - x_\nu T^{(s)}_{0\mu})(t,\mathbf{x}),$$

whose integrand is supported in an annulus of radius $\sim \max\{R, R'\}$, disjoint from $\operatorname{supp} f$. By almost locality and exponential clustering for flowed fields (Lemma 18.12 and Theorem 18.121), the commutator of this boundary term with $\widehat{A}(f)$ decays faster than any inverse power of R (uniformly in s on compact subsets of $(0,\infty)$). Thus, for every $N \in \mathbb{N}$ there exist κ and $C_{N,\kappa}$ such that

$$\| \, [J^{(s)}_{\mu\nu}[\chi_R] - J^{(s)}_{\mu\nu}[\chi_{R'}], \widehat{A}(f)] \, \| \le C_{N,\kappa}\, R^{-N}\, \|\widehat{A}(f)\|_\kappa,$$

and in particular this yields the R^{-1} part of (143).

It remains to identify the local action near $\operatorname{supp} f$. Since $\chi_R \equiv 1$ on a neighborhood of $\operatorname{supp} f$ for $R \ge 2r$, conservation $\partial^\lambda T^{(s)}_{\lambda\nu} = 0$ implies that integrating by parts in the equal-time commutator only produces derivatives of χ_R, hence boundary terms supported where $\nabla\chi_R \neq 0$; these have already been estimated above. Therefore, up to an error of order R^{-N}, we may replace χ_R by 1 in the commutator with $\widehat{A}(f)$.

On this neighborhood we insert the small–flow–time expansion

$$T^{(s)}_{\alpha\beta} \;=\; Z_T(s)\, T_{\alpha\beta} \;+\; Z_\theta(s)\, \eta_{\alpha\beta} \operatorname{tr}(F^2) \;+\; \partial^\rho \Xi_{\rho\alpha\beta}(s,\cdot) \;+\; R_{N,\kappa}(s;\cdot),$$

valid in GI correlators with separated insertions, from Proposition 18.27. The improvement term $\partial^\rho \Xi_{\rho\alpha\beta}$ integrates to a total derivative in space; after integrating by parts its contribution again involves $\nabla\chi_R$ and is thus a boundary term controlled as above. The remainder $R_{N,\kappa}$ satisfies $\|R_{N,\kappa}(s;\cdot)\| = O(s^{N/2})$ in matrix elements between vectors in $\mathcal{D}_{\text{poly}}$, so its contribution to the commutator is bounded by $C_{N,\kappa} s^{N/2} \|\widehat{A}(f)\|_\kappa$.

The only surviving leading term is therefore $Z_T(s)\, T_{0\nu}$ (and similarly $Z_T(s)\, T_{0\mu}$). For the unflowed stress tensor we have the standard equal-time commutator on $\mathcal{D}_{\text{poly}}$,

$$i[T_{0\nu}(t,\mathbf{x}), \widehat{A}(f)] = \partial_\nu \widehat{A}(f), \qquad i[T_{0\mu}(t,\mathbf{x}), \widehat{A}(f)] = \partial_\mu \widehat{A}(f),$$

so the leading piece of $i[J^{(s)}_{\mu\nu}[\chi_R], \widehat{A}(f)]$ is

$$Z_T(s)\, (x_\mu \partial_\nu - x_\nu \partial_\mu) \widehat{A}(f).$$

By the canonical charge normalization, $Z_T(s) \to 1$ as $s \downarrow 0$ (Proposition 18.30). Collecting the $O(R^{-1})$ and $O(s^{N/2})$ errors gives (143).

Finally, the estimate (143) implies that for each fixed $s > 0$ the family $J^{(s)}_{\mu\nu}[\chi_R]$ is Cauchy in R on $\mathcal{D}_{\text{poly}}$ in the graph norm of $(1+H)^\kappa$, hence converges strongly to a densely defined symmetric operator $J^{(s)}_{\mu\nu}$. The $s^{N/2}$ control of the commutator with local fields, together with the analogous bounds for the flowed translation charges (Lemma 18.29) and Nelson's commutator theorem, show that $J^{(s)}_{\mu\nu}$ converges in the strong resolvent sense as $s \downarrow 0$ to a self-adjoint operator $M_{\mu\nu}$ whose commutator with every local observable is the differential operator $x_\mu \partial_\nu - x_\nu \partial_\mu$. Thus $M_{\mu\nu}$ implements rotations/boosts on local fields as claimed. □

Theorem 18.32 (Rotation/boost generators and the Poincaré algebra)**.** *Let P_μ be the translation generators from Theorem 18.17. The limits*

$$M_{\mu\nu} \;:=\; \text{s-}\lim_{R\to\infty} \lim_{s\downarrow 0} J^{(s)}_{\mu\nu}[\chi_R]$$

exist on $\mathcal{D}_{\text{poly}}$, are essentially self-adjoint on this core, are independent of the time slice t, and satisfy

$$[M_{\mu\nu}, A] \;=\; i\,(x_\mu \partial_\nu - x_\nu \partial_\mu) A \qquad \text{on } \mathcal{D}_{\text{poly}}$$

for every local observable A. Moreover, on $\mathcal{D}_{\text{poly}}$,

$$[P_\rho, M_{\mu\nu}] \;=\; i\,(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu), \qquad [M_{\mu\nu}, M_{\rho\sigma}] \;=\; i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}),$$

which becomes the usual Poincaré Lie algebra after OS reconstruction.

Proof. By Lemma 18.31, for each fixed $s > 0$ the strong limit

$$J^{(s)}_{\mu\nu} := \text{s-}\lim_{R\to\infty} J^{(s)}_{\mu\nu}[\chi_R]$$

exists on $\mathcal{D}_{\text{poly}}$, and the family $J^{(s)}_{\mu\nu}$ converges in the strong resolvent sense as $s \downarrow 0$ to a self-adjoint operator $M_{\mu\nu}$. This justifies the iterated limit in the definition of $M_{\mu\nu}$.

The estimate (143) implies that for every local observable A and every N,

$$i[J^{(s)}_{\mu\nu}, A] = (x_\mu \partial_\nu - x_\nu \partial_\mu) A + O(s^{N/2})$$

in the graph norm of $(1+H)^\kappa$ on $\mathcal{D}_{\text{poly}}$. Passing to the strong limit $s \downarrow 0$ yields

$$[M_{\mu\nu}, A] = i\,(x_\mu \partial_\nu - x_\nu \partial_\mu) A \qquad \text{on } \mathcal{D}_{\text{poly}},$$

which is the stated implementer relation.

To see that $M_{\mu\nu}$ is independent of the time slice t, differentiate $J^{(s)}_{\mu\nu}[\chi_R](t)$ with respect to t and use conservation $\partial^\lambda T^{(s)}_{\lambda\nu} = 0$:

$$\frac{d}{dt} J^{(s)}_{\mu\nu}[\chi_R](t) = -\int_{\mathbb{R}^3} d^3\mathbf{x}\, \partial_i \chi_R(\mathbf{x})\, \big(x_\mu T^{(s)}_{i\nu}(t,\mathbf{x}) - x_\nu T^{(s)}_{i\mu}(t,\mathbf{x})\big).$$

The right–hand side is supported in the annulus where $\nabla\chi_R \neq 0$, hence its commutator with any local observable is $O(R^{-N})$ for every N by almost locality and exponential clustering of flowed fields (Lemma 18.12 and Theorem 18.121). Therefore the derivative vanishes in the limit $R \to \infty$, and the resulting operator $M_{\mu\nu}$ does not depend on t.

For the commutators with the translation generators, recall from Theorem 18.17 that $[P_\rho, A] = i\,\partial_\rho A$ on $\mathcal{D}_{\text{poly}}$ for every local A. Together with the already established relation

$$[M_{\mu\nu}, A] = i\,(x_\mu \partial_\nu - x_\nu \partial_\mu) A,$$

we compute, on $\mathcal{D}_{\text{poly}}$,

$$\begin{aligned}[[P_\rho, M_{\mu\nu}], A] &= [P_\rho, [M_{\mu\nu}, A]] - [M_{\mu\nu}, [P_\rho, A]] \\ &= i\big(P_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu)A - (x_\mu\partial_\nu - x_\nu\partial_\mu)P_\rho A\big) \\ &= i\big(\eta_{\rho\mu}\partial_\nu - \eta_{\rho\nu}\partial_\mu\big)A = i\big(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\big)A.\end{aligned}$$

Since $\mathcal{D}_{\text{poly}}$ is cyclic for the local algebra and a common core for all generators, this determines the commutator and yields

$$[P_\rho, M_{\mu\nu}] = i\big(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\big) \quad \text{on } \mathcal{D}_{\text{poly}}.$$

For the $[M, M]$ commutator, note that the differential operators

$$L_{\mu\nu} := x_\mu\partial_\nu - x_\nu\partial_\mu$$

satisfy the Lorentz Lie algebra relations

$$[L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\nu\rho}L_{\mu\sigma} + \eta_{\nu\sigma}L_{\mu\rho}.$$

Using $[M_{\mu\nu}, A] = i\, L_{\mu\nu}A$, a straightforward computation gives, for every local A,

$$\begin{aligned}[[M_{\mu\nu}, M_{\rho\sigma}], A] &= i\big(\eta_{\mu\rho}L_{\nu\sigma} - \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\nu\rho}L_{\mu\sigma} + \eta_{\nu\sigma}L_{\mu\rho}\big)A \\ &= i\big(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\big)A.\end{aligned}$$

Again, by cyclicity and core properties, this fixes the commutator on $\mathcal{D}_{\text{poly}}$ and yields the stated Poincaré Lie algebra.

Finally, essential self-adjointness of $M_{\mu\nu}$ on $\mathcal{D}_{\text{poly}}$ follows from Nelson's commutator theorem with the Hamiltonian H as control operator: the bounds of Lemma 18.31 provide the required graph-norm estimates, and $\mathcal{D}_{\text{poly}}$ is a common Nelson core. After OS reconstruction, the pair $(P_\mu, M_{\mu\nu})$ therefore gives the usual unitary representation of the Poincaré group. □

Proposition 18.33 (Global rotation Ward identity)**.** *Let $X_1, \dots, X_n$ be bounded functions of smeared point-local GI fields from $\mathfrak{A}(\mathcal{O})$ with supports strictly inside $\mathcal{O}$. Then for any antisymmetric $\omega^{\mu\nu}$,*

$$\begin{aligned}\sum_{k=1}^{n} \frac{d}{d\theta}\Big|_{\theta=0} \Big\langle \Omega,\, X_1 \cdots e^{\frac{i\theta}{2}\omega^{\mu\nu}M_{\mu\nu}}\, X_k\, e^{-\frac{i\theta}{2}\omega^{\mu\nu}M_{\mu\nu}} \cdots X_n\, \Omega \Big\rangle & \\ = \ \frac{i}{2}\,\omega^{\mu\nu}\int d^4x\, \langle \Omega,\, \partial^\lambda J_{\lambda\mu\nu}(x)\, X_1 \cdots X_n\, \Omega\rangle & \\ = \ 0. &\end{aligned}$$

In particular, $[M_{\mu\nu}, X] = i(x_\mu\partial_\nu - x_\nu\partial_\mu)X$ on $\mathcal{D}_{\text{poly}}$.

Proof. Let $\omega^{\mu\nu}$ be antisymmetric and set

$$U(\theta) \ := \ \exp\Big(\frac{i\theta}{2}\,\omega^{\mu\nu}M_{\mu\nu}\Big), \qquad X_k(\theta) \ := \ U(\theta)X_kU(\theta)^{-1}.$$

Differentiating at $\theta = 0$ and using the implementer relation from Theorem 18.32 gives, on $\mathcal{D}_{\text{poly}}$,

$$\frac{d}{d\theta}\Big|_{\theta=0} X_k(\theta) = \frac{i}{2}\,\omega^{\mu\nu}[M_{\mu\nu}, X_k] = -\frac{1}{2}\,\omega^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)X_k.$$

Summing over k and inserting between Ω and Ω yields

$$\sum_{k=1}^{n} \frac{d}{d\theta}\Big|_{\theta=0} \langle \Omega, X_1 \cdots X_k(\theta) \cdots X_n \Omega \rangle = \frac{i}{2}\, \omega^{\mu\nu} \sum_{k=1}^{n} \langle \Omega, X_1 \cdots [M_{\mu\nu}, X_k] \cdots X_n \Omega \rangle.$$

To relate the right–hand side to the divergence of the Noether current $J_{\lambda\mu\nu}$ we approximate the global rotation by a localized one, exactly as in the proof of the global translation Ward identity (Proposition 18.18). Smearing the conservation law

$$\partial^\lambda J_{\lambda\mu\nu} = 0$$

with a test function $\varphi \in C_c^\infty(\mathbb{R}^4)$ that is identically 1 on a neighborhood of the supports of all X_k, integrating by parts, and using the local equal-time commutator control of Lemma 18.31, one identifies the resulting boundary terms with the sum over k of commutators with the corresponding localized angular-momentum charges. Letting the support of φ tend to all of $\mathbb{R}^4$ and using exponential clustering and dominated convergence, one arrives at

$$\sum_{k=1}^{n} \frac{d}{d\theta}\Big|_{\theta=0} \langle \Omega, X_1 \cdots X_k(\theta) \cdots X_n \Omega \rangle = \frac{i}{2}\, \omega^{\mu\nu} \int d^4x \, \langle \Omega, \partial^\lambda J_{\lambda\mu\nu}(x) X_1 \cdots X_n \Omega \rangle.$$

Since $\partial^\lambda J_{\lambda\mu\nu} = 0$ holds as an operator-valued distribution (because $T_{\mu\nu}$ is symmetric and conserved), the last integral vanishes, giving the claimed identity and its vanishing.

The commutator statement $[M_{\mu\nu}, X] = i(x_\mu \partial_\nu - x_\nu \partial_\mu) X$ for X in the polynomial algebra is obtained by taking $n = 1$ and interpreting the derivative of $X_1(\theta)$ at $\theta = 0$ as the commutator with $M_{\mu\nu}$. □

Corollary 18.34 (Trace anomaly). *With the canonical normalization of $T_{\mu\nu}$ fixed by the charges,*

$$T^\mu{}_\mu(x) = \frac{\beta(g)}{2g} \, \mathrm{tr}\big(F_{\rho\sigma} F^{\rho\sigma}\big)(x) + \partial^\mu J_\mu(x),$$

where the divergence term is irrelevant in GI correlators at separated points.

Proof. Insert the small flow–time expansion of Proposition 18.27:

$$T^{(s)}_{\mu\nu} \;=\; Z_T(s)\, T_{\mu\nu} \;+\; Z_\theta(s)\, \eta_{\mu\nu}\, \mathrm{tr}(F_{\rho\sigma} F^{\rho\sigma}) \;+\; \partial^\rho \Xi_{\rho\mu\nu}(s, \cdot) \;+\; R_{N,\kappa}(s; \cdot),$$

valid in GI correlators with separated insertions and with $\|R_{N,\kappa}\| = O(s^{N/2})$. Taking the trace and using that improvements are traceless up to total derivatives in GI correlators,

$$T^{(s)\,\mu}{}_\mu \;=\; 4\, Z_\theta(s)\, \mathrm{tr}(F_{\rho\sigma} F^{\rho\sigma}) \;+\; \partial^\mu J^{(s)}_\mu \;+\; R^{\mathrm{tr}}_{N,\kappa}(s; \cdot),$$

for a suitable local current $J^{(s)}_\mu$.

By the charge normalization (Proposition 18.30), $\lim_{s\downarrow 0} Z_T(s) = 1$, while Proposition 18.27 yields $\lim_{s\downarrow 0} Z_\theta(s) = \beta(g)/(2g)$. Since $R^{\mathrm{tr}}_{N,\kappa}(s; \cdot) \to 0$ in matrix elements between vectors from the common Nelson core, $T^{(s)\,\mu}{}_\mu \to T^\mu{}_\mu$ in the distributional sense on GI correlators as $s \downarrow 0$. Passing to the limit $s \downarrow 0$ in the displayed identity and absorbing the limit of the improvement trace into $\partial^\mu J_\mu$, we conclude

$$T^\mu{}_\mu \;=\; \frac{\beta(g)}{2g}\, \mathrm{tr}(F_{\rho\sigma} F^{\rho\sigma}) \;+\; \partial^\mu J_\mu$$

in GI correlators at separated points. *This is precisely a corollary of Theorem 18.28, with the coefficient fixed by the OPE normalization* (136) *and* (137). □

18.3.4 YM short-distance identification of the GI sector

We now formulate the precise UV matching statement we will use subsequently.

Theorem 18.35 (YM short-distance identification in GI correlators)**.** *Let $\{\mathcal{O}_i\}_{i\in I}$ be a finite basis of renormalized GI local operators of canonical dimension ≤ 4, closed under Poincaré and discrete symmetries, containing $T_{\mu\nu}$ and $\operatorname{tr}(F_{\rho\sigma}F^{\rho\sigma})$. For each i define the flowed operator $\mathcal{O}_i^{(s)} := G_s * \mathcal{O}_i$ as in Lemma 18.24. Then, for any GI correlator with mutually separated insertions, one has the small-flow-time expansion*

$$\mathcal{O}_i^{(s)}(x) = \sum_{j\in I} Z_{ij}(s)\,\mathcal{O}_j(x) + \partial^\rho \Upsilon_\rho^{(i)}(s,x) + R_{N,\kappa}^{(i)}(s;x), \tag{144}$$

where:

(i) *the remainders obey the bound* (134) *uniformly in the spectators;*

(ii) *the coefficient matrix $Z(s) = (Z_{ij}(s))$ satisfies the RG equation*

$$\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + \beta(g)\frac{\mathrm{d}}{\mathrm{d}g} + \gamma^T\Big)\, Z(s) = 0,$$

with γ the anomalous-dimension matrix of the basis;

(iii) *$Z(s)$ is uniquely determined by the Ward identities of Section 18 together with the canonical normalization of $T_{\mu\nu}$ (Proposition 18.30) and the trace-anomaly matching (Proposition 18.27); in particular,*

$$Z_{T\to T}(s) \xrightarrow[s\downarrow 0]{} 1, \qquad Z_{T\to\eta\,\operatorname{tr}(F^2)}(s) \xrightarrow[s\downarrow 0]{} \frac{\beta(g)}{2g}, \tag{145}$$

and coefficients multiplying BRST-exact operators vanish in GI correlators by Theorem 18.23.

Moreover, upon inserting the Yang–Mills β-function and anomalous dimensions (pure YM: asymptotically free), $Z(s)$ coincides to all orders in the formal weak-coupling expansion with the Wilson-coefficient matrix of continuum YM at renormalization scale $\mu = (8s)^{-1/2}$.

Proof. Equation (144) with remainder control follows from Lemma 18.24 applied to each $\mathcal{O}_i$. The RG equation is the matrix form of the scalar RG equation in Lemma 18.24, using that the chosen basis closes under renormalization.

The Ward identities (Poincaré, BRST, and the trace anomaly) impose linear constraints on $Z(s)$ which fix the components in (145). Proposition 18.30 removes any residual finite normalization ambiguity for $T_{\mu\nu}$, and Theorem 18.23 eliminates BRST-exact admixtures in GI correlators, yielding uniqueness of $Z(s)$ on the GI quotient.

Finally, expanding the RG equation perturbatively at $\mu = (8s)^{-1/2}$ and solving with the same boundary/normalization conditions gives the YM Wilson coefficients order by order in $g(\mu)$. Uniqueness of solutions to the resulting first-order system ensures equality of the formal series. □

Remark 18.36. The improvement terms $\partial^\rho \Upsilon_\rho^{(i)}$ in (144) do not affect integrated charges or on-shell scattering and can be fixed by conventional choices (e.g. Belinfante). The identification in Theorem 18.35 is precisely what we need to transport YM short-distance information (trace anomaly, operator mixings, UV dimensions) into the nonperturbative GI sector built earlier.

18.3.5 Associativity of the GI OPE from the SFTE

Theorem 18.37 (Associativity of the gauge-invariant OPE)**.** *Let $\{Q_\alpha^{\rm ren}\}_{\alpha\in\mathcal{B}}$ be a finite symmetry-closed basis of renormalized GI local operators of canonical dimension ≤ 4 as in Theorem 18.35. Define the (point-local) OPE inside GI correlators with separated insertions by*

$$Q_i^{\rm ren}(x)\,Q_j^{\rm ren}(y) \;\sim\; \sum_{n\in\mathcal{B}} C_{ij}^{n}(x{-}y;\mu)\,Q_n^{\rm ren}(y) \qquad (x\to y),$$

where "$\sim$" means equality when paired with any GI test configuration whose other insertions are a positive distance away from $\{x,y\}$. Then, for hierarchical configurations $0<|x{-}y|\ll|y{-}z|$,

$$\sum_{m\in\mathcal{B}} C_{ij}^{m}(x{-}y;\mu)\,C_{mk}^{n}(y{-}z;\mu) = \sum_{m\in\mathcal{B}} C_{jk}^{m}(y{-}z;\mu)\,C_{im}^{n}(x{-}y;\mu), \tag{146}$$

as an identity of distributions on the off-diagonal region $\{(x,y,z): x\neq y\neq z\}$ inside GI correlators. Coefficients multiplying BRST-exact operators vanish in GI correlators (Theorem 18.23). Moreover, $\{C_{ij}^{n}\}$ obey the Callan–Symanzik equation with the anomalous-dimension matrix of the chosen basis (Theorem 18.35).

Proof. **1) Normalization functionals.** Fix GI, $O(4)$-invariant linear functionals $\{\mathcal{N}_\alpha\}_{\alpha\in\mathcal{B}}$ supported in a small ball around the origin as in Definition 16.2, with

$$M := \big(\mathcal{N}_\alpha(Q_\beta^{\rm ren})\big)_{\alpha,\beta\in\mathcal{B}} \quad \text{invertible.}$$

Translate by y via $\mathcal{N}_\alpha^{(y)}(X) := \mathcal{N}_\alpha(\tau_{-y}X\tau_y)$. All pairings below are well-defined by temperedness/tightness (Theorem 13.3, Corollary 16.26) together with the off-diagonal bounds (Lemma 13.8, Proposition 13.9).

2) Definition of C_{ij}^{n} by a projector equation. For $x\neq y$, define the coefficient vector $\mathbf{C}_{ij}(x{-}y;\mu) = (C_{ij}^{n}(x{-}y;\mu))_{n\in\mathcal{B}}$ by

$$\big(\mathcal{N}_\alpha^{(y)}(Q_i^{\rm ren}(x)\,Q_j^{\rm ren}(y))\big)_{\alpha\in\mathcal{B}} = M\,\mathbf{C}_{ij}(x{-}y;\mu). \tag{147}$$

Since M is invertible, $\mathbf{C}_{ij}$ exists and is unique as a vector-valued distribution on $\{x\neq y\}$. Equation (147) is equivalent to the asserted OPE in the sense of pairings with all $\mathcal{N}_\alpha^{(y)}$, hence inside any GI correlator with other insertions kept away from $\{x,y\}$. By Theorem 18.23, BRST-exact operators are invisible in GI correlators, so the coefficients are defined on the GI cohomology.

3) Associativity off the diagonals. Consider $Q_i^{\rm ren}(x)Q_j^{\rm ren}(y)Q_k^{\rm ren}(z)$ with $x\neq y\neq z$ and apply $\mathcal{N}_\nu^{(z)}$ for arbitrary $\nu\in\mathcal{B}$. Using (147) twice and algebra associativity,

$$\begin{aligned}\mathcal{N}_\nu^{(z)}(Q_i(x)Q_j(y)Q_k(z)) &= \sum_{m\in\mathcal{B}} C_{ij}^{m}(x{-}y;\mu)\;\mathcal{N}_\nu^{(z)}(Q_m(y)Q_k(z))\\ &= \sum_{m\in\mathcal{B}} C_{ij}^{m}(x{-}y;\mu)\;\big(M\,\mathbf{C}_{mk}(y{-}z;\mu)\big)_\nu,\end{aligned}$$

and similarly

$$\mathcal{N}_\nu^{(z)}(Q_i(x)Q_j(y)Q_k(z)) = \sum_{m\in\mathcal{B}} C_{jk}^{m}(y{-}z;\mu)\;\big(M\,\mathbf{C}_{im}(x{-}y;\mu)\big)_\nu.$$

Subtracting and using that this holds for all ν yields

$$\sum_m C_{ij}^{m}(x{-}y;\mu)\,M\,\mathbf{C}_{mk}(y{-}z;\mu) = \sum_m C_{jk}^{m}(y{-}z;\mu)\,M\,\mathbf{C}_{im}(x{-}y;\mu).$$

Left-multiplying by M^{-1} gives (146). All distributions are tested off the diagonals, justified by the cited temperedness and off-diagonal control.

4) Compatibility with SFTE and RG. Let $\mathcal{O}_i^{(s)} := G_s * Q_i^{\rm ren}$ be the flowed representatives (Lemma 18.24). For $s > 0$ in the SFTE window (Definition 16.22), Theorem 16.25 together with Theorem 18.35 gives, in separated correlators,

$$\mathcal{O}_i^{(s)}(x) = \sum_{\alpha\in\mathcal{B}} Z_{i\alpha}(s,\mu)\, Q_\alpha^{\rm ren}(x;\mu) + R_i^{(s)}(x), \qquad \|R_i^{(s)}\| = O(s^\varepsilon),$$

with $Z(s,\mu)$ analytic in $\log(s\mu^2)$ and invertible on the GI quotient. Define flowed coefficients by the same projector prescription:

$$\big(\mathcal{N}_\alpha^{(y)}(\mathcal{O}_i^{(s)}(x)\, \mathcal{O}_j^{(s)}(y))\big)_\alpha = M\, \widetilde{\mathbf{C}}_{ij}(x{-}y; s, \mu).$$

Expanding $\mathcal{O}^{(s)}$ twice and using the $O(s^\varepsilon)$ off-diagonal bounds (Proposition 13.9),

$$\widetilde{\mathbf{C}}_{ij}(x{-}y; s, \mu) = Z(s,\mu)\, \mathbf{C}_{ij}(x{-}y;\mu) + O(s^\varepsilon),$$

uniformly on compact off-diagonal sets. Exact associativity holds for $\widetilde{C}$ at fixed $s > 0$ by the same algebraic argument as above. Letting $s \downarrow 0$ inside the SFTE window and using invertibility of $Z(s,\mu)$ on the GI quotient yields (146). The Callan–Symanzik equation for $\mathbf{C}_{ij}$ follows from the RG for $Z(s,\mu)$ in Theorem 18.35. □

Remark 18.38. The proof uses only: (a) existence of a separating GI/$O(4)$-invariant family $\{\mathcal{N}_\alpha\}$ with invertible M (Definition 16.2); (b) associativity of the product on a common polynomial domain; (c) off-diagonal continuity/temperedness (Theorem 13.3, Lemma 13.8, Proposition 13.9); (d) SFTE reduction and YM UV identification (Lemma 18.24, Theorem 16.25, Theorem 18.35). Improvement terms contribute only contacts and do not affect (146) for separated insertions.

Lemma 18.39 (Flow preserves BRST conservation and almost locality in the extended algebra)**.** *Let $j_{\rm B}^\mu$ be the lattice BRST Noether current (ghosts included) and define its flowed version by convolution: $j_{\rm B}^{\mu,(s)} := G_s * j_{\rm B}^\mu$. Then $\partial_\mu j_{\rm B}^{\mu,(s)} = 0$ in the sense of operator–valued distributions, and the almost–locality bound of Lemma 18.12 holds verbatim with the graded commutator $[\,\cdot\,,\,\cdot\,]_{\rm gr}$ and with O_i replaced by arbitrary local composites in the extended (ghost) algebra with uniformly bounded engineering dimension.*

Proof. Conservation: $\partial_\mu j_{\rm B}^{\mu,(s)} = \partial_\mu(G_s * j_{\rm B}^\mu) = G_s * (\partial_\mu j_{\rm B}^\mu) = 0$. Almost locality: the proof of Lemma 18.12 only uses (i) the tail bound (132), (ii) off–diagonal graded commutator bounds for locals, and (iii) the flow/local–tail decomposition. These extend to the ghost sector with the graded commutator and the same dimension bookkeeping. □

Definition 18.40 (Localized flowed BRST charge)**.** Fix $s > 0$, $t \in \mathbb{R}$, and $\chi_R \in C_c^\infty(\mathbb{R}^3)$ with $\chi_R \equiv 1$ on $B_R(0)$ and $\|\partial^\alpha \chi_R\|_\infty \lesssim_\alpha R^{-|\alpha|}$. Set

$$Q_{\rm B}^{(s)}[\chi_R; t] \;:=\; \int_{\mathbb{R}^3} j_{\rm B}^{0,(s)}(t,\mathbf{x})\, \chi_R(\mathbf{x})\, {\rm d}^3\mathbf{x},$$

initially defined on the common polynomial core $\mathcal{D}_{\rm poly}^{\rm ext}(s)$ generated by flowed extended locals.

Proposition 18.41 (Implementer property, independence of cutoff, and closability)**.** *Let X be any local composite in the extended algebra and let $f \in \mathcal{S}(\mathbb{R}^4)$ have* $\operatorname{supp} f \subset \{t\} \times \mathbb{R}^3$*. Then, for every $N \in \mathbb{N}$, there exist κ and $C_{N,\kappa}(s) < \infty$ such that on $\mathcal{D}_{\rm poly}^{\rm ext}(s)$,*

$$\Big\| i\big[Q_{\rm B}^{(s)}[\chi_R; t],\, X^{(s)}(f)\big]_{\rm gr} \;-\; (sX)^{(s)}(f) \Big\| \;\le\; C_{N,\kappa}(s)\, (1+R)^{-N}\, \|(1+H)^\kappa X^{(s)}(f)(1+H)^\kappa\|. \tag{148}$$

Consequently, $\{Q_{\mathrm{B}}^{(s)}[\chi_R; t]\}_{R\to\infty}$ *is a Cauchy net in the strong operator topology on* $\mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s)$*, with limit* $Q_{\mathrm{B}}^{(s)}$ *independent of* χ_R *and* t*. The operator* $Q_{\mathrm{B}}^{(s)}$ *is closable,* $\mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s)$ *is a core for its closure, and*

$$i\,[Q_{\mathrm{B}}^{(s)},\, X^{(s)}(f)]_{\mathrm{gr}} \;=\; (sX)^{(s)}(f) \qquad \text{on } \mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s). \tag{149}$$

Moreover, $Q_{\mathrm{B}}^{(s)}\Omega = 0$ *and, on* $\mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s)$*,* $(Q_{\mathrm{B}}^{(s)})^2 = 0$*.*

Proof. Integrate the conservation law $\partial_\mu j_{\mathrm{B}}^{\mu,(s)} = 0$ against a spacetime test of the form $g_t \otimes \chi_R$ with $g_t \equiv 1$ near t and use graded locality to convert spatial derivatives to boundary terms supported where $\nabla\chi_R \neq 0$. Applying Lemma 18.39 yields the $(1+R)^{-N}$ decay of those boundary contributions. The local BRST Ward identity (Theorem 18.22(1) with the graded bracket) identifies the remaining contact term with $(sX)^{(s)}(f)$, giving (148). The Cauchy property and cutoff independence follow by taking $R \to \infty$. Closability is standard from (149) and the uniform energy bounds (Theorem 18.11(4)). The vacuum invariance $Q_{\mathrm{B}}^{(s)}\Omega = 0$ follows by testing the Ward identity with GI spectators and letting $R \to \infty$. Finally, on $\mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s)$, (149) and $s^2 = 0$ imply $-[Q_{\mathrm{B}}^{(s)}, [Q_{\mathrm{B}}^{(s)}, X^{(s)}(f)]_{\mathrm{gr}}]_{\mathrm{gr}} = (s^2X)^{(s)}(f) = 0$, and with $Q_{\mathrm{B}}^{(s)}\Omega = 0$ this yields $(Q_{\mathrm{B}}^{(s)})^2 = 0$ on the polynomial core. □

Corollary 18.42 (Operator–level Ward/ST identities at fixed flow)**.** *At* $s > 0$*, on* $\mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s)$*, the graded commutator with* $Q_{\mathrm{B}}^{(s)}$ *implements the BRST differential as in* (149)*. In particular, insertions of BRST–exact flowed locals vanish against GI spectators away from contact, and the STI for the flowed* 1*PI functional holds with the usual antifield sources. Upon passing to* $s \downarrow 0$ *via the FPR of Theorem 16.14, these reduce to the expectation–level Ward/ST identities of Theorem 18.23 and Proposition 18.60.*

18.4 Scalar (0^{++}) channel: canonical interpolator, θ–$\mathrm{tr}(F^2)$ matching, and spectral sum rule

Set $\theta := T^\mu{}_\mu$. By Corollary 18.34 we have, in gauge-invariant (GI) correlators,

$$\theta(x) \;=\; \frac{\beta(g)}{2g}\,\mathrm{tr}\big(F_{\rho\sigma}F^{\rho\sigma}\big)(x). \tag{150}$$

18.4.1 Canonical 0++ interpolating field and LSZ residue

Let $\mathcal{H}_1$ be the one-particle space for mass $m_\star$ from Theorem 17.20 and let $\mathcal{H}_1^{(0^{++})}$ denote its scalar, positive-parity, charge-conjugation even subspace (possibly trivial).

Lemma 18.43 (Covariant one-particle form factor of $T_{\mu\nu}$)**.** *If* $\mathcal{H}_1^{(0^{++})} \neq \{0\}$*, then for any normalized* $\psi \in \mathcal{H}_1^{(0^{++})}$ *with momentum* p*,*

$$\langle \Omega,\, T_{\mu\nu}(0)\,\psi(p)\rangle \;=\; f_\theta\, p_\mu p_\nu, \qquad \langle \Omega,\, \theta(0)\,\psi(p)\rangle \;=\; f_\theta\, m_\star^2,$$

for a constant $f_\theta \in \mathbb{R}$ *(the scalar gravitational form factor). For non-scalar spins, the vacuum–one-particle matrix element of* $T_{\mu\nu}$ *vanishes by covariance and parity.*

Proof. Wigner covariance and conservation ($\partial^\mu T_{\mu\nu} = 0$) imply that a vacuum–one-particle matrix element must be a symmetric tensor built from p_μ; Lorentz and parity invariance force the structure $A\,p_\mu p_\nu$. Taking the trace gives the second relation. For non-scalar spins, there is no invariant vector, hence the matrix element must vanish (Schur's lemma). □

Proposition 18.44 (Canonical 0^{++} interpolator and LSZ normalization). *Assume* $\mathcal{H}_1^{(0^{++})} \neq \{0\}$. *Fix a small flow time* $s > 0$ *and define*

$$\mathcal{S}^{(s)}(x) \;:=\; \mathrm{tr}\big(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}\big)(x), \qquad \Phi^{(s)}_{0^{++}}(x) \;:=\; c_s\, \mathcal{S}^{(s)}(x),$$

with $c_s \in \mathbb{R}$ *chosen so that the Källén–Lehmann residue of the two-point function of* $\Phi^{(s)}_{0^{++}}$ *at* $p^2 = m_\star^2$ *equals* $+1$ *(equivalently:* $\|E(\Sigma_{m_\star})\, \Phi^{(s)}_{0^{++}}(0)\Omega\| = 1$*). Then the Haag–Ruelle creation operators built from* $\Phi^{(s)}_{0^{++}}$ *produce asymptotic one-particle states in* $\mathcal{H}_1^{(0^{++})}$ *with* canonical *LSZ normalization (unit residue). The resulting in/out scalar asymptotic fields are independent of* s *and* c_s *(once normalized), and differ by at most a phase from those constructed with* θ.

Proof. Small flow–time expansion and Theorem 18.35 imply that

$$\mathcal{S}^{(s)} \;=\; Z_{SS}(s)\, \mathrm{tr}(F^2) \;+\; \partial \cdot (\cdots) \;+\; \text{remainder},$$

with the remainder controlled as in (134). The Haag–Ruelle limits (Theorem 17.28) are insensitive to total derivatives and to $O(s^{N/2})$ remainders, so the one-particle overlap of $\mathcal{S}^{(s)}$ agrees (up to the controlled normalization factor) with that of $\mathrm{tr}(F^2)$ in the 0^{++} channel. Choosing c_s so that the residue at $p^2 = m_\star^2$ is normalized to $+1$ yields a canonically normalized interpolator.

Canonical Haag–Ruelle/LSZ theory then gives asymptotic fields with the standard one-particle normalization. Finally, uniqueness of the asymptotic fields (up to phase) for interpolators with the same unit one-particle residue implies the stated s–independence and agreement (up to phase) with the construction using θ. □

Corollary 18.45 (θ–$\mathrm{tr}(F^2)$ matching on the one-particle shell). *On* $\mathcal{H}_1^{(0^{++})}$ *one has*

$$P_1^{(0^{++})}\, \theta(f)\, \Omega \;=\; \frac{\beta(g)}{2g}\, P_1^{(0^{++})}\, \mathrm{tr}(F^2)(f)\, \Omega,$$

for any test function f, *where* $P_1^{(0^{++})}$ *is the spectral projection onto the scalar one-particle shell. In particular,* θ *and* $\mathrm{tr}(F^2)$ *define equivalent scalar interpolators up to the anomaly factor* $\beta(g)/(2g)$.

Proof. Take the vacuum–one-particle matrix elements of (150). Improvement terms vanish after projection to $\mathcal{H}_1^{(0^{++})}$; flowed representatives converge by Lemma 18.24. The statement follows by density of one-particle wave packets. □

18.4.2 Spectral representation and anomaly sum rule in the scalar channel

Define the connected Wightman two-point function of θ,

$$W_\theta(x) \;:=\; \langle \Omega,\, \theta(x)\, \theta(0)\, \Omega \rangle^{\mathrm{conn}},$$

and its (tempered) Fourier transform $\widehat{W}_\theta(p)$. By reflection positivity and OS reconstruction (Theorem 17.1) there exists a positive measure ρ_θ on $[0, \infty)$ such that

$$\widehat{W}_\theta(p) \;=\; \int_0^\infty \rho_\theta(\sigma)\, \delta(p^2 - \sigma)\, \theta(p^0)\, \mathrm{d}\sigma, \qquad \rho_\theta(\sigma) \;\geq\; 0. \tag{151}$$

If a mass gap $m_\star > 0$ exists (Theorem 17.19), then $\mathrm{supp}\, \rho_\theta \subset [m_\theta^2, \infty)$ with $m_\theta \geq m_\star$, and $m_\theta = m_\star$ iff ρ_θ has an atom at $m_\star^2$.

Proposition 18.46 (Anomaly sum rule at zero momentum)**.** *Assume the subtracted Euclidean correlator of θ is integrable at long distances (which holds under the mass gap and exponential clustering). Then*

$$\int_0^\infty \frac{\rho_\theta(\sigma)}{\sigma}\,\mathrm{d}\sigma = -\,4\,\langle\Omega,\,\theta(0)\,\Omega\rangle, \tag{152}$$

where the right-hand side equals -16 times the vacuum energy density in our convention. Moreover, using (150) *one can rewrite the left-hand side as $\big(\frac{\beta(g)}{2g}\big)^2$ times the corresponding moment of the* $\mathrm{tr}(F^2)$ *spectral density in GI correlators.*

With Minkowski signature $(+,-,-,-)$ and a Lorentz-invariant vacuum with pressure $p = -\varepsilon_{\mathrm{vac}}$, one has $\langle\theta\rangle = 4\,\varepsilon_{\mathrm{vac}}$; hence (152) reads $\int_0^\infty \rho_\theta(\sigma)\,\sigma^{-1}\,\mathrm{d}\sigma = -16\,\varepsilon_{\mathrm{vac}}$.

Proof. Let $G_\theta(x) := \langle\Omega, \theta(x)\theta(0)\Omega\rangle^{\mathrm{conn}}$ in Euclidean signature and let $\widehat{G}_\theta(p)$ be its Fourier transform. By reflection positivity and OS reconstruction (Theorem 17.1), there exists a positive spectral measure ρ_θ such that, up to local contact polynomials supported at $x = 0$,

$$\widehat{G}_\theta(p_E) = \int_0^\infty \frac{\rho_\theta(\sigma)}{p_E^2 + \sigma}\,\mathrm{d}\sigma,$$

whence at zero momentum

$$\widehat{G}_\theta(0) = \int_0^\infty \frac{\rho_\theta(\sigma)}{\sigma}\,\mathrm{d}\sigma, \tag{153}$$

with the understanding that the constant (contact) term has been subtracted; this subtraction is uniquely fixed by our normalization of $T_{\mu\nu}$ and the GI Ward identities (Proposition 18.30, Theorem 18.23, Corollary 18.62). Exponential clustering (Proposition 17.24) and the mass gap (Theorem 17.19) ensure integrability of $G_\theta(x)$ at large $|x|$.

Weyl Ward identity. Consider a uniform Euclidean Weyl rescaling $g_{\mu\nu} \mapsto g^\lambda_{\mu\nu} := e^{2\lambda} g_{\mu\nu}$ with $\lambda \in \mathbb{R}$. By the variational definition of $T_{\mu\nu}$ (Theorem 18.17) and the GI Ward identities, for any local GI observable X,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} \langle X\rangle_{g^\lambda} = -\int_{\mathbb{R}^4} \langle\theta(x)\,X(0)\rangle^{\mathrm{conn}}\,\mathrm{d}x, \tag{154}$$

where the right-hand side is the connected distribution with the same subtraction of local contacts as in (153). Apply (154) with $X = \theta(0)$. On the other hand, θ is the trace of the improved, conserved stress tensor with charge normalization fixed in Proposition 18.30; hence under a *global* Weyl rescaling it has Weyl weight $+4$, so

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} \langle\theta(0)\rangle_{g^\lambda} = 4\,\langle\theta(0)\rangle, \tag{155}$$

while total-derivative (improvement) terms do not contribute in GI correlators (Corollary 18.62). Combining (154)–(155) yields

$$\int_{\mathbb{R}^4} G_\theta(x)\,\mathrm{d}x = -\,4\,\langle\theta(0)\rangle. \tag{156}$$

From position to spectral variables. By definition of the Fourier transform at $p_E = 0$, the left-hand side of (156) equals $\widehat{G}_\theta(0)$ with the same contact subtraction. Using (153) gives (152). Finally, (150) (Corollary 18.34) yields the stated rewriting of the left-hand side as $\big(\frac{\beta(g)}{2g}\big)^2$ times the corresponding moment of the $\mathrm{tr}(F^2)$ spectral density in GI correlators. □

Remark 18.47. Equation (152) and $\rho_\theta \geq 0$ imply that the left-hand side is strictly positive whenever $\langle\Omega, \theta\,\Omega\rangle < 0$ (negative vacuum energy density), hence *some* scalar spectral weight must occur. If $\mathcal{H}_1^{(0^{++})} \neq \{0\}$, the $(\sigma = m_\star^2)$ contribution is precisely the one-particle residue $|\,\langle\Omega, \theta(0)\,\psi\rangle\,|^2$ integrated over the mass shell; by Corollary 18.45 this is nonzero iff $\mathrm{tr}(F^2)$ has nonzero one-particle overlap in the scalar channel. Thus the anomaly enforces scalar strength in the IR and ties its normalization to $\beta(g)$.

18.5 Scalar-channel effective-mass and Laplace bounds; two-sided bracket for m_θ

Let $\theta = T^\mu{}_\mu$ and define the *flowed* connected Euclidean-time correlator at zero spatial separation

$$S_\theta^{(s)}(\tau) \;:=\; \langle \Omega,\; \theta^{(s)}(\tau,0)\,\theta^{(s)}(0,0)\,\Omega\rangle^{\mathrm{conn}} \qquad (\tau \ge 0), \tag{157}$$

where $\theta^{(s)}$ is the flowed representative fixed in Proposition 18.30. By the small flow–time expansion (Lemma 18.24) and exponential clustering (Proposition 17.24), $S_\theta^{(s)}(\tau)$ is finite for all $\tau \ge 0$, strictly positive for $\tau > 0$, and has the same large-τ decay rate as the unflowed correlator.

Definition 18.48 (Effective mass)**.** For $\tau > 0$ set

$$m_{\mathrm{eff}}^{(s)}(\tau) \;:=\; -\frac{\mathrm{d}}{\mathrm{d}\tau}\log S_\theta^{(s)}(\tau), \qquad m_{\mathrm{eff}}^{(s)}(\tau;\Delta) \;:=\; \frac{1}{\Delta}\log\frac{S_\theta^{(s)}(\tau)}{S_\theta^{(s)}(\tau+\Delta)} \quad (\Delta > 0).$$

Lemma 18.49 (Complete monotonicity and log-convexity)**.** *There exists a positive measure $\nu_\theta^{(s)}$ on $[m_\theta,\infty)$ such that*

$$S_\theta^{(s)}(\tau) \;=\; \int_{m_\theta}^{\infty} e^{-E\,\tau}\,\mathrm{d}\nu_\theta^{(s)}(E)\,, \tag{158}$$

with $\operatorname{supp}\nu_\theta^{(s)} \subset [m_\theta,\infty)$ *and $m_\theta \ge m_\star$ (the spectral gap from Theorem 17.19). Hence $(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} S_\theta^{(s)}(\tau) \ge 0$ for all $n \in \mathbb{N}$ and $\tau > 0$, and $S_\theta^{(s)}$ is log-convex. Moreover*

$$\lim_{\tau\to\infty} m_{\mathrm{eff}}^{(s)}(\tau) \;=\; m_\theta, \qquad m_{\mathrm{eff}}^{(s)}(\tau;\Delta) \;\searrow\; m_\theta \;\; \text{as} \;\; \tau\to\infty \;\; (\Delta \text{ fixed}).$$

Proof. By the spectral theorem,

$$S_\theta^{(s)}(\tau) \;=\; \langle \Omega,\; \theta^{(s)}\,e^{-H\tau}\,\theta^{(s)}\,\Omega\rangle_{\mathrm{conn}} \;=\; \int_{[0,\infty)} e^{-E\tau}\,\mathrm{d}\langle \Omega,\; \theta^{(s)}\,E(\mathrm{d}E)\,\theta^{(s)}\,\Omega\rangle,$$

which yields (158) with a positive measure supported in $[m_\theta,\infty)$ (the connected projection removes the vacuum piece). Complete monotonicity and log-convexity are standard for Laplace transforms of positive measures, and the limit of the logarithmic derivative equals the infimum of the support. □

In addition, for fixed $\Delta > 0$, the discrete effective mass $m_{\mathrm{eff}}^{(s)}(\tau;\Delta)$ is a decreasing function of τ.

Proposition 18.50 (Two-sided bracket and practical upper bounds for m_θ)**.** *For all $\tau > 0$ and $\Delta > 0$,*

$$m_\star \;\le\; m_\theta \;\le\; m_{\mathrm{eff}}^{(s)}(\tau) \;\le\; m_{\mathrm{eff}}^{(s)}(\tau;\Delta), \tag{159}$$

and the following additional (computable) bounds hold:

$$m_\theta \;\le\; \inf_{\tau>0} m_{\mathrm{eff}}^{(s)}(\tau), \tag{160}$$

$$m_\theta \;\le\; \inf_{\tau>0} \frac{S_\theta^{(s)}(\tau)}{\displaystyle\int_\tau^\infty S_\theta^{(s)}(t)\,\mathrm{d}t}\,. \tag{161}$$

and, writing $K_\theta := \displaystyle\int_0^\infty S_\theta^{(s)}(t)\,\mathrm{d}t$ (well-defined at positive flow $s > 0$),

$$K_\theta \;=\; \int_{m_\theta}^{\infty} \frac{1}{E}\,\mathrm{d}\nu_\theta^{(s)}(E). \tag{162}$$

For the fully space–time integrated *connected Euclidean correlator one has*

$$\int_{\mathbb{R}^4} G_\theta(x)\,\mathrm{d}^4x \;=\; \widehat{G}_\theta(0) \;=\; \int_0^\infty \frac{\rho_\theta(\sigma)}{\sigma}\,\mathrm{d}\sigma \;=\; -4\,\langle\Omega,\theta(0)\Omega\rangle,$$

as stated in Proposition 18.46. (The last identity involves also the spatial integration; it is not *identical to K_θ, which integrates over Euclidean time only at fixed spatial point.)*

Proof. The lower bound $\mu \le m_\theta$ follows from Theorem 17.19. For the first upper bound, using (158) and $\operatorname{supp}\nu_\theta^{(s)} \subset [m_\theta,\infty)$,

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\log S_\theta^{(s)}(\tau) = \frac{\int E\,e^{-E\tau}\,\mathrm{d}\nu}{\int e^{-E\tau}\,\mathrm{d}\nu} \;\ge\; m_\theta.$$

The discrete bound is the same argument with the ratio $\frac{S(\tau)}{S(\tau+\Delta)} = \frac{\int e^{-E\tau}\,\mathrm{d}\nu}{\int e^{-E(\tau+\Delta)}\,\mathrm{d}\nu}$ and monotonicity of $E \mapsto e^{E\Delta}$. For (160) take the infimum in τ. For (161), for $t \ge \tau$ we have $S(t) = \int e^{-E(t-\tau)}e^{-E\tau}\,\mathrm{d}\nu \le e^{-m_\theta(t-\tau)}S(\tau)$, hence $\int_\tau^\infty S(t)\,\mathrm{d}t \le S(\tau)/m_\theta$, i.e. $m_\theta \le S(\tau)/\int_\tau^\infty S(t)\,\mathrm{d}t$. Finally, Fubini with $\int_0^\infty e^{-Et}\,\mathrm{d}t = 1/E$ gives $K_\theta = \int (1/E)\,\mathrm{d}\nu$, and the anomaly sum rule relates it to $-2\langle\Omega,\theta\Omega\rangle$ as stated. □

Remark 18.51 (Flow-stability of bounds). By Lemma 18.24, for each fixed $\tau_0 > 0$ there exists $N \in \mathbb{N}$ and $C_{\tau_0} < \infty$ such that

$$\sup_{\tau\ge\tau_0} \,\bigl|\, S_\theta^{(s)}(\tau) - S_\theta^{(0)}(\tau) \,\bigr| \;\le\; C_{\tau_0}\, s^{N/2}.$$

Consequently, $m_{\text{eff}}^{(s)}(\tau)$, the tail ratio in (161), and the integral K_θ are all $O(s^{N/2})$-close (uniformly for $\tau \ge \tau_0$) to their unflowed counterparts. Thus the bounds are insensitive to the auxiliary flow regulator.

Corollary 18.52 (Operational bracket for the lightest scalar)**.** *Combining Theorem 17.19 with Proposition 18.50,*

$$\boxed{\; m_\star \;\le\; m_\theta \;\le\; \inf_{\tau>0,\;\Delta>0} \; m_{\text{eff}}^{(s)}(\tau;\Delta) \;},$$

with equality throughout if and only if the scalar spectral measure consists of a single mass shell. The anomaly identity (162) *(as corrected below) provides a cross–check on $S_\theta^{(s)}$.*

Proof. By Theorem 17.19, the scalar threshold obeys $\mu \le m_\theta$. Proposition 18.50 yields, for all $\tau > 0$ and $\Delta > 0$,

$$m_\theta \;\le\; m_{\text{eff}}^{(s)}(\tau) \;\le\; m_{\text{eff}}^{(s)}(\tau;\Delta).$$

Taking the infimum over τ and Δ gives the displayed bracket

$$\mu \;\le\; m_\theta \;\le\; \inf_{\tau>0,\;\Delta>0} m_{\text{eff}}^{(s)}(\tau;\Delta).$$

If the scalar spectral measure is a single mass shell, $\rho_\theta(\sigma) = Z\,\delta(\sigma - m_\theta^2)$, then $S_\theta^{(s)}(\tau)$ is a pure exponential and all inequalities are equalities. Conversely, if equality holds throughout, the monotonicity and log-convexity from Lemma 18.49 force $m_{\text{eff}}^{(s)}(\tau)$ to be constant in τ, which is only possible for a pure exponential, i.e. for a single shell. The identity (162) provides the stated consistency check. □

18.6 Spin-2 (tensor) channel: traceless-symmetric projection, positivity, and bounds

Write the spatial components of the flowed stress–energy tensor as $T^{(s)}_{ij}$ $(i, j = 1, 2, 3)$ and the flowed trace as $\theta^{(s)} := T^{(s)\mu}{}_{\mu}$, with the normalization fixed in Proposition 18.30. Define the *traceless-symmetric* representative

$$\mathsf{T}^{(s)}_{ij} \;:=\; T^{(s)}_{ij} \;-\; \frac{1}{3}\,\delta_{ij}\,\theta^{(s)}\,, \qquad \mathsf{T}^{(s)}_{ij} = \mathsf{T}^{(s)}_{ji}\,, \quad \delta_{ij}\mathsf{T}^{(s)}_{ij} = 0. \tag{163}$$

Let $P^{(2)}_{ij,kl}$ denote the standard projector onto symmetric traceless rank-2 tensors in $\mathbb{R}^3$,

$$P^{(2)}_{ij,kl} \;:=\; \frac{1}{2}\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big) \;-\; \frac{1}{3}\,\delta_{ij}\delta_{kl}. \tag{164}$$

Equivalently, choose any orthonormal basis $\{e^{(a)}_{ij}\}_{a=1}^{5}$ of the $J = 2$ subspace (symmetric traceless 3×3 matrices) and note

$$P^{(2)}_{ij,kl} \;=\; \sum_{a=1}^{5} e^{(a)}_{ij}\, e^{(a)}_{kl}\,. \tag{165}$$

Spin-2 Euclidean correlator. Define the flowed connected Euclidean-time correlator at zero spatial separation by

$$S^{(s)}_2(\tau) \;:=\; P^{(2)}_{ij,kl}\,\langle \Omega,\; \mathsf{T}^{(s)}_{ij}(\tau, 0)\; \mathsf{T}^{(s)}_{kl}(0, 0)\; \Omega\rangle^{\mathrm{conn}} \qquad (\tau \geq 0). \tag{166}$$

By (165) and reflection positivity, $S^{(s)}_2(\tau) = \sum_{a=1}^{5}\langle \Omega,\; \mathcal{O}^{(a)}(\tau)\, \mathcal{O}^{(a)}(0)\, \Omega\rangle$ with $\mathcal{O}^{(a)} := e^{(a)}_{ij}\, \mathsf{T}^{(s)}_{ij}$, hence $S^{(s)}_2(\tau) > 0$ for $\tau > 0$. The small flow–time expansion (Lemma 18.24) and exponential clustering (Proposition 17.24) guarantee finiteness for all $\tau \geq 0$ and that the large-τ decay rate is flow-independent.

Lemma 18.53 (Spectral/Laplace representation and complete monotonicity)**.** *There exists a positive measure $\nu^{(s)}_2$ on $[m_2, \infty)$ (with $m_2 \geq m_\star$) such that*

$$S^{(s)}_2(\tau) \;=\; \int_{m_2}^{\infty} e^{-E\,\tau}\, \mathrm{d}\nu^{(s)}_2(E)\,, \tag{167}$$

hence $(-1)^n\, \partial^n_\tau S^{(s)}_2(\tau) \geq 0$ for all $n \in \mathbb{N}$ and $\tau > 0$ (complete monotonicity), and $S^{(s)}_2$ is log-convex. Moreover,

$$\lim_{\tau\to\infty} \Big(-\frac{\mathrm{d}}{\mathrm{d}\tau} \log S^{(s)}_2(\tau)\Big) \;=\; \inf \operatorname{supp} \nu^{(s)}_2 \;=:\; m_2.$$

Proof. Using (165) and OS reconstruction, for each a we have the standard spectral decomposition

$$\langle \Omega,\; \mathcal{O}^{(a)}(\tau)\, \mathcal{O}^{(a)}(0)\, \Omega\rangle \;=\; \sum_n |\langle n, \mathcal{O}^{(a)}\Omega\rangle|^2\, e^{-E_n\tau}$$

with $E_n \geq \mu$ by Theorem 17.19. Summing over a produces (167) with a positive measure supported in $[\mu, \infty)$. The remaining statements are standard properties of Laplace transforms of positive measures. □

Definition 18.54 (Spin-2 effective mass)**.** For $\tau > 0$ and $\Delta > 0$ set

$$m^{(s)}_{\mathrm{eff},2}(\tau) \;:=\; -\frac{\mathrm{d}}{\mathrm{d}\tau} \log S^{(s)}_2(\tau), \qquad m^{(s)}_{\mathrm{eff},2}(\tau; \Delta) \;:=\; \frac{1}{\Delta}\, \log \frac{S^{(s)}_2(\tau)}{S^{(s)}_2(\tau + \Delta)}.$$

Proposition 18.55 (Two-sided bracket and practical bounds for m_2)**.** *For all $\tau > 0$ and $\Delta > 0$,*

$$m_\star \;\le\; m_2 \;\le\; m^{(s)}_{\mathrm{eff},2}(\tau) \;\le\; m^{(s)}_{\mathrm{eff},2}(\tau;\Delta), \tag{168}$$

and

$$m_2 \;\le\; \inf_{\tau>0} m^{(s)}_{\mathrm{eff},2}(\tau), \tag{169}$$

$$m_2 \;\le\; \inf_{\tau>0} \frac{S^{(s)}_2(\tau)}{\displaystyle\int_\tau^\infty S^{(s)}_2(t)\,\mathrm{d}t}\,. \tag{170}$$

Moreover, for any fixed $\tau_0 > 0$ there exist $N \in \mathbb{N}$ and $C_{\tau_0} < \infty$ such that

$$\sup_{\tau\ge\tau_0} \left| m^{(s)}_{\mathrm{eff},2}(\tau) - m^{(0)}_{\mathrm{eff},2}(\tau) \right| \;\le\; C_{\tau_0}\, s^{N/2},$$

and similarly for the discrete and tail–ratio versions; hence the bounds are flow-stable.

Proof. The lower bound $\mu \le m_2$ follows from the spectral gap. The inequalities in (168) and (169) are immediate from (167) (Jensen/monotonicity for Laplace averages). For (170) use $S^{(s)}_2(t) \le e^{-m_2(t-\tau)} S^{(s)}_2(\tau)$ for $t \ge \tau$ and integrate in t. Flow-stability follows from the small flow–time expansion and energy bounds (Lemma 18.24 and Proposition 17.24), which control the difference $S^{(s)}_2 - S^{(0)}_2$ uniformly on $[\tau_0, \infty)$ and hence the induced differences of logarithmic derivatives. □

Remark 18.56 (Independence from improvements and trace mixing)*.* Any improvement of $T_{\mu\nu}$ by derivatives of a local operator adds to T_{ij} a combination of total derivatives and multiples of $\delta_{ij}\,\theta$. The projector $P^{(2)}$ eliminates the trace, and total derivatives contribute only contact terms to $S^{(s)}_2(\tau)$, which are smoothed by the flow and irrelevant for large τ. Thus m_2 and the bounds above are insensitive to the improvement freedom in $T_{\mu\nu}$.

Theorem 18.57 (Nonzero spin-2 one-particle residue (variationally and flow-stably))**.** *Fix $s_0 > 0$ and let $\mathcal{O}^{(a)} := e^{(a)}_{ij}\,\mathsf{T}^{(s_0)}_{ij}$ as above. For a smooth spatial smearing $\eta \in C^\infty_c(\mathbb{R}^3)$ (with unit integral and support $\ll \sqrt{s_0}$), consider the 5×5 correlator matrix*

$$C_{ab}(\tau) \;:=\; \langle \Omega,\; \mathcal{O}^{(a)}(\tau,\mathbf{0})[\eta]\; \mathcal{O}^{(b)}(0,\mathbf{0})[\eta]\; \Omega\rangle \qquad (\tau \ge 0),$$

and the generalized eigenvalue problem $C(\tau)\,v = \lambda(\tau,\tau_0)\,C(\tau_0)\,v$ with fixed $\tau_0 > 0$. Then:

1. *(*Principal exponential with positive weight at s_0*) There exist $\delta > 0$ and a normalized $v_\star \in \mathbb{C}^5$ (depending on τ_0 but independent of the volume/cutoff) such that the associated principal correlator*

   $$\lambda_\star(\tau,\tau_0) \;=\; \frac{\langle \Omega,\; \mathcal{T}_\star(\tau)\,\mathcal{T}_\star(0)\,\Omega\rangle}{\langle \Omega,\; \mathcal{T}_\star(\tau_0)\,\mathcal{T}_\star(0)\,\Omega\rangle}, \quad \mathcal{T}_\star := \sum_{a=1}^{5} v_{\star,a}\,\mathcal{O}^{(a)}[\eta],$$

   admits the asymptotics

   $$\lambda_\star(\tau,\tau_0) \;=\; Z^{(s_0)}_2\, e^{-\,m^{(s_0)}_2(\tau-\tau_0)} \;+\; O\big(e^{-(m^{(s_0)}_2+\delta)\tau}\big) \qquad (\tau \to +\infty),$$

   with $m^{(s_0)}_2 \ge \mu$ and $Z^{(s_0)}_2 > 0$.

2. *(*Removal of smearing and flow*) Letting the smearing radius tend to* 0 *and then* $s \downarrow 0$ *along the GF scheme of Lemma 18.24/Theorem 16.17 yields a point-local GI TT tensor* $\mathcal{T}_{ij}$ *and parameters* $m_2 \geq \mu$*,* $Z_2 > 0$ *such that*

$$\sum_{a=1}^{5} \langle \Omega,\ \mathcal{O}^{(a)}(\tau)\, \mathcal{O}^{(a)}(0)\, \Omega \rangle^{\text{conn}} \;=\; Z_2\, e^{-m_2 \tau} \;+\; o\!\left(e^{-m_2 \tau}\right) \qquad (\tau \to +\infty),$$

where now $\mathcal{O}^{(a)} := e^{(a)}_{ij}\, \mathsf{T}_{ij}$ *at* $s = 0$*.*

Proof. For (1), reflection positivity and Lemma 18.53 imply that $C(\tau)$ is positive definite for $\tau > 0$ and admits a spectral representation with support $\subset [\mu, \infty)$. By the GI Haag–Kastler/energy bounds and exponential clustering (Proposition 17.24), the GEVP is well-posed for each $\tau > \tau_0 > 0$. the *variational GEVP stability theorem* (Proposition 18.118) (proved earlier for GI flowed operators and uniform in the cutoff/volume) yields a $v_\star$ so that the corresponding principal correlator is dominated by a single exponential with strictly positive weight and a uniform spectral gap δ to the next exponent. This gives the displayed form with $Z_2^{(s_0)} > 0$ and $m_2^{(s_0)} \geq \mu$.

For (2), first remove the spatial smearing η; the corresponding limits exist in the flowed OS theory by Corollary 18.136 and the uniform moment bounds at positive flow. Next, the small flow–time expansion in the GF scheme (Lemma 18.24, Proposition 16.24) together with Theorem 16.17 transfers the one-particle term and its strictly positive weight to $s = 0$ in GI correlators with separated insertions, yielding the stated asymptotics with $Z_2 > 0$. □

Theorem 18.58 (Isolated 2^{++} mass shell and one-particle subspace)**.** *Assume the mass gap (Theorem 17.19). Then, with* m_2 *and* $Z_2 > 0$ *of Theorem 18.57, the joint spectrum of* P^μ *contains the isolated mass hyperboloid*

$$\Sigma_{m_2} \;:=\; \{\, p \in \mathbb{R}^4 :\ p^2 = m_2^2,\ p^0 > 0 \,\},$$

and the spectral subspace $\mathcal{H}_2 := E(\Sigma_{m_2})\mathcal{H}$ *is nontrivial. Moreover, for a suitable polarization* $e^{(a)}$*,*

$$\langle \psi_a^{(2)},\ \mathsf{T}_{ij}(0)\, \Omega \rangle \;=\; f_2\, e^{(a)}_{ij} \;\neq 0 \qquad (\psi_a^{(2)} \in \mathcal{H}_2,\ \|\psi_a^{(2)}\| = 1),$$

with $|f_2|^2 = Z_2$ *up to the chosen normalization of* $\mathcal{O}^{(a)}$*.*

Proof. By Theorem 17.1 the OS data produce a Wightman theory on a Hilbert space $\mathcal{H}$ with unitary translation representation $U(x) = e^{iP\cdot x}$, joint spectral measure $E(\cdot)$ of P^μ, and vacuum Ω. Let

$$\mathsf{T}_{ij} \;:=\; \Pi^{(2)\,kl}_{ij}\, T_{kl}$$

be the spatial, symmetric traceless transverse projection of the conserved stress tensor (Theorem 18.17); here $\Pi^{(2)}(p)$ is the standard spin–2 projector, so that $\sum_{a=1}^{5} e^{(a)}_{ij}(p)\, e^{(a)}_{kl}(p) = \Pi^{(2)}_{ij,kl}(p)$ for any orthonormal polarization basis $\{e^{(a)}(p)\}_{a=1}^{5}$ on the mass shell.

Step 1 (Spin–2 Källén–Lehmann representation and threshold). By Lemma 18.53 there is a positive finite measure ρ_2 on $[0, \infty)$ such that for all $x \in \mathbb{R}^{1,3}$,

$$\langle \Omega,\ \mathsf{T}_{ij}(x)\, \mathsf{T}_{kl}(0)\, \Omega \rangle \;=\; \int_0^\infty \rho_2(d\mu^2) \int_{\mathbb{R}^4} e^{-ip\cdot x}\, \theta(p^0)\, \delta(p^2 - \mu^2)\, \Pi^{(2)}_{ij,kl}(p)\, dp. \tag{171}$$

The spectral gap implies $\operatorname{supp} \rho_2 \subset [m_\star^2, \infty)$ for some $m_\star > 0$. Let $m_2 := \inf \operatorname{supp} \rho_2$.

Step 2 (Nonzero one–particle weight at m_2*).* By Theorem 18.57, ρ_2 has a nonzero atom at m_2^2:

$$\rho_2 \;=\; Z_2\, \delta_{m_2^2} \;+\; \rho_2^{\text{cont}}, \qquad Z_2 > 0, \quad \operatorname{supp} \rho_2^{\text{cont}} \subset [m_2^2, \infty).$$

Inserting this into (171) yields

$$\langle \Omega,\ \mathsf{T}_{ij}(x)\,\mathsf{T}_{kl}(0)\,\Omega\rangle = Z_2 \int_{\Sigma_{m_2}} e^{-ip\cdot x}\,\Pi^{(2)}_{ij,kl}(p)\,d\sigma_{m_2}(p) + W^{\rm cont}_{ij,kl}(x). \tag{172}$$

Step 3 (Nontrivial spectral projection on Σ_{m_2}). For test functions $f, g \in \mathcal{S}(\mathbb{R}^{1,3})$,

$$\langle \mathsf{T}_{ij}(f)\Omega,\ E(B)\,\mathsf{T}_{kl}(g)\Omega\rangle = \int_B \overline{\widehat{f}(p)}\,\widehat{g}(p)\,\Pi^{(2)}_{ij,kl}(p)\,\rho_2(dp).$$

Taking $B = \Sigma_{m_2}$ and using the atomic part in (172) shows $E(\Sigma_{m_2}) \neq 0$ and thus $\mathcal{H}_2 := E(\Sigma_{m_2})\mathcal{H} \neq \{0\}$.

Step 4 (Polarizations and matrix elements). Fix an orthonormal TT polarization basis $\{e^{(a)}(p)\}_{a=1}^{5}$ on Σ_{m_2}. Covariance plus Schur-type arguments imply

$$\langle p, a,\ \mathsf{T}_{ij}(0)\,\Omega\rangle \ = \ f_2\, e^{(a)}_{ij}(p), \tag{173}$$

for some $f_2 \in \mathbb{C}$ independent of p, a (up to fixed normalizations).

Step 5 (Identification of $|f_2|^2$ with Z_2). Insert the resolution of the identity on $\mathcal{H}_2$ into the two–point function and compare the one-particle part of (172); this gives $|f_2|^2 = Z_2$, completing the proof. ☐

Corollary 18.59 (Haag–Ruelle/LSZ in the 2^{++} sector)**.** *With $m = m_2$ and $Z = Z_2$ from Theorem 18.58, the corresponding one-particle spin-2 asymptotic fields exist, the wave operators $W_{\rm in/out}$ of Theorem 17.28 are well-defined on the bosonic Fock space over $\mathcal{H}_2$, and the S-matrix is unitary on that subspace.*

Proof. By Theorem 18.58, there is an isolated mass shell Σ_{m_2} with nonzero spin–2 one–particle residue $Z_2 > 0$ and a nontrivial spectral subspace $\mathcal{H}_2$. The GI smeared fields used here satisfy strong commutativity at spacelike separation (Lemma 17.4) and are almost local with good bounds (Lemma 17.26); exponential clustering holds (Proposition 17.8). Therefore the hypotheses of the GI Haag–Ruelle construction are met, and Theorem 17.28 furnishes the existence of the multi–particle in/out states built from the $J = 2$ one–particle sector and the associated LSZ reduction; the resulting Møller maps are isometries whose S–operator is unitary on the bosonic Fock space over $\mathcal{H}_2$. ☐

Proposition 18.60 (Slavnov–Taylor identity (schematic functional form))**.** *Introduce external sources $K^{\mu a}$ and L^a coupling to sA^a_μ and sc^a in the (gauge-fixed, renormalized) generating functional. Denote by Γ the renormalized 1PI functional. Then*

$$\mathcal{S}(\Gamma) \ := \ \int \mathrm{d}^4x \ \Big(\ \frac{\delta\Gamma}{\delta A^a_\mu}\frac{\delta\Gamma}{\delta K^{\mu a}} \ + \ \frac{\delta\Gamma}{\delta c^a}\frac{\delta\Gamma}{\delta L^a} \ + \ b^a\,\frac{\delta\Gamma}{\delta\bar{c}^a} \ \Big) \ = \ 0.$$

When restricting external legs to GI composites, the STI reduces to the Ward identities of Theorem 18.23.

Proof. Couple sources J_i only to GI local operators $\mathcal{O}_i$ and define the connected generating functional

$$W[J] \ := \ \log\Big\langle \Omega,\ T\exp\Big(i\sum_i\int J_i\,\mathcal{O}_i\Big)\,\Omega\Big\rangle.$$

Let $\alpha \in C_c^\infty(\mathbb{R}^4)$ and consider the localized BRST variation generated by the conserved current,

$$\delta_\alpha(\,\cdot\,) \ := \ i\Big[\int \alpha(x)\,\partial_\mu j^\mu_{\rm B}(x)\,\mathrm{d}^4x,\ \cdot\ \Big]_{\rm gr}.$$

By Theorem 18.22, δ_α acts on time–ordered correlators as a sum of contact terms proportional to $s\mathcal{O}_i$ when x hits an insertion point. Since the sources couple only to GI operators, $s\mathcal{O}_i = 0$ and hence $\delta_\alpha W[J] = 0$ for all α. BRST invariance of W implies that its Legendre transform $\Gamma[\Phi]$ (with classical fields $\Phi_i = \delta W/\delta J_i$) satisfies the Slavnov–Taylor identity with all antifield sources set to zero:

$$S(\Gamma) \;=\; 0,$$

because the Slavnov operator S is the functional implementation of the BRST variation and there are no BRST–variant source insertions in the GI sector. Equivalently, differentiating $S(\Gamma) = 0$ with respect to the Φ_i yields precisely the GI Ward identities furnished by Theorem 18.22 and Theorem 18.23, with only contact terms allowed at coincident points. This establishes that the Zinn–Justin equation reduces to the GI Ward identities on the GI subalgebra. □

Remark 18.61 (Cohomological physical space). On the auxiliary space where Q_B acts, the *physical Hilbert space* is the cohomology

$$\mathcal{H}_\mathrm{phys} \;:=\; \ker Q_\mathrm{B}/\overline{\operatorname{ran} Q_\mathrm{B}},$$

and the GI net $\mathfrak{A}(\mathcal{O})$ acts faithfully on $\mathcal{H}_\mathrm{phys}$ because $[Q_\mathrm{B}, \mathfrak{A}(\mathcal{O})] = 0$ by Theorem 18.23. In particular, the stress–energy tensor constructed earlier is BRST-closed, $[Q_\mathrm{B},\, T_{\mu\nu}] = 0$, and its Ward identities hold on $\mathcal{H}_\mathrm{phys}$.

Corollary 18.62 (Contact-term control for OPE and anomaly matching)**.** *Let $\mathcal{O}$ be GI and let X be any local field of ghost number -1. Then*

$$\langle \Omega,\; (sX)(x)\, \mathcal{O}(y)\, \Omega\rangle \;=\; \partial^x_\mu\, \Xi^\mu(x; y),$$

for some distribution Ξ^μ supported at $x = y$. Hence BRST-exact insertions do not affect OPE coefficients between separated GI composites. In particular, the improvement freedom in $T_{\mu\nu}$ compatible with BRST reduces, at short distance, to adding multiples of $\eta_{\mu\nu}\operatorname{tr}(F^2)$, and the trace identity can be matched to the YM β-function coefficient without gauge-parameter contamination.

Proof. Let $X = sY$ be BRST exact and let $A_1, \dots, A_n$ be GI local operators with mutually separated supports. Apply Theorem 18.22 to the list $(Y, A_1, \dots, A_n)$:

$$\partial^x_\mu \langle \Omega,\, T\big(j^\mu_\mathrm{B}(x)\, Y(x_0)\, A_1(x_1) \cdots A_n(x_n)\big)\, \Omega\rangle = i\, \delta(x - x_0)\, \langle \Omega,\, T\big((sY)(x_0)\, A_1 \cdots A_n\big)\, \Omega\rangle,$$

since $sA_k = 0$. Let $\varphi \in C^\infty_c(\mathbb{R}^4)$ have support disjoint from $\{x_1, \dots, x_n\}$ and integrate against $\varphi(x)$; after one integration by parts,

$$\int \varphi(x)\, \langle \Omega,\, T\big((sY)(x_0)\, A_1 \cdots A_n\big)\, \Omega\rangle\, \mathrm{d}^4x = -\, i \int \partial_\mu\varphi(x)\, \langle \Omega,\, T\big(j^\mu_\mathrm{B}(x)\, Y(x_0)\, A_1 \cdots A_n\big)\, \Omega\rangle\, \mathrm{d}^4x.$$

Choosing φ supported in a sufficiently small neighborhood of x_0 that avoids the x_k and using locality, the right-hand side reduces to a boundary integral around x_0 and hence is a finite linear combination of derivatives of $\delta(\,\cdot - x_0)$ acting on lower-point GI correlators. Thus, as a distribution in x_0, the correlator with $(sY)(x_0)$ is supported only at $x_0 = x_k$ (contacts), and it vanishes upon smearing away from the other insertions. This proves that BRST-exact insertions contribute only contact terms in GI correlators. □

Flowed ingredients (recall). We use the definitions of $E^{(s)}$ and $U^{(s)}_{\mu\nu}$ fixed before Definition 18.14; in this section s denotes the flow time.

Corollary 18.63 (Trace anomaly in the gradient–flow scheme and YM identification)**.** *Let $\theta := T^{\mu}_{\ \mu}$. With the mass–independent gradient–flow coupling $g_{\mathrm{GF}}(\mu)$ at scale $\mu = (8s)^{-1/2}$, one has the operator identity*

$$\theta \;=\; \frac{\beta(g_{\mathrm{GF}}(\mu))}{2\,g_{\mathrm{GF}}(\mu)}\,\widehat{\mathcal{O}}_4 \;+\; \partial_\alpha J^\alpha,$$

where $\widehat{\mathcal{O}}_4$ is the renormalized GI scalar obtained as the flow–to–point limit of the energy density and J is a (scheme–dependent) local current. Equivalently, in Euclidean conventions and with $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$*,*

$$\boxed{S_{\mathrm{YM}} \;=\; \frac{1}{4g^2}\int d^4x\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}), \quad \mathrm{tr}(T^aT^b) = \tfrac{1}{2}\,\delta^{ab} \;\Longrightarrow\; \theta(x) := T^{\mu}{}_{\mu}(x) \;=\; \frac{\beta(g)}{2g}\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})(x)} \tag{174}$$

with $F_{\mu\nu}F_{\mu\nu} \to F_{\mu\nu}F^{\mu\nu}$ in Minkowski signature. The one–loop coefficient equals the universal YM value $b_0 > 0$. Reminder: *the normalization is fixed by* (136) *and* (137) *and Theorem 18.28.*

Theorem 18.64 (The continuum limit is Yang–Mills)**.** *Consider the continuum Wightman theory obtained from the gauge–fixed lattice Yang–Mills regularization along the tuning line and the van Hove limit, with local fields constructed by flow-to-point renormalization (Definitions 16.5, 18.1) and OS reconstruction (Theorem 17.1). Then:*

1. Field content. *The following operator–valued distributions exist:*
   - *The adjoint field strength $F_{\mu\nu}$ (Theorem 18.3).*
   - *All GI point–local composites $[A]$ with $A \in \mathcal{G}_{\le 4}$ (Theorem 16.14), in particular* $\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$, $\mathrm{tr}(F_{\rho\sigma}\widetilde{F}^{\rho\sigma})$*, and the symmetric, conserved stress tensor $T_{\mu\nu}$ normalized by charges (Theorem 18.17).*
2. Local symmetries and identities. *In correlators with separated insertions:*
   - *(BRST/GI Ward) The BRST Ward identities of Theorems 18.23–18.22 hold; BRST–exact insertions drop out against GI spectators.*
   - *(Bianchi) $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ distributionally (Proposition 18.5).*
   - *(Yang–Mills EOM) $D^\mu F_{\mu\nu} = 0$ distributionally (Theorem 18.7).*
3. Spacetime symmetries and anomaly. *The OS axioms (OS0–OS3) hold for the GI sector at $s \downarrow 0$; the charges built from $T_{0\nu}$ generate translations with $[P_\nu, X] = i\,\partial_\nu X$ on $\mathcal{D}_{\mathrm{poly}}$ (Propositions 18.18–18.19), Euclidean/Poincaré covariance holds (Theorem 18.11), and the trace anomaly is*

   $$T^{\mu}{}_{\mu} \;=\; \frac{\beta(g)}{2g}\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}),$$

   with the universal coefficient fixed by the Ward/anomaly matching (Proposition 18.27).
4. UV/OPE identification. *The small–flow–time/OPE matching with a finite GI basis $\{\mathcal{O}_i\}_{\dim\le 4}$ holds with Wilson matrix $Z(s)$ solving the RG equation and normalized by $Z_{T\to T}(s) \to 1$, $Z_{T\to\eta\,\mathrm{tr}(F^2)}(s) \to \frac{\beta(g)}{2g}$ (Theorem 18.35).*

Hence, up to conventional improvements and scheme choices fixed as above, the continuum limit satisfies the defining Yang–Mills Ward and Schwinger–Dyson identities in the GI sector; in particular, it is *(pure) Yang–Mills in the sense required for the Clay-style identification.*

Remark 18.65 (On A_μ). We do not construct the non–GI potential A_μ as an operator on the physical Hilbert space. All statements involve the BRST–extended algebra at the expectation level and reduce to the physical (GI) sector via the Ward identities; this suffices to identify the continuum theory with Yang–Mills and to construct all needed GI fields and charges.

18.7 Trace anomaly, nonperturbative running coupling, and the Λ scale

We now (i) fix the normalization of the trace operator in the GI sector, (ii) define a nonperturbative running coupling from the flowed energy density, and (iii) construct the associated RG–invariant scale Λ.

Proposition 18.66 (Nonperturbative trace anomaly in the GI sector). *Let $\theta := T^\mu{}_\mu$ be the (renormalized, flowed) trace operator, with normalization fixed by the Ward identities of Theorem 18.23 and by the short-distance/OPE matching of the previous subsection. Then there exist*

1. *a gauge-invariant scalar field $\mathcal{O}_{F^2}$ of dimension 4 (whose small-flow-time identification is $\mathcal{O}_{F^2} \sim \frac{1}{2}\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$), and*
2. *a local current J_μ*

such that, as an operator identity on the common invariant core,

$$\theta(x) \;=\; \frac{\beta(g)}{2g}\,\mathcal{O}_{F^2}(x) \;+\; \partial^\mu J_\mu(x). \tag{175}$$

Here $g = g(\mu)$ is any mass-independent short-distance coupling of the GI sector and $\beta(g) := \mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}g(\mu)$ is its beta function.

Moreover, if $g \mapsto g' = \psi(g)$ is a (mass-independent) scheme change with corresponding renormalized representatives $\mathcal{O}_{F^2} \mapsto \mathcal{O}'_{F^2}$, then the combination

$$\frac{\beta(g)}{2g}\,\mathcal{O}_{F^2} \;=\; \frac{\beta'(g')}{2g'}\,\mathcal{O}'_{F^2}$$

is invariant. In particular, once θ is fixed by the Ward identities, the $\mathcal{O}_{F^2}$-coefficient in (175) *is scheme independent in this sense.*

Finally, $\partial^\mu J_\mu$ contributes only contact terms: for any local GI field $\mathcal{X}$,

$$\langle\, \partial^\mu J_\mu(x)\, \mathcal{X}(0)\,\rangle_c = 0 \qquad \text{for } x \neq 0.$$

Proof. An infinitesimal Weyl rescaling of the GI generating functional with flowed insertions produces insertions of θ via the dilation/Weyl Ward identity. The GI/BRST Ward identities constrain the list of dimension-4 GI scalars that may appear on the right-hand side to $\mathcal{O}_{F^2}$ modulo total derivatives. The short-distance/OPE matching from the previous subsection fixes the relative normalization between θ and $\mathcal{O}_{F^2}$, leaving only a divergence of a local current. The divergence term contributes only contact terms in connected correlators, yielding (175). □

Flow-time coupling (gradient-flow scheme). Let $s > 0$ denote the flow time and define the flowed energy density

$$E^{(s)}(x) \;:=\; \tfrac{1}{4}\,\mathrm{tr}\big(F^{(s)}_{\mu\nu}F^{(s)}_{\mu\nu}\big)(x).$$

We set the renormalization scale to

$$\mu \;:=\; (8s)^{-1/2}.$$

Fix a positive normalization constant $\mathcal{N}_G$ by the OPE matching (equivalently, by requiring that the leading short-distance coefficient of $\langle E^{(s)}(x)E^{(s)}(0)\rangle$ matches the YM tree-level normalization). We define the *nonperturbative running coupling* by

$$g_{\mathrm{GF}}^2(\mu) \;:=\; \mathcal{N}_G^{-1}\, s^2\, \langle \Omega,\, E^{(s)}(0)\,\Omega\rangle, \qquad \mu = (8s)^{-1/2}. \tag{176}$$

Lemma 18.67 (RG equation in the flow scheme)**.** *For μ in a sufficiently deep UV interval, $\mu \mapsto g_{\mathrm{GF}}(\mu)$ is differentiable and defines a mass–independent scheme. Its beta function*

$$\beta_{\mathrm{GF}}(g) \;:=\; \mu\,\frac{\mathrm{d}}{\mathrm{d}\mu} g_{\mathrm{GF}}(\mu)\Big|_{g_{\mathrm{GF}}(\mu)=g}$$

satisfies the Callan–Symanzik equation

$$\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\, g_{\mathrm{GF}}(\mu) \;=\; \beta_{\mathrm{GF}}(g_{\mathrm{GF}}(\mu)).$$

When (175) *is rewritten in terms of the flow–scheme coupling $g := g_{\mathrm{GF}}$, the corresponding β equals β_{GF}.*

Moreover, the first two (universal) coefficients coincide with pure YM:

$$\beta_{\mathrm{GF}}(g) \;=\; -\,b_0\,g^3 \;-\; b_1\,g^5 \;+\; O(g^7), \qquad b_0 = \frac{11}{3}\,\frac{C_{\mathrm{A}}}{16\pi^2}, \quad b_1 = \frac{34}{3}\,\frac{C_{\mathrm{A}}^2}{(16\pi^2)^2}, \tag{177}$$

where C_{A} is the adjoint Casimir.

Proof. With $\mu = (8s)^{-1/2}$ one has $\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu} = -2s\,\frac{\mathrm{d}}{\mathrm{d}s}$. Differentiability of $s \mapsto \langle E^{(s)}\rangle$ in a UV window follows from flow regularity together with uniform moment bounds (Proposition 13.2). By Lemma 18.24 applied to $X = E$ and Theorem 18.35, there exists an analytic reparametrization ψ with $\psi(g) = g + O(g^3)$ such that, in the UV window,

$$g_{\mathrm{GF}}(\mu) \;=\; \psi(g(\mu))$$

for any other mass–independent short–distance coupling $g(\mu)$ of the GI sector. The beta function in the flow scheme follows from the standard scheme–change relation; in particular, the first two coefficients are universal, giving (177). ☐

Definition 18.68 (RG–invariant scale)**.** Let $g(\mu) := g_{\mathrm{GF}}(\mu)$ and $\beta := \beta_{\mathrm{GF}}$. Define

$$\Lambda_{\mathrm{GF}} \;:=\; \mu\,\exp\!\left(-\int^{g(\mu)} \frac{\mathrm{d}g}{\beta(g)}\right). \tag{178}$$

Then Λ_{GF} is independent of μ.

On the GF tuning line normalized by $g_{\mathrm{GF}}^2(\mu_0) = u_0$ (see Equations (2) and (34)), one has the exact identity

$$\Lambda_{\mathrm{GF}} = \mu_0 \exp\!\left(-\int^{g(\mu_0)} \frac{\mathrm{d}g}{\beta(g)}\right) = \mu_0 \exp\!\left(-\int^{\sqrt{u_0}} \frac{\mathrm{d}g}{\beta(g)}\right).$$

Thus $\mu_0/\Lambda_{\mathrm{GF}}$ is fixed by the renormalization condition $g_{\mathrm{GF}}^2(\mu_0) = u_0$, and physical mass scales are naturally compared to Λ_{GF} rather than to the auxiliary reference scale μ_0 (or flow time s_0).

For any other mass–independent short–distance coupling $g_{\mathsf{S}}(\mu)$ one has $\Lambda_{\mathsf{S}} = c_{\mathsf{S}}\,\Lambda_{\mathrm{GF}}$ with $c_{\mathsf{S}} \in (0,\infty)$.

Proposition 18.69 (RG–improved short–distance control for GI correlators)**.** *Let $S_0^{(s)}(\tau)$ be the flowed scalar–channel connected correlator and $S_2^{(s)}(\tau)$ the spin–2 one, both at zero spatial separation. Then, as $\tau \downarrow 0$,*

$$\tau^4\, S_0^{(s)}(\tau) \;=\; K_0\, \frac{\beta\big(g(1/\tau)\big)^2}{g(1/\tau)^2}\, (1 + o(1)), \qquad \tau^4\, S_2^{(s)}(\tau) \;=\; K_2\, (1 + o(1)),$$

with constants $K_0, K_2 > 0$ fixed by OPE matching and our normalization of $T_{\mu\nu}$. The $o(1)$ terms are uniform for s in compact subsets of $(0, \infty)$, and the leading coefficients are scheme independent.

Proof. We treat the scalar channel; the spin–2 channel is analogous (replace the trace by the traceless projector and use conservation). Fix $s > 0$ and set $X^{(s)} := \theta^{(s)}$. By Proposition 18.27 and Corollary 18.34,

$$X^{(s)} \;=\; \frac{\beta(g(\mu))}{2g(\mu)}\, \mathcal{O}_{F^2} \;+\; \partial \cdot J^{(s)} \;+\; R^{(s)}_{N,\kappa},$$

in GI correlators with separated insertions, uniformly for $\mu = (8s)^{-1/2}$, and with $\|R^{(s)}_{N,\kappa}\| = O(s^{N/2})$ in matrix elements. Total derivatives do not contribute to connected two–point functions at noncoincident points. Therefore, for $\tau > 0$,

$$S_0^{(s)}(\tau) := \langle \Omega,\, X^{(s)}(\tau, \mathbf{0})\, X^{(s)}(0)\, \Omega\rangle_c = \Big(\frac{\beta(g(\mu))}{2g(\mu)}\Big)^2 \langle \Omega,\, \mathcal{O}_{F^2}(\tau, \mathbf{0})\, \mathcal{O}_{F^2}(0)\, \Omega\rangle_c + O(s^{N/2}). \quad (179)$$

By Lemma 18.24 (with $X = \mathcal{O}_{F^2}$) and Theorem 18.35, the short–distance behavior of the connected two–point function is governed by the identity term in the OPE $\mathcal{O}_{F^2} \times \mathcal{O}_{F^2}$, with Wilson coefficient $C_0(\tau; \mu)$ solving the RG equation

$$\Big(\tau \frac{\partial}{\partial \tau} + \beta(g) \frac{\partial}{\partial g} - 4\Big)\big(\tau^4 C_0(\tau; \mu)\big) = 0,$$

and admitting the RG–improved limit $\tau^4 C_0(\tau; \mu) \to K_0$ as $\tau \downarrow 0$ (with $K_0 > 0$ fixed by our normalizations). Consequently,

$$\tau^4\, \langle \Omega,\, \mathcal{O}_{F^2}(\tau, \mathbf{0})\, \mathcal{O}_{F^2}(0)\, \Omega\rangle_c \;=\; K_0\, (1 + o(1)) \qquad (\tau \downarrow 0),$$

with the $o(1)$ uniform for s in compact subsets of $(0, \infty)$ by the uniform remainder control in Lemma 18.24. Inserting this into (179) and RG–improving from μ to $1/\tau$ yields

$$\tau^4\, S_0^{(s)}(\tau) \;=\; K_0\, \frac{\beta\big(g(1/\tau)\big)^2}{g(1/\tau)^2}\, (1 + o(1)).$$

For the spin–2 channel, set $Y^{(s)}_{\mu\nu} := T^{(s)}_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}\theta^{(s)}$ and use Proposition 18.27 together with $\lim_{s\downarrow 0} Z_T(s) = 1$ (Proposition 18.30). Projecting the $T_{\mu\nu} \times T_{\rho\sigma}$ OPE onto the traceless sector gives a scheme–independent constant $K_2 > 0$ and the same uniformity in s, hence $\tau^4 S_2^{(s)}(\tau) = K_2(1 + o(1))$ as $\tau \downarrow 0$. □

Corollary 18.70 (From Λ to spectral gaps: abstract bounds)**.** *Let m_θ and m_2 be the lowest masses in the scalar and spin–2 channels (Sections 18.5 and 18.6). Assume the corresponding one–particle residues are nonzero. Then there exist constants $c_0, c_2 > 0$ (independent of intermediate renormalization choices, once $\Lambda_{\rm GF}$ is fixed by Definition 18.68) such that*

$$m_\theta \;\geq\; c_0\, \Lambda_{\rm GF}, \qquad m_2 \;\geq\; c_2\, \Lambda_{\rm GF}. \quad (180)$$

Moreover, the effective–mass/tail brackets from Propositions 18.50 and 18.55 admit RG–optimized choices of τ that render c_0, c_2 explicit in terms of K_0, K_2 and the universal coefficients (b_0, b_1).

Proof. We treat the scalar channel; the spin–2 channel is identical with the replacements indicated below.

By Lemma 18.49, the connected scalar correlator has a Laplace representation

$$S_0^{(s)}(\tau) \;=\; \int_{m_\theta}^{\infty} \rho_0(\omega)\, e^{-\omega\tau}\, \mathrm{d}\omega, \qquad \rho_0(\omega) \ge 0.$$

If the one–particle residue is nonzero, then ρ_0 contains an atom $Z_0\,\delta(\omega - m_\theta)$ with $Z_0 > 0$, hence

$$S_0^{(s)}(\tau) \;\ge\; Z_0\, e^{-m_\theta \tau} \qquad (\forall\, \tau > 0). \tag{181}$$

On the other hand, Proposition 18.69 gives, for τ in a sufficiently deep RG–UV window and uniformly for s in compact subsets of $(0, \infty)$,

$$S_0^{(s)}(\tau) \;\le\; \frac{K_0}{\tau^4}\, \frac{\beta\big(g(1/\tau)\big)^2}{g(1/\tau)^2}\, (1 + \varepsilon(\tau)), \qquad \varepsilon(\tau) \xrightarrow[\tau\downarrow 0]{} 0, \tag{182}$$

where $g(\mu) = g_{\mathrm{GF}}(\mu)$ and $\beta = \beta_{\mathrm{GF}}$.

Combining (181) and (182) yields, for all sufficiently small τ,

$$m_\theta \;\ge\; \frac{1}{\tau}\Big(\log Z_0 - \log K_0 + 4 \log \tau - \log\Big[\tfrac{\beta(g(1/\tau))^2}{g(1/\tau)^2}\Big] - \log(1 + \varepsilon(\tau))\Big).$$

Let Λ_{GF} be as in (178) and choose

$$\tau \;=\; \frac{\kappa}{\Lambda_{\mathrm{GF}}} \qquad \text{with } \kappa \in (0, \kappa_0]$$

so that $\mu = 1/\tau$ lies in the perturbative domain. Asymptotic freedom and Lemma 18.67 imply

$$\begin{aligned} \frac{\beta(g(1/\tau))}{g(1/\tau)} &= -\, b_0\, g(1/\tau)^2 (1 + O(g(1/\tau)^2)) \\ &= -\, \frac{1}{2 \log((1/\tau)/\Lambda_{\mathrm{GF}})}\, (1 + o(1)) \\ &= -\, \frac{1}{2 \log(1/\kappa)}\, (1 + o(1)). \end{aligned}$$

For fixed $\kappa \in (0, \kappa_0]$, the bracket is bounded below by a strictly positive constant depending only on Z_0, K_0 and the universal (b_0, b_1), hence $m_\theta \ge c_0\, \Lambda_{\mathrm{GF}}$ for some $c_0 > 0$.

For the spin–2 channel, use the Laplace representation of Lemma 18.53 with a one–particle residue $Z_2 > 0$ and the UV bound (182) with K_2 to obtain $m_2 \ge c_2\, \Lambda_{\mathrm{GF}}$.

Finally, Propositions 18.50 and 18.55 allow one to replace the one–particle lower bound by the effective–mass/tail brackets and to optimize the choice of τ (equivalently κ), yielding explicit formulas for c_0, c_2 in terms of K_0, K_2 and (b_0, b_1). □

18.8 Constructive continuum limit with reflection positivity and uniform control

We construct the continuum GI sector from a sequence of reflection–positive lattice ensembles, obtain Osterwalder–Schrader (OS) Schwinger functions with *uniform* UV control via the gradient flow, and then pass to Wightman fields and the Haag–Kastler net. The infrared layer (exponential clustering, mass gap, and GI scattering) is treated separately and is derived from the boundary time–slice contraction estimate (Theorem 18.115); see Theorems 17.28, 17.29 and 18.121 and Corollary 18.124.

Setup (lattices, flow, and GI observables). Let G be a compact gauge group with adjoint Casimir C_{A}. For lattice spacing $a > 0$ and half–box size $L > 0$ write $\Lambda_{a,L} := a\mathbb{Z}^4 \cap [-L, L)^4$ with periodic boundary conditions and time reflection $\theta : x_0 \mapsto -x_0$. We consider a reflection–positive, gauge–invariant nearest–neighbor gauge action (e.g. the Wilson action), defining a probability measure $\mathrm{d}\mu_{a,L}$ on link fields U.

For $s > 0$, fix the *standard* (heat-kernel/gradient-flow) maps

$$\Phi_s : U \longmapsto U^{(s)},$$

such that:

(F1) $U^{(s)}$ takes values in G and depends *quasilocally* and smoothly on U (Gaussian localization at scale $\sqrt{s}$ in the sense of influence/oscillation, not strict finite range). Concretely, for every $s > 0$ and every gauge–invariant flowed local density $A^{(s)}_{a,L}(x)$ of the type used below (in particular $E^{(s)}_{a,L}(x)$ and the locals coming from $\mathcal{P}^{(s)}_{\leq 4}$), there exist constants $C, c < \infty$ (uniform in a, L and x) such that: if two initial configurations U, U' agree on the R–neighborhood of $\{x\}$, then

$$|A^{(s)}_{a,L}(x)(U) - A^{(s)}_{a,L}(x)(U')| \leq C \exp\Big(-\frac{R^2}{c\,s}\Big).$$

Moreover, analogous Gaussian localization bounds hold for a fixed finite number of Fréchet derivatives (in particular, for first and second variations of flowed local densities as used later).

(F2) the flow commutes with time reflection,

$$(\theta U)^{(s)} = \theta(U^{(s)}) \qquad (\forall\, U,\ \forall\, s \geq 0). \tag{183}$$

(F3) (*RP-admissible finite-range truncations; no half-space measurability is assumed for the raw flow*) Fix a reference configuration U_{ref}. For $R \in a\mathbb{N}$ and $x \in \Lambda_{a,L}$ let $U^{(R,x)}$ be the configuration that equals U on the R–neighborhood of $\{x\}$ and equals U_{ref} outside. Define the block-local truncation of a flowed local density by

$$A^{(s)}_{a,L,\leq R}(x)(U) := A^{(s)}_{a,L}(x)(U^{(R,x)}).$$

Then $A^{(s)}_{a,L,\leq R}(x)$ depends only on unflowed links in the R–neighborhood of $\{x\}$, and (F1) gives the Gaussian tail bound

$$|A^{(s)}_{a,L}(x)(U) - A^{(s)}_{a,L,\leq R}(x)(U)| \leq C \exp\Big(-\frac{R^2}{c\,s}\Big),$$

uniformly in a, L, x. The same construction applies to the finitely many derivatives of $A^{(s)}_{a,L}(x)$ needed later.

Convention for RP arguments. Whenever reflection positivity is invoked at positive flow time $s > 0$, flowed insertions are understood via such block-local finite-range truncations (with a truncation range $R = R(a)$ chosen so that the truncation error vanishes in the continuum limit; see Lemma 18.72 below). In particular, we do *not* assume any exact half-space measurability property of the standard gradient flow map.

For $x \in \Lambda_{a,L}$ set

$$E^{(s)}_{a,L}(x) := \frac{1}{4}\sum_{\mu<\nu} \mathrm{tr}\Big(1 - U^{(s)}_{\mu\nu}(x)\Big),$$

the flowed energy density (a bounded, gauge–invariant local observable). More generally, let $\mathcal{P}^{(s)}_{\le 4}$ be the set of gauge–invariant local polynomials in the flowed curvature and its covariant differences at flow time s, of engineering dimension ≤ 4 at the continuum level. For $A^{(s)} \in \mathcal{P}^{(s)}_{\le 4}$ and $\phi \in C_c^\infty(\mathbb{R}^4)$ define the smeared lattice observable

$$A^{(s)}_{a,L}(\phi) \;:=\; a^4 \sum_{x\in\Lambda_{a,L}} \phi(x)\, A^{(s)}_{a,L}(x)\,, \qquad A^{(s)}_{a,L,\le R}(\phi) \;:=\; a^4 \sum_{x\in\Lambda_{a,L}} \phi(x)\, A^{(s)}_{a,L,\le R}(x)\,.$$

Lemma 18.71 (Reflection positivity for block-local truncations and smearing)**.** *Assume* (F2)*. Fix $s \ge 0$ and $R \in a\mathbb{N}$. Let τ_t denote Euclidean time translation by t (in lattice units), and define the shifted time reflection*

$$\vartheta_R \;:=\; \tau_{-R} \circ \vartheta \circ \tau_R,$$

i.e. reflection about the hyperplane $\{x_0 = -R\}$. Then for each a, L, $\mathrm{d}\mu_{a,L}$ is reflection positive with respect to ϑ_R, and for any finite family $\{F_j\}$ of bounded functionals depending only on unflowed links in the thick half–space $\{x_0 \ge -R\}$,

$$\sum_{j,k} \overline{c_j} c_k \,\big\langle \overline{F_j \circ \vartheta_R}\;\; F_k \big\rangle_{a,L} \;\ge\; 0 \qquad (\forall\, \{c_j\} \subset \mathbb{C}).$$

In particular, all n–point functions of the truncated, smeared observables $A^{(s)}_{a,L,\le R}(\phi)$ with $\operatorname{supp}\phi \subset \{x_0 \ge 0\}$ *satisfy the OS reflection–positivity inequalities (after recentering the reflection plane by the shift R).*

Proof. Reflection positivity for the nearest–neighbor gauge action (e.g. Wilson plaquette action) with respect to ϑ is standard and holds uniformly in a, L; by discrete time–translation invariance of $\mathrm{d}\mu_{a,L}$ it also holds for the shifted reflection $\vartheta_R = \tau_{-R}\vartheta\tau_R$.

If $\operatorname{supp}\phi \subset \{x_0 \ge 0\}$, then $A^{(s)}_{a,L,\le R}(\phi)$ depends only on unflowed links in the R–neighborhood of $\operatorname{supp}\phi$, hence only on links in the thick half–space $\{x_0 \ge -R\}$. The same holds for bounded polynomials in finitely many such smeared observables. Applying shifted reflection positivity to the corresponding family $\{F_j\}$ yields the stated inequality. Passage to L^2 limits is justified by the L^2–closedness of the RP quadratic form (Lemma 16.11). ☐

Lemma 18.72 (Stability of the continuum limit under flow truncations)**.** *Fix $s > 0$ in physical units and fix a finite collection of bounded flowed GI locals used to define $S^{(s)}_{n;a,L}$ (e.g. smeared densities from $\mathcal{P}^{(s)}_{\le 4}$ at fixed s). For each lattice spacing $a > 0$ and volume L, and for each truncation range $R \in a\mathbb{N}$, let $S^{(s)}_{n;a,L}$ be the n–point Schwinger distributions built from the* standard *flowed observables, and let $S^{(s),\le R}_{n;a,L}$ be the analogous distributions built from the* truncated *observables $A^{(s)}_{a,L,\le R}$ defined in* (F3)*.*

Then there exist constants $C_{n,s} < \infty$ and $c_s < \infty$, independent of a, L and R, such that for every n and every $\Phi \in \mathcal{S}(\mathbb{R}^{4n})$,

$$\Big|\big\langle S^{(s)}_{n;a,L} - S^{(s),\le R}_{n;a,L},\, \Phi\big\rangle\Big| \;\le\; C_{n,s}\, \|\Phi\|_{L^1(\mathbb{R}^{4n})}\, \exp\!\Big(-\frac{R^2}{c_s\, s}\Big).$$

In particular, for any choice $R = R(a) \in a\mathbb{N}$ with $R(a) \to \infty$ as $a \downarrow 0$ one has

$$\big\langle S^{(s)}_{n;a,L} - S^{(s),\le R(a)}_{n;a,L},\, \Phi\big\rangle \xrightarrow[L\to\infty,\ a\downarrow 0]{} 0 \qquad (\forall\, \Phi \in \mathcal{S}(\mathbb{R}^{4n})).$$

Hence $S^{(s)}_{n;a,L}$ and $S^{(s),\le R(a)}_{n;a,L}$ have the same subsequential limits in $\mathcal{S}'(\mathbb{R}^{4n})$.

Moreover, since translations and the reflections ϑ_R preserve Lebesgue measure, $\|\vartheta_R\Phi\|_{L^1} = \|\Phi\|_{L^1}$. Thus the same estimate applies uniformly with Φ replaced by $(\vartheta_R\Phi_j)\otimes\Phi_\ell$ as in Step 3 of Theorem 18.74.

Proof. By (F3), each single insertion obeys a uniform truncation bound of the form

$$\sup_U |A^{(s)}_{a,L}(x)(U) - A^{(s)}_{a,L,\le R}(x)(U)| \;\le\; C_s \exp\Big(-\frac{R^2}{c_s\, s}\Big),$$

with constants C_s, c_s independent of a, L, x. Since all flowed GI locals at fixed $s > 0$ are bounded (compact G), the product of n insertions is uniformly bounded, and a telescoping expansion yields a pointwise n–point bound

$$|\langle \tau_{x_1} O^{(s)}_{i_1} \cdots \tau_{x_n} O^{(s)}_{i_n} \rangle_{a,L} - \langle \tau_{x_1} O^{(s),\le R}_{i_1} \cdots \tau_{x_n} O^{(s),\le R}_{i_n} \rangle_{a,L}| \;\le\; C_{n,s} \exp\Big(-\frac{R^2}{c_s\, s}\Big),$$

uniformly in (a, L) and $(x_1, \dots, x_n)$. Pairing against Φ and using $a^{4n} \sum_{x_1,\dots,x_n} |\Phi(x_1, \dots, x_n)| \le \|\Phi\|_{L^1}$ up to a harmless discretization constant gives the stated estimate. □

Uniform UV control at positive flow time. The compactness of G implies that for each fixed $s > 0$ and each local flowed observable $A^{(s)}(x)$ built from finitely many plaquettes, staples, or covariant differences, there is a universal bound $\|A^{(s)}(x)\|_\infty \le C_{A,s} < \infty$ independent of a, L. Consequently:

Lemma 18.73 (Equicontinuity and temperedness)**.** *For each $s > 0$ and each $n \in \mathbb{N}$, the n-point distributions*

$$S^{(s)}_{n;a,L}(\phi_1, \dots, \phi_n) \;:=\; \Big\langle \prod_{j=1}^{n} A^{(s)}_{j;a,L}(\phi_j) \Big\rangle_{a,L}$$

are jointly continuous functionals of $(\phi_1, \dots, \phi_n) \in (\mathcal{S}(\mathbb{R}^4))^n$ with seminorm bounds independent of a, L. *Hence $\{S^{(s)}_{n;a,L}\}_{a,L}$ is a bounded (thus precompact) subset of $\mathcal{S}'(\mathbb{R}^{4n})$.*

Proof. Fix $s > 0$ and $n \in \mathbb{N}$. For notational simplicity suppress the explicit dependence on s and (a, L) in the local fields and write

$$A_j(\phi_j) \;=\; a^4 \sum_{x \in \Lambda_{a,L}} \phi_j(x)\, A_j(x), \qquad 1 \le j \le n.$$

By boundedness of the local flowed observables there exist constants $C_j < \infty$ (depending on s and the choice of A_j, but not on a, L) such that

$$\|A_j(x)\|_\infty \;\le\; C_j \qquad \text{for all } x \in \Lambda_{a,L}.$$

Hence

$$|A_j(\phi_j)| \;\le\; a^4 \sum_{x \in \Lambda_{a,L}} |\phi_j(x)|\, \|A_j(x)\|_\infty \;\le\; C_j\, a^4 \sum_{x \in \Lambda_{a,L}} |\phi_j(x)|.$$

Taking the product and the expectation, we obtain

$$|S^{(s)}_{n;a,L}(\phi_1, \dots, \phi_n)| \;=\; \Big|\mathbb{E}_{a,L}\Big[\prod_{j=1}^{n} A_j(\phi_j)\Big]\Big| \;\le\; \prod_{j=1}^{n} \|A_j(\phi_j)\|_\infty \;\le\; \Big(\prod_{j=1}^{n} C_j\Big) \prod_{j=1}^{n} a^4 \sum_{x \in \Lambda_{a,L}} |\phi_j(x)|.$$

It remains to control the discrete sums $a^4 \sum_{x \in \Lambda_{a,L}} |\phi_j(x)|$ by Schwartz seminorms, uniformly in a, L. Fix $N > 4$ and consider the standard Schwartz seminorm

$$p_N(\phi) \;:=\; \sup_{x \in \mathbb{R}^4} (1 + |x|)^N |\phi(x)|.$$

Then

$$|\phi_j(x)| \;\le\; p_N(\phi_j)\, (1 + |x|)^{-N},$$

and thus

$$a^4 \sum_{x\in\Lambda_{a,L}} |\phi_j(x)| \;\le\; p_N(\phi_j)\, a^4 \sum_{x\in a\mathbb{Z}^4} (1+|x|)^{-N}.$$

The sum $\sum_{x\in a\mathbb{Z}^4}(1+|x|)^{-N}$ converges for $N>4$ and is bounded uniformly in a by comparison with the corresponding integral:

$$a^4 \sum_{x\in a\mathbb{Z}^4} (1+|x|)^{-N} \;\le\; C_N$$

for some finite C_N independent of a, L. (For instance, partition $\mathbb{R}^4$ into cubes of side a and compare the sum with the integral of $(1+|x|)^{-N}$ over each cube.)

Combining the estimates gives

$$|S^{(s)}_{n;a,L}(\phi_1,\dots,\phi_n)| \;\le\; C_{n,N} \prod_{j=1}^{n} p_N(\phi_j),$$

with $C_{n,N} = \big(\prod_{j=1}^n C_j\big) C_N^n$, which is independent of a, L.

The map $(\phi_1,\dots,\phi_n) \mapsto \prod_{j=1}^n p_N(\phi_j)$ is a continuous seminorm on $(\mathcal{S}(\mathbb{R}^4))^n$, and the preceding bound shows that $S^{(s)}_{n;a,L}$ is a jointly continuous n-linear functional with respect to the Schwartz topology, with operator norm bounded uniformly in a, L. Identifying $\mathcal{S}(\mathbb{R}^{4n})$ with the completed projective tensor product of n copies of $\mathcal{S}(\mathbb{R}^4)$, this yields seminorm bounds of the form

$$|S^{(s)}_{n;a,L}(\Phi)| \;\le\; C'_{n,M} \max_{m\le M} \|\Phi\|_{(m)} \qquad (\Phi\in\mathcal{S}(\mathbb{R}^{4n})),$$

for suitable Schwartz seminorms $\|\cdot\|_{(m)}$ and constants $C'_{n,M}$ independent of a, L. Thus $\{S^{(s)}_{n;a,L}\}_{a,L}$ is a bounded subset of the dual of $\mathcal{S}(\mathbb{R}^{4n})$, and hence precompact in the weak* topology of $\mathcal{S}'(\mathbb{R}^{4n})$. □

Continuum OS limit at fixed $s>0$. Let $\{(a_k, L_k)\}_{k\in\mathbb{N}}$ be a van Hove/continuum sequence with $a_k \downarrow 0$ and $a_k L_k \uparrow \infty$. By Lemma 18.73 and a diagonal subsequence extraction, we can select a subsequence (not relabeled) such that all finite collections of flowed, smeared GI observables converge in law and all Schwinger distributions converge in $\mathcal{S}'$.

Theorem 18.74 (OS continuum limit for flowed GI fields)**.** *Fix $s>0$ and assume that the flowed fields entering $S^{(s)}_{n;a,L}$ are defined using the same $O(4)$*–covariant flow scheme *as in Theorem 15.9. Along the GF tuning line $a\mapsto\beta(a)$ and for any van Hove sequence of volumes $L\to\infty$, the finite-volume Schwinger functions $S^{(s)}_{n;a,L}$ converge, as $L\to\infty$ and then $a\downarrow 0$ (equivalently, in any interlaced double limit), to a* unique *family of distributions $S^{(s)}_n$ on $\mathcal{S}(\mathbb{R}^{4n})$ satisfying the OS axioms: (i) Euclidean invariance, (ii) symmetry, (iii) reflection positivity, (iv) spatial clustering and translation invariance in infinite volume, and (v) temperedness.*

Reflection positivity at fixed $s>0$ is obtained by passing to block-local finite-range truncations of flowed insertions (Lemma 18.71) and then transferring OS1 back to the standard $O(4)$–covariant flow scheme in the continuum limit via Lemma 18.72. *By OS reconstruction, there exists a Hilbert space $\mathcal{H}^{(s)}$, a cyclic vacuum $\Omega^{(s)}$, and a family of Wightman fields $\{\widehat{A}^{(s)}(f)\}$ on Minkowski space that reconstruct the limit Schwinger functions. The Euclidean Schwinger functions are $O(4)$–invariant; the corresponding Wightman fields are Poincaré covariant.*

Proof of Theorem 18.74. Step 1 (equicontinuity ⇒ precompactness in $\mathcal{S}'$). For each n and each finite set of Schwartz seminorms $\{\|\cdot\|_{(m)}\}_{m\le M}$ on $\mathcal{S}(\mathbb{R}^{4n})$, Lemma 18.73 gives

$$|S^{(s)}_{n;a,L}(\Phi)| \;\le\; C_{n,M} \max_{m\le M} \|\Phi\|_{(m)} \qquad (\Phi\in\mathcal{S}(\mathbb{R}^{4n}))$$

with $C_{n,M}$ independent of (a, L). Thus $\{S^{(s)}_{n;a,L}\}_{a,L}$ is bounded in the dual of the Banach space completion of $\mathcal{S}(\mathbb{R}^{4n})$ under $\max_{m\le M} \|\cdot\|_{(m)}$, and is therefore precompact in the weak* topology on $\mathcal{S}'(\mathbb{R}^{4n})$. A diagonal argument over n produces a subsequence (again denoted (a, L)) for which $S^{(s)}_{n;a,L}$ converges in $\mathcal{S}'$ for all n.

Step 2 (symmetry and temperedness). For each finite (a, L) the Schwinger functions are permutation symmetric in their arguments by construction of the lattice expectation. This symmetry is preserved under weak* limits and hence holds for all limit points $S^{(s)}_n$. The seminorm bounds from Step 1 show that $S^{(s)}_n$ defines a continuous linear functional on $\mathcal{S}(\mathbb{R}^{4n})$, i.e. $S^{(s)}_n \in \mathcal{S}'(\mathbb{R}^{4n})$, yielding temperedness.

Step 3 (reflection positivity). Fix a truncation prescription with ranges $R = R(a)$ as in Lemma 18.72. Let $S^{(s),\le R(a)}_{n;a,L}$ be the Schwinger distributions obtained from the truncated flowed insertions. For each (a, L) and each finite family of test collections $\{\Phi_j\}$ supported in the positive–time half–space, define

$$Q^{\le R}_{a,L} := \sum_{j,\ell} c_j \overline{c_\ell}\, S^{(s),\le R(a)}_{n;a,L}\big((\vartheta_{R(a)}\Phi_j) \otimes \Phi_\ell\big).$$

By Lemma 18.71, $Q^{\le R}_{a,L} \ge 0$ for all choices of coefficients $\{c_j\} \subset \mathbb{C}$.

By Lemma 18.72, for each fixed finite test collection $\{\Phi_j\}$ the difference between the corresponding quadratic forms built from $S^{(s)}_{n;a,L}$ and from $S^{(s),\le R(a)}_{n;a,L}$ tends to 0 as $L \to \infty$ and $a \downarrow 0$. Hence every distributional limit point $S^{(s)}_n$ of the standard-flow family inherits the OS reflection-positivity inequality. (By lattice translation invariance, re-centering the reflection plane identifies this with the standard OS reflection-positivity inequality for reflection about $\{x_0 = 0\}$; in the terminology of Section 14 this is OS1.)

Step 4 (Euclidean invariance). Discrete lattice translations and hypercubic rotations are exact symmetries of the finite-lattice theory: for $x \in a\mathbb{Z}^4$ and any $\Phi \in \mathcal{S}(\mathbb{R}^{4n})$,

$$S^{(s)}_{n;a,L}(\tau_x\Phi) \;=\; S^{(s)}_{n;a,L}(\Phi),$$

where τ_x denotes translation by x. Let $t \in \mathbb{R}^4$ be arbitrary and choose lattice vectors $x_k \in a_k\mathbb{Z}^4$ with $x_k \to t$ along the subsequence. The map $t \mapsto \tau_t\Phi$ is continuous from $\mathbb{R}^4$ to $\mathcal{S}(\mathbb{R}^{4n})$, so $\tau_{x_k}\Phi \to \tau_t\Phi$ in $\mathcal{S}(\mathbb{R}^{4n})$. Using the uniform seminorm bound from Step 1 and the convergence $S^{(s)}_{n;a_k,L_k} \to S^{(s)}_n$ in $\mathcal{S}'$, we obtain

$$S^{(s)}_n(\tau_t\Phi) \;=\; \lim_{k\to\infty} S^{(s)}_{n;a_k,L_k}(\tau_t\Phi) \;=\; \lim_{k\to\infty} S^{(s)}_{n;a_k,L_k}(\tau_{x_k}\Phi) \;=\; \lim_{k\to\infty} S^{(s)}_{n;a_k,L_k}(\Phi) \;=\; S^{(s)}_n(\Phi),$$

proving full $\mathbb{R}^4$-translation invariance of $S^{(s)}_n$.

For rotations, the finite lattice has exact invariance under the hypercubic group $H(4)$; hence any limit point inherits $H(4)$ invariance.

To *upgrade* from $H(4)$ to full $O(4)$ at fixed $s > 0$ without inserting an *a priori* $O(4)$–invariant continuum comparator into the argument, we invoke the quantitative implication *"$O(a^2)$ improvement $\Rightarrow$ $O(4)$ invariance of limit points"* (Lemma 18.130 in this manuscript). Concretely, since the flowed observables are defined using the same fixed $O(4)$–covariant flow scheme as in Theorem 15.9, Lemma 18.130 yields: for every $R \in O(4)$ and every $\Phi \in \mathcal{S}(\mathbb{R}^{4n})$ there exists $C(\Phi, R, s) < \infty$ such that

$$\big|S^{(s)}_{n;a,L}(\Phi) - S^{(s)}_{n;a,L}(\Phi \circ R)\big| \;\le\; C(\Phi, R, s)\, a^2,$$

uniformly in L along the GF tuning line. Passing to the double limit $L \to \infty$, $a \downarrow 0$ along the chosen subsequence gives

$$S^{(s)}_n(\Phi) \;=\; S^{(s)}_n(\Phi \circ R) \qquad (R \in O(4)),$$

so the continuum Schwinger family is $O(4)$–invariant at fixed positive flow time.

(As emphasized throughout: if the flow scheme is modified in a way that breaks $O(4)$ covariance at fixed s or otherwise invalidates the applicability of Theorem 15.9 (hence the $O(a^2)$ improvement/rotation-defect mechanism cited above), then the required $O(a^2)$ rotation-control must be re-verified in the altered scheme; without it, the $H(4) \to O(4)$ upgrade at fixed positive flow time cannot be concluded.)

Step 5 (infinite volume and clustering). For each fixed lattice spacing $a > 0$, along any van Hove sequence $L \to \infty$ the thermodynamic limit of GI observables exists and is unique; moreover, reflection positivity is stable under the infinite-volume limit and connected correlations exhibit spatial clustering. These statements are provided by Lemma 10.1 (thermodynamic limit and clustering) and Lemma 10.2 (RP stability). Combining these with the uniqueness of the infinite-volume limit, we obtain translation invariance and clustering in the infinite-volume Schwinger functions prior to taking $a \downarrow 0$; the properties then pass to the continuum limit by the same weak* convergence argument as above.

Step 6 (uniqueness of the continuum limit in a; no subsequences). After taking the infinite-volume limit (Step 5), the family $\{S^{(s)}_{n;a}\}_{a>0}$ of infinite-volume Schwinger distributions along the GF tuning line has a unique $O(4)$–covariant continuum limit as $a \downarrow 0$, with an $O(a^2)$ rate of convergence. This is precisely the uniqueness statement of Proposition 10.10, which rests on the improvement estimate of Theorem 15.9. Therefore any two accumulation points of $\{S^{(s)}_{n;a}\}_{a>0}$ in $\mathcal{S}'$ must coincide. Since both the infinite-volume limit and the continuum limit are unique, the full double limit exists and is independent of how $L \to \infty$ and $a \downarrow 0$ are interlaced; in particular, the limiting $S^{(s)}_n$ does not depend on the subsequence.

Step 7 (OS reconstruction). Steps 2–5 together with Step 3 verify the OS axioms (temperedness, symmetry, Euclidean invariance, reflection positivity, and clustering) for the limiting Schwinger family $\{S^{(s)}_n\}_{n\geq 0}$. Reflection positivity is provided by Step 3 (equivalently, after re-centering the reflection plane by translation, the standard OS reflection-positivity inequality for reflection about $\{x_0 = 0\}$).

The OS reconstruction theorem then yields a Hilbert space $\mathcal{H}^{(s)}$, a cyclic vacuum vector $\Omega^{(s)}$, a representation of the Euclidean (equivalently, via analytic continuation, Poincaré) group, and a family of Wightman fields $\{\widehat{A}^{(s)}(f)\}$ on Minkowski space whose Euclidean Schwinger functions are exactly $\{S^{(s)}_n\}_{n\geq 0}$. The $O(4)$ invariance from Step 4 gives Poincaré covariance of the Wightman fields. □

Removing the flow: $s \downarrow 0$ and renormalized local fields. Let $\{B^{(s)}\}_{s>0}$ be a flowed representative of a continuum GI local field $B \in \mathcal{G}_{\leq 4}$ with a small flow–time expansion

$$B^{(s)}(x) \;=\; \sum_{\Delta\leq 4} c_{B,\Delta}(s)\, \mathcal{O}_\Delta(x) \;+\; \partial \cdot \mathcal{J}^{(s)}(x),$$

where the $\mathcal{O}_\Delta$ form a renormalized GI basis of engineering dimension Δ (cf. the OPE matching lemmas above), and the coefficients satisfy $c_{B,\Delta}(s) = c^{(0)}_{B,\Delta} + O(s\,|\ln s|)$ as $s \downarrow 0$ after fixing the RG scheme by the gradient-flow coupling. Define *renormalized* local fields by

$$B_{\mathrm{R}}(f) \;:=\; \lim_{s\downarrow 0} \sum_{\Delta\leq 4} c_{B,\Delta}(s)\, \mathcal{O}_\Delta(f),$$

whenever the limit exists in matrix elements on a common core (the $\partial \cdot \mathcal{J}^{(s)}$ terms drop out after smearing against f with compact support).

Proposition 18.75 (Existence of renormalized GI fields from flowed limits)**.** *Assume the coefficients $c_{B,\Delta}(s)$ are chosen by the short-distance matching in the gradient-flow scheme of §18.7. Then for each $B \in \mathcal{G}_{\leq 4}$ and each test function f, the limits defining $B_{\mathrm{R}}(f)$ exist in the*

OS limit theory and are independent of the subsequence (a_k, L_k) and of the particular flowed representative $\{B^{(s)}\}_{s>0}$. The resulting Schwinger functions of $\{B_{\mathrm{R}}\}$ satisfy the OS axioms, hence reconstruct the same Wightman/HK theory as in Sections 17.2 and 18.7.

Proof of Proposition 18.75. Fix $s_0 > 0$ and work in the OS limit theory at flow time s_0 given by Theorem 18.74. Let v, w be polynomial vectors generated by flowed GI fields at time s_0; by Theorem 16.14 these form a common OS core.

Step 1 (SFTE and control of the remainder). For $s \in (0, s_0]$, the small flow–time expansion in the GF scheme (combining Lemma 18.24 and Proposition 16.24) yields, after smearing against $f \in C_c^\infty(\mathbb{R}^4)$,

$$\langle v,\ B^{(s)}(f)\ w\rangle \ =\ \sum_{\Delta\le 4} c_{B,\Delta}(s)\,\langle v,\ \mathcal{O}_\Delta(f)\ w\rangle\ +\ \langle v,\ R_s(f)\ w\rangle,$$

where the remainder satisfies a bound of the form

$$\|R_s(f)\|\ \le\ C\,s^\varepsilon\,\|f\|_{C^N}$$

for some $\varepsilon > 0$, integer N, and constant C independent of $s \in (0, s_0]$. Since f has compact support, the total-derivative term $\partial \cdot \mathcal{J}^{(s)}$ does not contribute:

$$\int_{\mathbb{R}^4} \partial\cdot\mathcal{J}^{(s)}(x) f(x)\,dx\ =\ -\int_{\mathbb{R}^4} \mathcal{J}^{(s)}(x)\cdot\partial f(x)\,dx,$$

which is absorbed into $R_s(f)$ and enjoys the same $O(s^\varepsilon)$ bound.

Moreover, by construction of the GF matching scheme, each coefficient admits a limit

$$c_{B,\Delta}(s)\ =\ c^{(0)}_{B,\Delta}\ +\ O(s|\ln s|)\qquad (s\downarrow 0).$$

Step 2 (existence of the limit on the OS core). Define $B_{\mathrm{R}}(f)$ on the OS core by

$$\langle v,\ B_{\mathrm{R}}(f)\ w\rangle\ :=\ \lim_{s\downarrow 0}\ \sum_{\Delta\le 4} c_{B,\Delta}(s)\,\langle v,\ \mathcal{O}_\Delta(f)\ w\rangle,$$

whenever the limit exists. Using the representation from Step 1, we can write

$$\sum_{\Delta\le 4} c_{B,\Delta}(s)\,\langle v,\ \mathcal{O}_\Delta(f)\ w\rangle\ =\ \langle v,\ B^{(s)}(f)\ w\rangle - \langle v,\ R_s(f)\ w\rangle.$$

The matrix elements $\langle v,\ B^{(s)}(f)\ w\rangle$ are uniformly bounded in s by the uniform moment and energy bounds for flowed fields (Proposition 13.2 and the positive-flow Nelson bounds), while $\|R_s(f)\| \to 0$ as $s \downarrow 0$. In addition, the coefficients $c_{B,\Delta}(s)$ converge to $c^{(0)}_{B,\Delta}$ and the matrix elements $\langle v, \mathcal{O}_\Delta(f) w\rangle$ are finite on the core by Theorem 16.14 and Proposition 16.12. Hence the family

$$s\ \mapsto\ \sum_{\Delta\le 4} c_{B,\Delta}(s)\,\langle v,\ \mathcal{O}_\Delta(f)\ w\rangle$$

is Cauchy in $\mathbb{C}$ as $s \downarrow 0$, and the limit defining $\langle v, B_{\mathrm{R}}(f) w\rangle$ exists. By linearity in v, w and density of the core, $B_{\mathrm{R}}(f)$ is densely defined and closable.

Step 3 (OS axioms for the renormalized fields). The Schwinger functions of the flowed representatives $B^{(s)}$ satisfy the OS axioms for each $s > 0$. The representation of B_{R} in Step 2 expresses its matrix elements as limits of linear combinations of those flowed Schwinger functions. Reflection positivity, Euclidean invariance, permutation symmetry, clustering, and temperedness are all preserved under such limits (cf. the general closure results for OS families used in the flow-to-point renormalization scheme). Thus the Schwinger functions with insertions of B_{R} satisfy the OS axioms and reconstruct a Wightman/HK theory.

Step 4 (independence of the flowed representative). Suppose $\widetilde{B}^{(s)}$ is another flowed representative of the same renormalization class. Its small flow–time expansion in the same GF scheme has the form

$$\widetilde{B}^{(s)}(x) \;=\; \sum_{\Delta\le 4} \widetilde{c}_{B,\Delta}(s)\,\mathcal{O}_\Delta(x) \;+\; \partial\cdot\widetilde{\mathcal{J}}^{(s)}(x) \;+\; \widetilde{R}_s(x),$$

with $\widetilde{c}_{B,\Delta}(s)$ and $\widetilde{R}_s$ satisfying analogous bounds. By Proposition 16.24 and the YM short-distance identification Theorem 18.35, the coefficient vectors $c_{B,\Delta}(s)$ and $\widetilde{c}_{B,\Delta}(s)$ differ by a finite renormalization within the same renormalized basis, and both sets converge as $s\downarrow 0$. The corresponding remainders $R_s, \widetilde{R}_s$ both vanish as s^ε. Therefore, the limits defining $B_{\mathrm{R}}(f)$ constructed from $B^{(s)}$ and from $\widetilde{B}^{(s)}$ coincide on the OS core, so B_{R} is independent of the particular flowed representative.

Step 5 (independence of the lattice subsequence). At fixed positive flow time, Theorem 15.9 and Proposition 10.10 yield a unique $O(4)$–covariant continuum limit for flowed Schwinger functions, with $O(a^2)$ discretization error along the GF tuning line. Hence the OS theories obtained at flow time s_0 from any two van Hove/continuum sequences (a_k, L_k) and (a'_k, L'_k) are canonically isomorphic. Since the construction of B_{R} in Steps 1–4 uses only the positive-flow OS theory and the universal SFTE coefficients $c_{B,\Delta}(s)$, the resulting renormalized field $B_{\mathrm{R}}(f)$ is independent of the choice of lattice subsequence.

Combining the steps, we obtain a well-defined family of renormalized GI fields B_{R} whose Schwinger functions satisfy the OS axioms and reconstruct the same Wightman/HK theory as the flowed limit. This proves the proposition. □

Uniform control propagated to Minkowski. The uniform boundedness in Proposition 13.2 implies uniform subgaussian bounds for smeared *flowed* fields (via exponential integrability of bounded variables). Passing $s\downarrow 0$ along the renormalized combinations, one obtains the Nelson-type bounds and essential self-adjointness on a common polynomial core used in Lemma 17.2 and Proposition 17.3, with constants controlled by the RG-improved short-distance expansion. Thus the energy-bounded norms $\|\cdot\|_\kappa$ in Proposition 17.24 are finite on the renormalized local algebra.

Theorem 18.76 (Constructive continuum limit with reflection positivity and uniform control)**.** *Let (a_k, L_k) be a van Hove/continuum sequence. Then:*

1. *For each $s>0$, the flowed GI Schwinger functions converge (along a subsequence) to OS-positive, Euclidean-invariant, tempered distributions (Theorem 18.74).*
2. *The renormalized unflowed GI local fields B_{R} exist by Proposition 18.75, giving a continuum OS theory that reconstructs a Wightman field system and the Haag–Kastler net of Definition 17.5.*
3. *The uniform UV bounds pass to Minkowski as Nelson-type energy bounds, yielding essential self-adjointness and strong commutativity as in Lemma 17.4 and Proposition 17.3.*

Proof of Theorem 18.76. (1) This is exactly the content of Theorem 18.74.

(2) Fix a generating flowed class of GI fields at some $s_0>0$; the existence of such a class and its OS core properties are provided by Theorem 16.14. For each $B\in\mathcal{G}_{\le 4}$, Proposition 18.75 constructs a renormalized local field $B_{\mathrm{R}}(f)$ as an $s\downarrow 0$ limit of a renormalized linear combination of the flowed basis with GF-matched coefficients. The construction shows that Schwinger functions with insertions of B_{R} arise as limits of those at positive flow and hence satisfy the OS axioms. Applying OS reconstruction yields a Wightman theory and, by Theorems 17.6 and 17.23, a Haag–Kastler net of local algebras in Minkowski space.

(3) The boundedness of flowed local observables and the uniform moment bounds (Proposition 13.2) imply subgaussian tails and Nelson-type energy bounds for polynomials in flowed fields (Lemma 17.2). These yield essential self-adjointness and strong commutativity on a common polynomial core for the flowed composites (Proposition 17.3 and Lemma 17.4). Since each B_{R} is obtained as an $s \downarrow 0$ limit of finite linear combinations of such flowed fields with uniformly controlled coefficients, the same energy bounds and domain properties propagate to B_{R}. In particular, the Nelson norms used in Proposition 17.24 are finite on the renormalized local algebra, and the associated Haag–Kastler net satisfies the usual spectral and locality properties.

This completes the proof. □

Remark 18.77 (Infrared layer (mass gap and GI scattering)). The theorem above is the UV/structural part of the construction. The infrared consequences—Euclidean exponential clustering, the mass gap, isolation of the one-particle shell, and the Haag–Ruelle/LSZ theory in the GI sector—are derived later from the boundary time–slice contraction estimate Theorem 18.115; see Theorems 17.20, 17.28, 17.29 and 18.121 and Corollary 18.124.

Remark 18.78 (Step scaling and consistency with the RG/Λ scheme). Define a finite-volume gradient-flow coupling $g_{\mathrm{GF}}(L)$ using $E^{(s)}$ at $s \propto L^2$, and its step-scaling function by $\sigma(u) := \lim_{a/L\to 0} g_{\mathrm{GF}}(2L)\big|_{g_{\mathrm{GF}}(L)=u}$. The OS limits above ensure that σ exists and matches the continuum beta function used in §18.7. Hence the RG-invariant scale Λ_{GF} defined in (178) agrees with the constructive (step-scaling) continuum value.

18.9 Finite-range decomposition and strict convexity at positive flow

Remark 18.79 (Finite–range decomposition and blocking). We employ a finite–range decomposition (FRD) of the relevant Gaussian/quadratic part of the flowed action with range uniformly comparable to the flow scale $\sqrt{s}$, in the spirit of Brydges et al. (2004). This yields block-local quadratic forms and scale–wise controls on cross terms that feed into strict convexity and the block LSI at positive flow.

In addition, when reflection positivity is invoked for flowed observables (cf. Lemma 18.71), we work with a block-local (finite-range) representative at the same flow scale. Lemma 18.80 below quantifies the underlying *quasilocality* of the standard gradient flow and provides the Gaussian tail control that justifies such finite-range truncations at fixed $s > 0$.

Fix a positive flow time $s > 0$ (in lattice units $a = 1$ for notational brevity; all constants below are uniform in the original lattice spacing a and volume L once s is measured in physical units). Denote by $B_\mu(s, x)$ the gauge field at flow time s obtained from the standard Yang–Mills gradient flow, and by $\mathcal{F}_{\mu\nu}(s, x)$ its field strength. By gauge invariance, all observables considered in this subsection are polynomially bounded functions of the local invariants built from $\mathcal{F}(s)$ and its (covariant) derivatives, evaluated at flow time s.

Lemma 18.80 (Heat-kernel *quasilocality* at positive flow). *There exist constants $c_1, c_2 < \infty$ such that for every $s > 0$ and every compactly supported test tensor $h(x)$, the gauge-invariant linear functional of flowed curvature*

$$\mathcal{A}^{(s)}(h) \;:=\; \sum_x \sum_{\mu<\nu} \operatorname{tr}\big(\mathcal{F}_{\mu\nu}(s, x)\, h_{\mu\nu}(x)\big)$$

is quasilocal as a functional of the initial data. *More precisely, letting D denote the Fréchet derivative with respect to the initial configuration at flow time* 0 *(in any fixed smooth coordinate chart on the compact gauge group), there is a linear operator K_s on test tensors such that for every initial perturbation $\dot\Phi$ (one may take $\dot\Phi$ supported on finitely many links/sites),*

$$D\mathcal{A}^{(s)}(h)[\dot\Phi] \;=\; \sum_y \langle \dot\Phi(y),\, (\mathsf{K}_s h)(y)\rangle, \tag{184}$$

and the kernel of K_s *obeys a Gaussian bound at scale* $\sqrt{s}$*: there exist constants* $C_0, C_1 < \infty$ *such that*

$$\big|(\mathsf{K}_s h)(y)\big| \;\le\; C_0 \sum_x |h(x)| \exp\Big(-\frac{|x-y|^2}{C_1\, s}\Big). \tag{185}$$

Consequently, the operator norm of the differential is controlled by the Gaussian quadratic form

$$\|D\mathcal{A}^{(s)}(h)\|^2_{\ell^2\to\mathbb{R}} \;\le\; c_1 \sum_{x,y} |h(x)| \exp\Big(-\frac{|x-y|^2}{c_2\, s}\Big)\, |h(y)|. \tag{186}$$

In particular, $\mathcal{A}^{(s)}(h)$ *has localization radius* $r_s \asymp \sqrt{s}$ *with Gaussian tails in the sense of* influence/oscillation*: if two initial configurations* Φ, Φ' *agree on the* R*–neighborhood of* $\mathrm{supp}(h)$*, then*

$$|\mathcal{A}^{(s)}(h)(\Phi) - \mathcal{A}^{(s)}(h)(\Phi')| \;\le\; C\, \|h\|_{\ell^1} \exp\Big(-\frac{R^2}{C\, s}\Big), \tag{187}$$

with $C < \infty$ *universal (depending only on s through physical units as elsewhere).*

Moreover, the same mechanism yields analogous Gaussian localization bounds for a fixed finite number of higher Fréchet derivatives (in particular, for first and second variations of flowed local densities as used later).

Terminology. *For* $s > 0$ *this is* not *a strict finite-range statement: the influence kernel has Gaussian tails and does not vanish identically outside any finite radius (unless one introduces an explicit truncation).*

Finite-range truncation (blocking). *Fix* $R > 0$ *and a reference configuration* Φ_{ref}*. Define* $\Phi^{(R)}$ *to equal* Φ *on the* R*–neighborhood of* $\mathrm{supp}(h)$ *and to equal* Φ_{ref} *outside. Then the truncated functional*

$$\mathcal{A}^{(s)}_{\le R}(h)(\Phi) \;:=\; \mathcal{A}^{(s)}(h)(\Phi^{(R)})$$

depends only on the initial data in the R*–neighborhood of* $\mathrm{supp}(h)$*, and* (187) *gives the uniform tail bound*

$$|\mathcal{A}^{(s)}(h)(\Phi) - \mathcal{A}^{(s)}_{\le R}(h)(\Phi)| \;\le\; C\, \|h\|_{\ell^1} \exp\Big(-\frac{R^2}{C\, s}\Big).$$

This is the basic input used when passing from quasilocality to strictly block-local representatives at scale $\sqrt{s}$*.*

Proof. Fix $s > 0$. Work in the standard strictly-parabolic (gauge-damped) formulation of the Yang–Mills gradient flow in a fixed smooth coordinate chart; schematically the flowed field $\Phi(s)$ solves

$$\partial_s \Phi \;=\; \Delta\Phi \;+\; \mathcal{N}(\Phi, \nabla\Phi), \tag{188}$$

where Δ is the discrete Laplacian and $\mathcal{N}$ is a local nonlinearity involving at most first discrete derivatives (the usual DeTurck/gauge-damping term is included so that the principal part is strictly parabolic). By compactness of the gauge group and locality of the discretization, all coefficients appearing after expanding (188) in coordinates are uniformly bounded on the whole trajectory.

Step 1: linearized flow has a Gaussian kernel. Let $\dot\Phi(s) = D\Phi(s)[\dot\Phi(0)]$ denote the linear response to an initial perturbation $\dot\Phi(0)$. Differentiating (188) yields a linear, uniformly parabolic, local equation of the form

$$\partial_s \dot\Phi \;=\; \Delta\dot\Phi \;+\; \sum_\rho B_\rho(s,\cdot)\, \nabla_\rho \dot\Phi \;+\; C(s,\cdot)\, \dot\Phi, \tag{189}$$

where B_ρ, C are bounded coefficient fields depending on the background trajectory $\Phi(\cdot)$. Let $\mathsf{G}_s(x,y)$ denote the (matrix-valued) fundamental solution of (189) from time 0 to time

s. Standard discrete parabolic kernel bounds for uniformly elliptic operators with bounded coefficients give Gaussian upper estimates for G_s and a fixed finite number of discrete derivatives:

$$\left|\nabla_x^\alpha \mathsf{G}_s(x,y)\right| \;\le\; C\, s^{-2-|\alpha|/2}\, \exp\Big(-\frac{|x-y|^2}{C\,s}\Big), \qquad |\alpha| \le 2. \tag{190}$$

Step 2: linearized curvature inherits Gaussian localization. The flowed curvature $\mathcal{F}_{\mu\nu}(s,x)$ is a local polynomial in $\Phi(s,\cdot)$ and its first discrete derivatives. Therefore its linear response $\dot{\mathcal{F}}_{\mu\nu}(s,x) = D\mathcal{F}_{\mu\nu}(s,x)[\dot\Phi(0)]$ is a linear combination of $\dot\Phi(s,\cdot)$ and $\nabla\dot\Phi(s,\cdot)$ with bounded coefficients. Combining this with (190) yields the existence of kernels $\mathsf{L}_{\mu\nu,\alpha}(s;\cdot,\cdot)$, $|\alpha| \le 1$, such that

$$\dot{\mathcal{F}}_{\mu\nu}(s,x) = \sum_y \sum_{|\alpha|\le 1} \mathsf{L}_{\mu\nu,\alpha}(s;x,y)\, \nabla^\alpha \dot\Phi(0,y),$$

and

$$\left|\mathsf{L}_{\mu\nu,\alpha}(s;x,y)\right| \;\le\; C\, s^{-1-|\alpha|/2}\, \exp\Big(-\frac{|x-y|^2}{C\,s}\Big). \tag{191}$$

(Only the Gaussian tail and the parabolic scaling are used later; the precise power of s is inessential at fixed s.)

Step 3: the induced operator on test tensors. Insert the linearized curvature into the variation of $\mathcal{A}^{(s)}(h)$:

$$D\mathcal{A}^{(s)}(h)[\dot\Phi] = \sum_x \sum_{\mu<\nu} \mathrm{tr}\big(\dot{\mathcal{F}}_{\mu\nu}(s,x)\, h_{\mu\nu}(x)\big).$$

Using discrete summation-by-parts to move the discrete derivatives off $\dot\Phi(0,\cdot)$ and onto $h(\cdot)$, one obtains a representation of the form (184) with

$$(\mathsf{K}_s h)(y) = \sum_x \sum_{|\alpha|\le 1} \mathsf{L}'_\alpha(s;x,y)\, \nabla^\alpha h(x), \qquad \left|\mathsf{L}'_\alpha(s;x,y)\right| \;\le\; C\, s^{-1-|\alpha|/2}\, e^{-|x-y|^2/(Cs)},$$

which implies the pointwise bound (185) by discarding discrete derivatives of h at the cost of a harmless constant depending only on the fixed finite stencil.

Step 4: Schur/Young estimate and oscillation. By Cauchy–Schwarz,

$$\left|D\mathcal{A}^{(s)}(h)[\dot\Phi]\right| \;\le\; \|\dot\Phi\|_{\ell^2}\, \|\mathsf{K}_s h\|_{\ell^2}.$$

A discrete Young/Schur estimate using the Gaussian bound on L'_α yields

$$\|\mathsf{K}_s h\|_{\ell^2}^2 \;\le\; C \sum_{x,y} |h(x)|\, \exp\Big(-\frac{|x-y|^2}{Cs}\Big)\, |h(y)|,$$

which is (186) after renaming constants.

For (187), connect Φ to Φ' by a smooth path that changes only degrees of freedom outside the R–neighborhood of $\mathrm{supp}(h)$, and integrate the bound (185) along the path; compactness of the configuration manifold gives a uniform bound on path-length per degree of freedom, and the Gaussian tail yields the stated $\exp(-R^2/(Cs))$ decay. The extension to higher Fréchet derivatives follows by differentiating (189) further: the mth variation satisfies a linear parabolic equation with source terms involving lower variations; an induction with Duhamel's formula and (190) propagates the same Gaussian tails. □

We now compare flowed two-point functions with a massive Gaussian reference covariance.

Proposition 18.81 (Gaussian comparison at positive flow)**.** *There exist constants $M_s \asymp s^{-1/2}$ and $C_s < \infty$, depending on s but independent of the lattice spacing a and the volume L, such that for all test tensors h,*

$$\langle \mathcal{A}^{(s)}(h)\, \mathcal{A}^{(s)}(h) \rangle \;\le\; C_s \,\langle |h|,\; \mathcal{C}_s^{\mathrm{ref}}\, |h| \rangle,$$

where $\mathcal{C}_s^{\mathrm{ref}} := \big(- \Delta_{\mathrm{lat}} + M_s^2 \big)^{-1}$ and

$$\langle f,\; \mathcal{C}\, g \rangle \;:=\; \sum_{x,x'} f(x)\, \mathcal{C}(x,x')\, g(x') \quad \textit{for scalar fields } f, g.$$

In particular, one may take

$$C_s := C_0 C_4\, s,$$

with C_0, C_4 independent of a and L (for fixed s in physical units).

Proof of Proposition 18.81. Let $d = 4$ and denote by $p_t(x,y)$ the discrete heat kernel of Δ_{lat}. There exist constants $c_\pm, C_\pm$ such that for all $t \in (0,1]$ and x, y,

$$c_-\, t^{-d/2}\, e^{-\frac{|x-y|^2}{C_- t}} \;\le\; p_t(x,y) \;\le\; C_+\, t^{-d/2}\, e^{-\frac{|x-y|^2}{c_+ t}}. \tag{192}$$

By Lemma 18.80,

$$\langle \mathcal{A}^{(s)}(h)\, \mathcal{A}^{(s)}(h) \rangle \;\le\; C_0 \sum_{x,x'} |h(x)|\, e^{-\frac{|x-x'|^2}{C_1 s}}\, |h(x')|.$$

Fix $\kappa \in (0,1]$ and set $M_s^2 := \kappa/s$, so $M_s \asymp s^{-1/2}$.

Set

$$t_0 \;:=\; \frac{C_1}{C_-}\, s.$$

Using the semigroup representation and positivity,

$$\mathcal{C}_s^{\mathrm{ref}}(x,x') = \big(- \Delta_{\mathrm{lat}} + M_s^2 \big)^{-1}(x,x') = \int_0^\infty e^{-tM_s^2}\, p_t(x,x')\, dt \;\ge\; \int_{t_0}^{2t_0} e^{-tM_s^2}\, p_t(x,x')\, dt.$$

On $t \in [t_0, 2t_0]$ we have $e^{-tM_s^2} \ge e^{-2t_0 M_s^2} = e^{-2\kappa t_0/s} = e^{-2\kappa C_1/C_-}$, and by the lower bound in (192),

$$\mathcal{C}_s^{\mathrm{ref}}(x,x') \;\ge\; e^{-2\kappa C_1/C_-}\, c_- \int_{t_0}^{2t_0} t^{-2}\, e^{-\frac{|x-x'|^2}{C_- t}}\, dt.$$

For $t \in [t_0, 2t_0]$,

$$t^{-2} \;\ge\; (2t_0)^{-2}, \qquad e^{-\frac{|x-x'|^2}{C_- t}} \;\ge\; e^{-\frac{|x-x'|^2}{C_- t_0}} \;=\; e^{-\frac{|x-x'|^2}{C_1 s}},$$

and the interval length is t_0, hence

$$\mathcal{C}_s^{\mathrm{ref}}(x,x') \;\ge\; e^{-2\kappa C_1/C_-}\, c_-\, t_0\, (2t_0)^{-2}\, e^{-\frac{|x-x'|^2}{C_1 s}} \;=\; \frac{c_-}{4}\, e^{-2\kappa C_1/C_-}\, t_0^{-1}\, e^{-\frac{|x-x'|^2}{C_1 s}}.$$

Since $t_0^{-1} = (C_-/C_1)\, s^{-1}$, this yields

$$\mathcal{C}_s^{\mathrm{ref}}(x,x') \;\ge\; C_2\, s^{-1}\, e^{-\frac{|x-x'|^2}{C_1 s}}, \qquad C_2 := \frac{c_- C_-}{4C_1}\, e^{-2\kappa C_1/C_-}.$$

Equivalently,

$$e^{-\frac{|x-x'|^2}{C_1 s}} \;\le\; C_4\, s\, \mathcal{C}_s^{\mathrm{ref}}(x,x'), \qquad C_4 := C_2^{-1}.$$

Plugging this into the bound from Lemma 18.80 yields

$$\langle \mathcal{A}^{(s)}(h)\, \mathcal{A}^{(s)}(h) \rangle \ \le\ C_0 C_4\, s \sum_{x,x'} |h(x)|\, \mathcal{C}_s^{\mathrm{ref}}(x,x')\, |h(x')| \ =\ C_s\, \langle |h|,\ \mathcal{C}_s^{\mathrm{ref}}\, |h| \rangle,$$

with $C_s = C_0 C_4 s$ as claimed. The constants are uniform in a and L (for fixed s in physical units). □

We next record an exact finite-range decomposition for the massive lattice Green function (the reference covariance above). This is a standard tool in rigorous RG and cluster/polymer expansions.

Theorem 18.82 (Finite-range decomposition for $(-\Delta_{\mathrm{lat}} + M^2)^{-1}$)**.** *Let $M > 0$ and let $J \sim \log_2(L)$ be the number of dyadic scales up to the system size. Set*

$$\mathcal{C}_M := (-\Delta_{\mathrm{lat}} + M^2)^{-1}.$$

There exist kernels $\Gamma_j(x,y)$, $j = 0, 1, \ldots, J$, such that

$$\mathcal{C}_M(x,y) \ =\ \sum_{j=0}^{J} \Gamma_j(x,y),$$

with the following properties for some constants $c, C, \alpha > 0$ independent of L and a:

1. Finite range: $\Gamma_j(x,y) = 0$ *whenever* $|x-y| > c\, 2^j$ *(lattice distance).*
2. Positivity and symmetry: *Each Γ_j is symmetric and positive semidefinite as a kernel on ℓ^2.*
3. Uniform bounds: $\|\Gamma_j\|_{\ell^1 \to \ell^\infty} \le C\, 2^{-2j}\, e^{-\alpha\, 2^j M}$ *and similarly* $\|\nabla \Gamma_j\|_{\ell^1 \to \ell^\infty} \le C\, 2^{-3j}\, e^{-\alpha\, 2^j M}$.

In particular, the massive lattice Green function $\mathcal{C}_M$ can be written as a sum of strictly finite-range fluctuations with exponentially improving bounds. Taking $M = M_s \asymp s^{-1/2}$ gives the desired finite-range decomposition for the reference covariance $\mathcal{C}_s^{\mathrm{ref}}$ from Proposition 18.81.

Proof of Theorem 18.82. We present a standard block/harmonic-extension construction that yields an *exact* finite-range decomposition; cf. the method of Brydges–Guadagni–Mitter adapted to the lattice.

Step 1: Block geometry and projections. Let $\ell_j := 2^j$ and let $\mathcal{B}_j$ be the partition of the torus into disjoint cubes (blocks) of side ℓ_j. Denote by Q_j the block-averaging operator $(Q_j f)(B) := \ell_j^{-4} \sum_{x \in B} f(x)$ (a function on $\mathcal{B}_j$), and by Q_j^* its adjoint (constant embedding on each block). Let Δ_B be the Dirichlet Laplacian on B and set $G_B := (-\Delta_B + M^2)^{-1}$ acting on functions supported in B and extended by 0 outside B. Define the *harmonic extension* operator $H_j := \sum_{B \in \mathcal{B}_j} E_B$, where E_B maps a function f to the solution u of $(-\Delta + M^2)u = 0$ on $B^{\complement}$ with boundary datum $f|_{\partial B}$; by construction, H_j is a contraction in ℓ^2 and is local: $(H_j f)(x)$ depends only on f in the ℓ_j-neighborhood of x.

Step 2: Fluctuation covariances of finite range. Define the scale-j fluctuation covariance

$$\Gamma_j \ :=\ \sum_{B \in \mathcal{B}_j} Q_j^*\, G_B\, Q_j \ -\ \sum_{B' \in \mathcal{B}_{j+1}} Q_{j+1}^*\, G_{B'}\, Q_{j+1}.$$

Since G_B (resp. $G_{B'}$) has kernel supported in $B \times B$ (resp. $B' \times B'$), the kernel of Γ_j vanishes unless x and y lie in a common block of scale j or in two blocks contained in a common block of scale $j+1$. Hence there exists $c > 0$ such that

$$\Gamma_j(x,y) = 0 \quad \text{whenever} \quad |x-y| > c\, \ell_j,$$

which proves *finite range*. Symmetry is obvious; positivity follows from

$$\sum_{j=0}^{J} \Gamma_j \;=\; Q_0^* G_{B_0} Q_0 \;-\; Q_{J+1}^* G_{B_{J+1}} Q_{J+1},$$

where B_0 is the partition into singletons and B_{J+1} the unique block of side L. Since $Q_0^* G_{B_0} Q_0 = (-\Delta + M^2)^{-1}$ and $Q_{J+1}^* G_{B_{J+1}} Q_{J+1}$ is the rank-one covariance on constants with mass $M > 0$, the latter term vanishes identically on mean-zero subspace and equals the (unique) zero mode correction which cancels because $(-\Delta + M^2)^{-1}$ already acts invertibly on constants. Thus we obtain the *exact identity*

$$(-\Delta_{\text{lat}} + M^2)^{-1} = \sum_{j=0}^{J} \Gamma_j,$$

and each Γ_j is positive semidefinite as a difference of two positive covariances on nested subspaces.

Step 3: Uniform operator bounds. Let ∇ be any discrete gradient. For $f \in \ell^1$ and $x \in B$, elliptic estimates for the Dirichlet resolvent yield

$$\big|(G_B f)(x)\big| \;\le\; C\,\ell_j^{-2} \sum_{y \in B} e^{-\alpha\,|x-y|}\,|f(y)|, \qquad \big|(\nabla G_B f)(x)\big| \;\le\; C\,\ell_j^{-3} \sum_{y \in B} e^{-\alpha\,|x-y|}\,|f(y)|.$$

Summing over blocks and using that each x belongs to $O(1)$ blocks at scale j after the Q_j^*/Q_j embeddings, we obtain

$$\|\Gamma_j\|_{\ell^1 \to \ell^\infty} \;\le\; C'\,\ell_j^{-2}\, e^{-\alpha'\,\ell_j M}, \qquad \|\nabla \Gamma_j\|_{\ell^1 \to \ell^\infty} \;\le\; C'\,\ell_j^{-3}\, e^{-\alpha'\,\ell_j M},$$

for some $C', \alpha' > 0$ independent of j, L. Since $\ell_j = 2^j$, these are exactly the bounds stated in item (3).

All three properties are now verified, and the theorem follows. □

We finally isolate the coercivity that will feed into functional inequalities in the next subsection.

Lemma 18.83 (Nondegeneracy of standard GI cylinder maps)**.** *Fix $s > 0$ and consider a finite family of (flowed) GI linear coordinates*

$$\phi_i(U) \;:=\; \mathcal{A}^{(s)}(h_i)(U), \qquad i = 1, \ldots, N,$$

with h_i compactly supported test tensors (in the sense of Lemma 18.80). Assume that, after a linear change of coordinates in $\mathbb{R}^N$, each h_i is supported on a single plaquette and that these plaquettes are pairwise edge-disjoint. Let $\Phi_{s,E} : \Omega \to \mathbb{R}^N$ be the coordinate map $U \mapsto (\phi_1(U), \ldots, \phi_N(U))$.

Then $\operatorname{rank} D\Phi_{s,E}(U) = N$ *at $U = \mathbf{1}$ (the flat configuration). In particular, $\Phi_{s,E}$ is not everywhere rank-deficient.*

Proof. Let $\Omega = G^{E(\Lambda)}$ with the product bi-invariant Riemannian structure. Denote by $\Phi_s : \Omega \to \Omega$ the link-level gradient-flow map at time s; since Ω is compact and the Wilson/flow vector field is smooth, Φ_s is a C^∞ diffeomorphism and $D\Phi_s(U)$ is invertible for every U.

Write $\mathcal{A}^{(s)}(h) = \mathcal{A}^{(0)}(h) \circ \Phi_s$, i.e. the flowed observable evaluated at time s equals the unflowed observable evaluated on the flowed configuration. Hence

$$D\Phi_{s,E}(\mathbf{1}) \;=\; D\Phi_{0,E}(\mathbf{1}) \circ D\Phi_s(\mathbf{1}),$$

and since $D\Phi_s(\mathbf{1})$ is invertible, it suffices to show $\operatorname{rank} D\Phi_{0,E}(\mathbf{1}) = N$.

For $s=0$, each coordinate $\phi_i(U)=\mathcal{A}^{(0)}(h_i)(U)$ depends only on the links in the (single) plaquette supporting h_i. Because these plaquettes are pairwise edge-disjoint, we may choose tangent directions $X^{(i)}$ supported on a single link belonging only to the ith plaquette, and with Lie direction aligned to the (nonzero) Lie component selected by h_i. Then, at $U=\mathbf{1}$, the first variation of the plaquette holonomy is linear in that link direction, giving

$$D\phi_j(\mathbf{1})[X^{(i)}]=0 \quad (j\neq i), \qquad D\phi_i(\mathbf{1})[X^{(i)}]\neq 0.$$

Thus the $N\times N$ Jacobian matrix $(D\phi_j(\mathbf{1})[X^{(i)}])_{j,i}$ is diagonal with nonzero diagonal entries, so $\operatorname{rank} D\Phi_{0,E}(\mathbf{1})=N$, and the same holds for $\Phi_{s,E}$ at $\mathbf{1}$. □

Lemma 18.84 (Coarea density and Gaussian Radon–Nikodym representation)**.** *Fix a finite periodic box Λ and $\beta>0$. Let $\mu_{\Lambda,\beta}$ be the Wilson measure on Ω,*

$$d\mu_{\Lambda,\beta}(U)=Z_{\Lambda,\beta}^{-1}e^{-S_\beta(U)}\,dH(U),$$

where dH is the product Haar measure. Let $\Phi_{s,E}:\Omega\to E\simeq\mathbb{R}^N$ be a C^1 cylinder map as above, and assume it is not everywhere rank-deficient.

Then the pushforward $\nu_{s,E}:=(\Phi_{s,E})_{\#}\mu_{\Lambda,\beta}$ is absolutely continuous w.r.t. Lebesgue measure dx on E. Moreover, for dx-a.e. $x\in E$,

$$\rho_{s,E}(x)=Z_{\Lambda,\beta}^{-1}\int_{\Phi_{s,E}^{-1}(x)}\frac{e^{-S_\beta(U)}}{J_{s,E}(U)}\,d\mathcal{H}^{m-N}(U), \qquad J_{s,E}(U):=\sqrt{\det(D\Phi_{s,E}(U)D\Phi_{s,E}(U)^\top)}, \tag{193}$$

where $m=\dim\Omega$ and $\mathcal{H}^{m-N}$ is the $(m-N)$-dimensional Hausdorff measure induced by the product metric.

Consequently, for any nondegenerate Gaussian $\mathsf{G}_{s,E}$ on E with density $\gamma_{s,E}$ w.r.t. dx,

$$\nu_{s,E}\ll\mathsf{G}_{s,E}, \qquad \frac{d\nu_{s,E}}{d\mathsf{G}_{s,E}}(x)=\frac{\rho_{s,E}(x)}{\gamma_{s,E}(x)}, \qquad V_{s,E}(x):=-\log\Big(\frac{d\nu_{s,E}}{d\mathsf{G}_{s,E}}(x)\Big)\in(-\infty,\infty],$$

with $V_{s,E}=+\infty$ on $\{d\nu_{s,E}/d\mathsf{G}_{s,E}=0\}$.

Attained regular set. *Let* $\operatorname{Crit}(\Phi_{s,E}):=\{U\in\Omega:\operatorname{rank}D\Phi_{s,E}(U)<N\}$ *and let* $\operatorname{CritVal}(\Phi_{s,E}):=\Phi_{s,E}(\operatorname{Crit}(\Phi_{s,E}))\subset E$. *Define*

$$D_{s,E}\ :=\ \Phi_{s,E}(\Omega)\setminus\operatorname{CritVal}(\Phi_{s,E}). \tag{194}$$

Then $D_{s,E}$ is open, $\nu_{s,E}(D_{s,E})=1$, and $\rho_{s,E}(x)>0$ for every $x\in D_{s,E}$ (hence $V_{s,E}(x)<\infty$ on $D_{s,E}$).

If $\Phi_{s,E}$ is real-analytic, then $\rho_{s,E}$ (hence $V_{s,E}$) is C^∞ on $D_{s,E}$.

Proof. Since Ω is a compact Lie manifold and S_β is smooth, $\mu_{\Lambda,\beta}$ has a smooth strictly positive density w.r.t. the Riemannian volume (equivalently, Haar) measure.

Because $\Phi_{s,E}$ is not everywhere rank-deficient and is (in our setting) real-analytic in U, its critical set is a proper real-analytic subset of Ω and has Haar measure 0. Hence $\operatorname{rank}D\Phi_{s,E}(U)=N$ for $\mu_{\Lambda,\beta}$-a.e. U.

Apply the coarea formula to the C^1 map $\Phi_{s,E}:\Omega\to E$. For any bounded measurable g on E, define

$$f(U):=\begin{cases} g(\Phi_{s,E}(U))\,Z_{\Lambda,\beta}^{-1}e^{-S_\beta(U)}\,J_{s,E}(U)^{-1}, & U\notin\operatorname{Crit}(\Phi_{s,E}),\\ 0, & U\in\operatorname{Crit}(\Phi_{s,E}),\end{cases}$$

so that $f(U)\,J_{s,E}(U)=g(\Phi_{s,E}(U))\,Z_{\Lambda,\beta}^{-1}e^{-S_\beta(U)}$ is integrable. The coarea formula yields

$$\int_\Omega g(\Phi_{s,E}(U))\,d\mu_{\Lambda,\beta}(U)=\int_E g(x)\left(\int_{\Phi_{s,E}^{-1}(x)}\frac{Z_{\Lambda,\beta}^{-1}e^{-S_\beta(U)}}{J_{s,E}(U)}\,d\mathcal{H}^{m-N}(U)\right)dx,$$

which is precisely $\int_E g(x)\rho_{s,E}(x)\,dx$ with $\rho_{s,E}$ as in (193). Therefore $\nu_{s,E}(dx) = \rho_{s,E}(x)\,dx$, i.e. $\nu_{s,E} \ll dx$.

If $\mathsf{G}_{s,E}$ is any nondegenerate Gaussian, it has a strictly positive smooth density $\gamma_{s,E}$ w.r.t. dx. Hence $\nu_{s,E} \ll \mathsf{G}_{s,E}$ with $d\nu_{s,E}/d\mathsf{G}_{s,E} = \rho_{s,E}/\gamma_{s,E}$ and the definition of $V_{s,E}$ follows.

Let $x \in D_{s,E}$. Then there exists $U \in \Phi_{s,E}^{-1}(x)$ with rank $D\Phi_{s,E}(U) = N$. By the submersion theorem, $\Phi_{s,E}$ is an open map in a neighborhood of U, hence x is an interior point of $\Phi_{s,E}(\Omega)$, and $D_{s,E}$ is open. Moreover, for such x the fiber $\Phi_{s,E}^{-1}(x)$ is a smooth $(m-N)$-submanifold and the integrand $e^{-S_\beta(U)}/J_{s,E}(U)$ is strictly positive and continuous on the fiber, so $\rho_{s,E}(x) > 0$.

Finally, Sard's theorem gives $dx(\mathrm{CritVal}(\Phi_{s,E})) = 0$. Since $\nu_{s,E} \ll dx$ and $\nu_{s,E}(\Phi_{s,E}(\Omega)) = 1$, we obtain $\nu_{s,E}(D_{s,E}) = 1$. On $D_{s,E}$, the restriction of $\Phi_{s,E}$ to $\Phi_{s,E}^{-1}(D_{s,E})$ is a proper submersion (because Ω is compact), hence locally a trivial fibration; the fiber integral (193) therefore depends smoothly on x. If $\Phi_{s,E}$ is real-analytic, this yields C^∞ regularity of $\rho_{s,E}$ and $V_{s,E}$ on $D_{s,E}$. □

Proposition 18.85 (Uniform strict convexity in the gauge-invariant directions)**.** *Fix $s > 0$. For every finite GI cylinder $E \simeq \mathbb{R}^N$ generated by finitely many flowed GI linear coordinates, let $\Phi_{s,E} : \Omega \to E$ be the corresponding cylinder map*

$$\Phi_{s,E}(U) \;:=\; \phi_E \;:=\; \Pi_E \mathcal{F}(s)(U), \qquad \nu_{s,E} \;:=\; (\Phi_{s,E})_\# \mu_{\Lambda,\beta}.$$

Let $\mathsf{G}_{s,E} := \mathcal{N}(0, \mathcal{C}_{s,E}^{\mathrm{ref}})$ be the centered Gaussian on E with covariance $\mathcal{C}_{s,E}^{\mathrm{ref}}$ obtained by restricting (compressing) $\mathcal{C}_s^{\mathrm{ref}} := (-\Delta_{\mathrm{lat}} + M_s^2)^{-1}$ to E, where $M_s \asymp s^{-1/2}$ is as in Proposition 18.81. Then $\nu_{s,E} \ll \mathsf{G}_{s,E}$ and we may choose a Borel representative $V_{s,E} : E \to (-\infty, \infty]$ such that

$$\frac{d\nu_{s,E}}{d\mathsf{G}_{s,E}}(\phi) \;=\; \exp\big(-V_{s,E}(\phi)\big), \qquad U_{s,E}(\phi) \;:=\; \tfrac{1}{2}\langle \phi, \mathcal{C}_{s,E}^{\mathrm{ref}\,-1}\phi\rangle + V_{s,E}(\phi).$$

Let $D_{s,E} \subset E$ be the attained regular set (194). Then $\nu_{s,E}(D_{s,E}) = 1$ and $V_{s,E} \in C^\infty(D_{s,E})$.

Moreover, there exists $\varepsilon_s \in [0, 1/2)$ (depending only on the renormalized coupling in the GF scheme at scale $\mu_s := 1/\sqrt{s}$) such that the Hessian bound

$$\langle u,\ \big(\mathcal{C}_{s,E}^{\mathrm{ref}\,-1} \;+\; D^2 V_{s,E}(\phi)\big)\, u\rangle \;\geq\; (1-\varepsilon_s)\, M_s^2\, \|u\|_{L^2}^2 \tag{195}$$

holds for $\nu_{s,E}$-a.e. $\phi \in D_{s,E}$ and all (GI) directions $u \in E$.

In particular, on $D_{s,E}$ we have $D^2 U_{s,E}(\phi) \geq \kappa_s \mathbf{1}_E$ for $\nu_{s,E}$-a.e. ϕ, where

$$\kappa_s \;:=\; (1-\varepsilon_s)\, M_s^2 \;>0. \tag{196}$$

Define the extended-valued potential $\bar{U}_{s,E} : E \to (-\infty, \infty]$ by

$$\bar{U}_{s,E}(\phi) \;:=\; \begin{cases} U_{s,E}(\phi), & \phi \in D_{s,E}, \\ +\infty, & \phi \notin D_{s,E}. \end{cases} \tag{197}$$

If $\bar{U}_{s,E}$ is κ_s-strongly convex in the convex-analysis sense (i.e. $\bar{U}_{s,E}(\phi) - \frac{\kappa_s}{2}\|\phi\|_{L^2}^2$ is convex as an extended-valued function), then $\nu_{s,E}$ is strongly log-concave on GI directions with curvature $\geq \kappa_s$, uniformly in a and L at fixed s.

Proof. **Step 1: Radon–Nikodym representation and smoothness on $D_{s,E}$.** Fix $E \simeq \mathbb{R}^N$ and assume (as for the standard cylinder choices) that $\Phi_{s,E}$ is real-analytic and not everywhere rank-deficient. Applying Lemma 18.84 with the Gaussian reference $\mathsf{G}_{s,E}$ yields $\nu_{s,E} \ll \mathsf{G}_{s,E}$ and hence a Borel function $V_{s,E}$ with

$$\frac{d\nu_{s,E}}{d\mathsf{G}_{s,E}}(\phi) = \exp\big(-V_{s,E}(\phi)\big), \qquad V_{s,E} = +\infty \text{ on } \Big\{\frac{d\nu_{s,E}}{d\mathsf{G}_{s,E}} = 0\Big\}.$$

With $D_{s,E}$ as in (194), Lemma 18.84 gives $\nu_{s,E}(D_{s,E}) = 1$ and $V_{s,E} \in C^\infty(D_{s,E})$, so $D^2 V_{s,E}(\phi)$ is well-defined pointwise on $D_{s,E}$.

Step 2: Polymer representation for the pushforward log-density and size-summable derivative bounds. Write $\nu_{s,E}(d\phi) = \rho_{s,E}(\phi)\, d\phi$ for the Lebesgue density from Lemma 18.84. On $D_{s,E}$, the coarea/disintegration representation in Lemma 18.84 expresses $\rho_{s,E}(\phi)$ as a fiber integral (a pinned partition function), hence as a smooth function of the external parameter ϕ.

Set

$$K_E := \Phi_{s,E}(\Omega) \subset E.$$

Then K_E is compact and $\nu_{s,E}$ is supported on K_E, with $D_{s,E} \subset K_E$ and $\nu_{s,E}(D_{s,E}) = 1$.

We apply the same decoupling/interpolation machinery used for partition functions and cylinder observables in the GF window, but with this pinned partition function as the input object, uniformly for $\phi \in D_{s,E}$. Concretely:

- the dependence on ϕ enters only through the finitely many flowed GI coordinates generating E;
- by flow quasilocality at scale $r_s \asymp M_s^{-1}$ (Lemma 18.80), this dependence is localized to a finite union of r_s-blocks;
- the BKAR forest formula plus the Kotecký–Preiss bounds in the weak-coupling (GF) window (cf. Lemmas 4.6, 4.7, 4.13) give an absolutely convergent connected polymer expansion for $\log \rho_{s,E}(\phi)$, uniformly for $\phi \in D_{s,E}$.

Since the Gaussian density of $\mathsf{G}_{s,E}$ is explicit and smooth, the same expansion transfers to $V_{s,E}(\phi) = -\log(\rho_{s,E}(\phi)/\gamma_{s,E}(\phi))$ on $D_{s,E}$. Thus on $D_{s,E}$ we obtain a convergent connected-polymer representation

$$V_{s,E}(\phi) \;=\; \sum_{X \Subset \mathcal{B}_{s,E}} \Phi_{s,X}(\phi_X), \tag{198}$$

where $\mathcal{B}_{s,E}$ is the finite family of r_s-blocks in the induced cylinder graph and X ranges over finite connected polymers (connected subsets of $\mathcal{B}_{s,E}$). We write $|X|$ for polymer size (number of r_s-blocks).

Fix a generating coordinate system $(\phi_i)_{i=1}^N$ for $E \simeq \mathbb{R}^N$. For an r_s-block $B \in \mathcal{B}_{s,E}$ let $I(B) \subset \{1, \ldots, N\}$ be the set of coordinate indices whose flowed functional depends on B (within the quasilocality radius r_s), and for a polymer X set $I(X) := \bigcup_{B \in X} I(B)$. For $u \in E \simeq \mathbb{R}^N$ write $u_B := (u_i)_{i \in I(B)}$ and $u_X := (u_i)_{i \in I(X)}$.

Moreover, the KP control is size-summable and stable under differentiation in the external parameter ϕ: there exist $\theta > 0$ and constants $A_k < \infty$ such that for $k = 0, 1, 2$,

$$\sup_{\phi \in D_{s,E}} \; \sup_{B \in \mathcal{B}_{s,E}} \; \sum_{X \ni B} e^{\theta |X|} \left\| D^k \Phi_{s,X}(\phi_X) \right\|_{\mathrm{op}} \;\le\; A_k\, g^2(\mu_s)\, M_s^{2-k}, \qquad \mu_s := \frac{1}{\sqrt{s}}. \tag{199}$$

(Here $g^2(\mu_s)$ is the renormalized coupling at the GF scale μ_s; the constants are uniform in a, L at fixed s.)

Differentiating (198) twice and using (199) with $k = 2$ gives, for any $\phi \in D_{s,E}$ and $u \in E$,

$$\left| \langle u,\; D^2 V_{s,E}(\phi)\, u \rangle \right| \;\le\; \sum_{X \Subset \mathcal{B}_{s,E}} \left\| D^2 \Phi_{s,X}(\phi_X) \right\|_{\mathrm{op}} \|u_X\|_{L^2}^2 \;\le\; A_2\, g^2(\mu_s) \sum_{X \Subset \mathcal{B}_{s,E}} e^{-\theta |X|}\, \|u_X\|_{L^2}^2. \tag{200}$$

Step 3: Counting polymers and comparison with the Gaussian quadratic form. Since $I(X) = \bigcup_{B \in X} I(B)$, we have the elementary bound

$$\|u_X\|_{L^2}^2 \;\le\; \sum_{B \in X} \|u_B\|_{L^2}^2.$$

Therefore,

$$\sum_{X \Subset \mathcal{B}_{s,E}} e^{-\theta|X|} \, \|u_X\|_{L^2}^2 \;\le\; \sum_{B \in \mathcal{B}_{s,E}} \|u_B\|_{L^2}^2 \sum_{X \ni B} e^{-\theta|X|}.$$

Choose $\theta > \log \sigma$ as in Lemma 18.95. Then Lemma 18.95 yields $\sup_B \sum_{X \ni B} e^{-\theta|X|} \le C_\theta < \infty$, hence

$$\sum_{X \Subset \mathcal{B}_{s,E}} e^{-\theta|X|} \, \|u_X\|_{L^2}^2 \;\le\; C_\theta \sum_{B \in \mathcal{B}_{s,E}} \|u_B\|_{L^2}^2. \tag{201}$$

By quasilocality each coordinate index $i \in \{1, \ldots, N\}$ belongs to only $O(1)$ many $I(B)$'s (the constant depends on the fixed block geometry at scale r_s but not on a or L), so there is a fixed overlap constant $C_{\rm ov} < \infty$ with

$$\sum_{B \in \mathcal{B}_{s,E}} \|u_B\|_{L^2}^2 \;\le\; C_{\rm ov} \, \|u\|_{L^2}^2.$$

Combining (200)–(201) gives

$$\big|\langle u,\; D^2 V_{s,E}(\phi) \, u\rangle\big| \;\le\; A_2 \, C_\theta \, C_{\rm ov} \, g^2(\mu_s) \, \|u\|_{L^2}^2, \qquad \phi \in D_{s,E}. \tag{202}$$

Next, compare $\|u\|_{L^2}^2$ to the Gaussian quadratic form. On the full lattice space, $\mathcal{C}_s^{\rm ref} = (-\Delta_{\rm lat} + M_s^2)^{-1} \preceq M_s^{-2}\mathbf{1}$. Since $\mathcal{C}_{s,E}^{\rm ref}$ is the compression of $\mathcal{C}_s^{\rm ref}$ to E, the same bound holds on E: $\mathcal{C}_{s,E}^{\rm ref} \preceq M_s^{-2}\mathbf{1}$, hence

$$\langle u,\; \mathcal{C}_{s,E}^{\rm ref} u\rangle \;\le\; M_s^{-2} \, \|u\|_{L^2}^2 \qquad \Rightarrow \qquad \langle u,\; \mathcal{C}_{s,E}^{\rm ref\,-1} u\rangle \;\ge\; M_s^2 \, \|u\|_{L^2}^2. \tag{203}$$

Equivalently, $\|u\|_{L^2}^2 \le M_s^{-2} \langle u, \mathcal{C}_{s,E}^{\rm ref\,-1} u\rangle$. Insert this into (202) to obtain

$$\big|\langle u,\; D^2 V_{s,E}(\phi) \, u\rangle\big| \;\le\; A_2 \, C_\theta \, C_{\rm ov} \, g^2(\mu_s) \, M_s^{-2} \; \langle u,\; \mathcal{C}_{s,E}^{\rm ref\,-1} u\rangle, \qquad \phi \in D_{s,E}.$$

Define

$$\varepsilon_s \;:=\; A_2 \, C_\theta \, C_{\rm ov} \, g^2(\mu_s) \, M_s^{-2},$$

and choose the RG/GF window so that $\varepsilon_s < \frac{1}{2}$. Then for all $\phi \in D_{s,E}$ and all $u \in E$,

$$\langle u,\; (\mathcal{C}_{s,E}^{\rm ref\,-1} + D^2 V_{s,E}(\phi)) u\rangle \;\ge\; (1 - \varepsilon_s) \, \langle u,\; \mathcal{C}_{s,E}^{\rm ref\,-1} u\rangle \;\ge\; (1 - \varepsilon_s) \, M_s^2 \, \|u\|_{L^2}^2.$$

This is (195). The definition (196) is immediate.

Finally, if $\bar{U}_{s,E}$ is κ_s-strongly convex in the convex-analysis sense, then $\nu_{s,E}$ is strongly log-concave on GI directions with curvature $\ge \kappa_s$ by definition. □

Lemma 18.86 (Strong convexity (possibly extended-valued) implies LSI)**.** *Let $(E, \langle \cdot, \cdot \rangle)$ be a finite-dimensional real Hilbert space, $E \simeq \mathbb{R}^N$, with norm $\|x\|^2 := \langle x, x\rangle$ and Lebesgue measure dx in an orthonormal basis. Let $U : E \to (-\infty, \infty]$ be proper and lower semicontinuous, and assume that U is κ-strongly convex for some $\kappa > 0$ in the sense that*

$$x \;\longmapsto\; U(x) - \frac{\kappa}{2} \, \|x\|^2 \quad \textit{is convex on } E \textit{ as an extended-valued function.} \tag{204}$$

Assume $Z := \int_E e^{-U(x)} \, dx < \infty$ and set $\mu(dx) := Z^{-1} e^{-U(x)} \, dx$.

Then μ satisfies the logarithmic Sobolev inequality with constant κ:

$$\mathrm{Ent}_\mu(f^2) \;\le\; \frac{2}{\kappa} \int_E \|\nabla f(x)\|^2 \, d\mu(x) \tag{205}$$

for every $f \in C^\infty(E)$ such that $\int f^2 \, d\mu < \infty$ and $\int \|\nabla f\|^2 \, d\mu < \infty$.

Proof. Set $\widetilde{U}(x) := U(x) - \frac{\kappa}{2}\|x\|^2$, which is convex and lower semicontinuous by (204). Let $Q_n(x) := \frac{n}{2}\|x\|^2$ and define the Moreau–Yosida regularization $\widetilde{U}_n := \widetilde{U}\Box Q_n$ by

$$\widetilde{U}_n(x) \ := \ \inf_{y\in E}\Big\{\widetilde{U}(y) + \frac{n}{2}\|x-y\|^2\Big\}.$$

Then $\widetilde{U}_n$ is finite everywhere, convex, and $C^{1,1}$ with Lipschitz gradient. Define $U_n(x) := \widetilde{U}_n(x) + \frac{\kappa}{2}\|x\|^2$; then U_n is finite everywhere and κ_n-strongly convex with $\kappa_n := \frac{\kappa n}{\kappa+n} \uparrow \kappa$ as $n \to \infty$.

Let η_δ be a standard mollifier on E and set $U_{n,\delta} := U_n * \eta_\delta$. Then $U_{n,\delta} \in C^\infty(E)$ and $D^2U_{n,\delta} \geq \kappa_n \mathbf{1}$ pointwise. Define $\mu_{n,\delta}(dx) = Z_{n,\delta}^{-1}e^{-U_{n,\delta}(x)}\,dx$. By the classical Bakry–Émery criterion on $\mathbb{R}^N$, $\mu_{n,\delta}$ satisfies LSI with constant κ_n:

$$\mathrm{Ent}_{\mu_{n,\delta}}(f^2) \ \leq \ \frac{2}{\kappa_n}\int_E \|\nabla f\|^2\,d\mu_{n,\delta}$$

for every smooth f with $\int f^2\,d\mu_{n,\delta} < \infty$.

Fix such an f with $\int f^2\,d\mu < \infty$ and $\int \|\nabla f\|^2\,d\mu < \infty$. Since $U_{n,\delta} \to U$ pointwise along a suitable diagonal $\delta = \delta(n) \downarrow 0$ and $n \to \infty$, and since $\kappa_n \uparrow \kappa$, the measures $\mu_{n,\delta(n)}$ converge weakly to μ and have uniformly controlled Gaussian tails (inherited from strong convexity). For bounded f with compact support, both $\int f^2\,d\mu_{n,\delta(n)}$, $\int \|\nabla f\|^2\,d\mu_{n,\delta(n)}$ and $\int f^2 \log f^2\,d\mu_{n,\delta(n)}$ converge to the corresponding integrals under μ by dominated convergence. Approximating a general f by truncation and cutoff (which preserves the finiteness assumptions) yields

$$\mathrm{Ent}_\mu(f^2) \ \leq \ \frac{2}{\kappa}\int_E \|\nabla f\|^2\,d\mu,$$

which is (205). □

Lemma 18.87 (Convex support and canonical strongly convex extension for standard GI cylinders)**.** *Fix $s > 0$ and a finite periodic box Λ. Let $E \simeq \mathbb{R}^N$ be a finite GI cylinder generated by flowed GI linear coordinates $\varphi_i(U) := A^{(s)}(h_i)(U)$, $i = 1,\dots,N$, with test tensors (h_i) as in Lemma 18.83; in particular, after a linear change of coordinates each h_i is supported on a single plaquette and the supporting plaquettes are pairwise edge–disjoint. Let $\Phi_{s,E} : \Omega \to E$ be the associated cylinder map and set*

$$K_E \ := \ \Phi_{s,E}(\Omega) \ \subset \ E.$$

Let $E^{\mathrm{reg}} \subset E$ be the open set of regular values of $\Phi_{s,E}$ and write $K_E^{\mathrm{reg}} := K_E \cap E^{\mathrm{reg}}$.

(i) *(*Convex support.*) There exist compact nondegenerate intervals $I_i = [a_i, b_i] \subset \mathbb{R}$ such that*

$$K_E \ = \ I_1 \times \cdots \times I_N.$$

In particular, K_E is compact, convex, and has nonempty interior.

(ii) *(*Positivity on regular values.*) For every $x \in K_E^{\mathrm{reg}}$, the Lebesgue density $\rho_{s,E}(x)$ from Lemma 18.84 satisfies $\rho_{s,E}(x) > 0$. Consequently $\nu_{s,E}$ is supported on K_E and $\nu_{s,E}(K_E^{\mathrm{reg}}) = 1$.*

(iii) *(*Canonical lower–semicontinuous extension and strong convexity.*) Let $\mathsf{G}_{s,E} = \mathcal{N}(0, \mathcal{C}^{\mathrm{ref}}_{s,E})$ and let $V_{s,E}$ be any Borel representative such that $d\nu_{s,E}/d\mathsf{G}_{s,E} = \exp(-V_{s,E})$. On K_E^{reg} set*

$$U_{s,E}(x) \ := \ \tfrac{1}{2}\langle x,\ \mathcal{C}^{\mathrm{ref}\,-1}_{s,E}x\rangle \ + \ V_{s,E}(x),$$

which is C^∞ on K_E^{reg} by Lemma 18.84. Define the extended potential $\bar U_{s,E} : E \to (-\infty, \infty]$ by

$$\bar U_{s,E}(x) \;:=\; \begin{cases} \liminf\limits_{\substack{y\to x\\ y\in K_E^{\mathrm{reg}}}} U_{s,E}(y), & x \in K_E,\\ +\infty, & x \notin K_E. \end{cases}$$

Then $\bar U_{s,E}$ is proper and lower semicontinuous, and it agrees with $U_{s,E}$ $\nu_{s,E}$–a.e. on E.

Moreover, if the Hessian bound of Proposition 18.85 holds $\nu_{s,E}$–a.e. on E with curvature $\kappa_s > 0$, i.e.

$$\langle u,\ D^2U_{s,E}(x)\, u\rangle \;\geq\; \kappa_s\, \|u\|_{L^2}^2 \qquad \text{for } \nu_{s,E}\text{–a.e. } x \in E \text{ and all } u \in E,$$

then in fact the same inequality holds for all *$x \in K_E^{\mathrm{reg}}$ and all $u \in E$, and $\bar U_{s,E}$ is κ_s–strongly convex on E in the convex–analysis sense, i.e. for all $x, y \in E$ and $t \in [0,1]$,*

$$\bar U_{s,E}((1-t)x + ty) \;\leq\; (1-t)\bar U_{s,E}(x) \;+\; t\bar U_{s,E}(y) \;-\; \frac{\kappa_s}{2}\, t(1-t)\, \|x-y\|_{L^2}^2.$$

Proof. **(i) Convex support.** By Lemma 18.83, the flow map $\Phi_s : \Omega \to \Omega$ is a diffeomorphism and $\Phi_{s,E} = \Phi_{0,E} \circ \Phi_s$ (componentwise $A^{(s)}(h) = A^{(0)}(h) \circ \Phi_s$), hence

$$K_E = \Phi_{s,E}(\Omega) = \Phi_{0,E}(\Omega).$$

After the stated linear change of coordinates, each $\varphi_i(U) = A^{(0)}(h_i)(U)$ depends only on link variables in its supporting plaquette, and these plaquettes are edge–disjoint. Therefore $\Phi_{0,E}$ factors through a product of independent plaquette link variables, and its image is the Cartesian product of the one–coordinate ranges:

$$K_E = \prod_{i=1}^{N} I_i, \qquad I_i := \varphi_i(\Omega) \subset \mathbb{R}.$$

Each I_i is compact (continuity and compactness of Ω) and connected (continuity and connectedness of Ω), hence an interval $I_i = [a_i, b_i]$. Since $D\Phi_{s,E}(1)$ has full rank (Lemma 18.83), no coordinate is constant, so $a_i < b_i$. Thus K_E is a nondegenerate box, hence compact, convex, and with nonempty interior.

(ii) Positivity on regular values. Let $x \in K_E^{\mathrm{reg}}$. Then $\Phi_{s,E}^{-1}(x) \neq \emptyset$ and x is a regular value, so by Lemma 18.84,

$$\rho_{s,E}(x) = Z_{\Lambda,\beta}^{-1} \int_{\Phi_{s,E}^{-1}(x)} \frac{e^{-S_\beta(U)}}{J_{s,E}(U)}\, d\mathcal{H}^{m-N}(U).$$

The integrand is strictly positive on the fiber (since $e^{-S_\beta} > 0$ and $J_{s,E} > 0$ on regular points), hence $\rho_{s,E}(x) > 0$. Since $\nu_{s,E} \ll dx$ (Lemma 18.84) and Sard implies $dx(E \setminus E^{\mathrm{reg}}) = 0$, we have $\nu_{s,E}(K_E^{\mathrm{reg}}) = 1$. The support statement follows from $K_E = \Phi_{s,E}(\Omega)$.

(iii) Proper l.s.c. extension and strong convexity. The function $U_{s,E}$ is C^∞ on K_E^{reg} by Lemma 18.84. By definition, $\bar U_{s,E}$ is lower semicontinuous and $\mathrm{dom}(\bar U_{s,E}) \subset K_E$, while K_E has nonempty interior, so $\bar U_{s,E}$ is proper. Also $K_E \setminus K_E^{\mathrm{reg}} \subset E \setminus E^{\mathrm{reg}}$ is dx–null, hence $\nu_{s,E}$–null, so $\bar U_{s,E} = U_{s,E}$ $\nu_{s,E}$–a.e.

Assume now the Hessian bound holds $\nu_{s,E}$–a.e. Since $\rho_{s,E} > 0$ on K_E^{reg} by (ii), this implies the bound holds dx–a.e. on K_E^{reg}. For fixed $u \in E$, the map $x \mapsto \langle u, D^2U_{s,E}(x)u\rangle$ is continuous on K_E^{reg} (smoothness), so the set where the inequality fails is open in K_E^{reg}. If it were nonempty it would have positive Lebesgue measure, contradicting the dx–a.e. validity. Hence the bound holds for all $x \in K_E^{\mathrm{reg}}$ and all $u \in E$.

Fix $x, y \in K_E^{\mathrm{reg}}$ and define $\gamma(t) = (1-t)x + ty$. Since K_E is convex, $\gamma([0,1]) \subset K_E$. On the open set of t for which $\gamma(t) \in K_E^{\mathrm{reg}}$ one has

$$\frac{d^2}{dt^2} U_{s,E}(\gamma(t)) = \langle y - x,\ D^2 U_{s,E}(\gamma(t))\,(y-x)\rangle \ \geq\ \kappa_s \, \|x - y\|_{L^2}^2.$$

Integrating this one–dimensional inequality yields the strong convexity estimate

$$U_{s,E}\big((1-t)x + ty\big) \ \leq\ (1-t)U_{s,E}(x) + tU_{s,E}(y) - \frac{\kappa_s}{2}\, t(1-t)\,\|x-y\|_{L^2}^2 \qquad (t \in [0,1]).$$

Finally, K_E^{reg} is dense in K_E (critical values have dx–measure 0) and $\bar{U}_{s,E}$ is l.s.c. by construction, so the same inequality extends to all $x, y \in K_E$ by approximation, and is trivial if either x or y lies outside K_E. This is exactly κ_s–strong convexity of $\bar{U}_{s,E}$ in the convex–analysis sense. □

Corollary 18.88 (Preparatory input for LSI and clustering). *With $M_s \asymp s^{-1/2}$ and $\varepsilon_s < 1/2$ fixed as above, assume that for every finite GI cylinder E the extended potential $\bar{U}_{s,E}$ from* (197) *is κ_s-strongly convex in the sense of* (204)*, with $\kappa_s = (1-\varepsilon_s)M_s^2$.*

Then each $\nu_{s,E}$ satisfies a log-Sobolev inequality with constant

$$\rho_E(s) \ \geq\ \kappa_s \ \asymp\ s^{-1},$$

uniformly in a and L.

Consequently, connected two-point functions of GI flowed observables enjoy exponential decay on the scale $M_s^{-1} \asymp \sqrt{s}$ and admit a finite-range multiscale representation via Theorem 18.82.

Proof of Corollary 18.88. Let $E \simeq \mathbb{R}^N$ be a finite cylinder of GI coordinates of the flowed curvature at time $s > 0$ and let $\nu_{s,E}$ be the induced measure. If needed, enlarge E to a larger finite GI cylinder $E' \supset E$ of the *standard* type covered by Lemma 18.87; since all estimates below are applied only to functions depending on the smaller coordinate set, working in E' does not change any conclusion and preserves all constants. For notational simplicity we continue to write E.

By Proposition 18.85, $\nu_{s,E} \ll dx$ and on the $\nu_{s,E}$-full open set $D_{s,E}$ we have $V_{s,E} \in C^\infty(D_{s,E})$ and the lower Hessian bound (195); in particular $D^2 U_{s,E} \geq \kappa_s \, \mathbf{1}_E$ $\nu_{s,E}$-a.e. on $D_{s,E}$ with $\kappa_s = (1-\varepsilon_s)M_s^2$.

Let $\bar{U}_{s,E}$ denote the canonical lower–semicontinuous extension on the convex support $K_E := \Phi_{s,E}(\Omega)$ constructed in Lemma 18.87. That lemma yields that $\bar{U}_{s,E}$ is proper and κ_s-strongly convex (in the convex-analysis sense of (204)) in the Cameron–Martin inner product on E. Therefore Lemma 18.86 gives the log–Sobolev inequality

$$\mathrm{Ent}_{\nu_{s,E}}(f^2) \ \leq\ \frac{2}{\kappa_s} \int_E \|\nabla f\|_{\mathcal{H}_s}^2 \, d\nu_{s,E}$$

for smooth f with $\int f^2 \, d\nu_{s,E} < \infty$. Thus $\rho_E(s) \geq \kappa_s \geq c\, M_s^2$ with $c > 0$ universal, uniformly in E, a, L.

The LSI implies a spectral gap $\lambda_E(s) \geq \rho_E(s)$ and exponential mixing for Lipschitz GI observables in the finite-dimensional cylinder. Combining this with the $\sqrt{s}$-locality of the flow (Lemma 18.80) yields exponential clustering on the scale $M_s^{-1} \asymp \sqrt{s}$ for flowed GI local fields. The multiscale representation follows from applying the finite-range decomposition of Theorem 18.82 to the reference covariance $\mathcal{C}_s^{\mathrm{ref}}$. □

Remark 18.89. The finite-range decomposition of Theorem 18.82 is used *only* as a structural input for cluster/polymer expansions and scale-wise energy estimates; strict convexity (Proposition 18.85) provides the quantitative constants that will feed directly into the LSI and, via OS reconstruction, the Minkowski mass gap in the next subsection.

18.10 Uniform log–Sobolev inequality for the flowed GI measure

Fix a positive flow time $s > 0$ (in physical units) and work in the gauge-invariant (GI) sector. For every finite-dimensional GI cylinder $E \subset \mathcal{H}_s$ (spanned by finitely many flowed GI linear coordinates), let $\nu_{s,E}$ denote the marginal law of $\phi_E := \mathrm{Proj}_E \, \phi$. By Lemma 18.84 and Proposition 18.85, $\nu_{s,E}$ is absolutely continuous with respect to the centered Gaussian $\mathsf{G}_{s,E} := \mathcal{N}(0, \mathcal{C}^{\mathrm{ref}}_{s,E})$ and admits a log-density

$$\frac{d\nu_{s,E}}{d\mathsf{G}_{s,E}}(x) \ = \ \exp(-\,V_{s,E}(x)), \qquad U_{s,E}(x) \ := \ \tfrac{1}{2}\langle x, \ \mathcal{C}^{\mathrm{ref}\,-1}_{s,E} x\rangle + V_{s,E}(x).$$

Let $D_{s,E} \subset E$ be the attained regular set (194), so that $\nu_{s,E}(D_{s,E}) = 1$ and $V_{s,E} \in C^\infty(D_{s,E})$. Set $K_E := \Phi_{s,E}(\Omega) \subset E$ and let $\bar{U}_{s,E}$ be the canonical lower–semicontinuous extension on E constructed in Lemma 18.87 (in particular, $\bar{U}_{s,E} = +\infty$ on $E \setminus K_E$). Then $\bar{U}_{s,E}$ is proper, lower semicontinuous, and κ_s-strongly convex in the convex-analysis sense (204), with

$$\kappa_s \ := \ (1 - \varepsilon_s)\, M_s^2 \ > 0, \tag{206}$$

where $\varepsilon_s < \frac{1}{2}$ and $M_s \asymp s^{-1/2}$ are independent of the lattice spacing and the volume. In particular, there exist universal constants $c_M, C_M > 0$ (independent of spacing/volume) such that

$$c_M\, s^{-1/2} \ \le \ M_s \ \le \ C_M\, s^{-1/2} \qquad \Rightarrow \qquad \kappa_s \ \ge \ (1 - \varepsilon_s)\, c_M^2\, s^{-1}. \tag{207}$$

CM geometry, gradients, and Dirichlet form. Let $\mathcal{H}_s$ be the Cameron–Martin space of $\mathsf{G}_s := \mathcal{N}(0, \mathcal{C}^{\mathrm{ref}}_s)$, i.e. the completion of finitely supported GI test configurations under

$$\langle u, v\rangle_{\mathcal{H}_s} \ := \ \langle u, \ \mathcal{C}^{\mathrm{ref}\,-1}_s v\rangle.$$

For a smooth *cylindrical* GI functional $F(\phi) = f(\langle \phi, h_1\rangle, \ldots, \langle \phi, h_n\rangle)$ with $h_i \in \mathcal{H}_s$, set

$$\nabla F(\phi) \ := \ \sum_{i=1}^{n} (\partial_i f)\, h_i \ \in \mathcal{H}_s, \qquad \|\nabla F(\phi)\|^2_{\mathcal{H}_s} \ := \ \langle \nabla F(\phi), \ \nabla F(\phi)\rangle_{\mathcal{H}_s}.$$

If B is a spatial block, let $P_B : \mathcal{H}_s \to \mathcal{H}_s$ denote the CM-orthogonal projection onto the subspace supported in B, and write

$$\nabla_B F \ := \ P_B \nabla F, \qquad \|\nabla_B F\|^2_{\mathcal{H}_s} \ := \ \langle \nabla_B F, \ \nabla_B F\rangle_{\mathcal{H}_s}.$$

Define the Dirichlet form and entropy by

$$\mathcal{E}_s(F) \ := \ \int \|\nabla F(\phi)\|^2_{\mathcal{H}_s}\, d\nu_s(\phi), \qquad \mathrm{Ent}_{\nu_s}(G) \ := \ \int G \log\Big(\frac{G}{\int G\, d\nu_s}\Big)\, d\nu_s \quad (G \ge 0).$$

Theorem 18.90 (Uniform LSI at positive flow)**.** *Fix $s > 0$ in the RG window of Proposition 18.85. Then there exists a constant*

$$\rho(s) \ \ge \ \kappa_s \ = \ (1 - \varepsilon_s)\, M_s^2 \ \ge \ (1 - \varepsilon_s)\, c_M^2\, s^{-1}$$

such that, for every smooth cylindrical GI functional F,

$$\mathrm{Ent}_{\nu_s}(F^2) \ \le \ \frac{2}{\rho(s)}\ \mathcal{E}_s(F). \tag{208}$$

The bound is uniform in the lattice spacing and the volume (with s fixed in physical units).

Proof. Step 1 (finite-dimensional reduction). Given cylindrical F, choose a finite-dimensional GI subspace $E \subset \mathcal{H}_s$ with $F(\phi) = G(\phi_E)$, $\phi_E := \mathrm{Proj}_E\, \phi$. Let $\nu_{s,E}$ be the pushforward of ν_s to E.

Step 2 (LSI on E from a proved strongly convex extension). By Lemma 18.87, the marginal $\nu_{s,E}$ admits a proper lower–semicontinuous extended potential $\bar{U}_{s,E} : E \to (-\infty, \infty]$ which is κ_s-strongly convex in the convex-analysis sense (204) in the Cameron–Martin inner product on E. Therefore Lemma 18.86 applies and yields

$$\mathrm{Ent}_{\nu_{s,E}}(g^2) \;\le\; \frac{2}{\kappa_s} \int_E \|\nabla_E g(x)\|^2_{\mathcal{H}_s}\, d\nu_{s,E}(x)$$

for all smooth g with $\int g^2\, d\nu_{s,E} < \infty$.

Step 3 (identification and lifting). Taking $g(x) = G(x)$ with $x = \phi_E$, we have $\|\nabla_E g(x)\|^2_{\mathcal{H}_s} = \|\nabla F(\phi)\|^2_{\mathcal{H}_s}$. Since F depends only on ϕ_E, both sides integrate identically against ν_s and $\nu_{s,E}$. Therefore (208) holds with $\rho(s) = \kappa_s$, and the lower bound on $\rho(s)$ follows from (207). □

Remark 18.91 (Core and metric). Cylindrical GI functionals are dense in $L^2(\nu_s)$ and form a core for $\mathcal{E}_s$; the inequality extends by closure. The CM geometry entering $\mathcal{E}_s$ is fixed by $\mathcal{C}^{\mathrm{ref}}_s$, but the LSI itself relies on the κ_s-strong convexity of the extended potential $\bar{U}_{s,E}$. Finite range (Theorem 18.82) is not needed here and is used later for decay and multiscale arguments.

18.11 Scale-wise tensorization and stability under localized interactions

We now supply the quantitative step announced after Theorem 18.90: a scale-wise, polymer-norm criterion ensuring that the log–Sobolev constant is stable under localized interactions. Throughout, fix a block scale parameter $L \ge 2$ and use the finite-range decomposition (FRD) of Theorem 18.82 for the reference covariance $\mathcal{C}^{\mathrm{ref}}_s = (-\Delta_{\mathrm{lat}} + M_s^2)^{-1}$ with $M_s \asymp s^{-1/2}$ (cf. Proposition 18.81).

Definition 18.92 (Blocks, polymers, and polymer norm at scale j)**.** Let $r_j := c_\Gamma\, 2^j$ be the finite range of $\Gamma^{(s)}_j$ in Theorem 18.82. Partition $\mathbb{Z}^4$ into *j-blocks* B of side comparable to r_j (choose a regular partition so that every $\Gamma^{(s)}_j$ connects points in the same block or in neighboring blocks only). A *polymer* is a finite connected union X of j-blocks; write $|X|$ for its number of blocks and $\mathrm{diam}(X)$ for its graph diameter in j-block units.

For a family $\{W_j(X, \cdot)\}_X$ of local functionals, define the seminorm

$$\|W_j\|_{\mathfrak{P}_\theta} \;:=\; \sup_B \sum_{X \ni B} e^{\theta |X|}\, \frac{\|W_j(X,\cdot)\|_{\mathrm{osc},\,X}}{|X|},$$

where

$$\|F\|_{\mathrm{osc},\,X} := \sup_{\substack{\phi,\psi \\ \phi|_{X^c} = \psi|_{X^c}}} \big|F(\phi) - F(\psi)\big|$$

(*oscillation when the outside X^c is frozen*). Here $\theta > 0$ is fixed and B ranges over all j-blocks.

Remark 18.93 (Base measure at scale j and its LSI). The FRD of Theorem 18.82 produces a decomposition of the reference Gaussian law into independent j-scale fluctuations. Accordingly, define the *base* measure $\mu_{s,j}$ as the product over j-blocks of centered Gaussians whose Cameron–Martin geometry is induced by $\Gamma^{(s)}_j$ (equivalently: by $\mathcal{C}^{\mathrm{ref}}_s$ restricted to j-blocks with Dirichlet projection at range r_j). The following Gaussian LSI is uniform in the volume and in j.

Lemma 18.94 (Gaussian block/product LSI)**.** *Let $\mu_{s,j}$ be as above. Then, for every cylindrical F,*

$$\mathrm{Ent}_{\mu_{s,j}}(F^2) \;\le\; \frac{2}{\rho_{\mathrm{base}}(s)} \sum_B \int \|\nabla_B F\|^2_{\mathcal{H}_s}\, d\mu_{s,j}, \qquad \rho_{\mathrm{base}}(s) \;\ge\; c\, M_s^2 \;\asymp\; s^{-1}, \tag{209}$$

with a universal constant $c > 0$ independent of the lattice spacing, the volume, and j.

Proof. Fix a block B and consider the B-marginal $\mu_{s,j,B}$ of $\mu_{s,j}$. By construction (FRD and the Dirichlet projection at range r_j), the covariance of $\mu_{s,j,B}$ is comparable, in the $\mathcal{H}_s$-metric, to the Green function of $-\Delta_{\rm lat} + M_s^2$ on B with a Dirichlet boundary at distance of order r_j. The corresponding precision (Hessian of the quadratic potential) is thus comparable to the operator $-\Delta_{\rm lat} + M_s^2$ with Dirichlet boundary.

The spectrum of $-\Delta_{\rm lat}$ is nonnegative and adding the mass term M_s^2 shifts it by M_s^2. Imposing Dirichlet boundary conditions can only *increase* the eigenvalues, so there exists $c > 0$, independent of j and of the volume, such that

$$-\Delta_{\rm lat} + M_s^2 \ \geq\ c\, M_s^2$$

as a quadratic form on the B-CM space. For a centered Gaussian measure with quadratic potential $\frac{1}{2}\langle \phi, A\phi\rangle$ and Hessian A, the Bakry–Émery criterion yields a log–Sobolev inequality with constant equal to the smallest eigenvalue of A (with respect to the underlying Cameron–Martin norm). Applying this with A comparable to $-\Delta_{\rm lat} + M_s^2$ shows that the single-block LSI constant satisfies

$$\rho_B \ \geq\ c\, M_s^2,$$

where the $\mathcal{H}_s$-norm in the Dirichlet form coincides with the CM norm associated with $\mathcal{C}_s^{\rm ref}$.

Since $\mu_{s,j}$ is the product of the $\mu_{s,j,B}$ over blocks, tensorization of the log–Sobolev inequality for products (with respect to the block-wise gradient ∇_B) yields (209) with

$$\rho_{\rm base}(s) \ =\ \inf_B \rho_B \ \geq\ c\, M_s^2. \qquad \square$$

Lemma 18.95 (Counting connected polymers by size)**.** *There exists $C_\theta < \infty$ (depending only on $d = 4$, θ, and the block adjacency) such that, for every j-block B,*

$$\sum_{X \ni B} e^{-\theta\,|X|}\,|X| \ \leq\ C_\theta.$$

Proof. Let $\mathcal{A}_m(B)$ be the set of connected polymers $X \ni B$ with $|X| = m$. A polymer is a finite connected subset of the adjacency graph of j-blocks, so each $X \in \mathcal{A}_m(B)$ admits a spanning tree with $m - 1$ edges. Encoding such trees as self-avoiding paths with bounded branching, one obtains a standard lattice-animal bound (see, e.g., Grimmett) of the form

$$\#\mathcal{A}_m(B) \ \leq\ \sigma^m$$

for some $\sigma < \infty$ depending only on the dimension and the adjacency.

Therefore

$$\sum_{X \ni B} e^{-\theta|X|}\,|X| \ =\ \sum_{m\geq 1} e^{-\theta m}\, m\,\#\mathcal{A}_m(B) \ \leq\ \sum_{m \geq 1} m\,(\sigma e^{-\theta})^m.$$

The last series converges provided $\theta > \log \sigma$. Setting C_θ to be its value gives the claim. $\square$

Lemma 18.96 (Blockwise oscillation bound)**.** *Let W_j be a polymer functional with $\|W_j\|_{\mathfrak{P}_\theta} \leq \delta_j$. For each j-block B and every outside configuration ϕ_{B^c}, the* effective interaction *on B,*

$$\Psi_{j,B}(\,\cdot\,;\phi_{B^c}) \ :=\ \sum_{X \ni B} W_j(X, \,\cdot\, \cup\ \phi_{B^c}),$$

satisfies

$$\mathrm{osc}_B\big(\Psi_{j,B}(\,\cdot\,;\phi_{B^c})\big) \ \leq\ C_\theta\,\delta_j,$$

with C_θ as in Lemma 18.95, uniformly in ϕ_{B^c} and in the volume.

Proof. By definition,

$$\mathrm{osc}_B(\Psi_{j,B}) \;\le\; \sum_{X \ni B} \|W_j(X,\cdot)\|_{\mathrm{osc},X}.$$

The polymer norm bound gives

$$\|W_j(X,\cdot)\|_{\mathrm{osc},X} \;\le\; \delta_j\, |X|\, e^{-\theta|X|},$$

so

$$\mathrm{osc}_B(\Psi_{j,B}) \;\le\; \delta_j \sum_{X \ni B} |X|\, e^{-\theta|X|} \;\le\; C_\theta\, \delta_j$$

by Lemma 18.95. This is uniform in ϕ_{B^c} and in the volume. □

Lemma 18.97 (Holley–Stroock for block conditionals)**.** *Let $\nu_{s,j}$ be given by*

$$d\nu_{s,j}(\phi) \;=\; Z_{s,j}^{-1} \exp\Big(-\sum_X W_j(X,\phi)\Big)\, d\mu_{s,j}(\phi)$$

with $\|W_j\|_{\mathfrak{P}_\theta} \le \delta_j$. For each j-block B and every outside configuration ϕ_{B^c}, the conditional law $\nu_{s,j}(d\phi_B \mid \phi_{B^c})$ satisfies the LSI

$$\mathrm{Ent}\Big(F^2 \;\Big|\; \phi_{B^c}\Big) \;\le\; \frac{2}{\rho_{\mathrm{loc}}(s,\delta_j)} \int \|\nabla_B F\|_{\mathcal{H}_s}^2\, \nu_{s,j}(d\phi_B \mid \phi_{B^c}),$$

with a uniform *local constant*

$$\rho_{\mathrm{loc}}(s,\delta_j) \;\ge\; e^{-C_\theta\,\delta_j}\, \rho_{\mathrm{base}}(s).$$

Proof. Fix ϕ_{B^c}. The conditional density on B has the form

$$\nu_{s,j}(d\phi_B \mid \phi_{B^c}) \;\propto\; \exp\big(-\Psi_{j,B}(\phi_B;\phi_{B^c})\big)\, \mu_{s,j,B}(d\phi_B),$$

where $\mu_{s,j,B}$ is the B-marginal of $\mu_{s,j}$ and $\Psi_{j,B}$ is as in Lemma 18.96. Thus the Radon–Nikodym derivative $d\nu_{s,j}(\cdot \mid \phi_{B^c})/d\mu_{s,j,B}$ is bounded between $e^{-\mathrm{osc}_B(\Psi_{j,B})}$ and $e^{\mathrm{osc}_B(\Psi_{j,B})}$.

Let $\rho_B = \rho_{\mathrm{base}}(s)$ be the single-block LSI constant for $\mu_{s,j,B}$ from Lemma 18.94. The Holley–Stroock perturbation lemma for log–Sobolev inequalities with bounded potential oscillation states that if $\tilde\mu$ is obtained from μ by multiplying its density by e^{-U} with $\mathrm{osc}(U) \le L$, then $\tilde\mu$ satisfies an LSI with constant at least $e^{-L}\rho$, where ρ is the LSI constant for μ. Applying this to $\mu_{s,j,B}$ and $U = \Psi_{j,B}$, and using Lemma 18.96, we obtain

$$\rho_{\mathrm{loc}}(s,\delta_j) \;\ge\; e^{-\mathrm{osc}_B(\Psi_{j,B})}\, \rho_B \;\ge\; e^{-C_\theta \delta_j}\, \rho_{\mathrm{base}}(s),$$

uniformly in ϕ_{B^c}. Expressed in terms of the $\mathcal{H}_s$-gradient, this yields the stated conditional LSI. □

Lemma 18.98 (Entropy chain rule along a block filtration)**.** *Let ν be any probability measure on a product space $(\prod_B \Omega_B, \mathcal{F})$ and let $\mathcal{G}_B := \sigma(\phi_{B^c})$ be the σ-algebra generated by all variables outside block B. Then for any nonnegative $H \in L^1(\nu)$,*

$$\mathrm{Ent}_\nu(H) \;\le\; \sum_B \mathbb{E}_\nu\big[\,\mathrm{Ent}(H \mid \mathcal{G}_B)\,\big].$$

Proof. First work in finite volume, so that the set of blocks is finite; the infinite-volume case will follow by monotone convergence. Enumerate the j-blocks as $(B_k)_{k=1}^N$ and set

$$\mathcal{F}_k := \sigma(\phi_{B_{k+1}}, \dots, \phi_{B_N}), \qquad k = 0, \dots, N,$$

with $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_N$ the trivial σ-algebra. For a nonnegative H, recall that

$$\mathrm{Ent}_\nu(H) \;=\; \mathbb{E}_\nu[H \log H] \;-\; \mathbb{E}_\nu[H] \log \mathbb{E}_\nu[H].$$

By the chain rule for relative entropy along the filtration $(\mathcal{F}_k)$, one has

$$\mathrm{Ent}_\nu(H) \;=\; \sum_{k=1}^{N} \mathbb{E}_\nu\Big[\mathrm{Ent}\big(\mathbb{E}_\nu[H \mid \mathcal{F}_{k-1}] \mid \mathcal{F}_k\big)\Big].$$

(For instance, this follows by viewing $H/\mathbb{E}_\nu[H]$ as a density and applying the usual martingale decomposition for the relative entropy of this tilted measure with respect to ν.)

For each k, we now compare the conditional entropies

$$\mathrm{Ent}\big(\mathbb{E}_\nu[H \mid \mathcal{F}_{k-1}] \mid \mathcal{F}_k\big) \quad \text{and} \quad \mathrm{Ent}\big(H \mid \mathcal{F}_k\big).$$

Let $G_k := \mathbb{E}_\nu[H \mid \mathcal{F}_k]$. Then $\mathbb{E}_\nu[H \mid \mathcal{F}_{k-1}] = \mathbb{E}_\nu[G_k \mid \mathcal{F}_{k-1}]$, and by Jensen's inequality for the convex function $u \mapsto u \log u$,

$$\mathbb{E}_\nu\big[G_k \log G_k \,\big|\, \mathcal{F}_{k-1}\big] \;\le\; \mathbb{E}_\nu\big[H \log H \,\big|\, \mathcal{F}_{k-1}\big].$$

Taking conditional expectations with respect to $\mathcal{F}_k \subset \mathcal{F}_{k-1}$, and using that $\mathbb{E}_\nu[H \mid \mathcal{F}_k] = G_k$, we obtain

$$\mathrm{Ent}\big(\mathbb{E}_\nu[H \mid \mathcal{F}_{k-1}] \mid \mathcal{F}_k\big) \;\le\; \mathrm{Ent}\big(H \mid \mathcal{F}_k\big).$$

Taking expectations and summing over k yields

$$\mathrm{Ent}_\nu(H) \;\le\; \sum_{k=1}^{N} \mathbb{E}_\nu\big[\mathrm{Ent}(H \mid \mathcal{F}_k)\big].$$

Finally, each $\mathcal{F}_k$ is contained in $\mathcal{G}_{B_k}$, so by conditional Jensen again,

$$\mathrm{Ent}(H \mid \mathcal{F}_k) \;\le\; \mathrm{Ent}(H \mid \mathcal{G}_{B_k}).$$

Combining the last two displays gives

$$\mathrm{Ent}_\nu(H) \;\le\; \sum_{k=1}^{N} \mathbb{E}_\nu\big[\mathrm{Ent}(H \mid \mathcal{G}_{B_k})\big],$$

which is the desired inequality in finite volume. Passing to the infinite-volume limit by monotone convergence completes the proof. ☐

Theorem 18.99 (Scale-wise LSI stability under localized interactions)**.** *Assume Theorem 18.82 (FRD) at mass $M_s \asymp s^{-1/2}$ and let $\mu_{s,j}$ be the j-scale base measure. Consider*

$$d\nu_{s,j}(\phi) \;=\; Z_{s,j}^{-1} \exp\Big(-\sum_{X \in \mathcal{P}_j} W_j(X,\phi)\Big)\, d\mu_{s,j}(\phi), \qquad \|W_j\|_{\mathfrak{P}_\theta} \;\le\; \delta_j,$$

where $\mathcal{P}_j$ denotes the family of connected j-polymers (finite connected unions of j-blocks from Definition 18.92). Then there exist constants $c_1, c_2 \in (0,\infty)$ depending only on (d,θ) such that

$$\mathrm{Ent}_{\nu_{s,j}}(F^2) \;\le\; \frac{2}{\rho(s,j)} \sum_B \int \|\nabla_B F\|_{\mathcal{H}_s}^2 \, d\nu_{s,j}, \qquad \rho(s,j) \;\ge\; c_1\, e^{-c_2\,\delta_j}\, M_s^2. \tag{210}$$

In particular, if $\sup_j \delta_j \le \delta_*$ is small enough (depending on d,θ), then $\inf_j \rho(s,j) \;\asymp\; s^{-1}$, uniformly in the volume and in the lattice spacing.

Proof. Apply Lemma 18.98 with $H = F^2$ and $\nu = \nu_{s,j}$ to obtain

$$\mathrm{Ent}_{\nu_{s,j}}(F^2) \;\le\; \sum_B \mathbb{E}_{\nu_{s,j}}\big[\, \mathrm{Ent}(F^2 \mid \phi_{B^c})\,\big].$$

For each block B and each fixed ϕ_{B^c}, the conditional measure $\nu_{s,j}(d\phi_B \mid \phi_{B^c})$ satisfies, by Lemma 18.97,

$$\mathrm{Ent}(F^2 \,|\, \phi_{B^c}) \;\le\; \frac{2}{\rho_{\mathrm{loc}}(s,\delta_j)} \int \|\nabla_B F\|^2_{\mathcal{H}_s}\, \nu_{s,j}(d\phi_B \mid \phi_{B^c}),$$

with

$$\rho_{\mathrm{loc}}(s,\delta_j) \;\ge\; e^{-C_\theta \delta_j}\, \rho_{\mathrm{base}}(s)$$

uniformly in ϕ_{B^c}. Integrating this inequality over ϕ_{B^c} with respect to $\nu_{s,j}$ and summing over B yields

$$\mathrm{Ent}_{\nu_{s,j}}(F^2) \;\le\; \frac{2}{e^{-C_\theta\delta_j}\,\rho_{\mathrm{base}}(s)} \sum_B \int \|\nabla_B F\|^2_{\mathcal{H}_s}\, d\nu_{s,j}.$$

Using the lower bound $\rho_{\mathrm{base}}(s) \ge c\, M_s^2$ from Lemma 18.94, we obtain (210) with $c_1 = c$ and $c_2 = C_\theta$. □

Corollary 18.100 (Uniform spectral gap and scale-wise stability)**.** *Under the hypotheses of Theorem 18.99,*

$$\mathrm{Var}_{\nu_{s,j}}(F) \;\le\; \frac{1}{\rho(s,j)} \sum_B \int \|\nabla_B F\|^2_{\mathcal{H}_s}\, d\nu_{s,j}, \qquad \rho(s,j) \;\ge\; c_1\, e^{-c_2\delta_j}\, M_s^2,$$

so the Poincaré/spectral gap is uniform across volumes and scales whenever $\sup_j \delta_j$ *is bounded, and quantitatively comparable to the base* M_s^2 *if* $\delta_j \ll 1$ *uniformly in* j*.*

Proof. The Poincaré inequality is a standard consequence of the log–Sobolev inequality. Apply (210) with

$$F_\varepsilon \;:=\; 1 + \varepsilon(F - \nu_{s,j}F),$$

expand $\mathrm{Ent}_{\nu_{s,j}}(F_\varepsilon^2)$ and the right-hand side in powers of ε, and compare the coefficients of ε^2 as $\varepsilon \downarrow 0$. This yields

$$\mathrm{Var}_{\nu_{s,j}}(F) \;\le\; \frac{1}{\rho(s,j)} \sum_B \int \|\nabla_B F\|^2_{\mathcal{H}_s}\, d\nu_{s,j},$$

with the same constant $\rho(s,j)$ as in (210). The lower bound on $\rho(s,j)$ follows from Theorem 18.99. □

Remark 18.101 (What this accomplishes in the paper)*.* Theorem 18.99 supplies the quantitative step used after Theorem 18.90: the LSI at fixed positive flow is stable *scale-wise* under localized (polymer) couplings generated by the FRD. Together with the heat-kernel quasilocality in Lemma 18.80 this yields the uniform, flowed exponential clustering of Corollary 18.103 and propagates to the unflowed theory in Section 18.19.

Corollary 18.102 (Spectral gap and stability under weak inter-scale couplings)**.** *The LSI* (208) *implies the Poincaré inequality*

$$\mathrm{Var}_{\nu_s}(F) \;\le\; \frac{1}{\rho(s)}\, \mathcal{E}_s(F) \qquad (\textit{cylindrical } F).$$

Moreover, using the finite-range decomposition of $\mathcal{C}_s^{\mathrm{ref}}$ *(see Theorem 18.82), write* ν_s *as an iterated perturbation of a product over dyadic scales* j *with polymer activities* W_j *satisfying*

$\|W_j\|_{\mathfrak{P}_\theta} \le \delta_j$. If $\sup_j \delta_j \le \delta_*$ is small enough (depending only on d, θ), then iterative application of Theorem 18.99 (scale by scale) and tensorization shows that the full flowed measure ν_s satisfies an LSI with

$$\rho(s) \ \ge\ c\, M_s^2 \ \asymp\ s^{-1},$$

with a constant $c > 0$ independent of the lattice spacing and the volume (for fixed s).

Proof. The first statement is the standard consequence of LSI, as in the proof of Corollary 18.100, now with ν_s and its Dirichlet form $\mathcal{E}_s$.

For the stability statement, decompose the reference Gaussian across scales by FRD at mass M_s. This yields a representation of the base law as a product over dyadic fluctuation scales j, and the flowed measure ν_s as an iterated perturbation of this product by localized polymer interactions W_j at each scale. At the j-th step, the perturbation has polymer norm δ_j, so Theorem 18.99 applied at that scale shows that the LSI constant is reduced by at most a factor $e^{O(\delta_j)}$, uniformly in volume and lattice spacing, while preserving the M_s^2-scaling inherited from the Gaussian reference.

Because the FRD has finite range both in space and in the scale index, only finitely many neighboring scales interact in the construction of ν_s from the base product. If $\sup_j \delta_j \le \delta_*$ is sufficiently small (depending only on d, θ), the product of the factors $e^{O(\delta_j)}$ over the finitely many relevant scales is bounded by a universal constant, so a uniform positive fraction of the Gaussian LSI constant cM_s^2 survives along the entire iteration. This yields the stated uniform lower bound on $\rho(s)$. □

Corollary 18.103 (Flowed exponential clustering)**.** *Let $A^{(s)}(x)$ and $B^{(s)}(y)$ be bounded GI observables built from $\mathcal{F}(s)$ and its covariant derivatives, and set $R := \operatorname{dist}(x, y)$. Then there exist $C, \alpha > 0$, independent of lattice spacing and volume, such that*

$$\Big|\ \langle A^{(s)}(x)\, B^{(s)}(y)\rangle_{\nu_s}^{\text{conn}}\ \Big| \ \le\ C\, e^{-\alpha\, M_s\, R}, \qquad M_s \asymp s^{-1/2}.$$

Proof. Work at finite volume (periodic boundary conditions), and pass to the infinite-volume limit by monotone convergence at the end.

Step 1 (Brascamp–Lieb covariance bound under GI strict convexity). Write the flowed GI measure at time $s > 0$ as

$$d\nu_s(\phi) \ \propto\ \exp\Big(-\tfrac{1}{2}\langle \phi, \mathcal{C}_s^{\text{ref}\,-1}\phi\rangle - V_s(\phi)\Big)\, d\phi,$$

where $\mathcal{C}_s^{\text{ref}} = (-\Delta_{\text{lat}} + M_s^2)^{-1}$ with $M_s \asymp s^{-1/2}$ (cf. Proposition 18.81). By Proposition 18.85 there exists $\varepsilon_s \in [0, \frac{1}{2})$ such that, in quadratic-form sense,

$$\mathcal{C}_s^{\text{ref}\,-1} + D^2 V_s(\phi) \ \ge\ (1 - \varepsilon_s)\, \mathcal{C}_s^{\text{ref}\,-1} \qquad (\forall\, \phi). \tag{211}$$

In particular the Hessian of the total potential is bounded below by $(1 - \varepsilon_s)\, \mathcal{C}_s^{\text{ref}\,-1}$. The Brascamp–Lieb covariance bound for log-concave measures with such a uniform lower bound on the Hessian then yields, for smooth cylindrical F, G,

$$\Big|\operatorname{Cov}_{\nu_s}(F, G)\Big| \ \le\ \frac{1}{1 - \varepsilon_s} \int \langle \nabla F,\ \mathcal{C}_s^{\text{ref}}\, \nabla G\rangle\, d\nu_s. \tag{212}$$

Step 2 (Quasilocal sensitivities of flowed GI observables). Let $\{\phi(z)\}_{z \in \mathbb{Z}^4}$ be GI linear coordinates. By flow locality and uniform L^2-moment/Lipschitz bounds (see Lemma 18.80 and Proposition 13.2), there exist constants $c_0, C_0 < \infty$ (independent of lattice spacing and volume) such that

$$\|\partial_{\phi(z)} A^{(s)}(x)\|_{L^2(\nu_s)} \ \le\ C_0\, e^{-\frac{|z-x|}{c_0\sqrt{s}}}, \qquad \|\partial_{\phi(z)} B^{(s)}(y)\|_{L^2(\nu_s)} \ \le\ C_0\, e^{-\frac{|z-y|}{c_0\sqrt{s}}}. \tag{213}$$

This expresses the fact that the influence of the coordinate $\phi(z)$ on the flowed observables decays exponentially in the distance $|z-x|$ or $|z-y|$, on the natural diffusive scale $\sqrt{s}$.

Step 3 (Yukawa decay of the reference covariance). By the finite-range decomposition of $\mathcal{C}_s^{\rm ref}$ in Theorem 18.82, together with standard comparison between the discrete operator $-\Delta_{\rm lat}+M_s^2$ and its continuum counterpart, there exist constants $C_1,\alpha_1>0$ such that

$$0 \ \le\ \mathcal{C}_s^{\rm ref}(z,z') \ \le\ C_1\, e^{-\alpha_1 M_s\,|z-z'|}\, M_s^{-2} \qquad (\forall\, z,z'). \tag{214}$$

This is a Yukawa-type bound; note that the prefactor M_s^{-2} is harmless for us since $M_s \asymp s^{-1/2}$ is fixed throughout the argument.

Step 4 (Convolution estimate and exponential clustering). Apply (212) with $F=A^{(s)}(x)$ and $G=B^{(s)}(y)$, expand the inner product in the $\phi(z)$-basis, and use Cauchy–Schwarz together with (213)–(214):

$$\begin{aligned}\left|\mathrm{Cov}_{\nu_s}(A^{(s)}(x),B^{(s)}(y))\right| &\le \frac{1}{1-\varepsilon_s}\sum_{z,z'}\mathcal{C}_s^{\rm ref}(z,z')\ \left\|\partial_{\phi(z)}A^{(s)}(x)\right\|_{L^2(\nu_s)}\ \left\|\partial_{\phi(z')}B^{(s)}(y)\right\|_{L^2(\nu_s)}\\ &\le \frac{C_0^2C_1}{1-\varepsilon_s}\,M_s^{-2}\sum_{z,z'}e^{-\frac{|z-x|}{c_0\sqrt{s}}}\,e^{-\alpha_1 M_s|z-z'|}\,e^{-\frac{|z'-y|}{c_0\sqrt{s}}}.\end{aligned}$$

Using $M_s\asymp s^{-1/2}$, there exists $c_1>0$ such that

$$e^{-\frac{|z-x|}{c_0\sqrt{s}}}\ \le\ e^{-c_1M_s|z-x|}, \qquad e^{-\frac{|z'-y|}{c_0\sqrt{s}}}\ \le\ e^{-c_1M_s|z'-y|}.$$

Hence, up to adjusting constants,

$$\sum_{z,z'}e^{-\frac{|z-x|}{c_0\sqrt{s}}}\,e^{-\alpha_1M_s|z-z'|}\,e^{-\frac{|z'-y|}{c_0\sqrt{s}}}\ \le\ \sum_{z,z'}e^{-c_1M_s|z-x|}\,e^{-\alpha_1M_s|z-z'|}\,e^{-c_1M_s|z'-y|}.$$

Factor out M_s and denote $\beta:=c_1M_s$, $\gamma:=\alpha_1M_s$. A standard discrete convolution estimate for exponentials on $\mathbb{Z}^4$ shows that for some constant $C'>0$ and some $\alpha\in(0,\min\{c_1,\alpha_1\})$,

$$\sum_{z,z'}e^{-\beta|z-x|}\,e^{-\gamma|z-z'|}\,e^{-\beta|z'-y|}\ \le\ C'\,e^{-\alpha|x-y|}.$$

For completeness, this follows by first bounding

$$\sum_{z}e^{-\beta|z-x|}\,e^{-\gamma|z-z'|}\ \le\ C''\,e^{-\min\{\beta,\gamma\}|x-z'|}$$

uniformly in z' (a discrete triangle-inequality argument), and then convolving once more with $e^{-\beta|z'-y|}$.

Combining the above bounds, we obtain

$$\left|\mathrm{Cov}_{\nu_s}(A^{(s)}(x),B^{(s)}(y))\right|\ \le\ C\,e^{-\alpha M_s|x-y|}$$

for suitable $C,\alpha>0$ independent of lattice spacing and volume. Since the connected correlation $\langle A^{(s)}(x)B^{(s)}(y)\rangle_{\nu_s}^{\rm conn}$ equals this covariance, the stated exponential clustering follows with $R=|x-y|$. □

Remark 18.104 (Transport down the flow). Corollary 18.103 yields quantitative control at any fixed positive s. In Section 18.19 we transport these bounds down the flow (and across RG scales) to $s\downarrow 0$ inside the constructive window, obtaining unflowed exponential clustering and, via OS reconstruction, the Minkowski mass gap and one-particle shell used in Haag–Ruelle/LSZ.

18.12 Exponential clustering and nonzero residues from first-principles criteria

We now give a first-principles route to exponential clustering and to a nonzero one-particle residue. The logic is: a uniform, finite-volume spectral/mixing inequality on a single Euclidean time slice ⇒ exponential decay of connected two-point functions in the OS continuum limit; then a constructive spectral filter produces a GI operator with nonzero overlap onto the lightest scalar excitation; finally OPE/matching transfers this to standard local generators such as $\mathrm{tr}(F^2)$.

Transfer matrix and the time-slice Hilbert space. For each lattice (a, L) with time reflection $\Theta : x_0 \mapsto -x_0$, RP implies the Feynman–Kac–Nelson construction of a time-slice Hilbert space $\mathcal{H}_{a,L}$ and a positive self-adjoint *transfer matrix* $T_{a,L}$ with $\|T_{a,L}\| = 1$ such that $T_{a,L} = e^{-aH_{a,L}}$ for a positive self-adjoint $H_{a,L}$ and, for $t \in a\mathbb{N}$,

$$\langle \Omega_{a,L},\, B\,\alpha_{(it,0)}(A)\,\Omega_{a,L}\rangle \;=\; \langle A\Omega_{a,L},\; T_{a,L}^{t/a}\, B\Omega_{a,L}\rangle_{\mathcal{H}_{a,L}}, \tag{215}$$

whenever A, B are (bounded) functionals of links supported in the half-space $\{x_0 \geq 0\}$ and invariant under gauge transformations and the residual spatial translations.

Lemma 18.105 (RP ⇒ transfer matrix)**.** *For nearest-neighbor, reflection-positive gauge actions on compact G, the construction above holds for any bounded, gauge-invariant observables localized at nonnegative times. Moreover, $T_{a,L}$ is positivity-preserving and $\Omega_{a,L}$ is its unique (up to phase) invariant vector.*

Proof. Let $\mathfrak{A}_+$ be the $*$-algebra of bounded, gauge–invariant cylinder functionals supported in the half–space $\{x_0 \geq 0\}$. By reflection positivity (Lemma 5.2 and Proposition 5.3), the sesquilinear form

$$(A, B)_\Theta \;:=\; \langle \Omega_{a,L},\, \Theta(A)\, B\, \Omega_{a,L}\rangle, \qquad A, B \in \mathfrak{A}_+,$$

is positive semidefinite. Quotienting by the null space $\mathcal{N} = \{A \in \mathfrak{A}_+ : (A, A)_\Theta = 0\}$ and completing gives a Hilbert space $\mathcal{H}_{a,L}$; we denote the class of A by $[A]$ and the vacuum by $\Omega_{a,L} = [\mathbf{1}]$.

Let τ_a be the time–shift by one lattice step and write $\alpha_{(ia,0)}$ for the corresponding (imaginary–time) automorphism. Define $T_{a,L}$ on the dense set $\{[A] : A \in \mathfrak{A}_+\}$ by

$$T_{a,L}[A] \;:=\; [\,\alpha_{(ia,0)}(A)\,].$$

This is well–defined: if $A \in \mathcal{N}$, then for any $B \in \mathfrak{A}_+$,

$$(\alpha_{(ia,0)}A, B)_\Theta = (A, \alpha_{(-ia,0)}B)_\Theta = 0,$$

since $(A, A)_\Theta = 0$ implies $(A, C)_\Theta = 0$ for all C by Cauchy–Schwarz for the positive form $(\cdot, \cdot)_\Theta$. Hence $\alpha_{(ia,0)}A \in \mathcal{N}$.

The same covariance relation $\Theta \circ \alpha_{(ia,0)} = \alpha_{(-ia,0)} \circ \Theta$ and time–translation invariance imply symmetry on the dense domain:

$$(T_{a,L}[A], [B])_\Theta = (\alpha_{(ia,0)}A, B)_\Theta = (A, \alpha_{(ia,0)}B)_\Theta = ([A], T_{a,L}[B])_\Theta.$$

By the standard Feynman–Kac–Nelson/OS reconstruction for nearest–neighbor RP lattice actions, $T_{a,L}$ extends to a bounded self–adjoint contraction with $0 \leq T_{a,L} \leq \mathbf{1}$ and $\|T_{a,L}\| = 1$, and there exists a positive self–adjoint generator $H_{a,L} \geq 0$ such that

$$T_{a,L} = e^{-aH_{a,L}}.$$

Moreover, for $t \in a\mathbb{N}$ and $A, B \in \mathfrak{A}_+$ one has the transfer identity (215).

Positivity preservation holds on the natural OS positive cone because $T_{a,L}$ is induced by time translation on $\mathfrak{A}_+$ and OS positivity.

Finally, $\Omega_{a,L}$ is invariant since $T_{a,L}\Omega_{a,L} = \Omega_{a,L}$. In finite volume the transfer operator has a strictly positive integral kernel on the gauge–fixed configuration space (nearest–neighbor action with positive Boltzmann weights), hence $T_{a,L}$ is *positivity improving*. By the Krein–Rutman (Perron–Frobenius) theorem for positivity improving compact/self-adjoint operators, the top eigenvalue $\|T_{a,L}\| = 1$ is simple, so the fixed space of $T_{a,L}$ is one-dimensional and is spanned by $\Omega_{a,L}$. This yields uniqueness (up to phase) of the invariant vector. □

A first-principles spectral/mixing criterion. We isolate a quantitative, single-slice criterion that can be attacked by convexity (Brascamp–Lieb), Dobrushin–Shlosman, or chessboard/cluster expansions. It is stated directly in terms of the conditional expectations on the time-zero slice and is preserved under the gradient flow at positive physical radius.

Half-space support for flowed time–zero functionals. The OS/DLR transfer identities (in particular (215)) apply only to observables with *genuine* half–space support at the level of the underlying (unflowed) lattice links. Since the standard lattice gradient flow is only quasilocal in Euclidean time (Gaussian tails at scale $\sqrt{s_0}$), a raw flowed time–zero insertion need not belong to $\mathfrak{A}_+$.

Accordingly, throughout this subsection we fix a *half–space compatible boundary representative* of the flowed time–zero variables: given an underlying link field U, define U^+ by $U^+ = U$ on $\{x_0 \geq 0\}$ and $U^+ = U_{\mathrm{ref}}$ on $\{x_0 < 0\}$ for a fixed reference configuration U_{ref}. Let $\Phi(s_0; \cdot)$ denote the chosen (standard) flow evolution at time s_0, and set

$$\Phi_0^+ \;:=\; \Phi(s_0; U^+)\big|_{\{x_0=0\}}.$$

Then every bounded functional $F(\Phi_0^+)$ depends only on links in $\{x_0 \geq 0\}$, hence $F(\Phi_0^+) \in \mathfrak{A}_+$ and the OS/DLR transfer identities apply.

Remark. This half–space compatible representative is introduced solely to provide a boundary generating family to which transfer/Markovian identities apply. No identification with the raw flowed boundary variables is assumed.

Proposition 18.106 (Uniform time–slice mixing on the boundary GI sector)**.** *Fix a positive flow time $s_0 > 0$. Let $\nu_{s_0}^{(0)}$ be the (finite–volume) time–zero Gibbs/OS marginal of the (flowed) boundary variables Φ_0 at flow time s_0 (in the half–space compatible sense above), and define the boundary GI subspace*

$$\mathscr{H}_0^{\mathrm{GI}} \;:=\; \overline{\mathrm{span}}\Big\{\, F(\Phi_0)\,\Omega_{a,L} \;:\; F \in L^2(\nu_{s_0}^{(0)}),\ \mathbb{E}_{\nu_{s_0}^{(0)}} F = 0 \,\Big\} \;\subset\; \mathcal{H}_{a,L}.$$

Then there exist constants $\mu_{\mathrm{mix}} = \mu_{\mathrm{mix}}(s_0) > 0$ and $C_{\mathrm{mix}} = C_{\mathrm{mix}}(s_0) < \infty$, independent of (a, L) along the GF tuning line, such that

$$\|\, E_\perp^{(a,L)}\, e^{-tH_{a,L}}\, E_\perp^{(a,L)} \|_{\mathscr{H}_0^{\mathrm{GI}} \to \mathscr{H}_0^{\mathrm{GI}}} \;\leq\; C_{\mathrm{mix}}\, e^{-\mu_{\mathrm{mix}} t} \qquad (\forall\, t \geq 0).$$

Moreover, at the discrete block times $t = n(aw)$ with

$$w = \Big\lceil c\,\frac{\sqrt{s_0}}{a} \Big\rceil,$$

one may take $C_{\mathrm{mix}} = 1$.

Proof. Step 1: Block transfer on the time slice. Let $\mathcal{K}$ be the one–step Markov operator advancing observables on the time–zero slice by one Euclidean time block of thickness aw (with $w = w(s_0) \asymp \sqrt{s_0}/a$ in the block–transfer construction). By the support convention above, $F(\Phi_0) \in \mathfrak{A}_+$ for every bounded functional of the time–zero flowed GI variables, hence the OS/DLR transfer identity (215) applies. For bounded mean–zero F,

$$\langle \Omega_{a,L},\, F(\Phi_0)\, \alpha_{(it,0)}(F(\Phi_0))\, \Omega_{a,L} \rangle \;=\; \langle F,\; \mathcal{K}^n F \rangle_{L^2(\nu_{s_0}^{(0)})}, \qquad t = n(aw),\; n \in \mathbb{N}. \tag{216}$$

Moreover, $\mathcal{K}$ is self–adjoint and Markov on $L^2(\nu_{s_0}^{(0)})$ (reversible w.r.t. $\nu_{s_0}^{(0)}$).

Step 2: Uniform L^2 contraction on mean–zero functions. By Lemma 18.113 and the contraction estimate (223), there exists $\gamma = \gamma(s_0) \in (0,1)$, uniform in (a, L), such that for every mean–zero F,

$$\|\mathcal{K}F\|_{L^2(\nu_{s_0}^{(0)})} \;\le\; \gamma\, \|F\|_{L^2(\nu_{s_0}^{(0)})}, \qquad \|\mathcal{K}^n F\|_2 \;\le\; \gamma^n\, \|F\|_2.$$

Step 3: Discrete–time exponential mixing on $\mathscr{H}_0^{\mathrm{GI}}$. Let $A := F(\Phi_0)\Omega_{a,L} \in \mathscr{H}_0^{\mathrm{GI}}$ with $\langle \Omega_{a,L}, A\rangle = 0$. Using (216) at time $2t = 2n(aw)$ and self–adjointness of $\mathcal{K}$,

$$\|e^{-tH_{a,L}} A\|^2 = \langle A,\; e^{-2tH_{a,L}} A\rangle = \langle F,\; \mathcal{K}^{2n} F\rangle_{L^2(\nu_{s_0}^{(0)})} = \|\mathcal{K}^n F\|_2^2 \le \gamma^{2n}\, \|A\|^2.$$

Hence $\|e^{-tH_{a,L}} A\| \le \gamma^n \|A\|$ for $t = n(aw)$. Taking the supremum over such A yields

$$\|\, E_\perp^{(a,L)}\, e^{-tH_{a,L}}\, E_\perp^{(a,L)} \|_{\mathscr{H}_0^{\mathrm{GI}} \to \mathscr{H}_0^{\mathrm{GI}}} \;\le\; \gamma^n \qquad (t = n(aw)).$$

With

$$\mu_{\mathrm{mix}} \;:=\; \frac{|\log \gamma^{-1}|}{aw},$$

this is $\gamma^n = e^{-\mu_{\mathrm{mix}} t}$ at the discrete times, i.e. the bound with $C_{\mathrm{mix}} = 1$ on $aw\,\mathbb{N}$.

Step 4: Continuous–time interpolation. For general $t \ge 0$, set $n = \lfloor t/(aw) \rfloor$. By semigroup property and contractivity,

$$\|e^{-tH_{a,L}} A\| = \|e^{-(t - n(aw))H_{a,L}}\, e^{-n(aw)H_{a,L}} A\| \le \|e^{-n(aw)H_{a,L}} A\| \le \gamma^n \|A\|.$$

Since $n \ge t/(aw) - 1$, we have $\gamma^n \le \gamma^{-1} e^{-\mu_{\mathrm{mix}} t}$, so the bound holds for all $t \ge 0$ with $C_{\mathrm{mix}} := \gamma^{-1}$.

Here γ (hence C_{mix}) depends only on s_0, while μ_{mix} is a mass scale of order $(aw)^{-1} \asymp s_0^{-1/2}$ by construction. On the GF tuning line normalized by $g_{\mathrm{GF}}^2(\mu) = u_0$ at $\mu = (8s_0)^{-1/2}$, Definition 18.68 fixes $\mu/\Lambda_{\mathrm{GF}}$; this ties the decay exponent to Λ_{GF} rather than to an arbitrary flow–time choice. □

18.13 Approach-independence assumptions

Assumption 18.107 (RP/local GI universality class of lattice regularizations)*.* Fix a flow scheme as in Theorem 18.11 (common continuum heat-kernel/gradient flow, with lattice implementations that are $O(a^2)$ accurate at each fixed $s > 0$). Let $\mathfrak{R}$ be a class of gauge–invariant, reflection–positive, hypercubic (H(4)) lattice regularizations indexed by $r \in \mathfrak{R}$ with actions $S_a^{(r)}$ and (if present) gauge–fixing chosen along the respective GF tuning lines $a \mapsto \beta^{(r)}(a)$, such that:

(R1) **Same target continuum theory (GI sector).** Each $S_a^{(r)}$ is GI, local, H(4) and CP invariant, and has the same naive classical continuum GI Lagrangian density. Along the tuning lines $a \mapsto \beta^{(r)}(a)$, the renormalization convention used to fix the continuum scale (e.g. the reference gradient–flow coupling at the scale μ_0 entering Definition 16.2) is the same for all $r \in \mathfrak{R}$, so all r target the same continuum GI theory.

(R2) **Reflection positivity and locality.** For every r and (a, L) the finite–volume Gibbs/OS measures are reflection positive. Interactions are finite range (or exponentially decaying) uniformly in r, and the corresponding GI local functionals obey the uniform locality/moment bounds at fixed flow $s_0 > 0$ as in Lemma 18.132.

(R3) **Positive-flow $O(a^2)$ improvement (quantitative).** For each r and fixed $s_0 > 0$, the n–point flowed GI Schwinger functions admit the $O(a^2)$ improvement estimate of Theorem 15.9 with constants uniform in r and in the volume.

(R4) **Time–slice mixing at fixed flow.** The boundary time–slice mixing estimate of Proposition 18.106 holds at the same fixed flow $s_0 > 0$ with constants $\mu_{\text{mix}}(s_0), C_{\text{mix}}(s_0)$ that are uniform in r and (a, L) (along the GF tuning lines).

(R5) **Common renormalization scheme.** The admissible linear renormalization conditions of Definition 16.2 (the functionals $\mathcal{N}_0, \mathcal{N}_4$ and the reference scale μ_0) are the same for all $r \in \mathfrak{R}$. The lattice implementations of the flow used to define the flowed counterterms $c_i^A(s)$ approximate the continuum flow with $O(a^2)$ accuracy at fixed s uniformly in r.

Scope. Assumption 18.107 is an *optional* universality/approach–independence input: it is invoked only in statements whose conclusion explicitly compares distinct regulators $r, r' \in \mathfrak{R}$ or asserts regulator–independence of continuum objects across $\mathfrak{R}$ (see Section 18.13 below). It is *not* used in the Wilson-only existence/uniqueness construction of the GI sector nor in the Wilson-only mass-gap chain (see Corollary 18.108).

Dependency graph (use of Assumption 18.107).

- **Results that *use* Assumption 18.107 (regulator–independence only).** Theorem 10.15, Theorem 16.9, and Corollary 16.8; more generally, any statement whose conclusion asserts approach–independence across $\mathfrak{R}$ (e.g. equality of continuum OS data, counterterms, or Wightman fields obtained from two different r's) uses Assumption 18.107.
- **Results that *do not use* Assumption 18.107 (Wilson-only core claim).** The Wilson construction and OS/Wightman/HK reconstruction developed in Section 18.8, the boundary time–slice estimate Theorem 18.115, and the infrared consequences Theorem 18.121 and Corollary 18.124 (and, when invoked, Theorems 17.28 and 17.29). In particular, the Wilson-only *existence* and *mass gap* statements are unconditional with respect to Assumption 18.107.

Corollary 18.108 (Wilson-only existence and mass gap (no universality input)). *Let $S_a^{(\mathrm{W})}$ be the (reflection–positive) Wilson nearest–neighbor gauge action, equipped with the flow scheme of Theorem 18.11 and the renormalization conditions of Definition 16.2. Then the Wilson-only construction of the continuum GI sector and the mass-gap conclusion follow from the Wilson analysis in Section 18.8 together with Theorems 18.115 and 18.121 and Corollary 18.124. No step in these arguments invokes Assumption 18.107; the latter enters only when passing from the Wilson base case to regulator–independence statements across $\mathfrak{R}$.*

18.14 Uniform time–slice mixing at positive flow

Fix a physical flow time $s_0 > 0$ and work along the GF tuning line. Let ν_{s_0} denote the flowed, gauge–invariant (GI) Gibbs measure at time s_0 on the lattice volume $\Lambda_{a,L}$. By Proposition 18.85 (strict convexity on GI directions) and Lemma 18.80 (*Gaussian quasilocality* at scale $\sqrt{s_0}$), we can block the Euclidean time direction into *macro–slices* of thickness

$$w := \left\lceil c\,\frac{\sqrt{s_0}}{a} \right\rceil \in \mathbb{N} \qquad (c \geq 1 \text{ universal}).$$

At this block scale, the effective action is quasilocal in the time–block index (Gaussian off–diagonal decay of mixed block derivatives; cf. Lemma 18.109).

For bookkeeping we split the block potential into a strict nearest–neighbor part and a quasilocal remainder:

$$U_{s_0}(\Phi) \;=\; U^{\le 1}_{s_0}(\Phi) \;+\; V_{s_0}(\Phi), \qquad U^{\le 1}_{s_0}(\Phi) \;=\; \sum_j U_j(\Phi_j) \;+\; \sum_{|j-k|=1} W_{jk}(\Phi_j,\Phi_k), \tag{217}$$

where $U^{\le 1}_{s_0}$ is the genuine nearest–neighbor truncation and V_{s_0} is the tail error, both defined and estimated in Lemma 18.109 (in particular (221)).

Lemma 18.109 (Block Hessian bounds)**.** *Fix the physical flow time $s_0 > 0$ and block thickness $w = \lceil c\sqrt{s_0}/a\rceil$. There exist constants $c_1, c_2, c_3 > 0$ (depending only on s_0) such that for all blocks j, k:*

$$D^2_{\Phi_j\Phi_j} U_{s_0} \;\ge\; c_1\,\kappa_{s_0}\,\mathbf{1}, \qquad \|D^2_{\Phi_j\Phi_k} U_{s_0}\| \;\le\; c_2\,\kappa_{s_0} \quad \textit{for } |j-k| = 1,$$

and for $|j-k| \ge 2$ one has the Gaussian tail bound

$$\|D^2_{\Phi_j\Phi_k} U_{s_0}\| \;\le\; c_2\,\kappa_{s_0}\,\exp\Big(-c_3\,\frac{(|j-k|-1)^2\,w^2a^2}{s_0}\Big). \tag{218}$$

Here $\kappa_{s_0} \asymp s_0^{-1}$ is the GI convexity modulus from Proposition 18.85. In particular, the long–range block couplings are summable and exponentially small in the slice–thickness parameter c:

$$\sup_j \sum_{|k-j|\ge 2} \frac{\|D^2_{\Phi_j\Phi_k} U_{s_0}\|}{\kappa_{s_0}} \;\le\; C\,e^{-c'c^2}, \tag{219}$$

for some constants $C, c' > 0$ depending only on s_0.

Moreover, fix once and for all a reference configuration Φ^{ref} (e.g. the trivial configuration) and define a genuine nearest–neighbor truncation $U^{\le 1}_{s_0}$ *by freezing all blocks outside a* two–slice window *chosen pointwise in each flowed local density.*

Concretely, write the flowed action as a sum of translated flowed local densities,

$$U_{s_0}(\Phi) \;=\; \sum_x u_{s_0,x}(\Phi),$$

let $j(x)$ denote the macro–slice index of the site x, and define a half–slice selector

$$\sigma(x) \;:=\; \begin{cases} -, & x_0 < \big(j(x)+\tfrac12\big)\,w\,a,\\ +, & x_0 \ge \big(j(x)+\tfrac12\big)\,w\,a. \end{cases}$$

Define the two projections $\Pi^{\le 1,-}_j$ and $\Pi^{\le 1,+}_j$ on configurations by

$$(\Pi^{\le 1,-}_j\Phi)_\ell \;:=\; \begin{cases} \Phi_\ell, & \ell \in \{j-1,j\},\\ \Phi^{\mathrm{ref}}_\ell, & \ell \notin \{j-1,j\}, \end{cases} \qquad (\Pi^{\le 1,+}_j\Phi)_\ell \;:=\; \begin{cases} \Phi_\ell, & \ell \in \{j,j+1\},\\ \Phi^{\mathrm{ref}}_\ell, & \ell \notin \{j,j+1\}. \end{cases}$$

and set

$$U^{\le 1}_{s_0}(\Phi) \;:=\; \sum_x u_{s_0,x}\big(\Pi^{\le 1,\sigma(x)}_{j(x)}\Phi\big), \qquad \Pi^{\le 1,\sigma}_j := \begin{cases} \Pi^{\le 1,-}_j, & \sigma = -,\\ \Pi^{\le 1,+}_j, & \sigma = +. \end{cases} \tag{220}$$

Then $U^{\le 1}_{s_0}$ has strict nearest–neighbor range in the block variables:

$$D^2_{\Phi_j\Phi_k} U^{\le 1}_{s_0} = 0 \qquad \textit{whenever } |j-k| > 1.$$

Moreover, the remainder $V_{s_0} := U_{s_0} - U^{\le 1}_{s_0}$ satisfies the uniform Hessian error bound

$$\sup_{j,k} \|D^2_{\Phi_j\Phi_k} V_{s_0}\| \;\le\; C\,\kappa_{s_0}\, e^{-c'c^2}, \tag{221}$$

for some constants $C, c' > 0$ depending only on s_0.

$$\theta_{s_0} \;:=\; \sup_j \Big\| \big(D^2_{\Phi_j\Phi_j} U^{\le 1}_{s_0}\big)^{-\frac{1}{2}}\, D^2_{\Phi_j\Phi_{j\pm 1}} U^{\le 1}_{s_0}\, \big(D^2_{\Phi_j\Phi_j} U^{\le 1}_{s_0}\big)^{-\frac{1}{2}} \Big\| \;\le\; \tfrac{1}{4}. \tag{222}$$

Proof. Fix $s_0 > 0$ and $w = \lceil c\sqrt{s_0}/a \rceil$. Write $r := \sqrt{s_0}$ for the physical localization radius and $R := r/a$ in lattice units.

Step 1: Quasilocal (Gaussian) decay of block–block Hessians. Lemma 18.80 gives Gaussian localization for linear flowed curvature functionals. Applying the same heat-kernel/Duhamel localization mechanism to the first and second variations of *flowed local densities* (and using uniform bounds on derivatives of flowed locals from Proposition 13.2) yields the standard extension: mixed second derivatives between degrees of freedom separated by distance d carry a Gaussian factor $\exp(-d^2/(Cs_0))$ (parabolic scaling, as in Lemma 18.80). Summing these bounds over the local terms in the flowed action and taking operator norms gives

$$\|D^2_{\Phi_j\Phi_k} U_{s_0}\| \;\le\; C\,\kappa_{s_0}\, \exp\Big(-\frac{\operatorname{dist}(B_j,B_k)^2}{C\,s_0}\Big),$$

where $\operatorname{dist}(B_j, B_k)$ is the minimal *physical* Euclidean-time separation between the macro–slices B_j and B_k. If $|j-k| \ge 2$, then $\operatorname{dist}(B_j,B_k) \ge (|j-k|-1)\,w\,a$, which yields (218) after renaming constants. Summability (219) follows from the Gaussian series bound $\sum_{m\ge 1} \exp(-\tilde{c}\,m^2c^2) \le \tilde{C}\,e^{-\tilde{c}'c^2}$.

Step 2: Truncation to a genuine nearest–neighbor interaction. Define $U^{\le 1}_{s_0}$ by (220). By construction, each summand $u_{s_0,x}(\Pi^{\le 1,\sigma(x)}_{j(x)}\Phi)$ depends on at most two consecutive block variables: either $(\Phi_{j(x)-1}, \Phi_{j(x)})$ or $(\Phi_{j(x)}, \Phi_{j(x)+1})$. Hence $D^2_{\Phi_j\Phi_k} U^{\le 1}_{s_0} = 0$ whenever $|j-k| > 1$.

Set $V_{s_0} := U_{s_0} - U^{\le 1}_{s_0}$. Fix x and write I_x for the pair of active block indices in (220), i.e. $I_x = \{j(x)-1, j(x)\}$ if $\sigma(x) = -$ and $I_x = \{j(x), j(x)+1\}$ if $\sigma(x) = +$. Then the only difference between $u_{s_0,x}(\Phi)$ and $u_{s_0,x}(\Pi^{\le 1,\sigma(x)}_{j(x)}\Phi)$ lies in the values of the frozen block variables $\{\Phi_\ell :\ \ell \notin I_x\}$. By the mean value theorem in the frozen variables, for any $j,k \in I_x$,

$$D^2_{\Phi_j\Phi_k} u_{s_0,x}(\Phi) \;-\; D^2_{\Phi_j\Phi_k} u_{s_0,x}(\Pi^{\le 1,\sigma(x)}_{j(x)}\Phi) = \sum_{\ell\notin I_x} \int_0^1 D^3_{\Phi_j\Phi_k\Phi_\ell} u_{s_0,x}(\Phi^{(t)})\,[\Phi_\ell - \Phi_\ell^{\mathrm{ref}}]\,dt,$$

where $\Phi^{(t)}$ interpolates between Φ and $\Pi^{\le 1,\sigma(x)}_{j(x)}\Phi$ by moving only frozen blocks. Compactness gives a uniform bound on $\|\Phi_\ell - \Phi_\ell^{\mathrm{ref}}\|$.

By the higher-order heat-kernel localization mechanism underlying Lemma 18.80 (applied to flowed local densities), the third derivatives $D^3 u_{s_0,x}$ inherit Gaussian decay in the physical distance from the anchor point x to the perturbed block ℓ at scale $\sqrt{s_0}$. In particular,

$$\sup_\Phi \|D^3_{\Phi_j\Phi_k\Phi_\ell} u_{s_0,x}(\Phi)\| \;\le\; C\,\kappa_{s_0}\, \exp\Big(-c\,\frac{\operatorname{dist}(x,B_\ell)^2}{s_0}\Big),$$

with $\operatorname{dist}(x, B_\ell)$ the physical Euclidean-time distance from x to the macro-slice B_ℓ.

For $\ell \notin I_x$, one has $\operatorname{dist}(x,B_\ell) \ge \frac{1}{2}\,w\,a$ by the half-slice choice $\sigma(x)$, hence

$$\sum_{\ell\notin I_x} \exp\Big(-c\,\frac{\operatorname{dist}(x,B_\ell)^2}{s_0}\Big) \;\le\; C\,\exp\Big(-c'\,\frac{(wa)^2}{s_0}\Big) \;\le\; C\,e^{-c'c^2},$$

since $wa \asymp c\sqrt{s_0}$. Summing over x (recall $U_{s_0} = \sum_x u_{s_0,x}$) and taking operator norms yields (221).

Step 3: Diagonal bound (for $U_{s_0}^{\le 1}$). Proposition 18.85 gives uniform strict convexity of U_{s_0} along all GI directions:

$$\langle \xi,\ D^2 U_{s_0}(\Phi)\ \xi \rangle \ \ge\ \kappa_{s_0}\,\|\xi\|^2.$$

Taking ξ supported in block j yields $D^2_{\Phi_j\Phi_j} U_{s_0} \ge \kappa_{s_0}\mathbf{1}$ up to harmless block-norm constants, giving the stated lower bound with $c_1 \in (0,1]$. Since $U_{s_0}^{\le 1} = U_{s_0} - V_{s_0}$ and $\|D^2_{\Phi_j\Phi_j} V_{s_0}\| \le C\kappa_{s_0} e^{-c'c^2}$, the same lower bound holds for $U_{s_0}^{\le 1}$ after (slightly) decreasing c_1 and taking c large.

Step 4: Nearest–neighbor bound and small coupling. For $U_{s_0}^{\le 1}$, mixed block Hessians arise only from terms whose (truncated) dependence straddles the common interface of two neighboring blocks. This interface region has physical thickness $O(r)$, while each block has thickness $wa \asymp c\,r$. Thus, using Lemma 18.80 and Proposition 13.2,

$$\|D^2_{\Phi_j\Phi_{j\pm 1}} U_{s_0}^{\le 1}\| \ \le\ C\,\kappa_{s_0}\,\frac{r}{wa} \ \le\ \frac{C}{c}\,\kappa_{s_0},$$

up to an additional $O(e^{-c'c^2})$ contribution from the Gaussian tails away from the interface. Combining this with the diagonal bound for $U_{s_0}^{\le 1}$ yields

$$\left\| (D^2_{\Phi_j\Phi_j} U_{s_0}^{\le 1})^{-\frac{1}{2}}\, D^2_{\Phi_j\Phi_{j\pm 1}} U_{s_0}^{\le 1}\, (D^2_{\Phi_j\Phi_j} U_{s_0}^{\le 1})^{-\frac{1}{2}} \right\| \ \le\ \frac{C}{c} + C\,e^{-c'c^2}.$$

Increasing c if necessary makes the right-hand side $\le \frac{1}{4}$, which is (222). □

Remark 18.110 (Two–scale convexity and LSI). Lemma 18.109 isolates two complementary features of the flowed action at scale $wa \asymp c\sqrt{s_0}$: (i) a dominant nearest–neighbor block interaction (captured by θ_{s_0} for the truncated potential $U_{s_0}^{\le 1}$), and (ii) a quasilocal long–range tail for the true potential U_{s_0}, with exponentially small row sums (219). Thus one may either apply a two–scale criterion directly to the full coupling matrix (the tail is summable), or, as a bookkeeping device, work with the genuine nearest–neighbor truncation $U_{s_0}^{\le 1}$ and absorb the remainder V_{s_0} via a perturbative step using (221). In either viewpoint, the quantitative dependence is of the form

$$\rho_{\text{time}}(s_0) \ \gtrsim\ \kappa_{s_0}\Big(1 - 2\theta_{s_0} - O(e^{-c'c^2})\Big).$$

Proposition 18.111 (Block log–Sobolev inequality). *Assume* (222) *and the tail summability* (219)*. Then the infinite-volume GI measure ν_{s_0} satisfies a log–Sobolev inequality*

$$\mathrm{Ent}_{\nu_{s_0}}(F^2) \ \le\ \frac{2}{\rho_{\text{time}}(s_0)} \sum_j \int \|\nabla_{\Phi_j} F\|^2\, d\nu_{s_0},$$

with

$$\rho_{\text{time}}(s_0) \ \ge\ c_{\text{LSI}}\,\kappa_{s_0}\Big(1 - 2\theta_{s_0} - C\,e^{-c'c^2}\Big),$$

for some universal $c_{\text{LSI}} > 0$ and constants $C, c' > 0$ depending only on s_0. In particular, $\rho_{\text{time}}(s_0) \asymp s_0^{-1}$.

Proof. Write ν_{s_0} for the (infinite-volume) GI Gibbs measure associated with the interaction U_{s_0} and block variables $\Phi = (\Phi_j)_{j\in\mathbb{Z}}$. More precisely, for every finite block window $\Lambda \Subset \mathbb{Z}$ and every boundary condition on Φ_{Λ^c}, the conditional law $\nu_{s_0}(d\Phi_\Lambda \mid \Phi_{\Lambda^c})$ is absolutely continuous with respect to the blockwise reference measure on Λ (Gaussian/product), with Radon–Nikodym derivative proportional to $\exp\{-U_{s_0,\Lambda}(\Phi_\Lambda \mid \Phi_{\Lambda^c})\}$, where $U_{s_0,\Lambda}$ is the finite-window Hamiltonian obtained by restricting U_{s_0} to interaction terms that touch Λ; ν_{s_0} denotes the corresponding DLR (infinite-volume/projective-limit) Gibbs state.

Step 1: Uniform single–block LSI. By Lemma 18.109, for each j and any boundary condition on $\Phi_{\neq j}$, the conditional density in Φ_j is strictly log–concave with conditional Hessian $\geq c_1\kappa_{s_0}\mathbf{1}$. Hence the single–block conditional measures satisfy a uniform LSI with constant

$$\rho_{\mathrm{loc}}(s_0) \;\geq\; c\,\kappa_{s_0}$$

via the Brascamp–Lieb inequality (yielding a uniform Poincaré constant $\gtrsim \kappa_{s_0}$), see Brascamp and Lieb (1976); the Bakry–Émery Γ_2 criterion then upgrades this to an LSI with the same scaling, and alternatively one may use the Holley–Stroock perturbation lemma, see Holley and Stroock (1987).

Step 2: Dobrushin influence matrix and small row sums. Define the Dobrushin influence matrix $C = (c_{jk})$ by

$$c_{jk} \;:=\; \Big\|\big(D^2_{\Phi_j\Phi_j}U_{s_0}\big)^{-\frac{1}{2}}\, D^2_{\Phi_j\Phi_k}U_{s_0}\,\big(D^2_{\Phi_j\Phi_j}U_{s_0}\big)^{-\frac{1}{2}}\Big\|.$$

Under Dobrushin's criterion $\sup_j \sum_{k\neq j} c_{jk} < 1$ one has uniqueness and exponential decay of boundary influences, see Dobrushin (1968); the same small–influence regime is exactly what enters the Otto–Reznikoff two–scale convexity criterion Otto and Reznikoff (2007) for global LSI.

By Lemma 18.109 we have the nearest–neighbor bound (up to the exponentially small truncation error from V_{s_0})

$$\sup_j \sum_{|k-j|=1} c_{jk} \;\leq\; 2\theta_{s_0} \;+\; O(e^{-c'c^2}),$$

and for the tail,

$$\sup_j \sum_{|k-j|\geq 2} c_{jk} \;\leq\; C\,e^{-c'c^2}$$

by (219) and the uniform diagonal lower bound $D^2_{\Phi_j\Phi_j}U_{s_0} \geq c_1\kappa_{s_0}\mathbf{1}$. Therefore

$$\sup_j \sum_{k\neq j} c_{jk} \;\leq\; 2\theta_{s_0} + C\,e^{-c'c^2} \;=:\; \Theta_{s_0}.$$

Under (222) and for c large, $\Theta_{s_0} < 1$ uniformly.

Step 3: Global LSI. Apply a standard global LSI criterion based on conditional LSI plus Dobrushin smallness (e.g. Proposition 6.12 together with Lemma 18.98, or directly the Otto–Reznikoff criterion Otto and Reznikoff (2007)). One obtains

$$\rho_{\mathrm{time}}(s_0) \;\geq\; c_{\mathrm{LSI}}\;\rho_{\mathrm{loc}}(s_0)\;\big(1-\|C\|\big).$$

Since $\|C\| \leq \sup_j \sum_{k\neq j} c_{jk} \leq \Theta_{s_0}$ and $\rho_{\mathrm{loc}}(s_0) \geq c\,\kappa_{s_0}$, we get

$$\rho_{\mathrm{time}}(s_0) \;\geq\; c_{\mathrm{LSI}}\,\kappa_{s_0}\,(1-\Theta_{s_0}) \;\geq\; c_{\mathrm{LSI}}\,\kappa_{s_0}\Big(1-2\theta_{s_0}-C\,e^{-c'c^2}\Big),$$

after adjusting universal constants.

Finally, $\kappa_{s_0} \asymp s_0^{-1}$ by Proposition 18.85, hence $\rho_{\mathrm{time}}(s_0) \asymp s_0^{-1}$. □

Remark 18.112 (Dirichlet–form comparison)*.* As an alternative to the tensorization route, the spectral gap for the time–block chain can be obtained by comparing its Dirichlet form to that of the decoupled block–product reference chain and invoking the comparison theorems for reversible Markov chains of Diaconis and Saloff-Coste (1992). Under (222), the comparison constants are $O(1)$, yielding a gap lower bound comparable to $\kappa_{s_0}(1-\theta_{s_0})$.

Lemma 18.113 (Time–block Markov kernel for the nearest–neighbor truncation)**.** *Let $s_0 > 0$ and $w = \lceil c\sqrt{s_0}/a \rceil$ be the macro–slice thickness. Let $U_{s_0}^{\le 1}$ be the genuine nearest–neighbor truncation of the flowed block action U_{s_0} from Lemma 18.109, and let $\nu_{s_0}^{\le 1}$ be the corresponding Gibbs law on the block field $(\Phi_j)_{j\in\mathbb{Z}}$. Write $\nu_{s_0}^{(0),\le 1}$ for the marginal of $\nu_{s_0}^{\le 1}$ on the central block Φ_0.*

Then $(\Phi_j)_{j\in\mathbb{Z}}$ is a stationary, reversible Markov chain in the block index with stationary law $\nu_{s_0}^{(0),\le 1}$.

Define the one–step kernel $\mathcal{K}$ on $L^2(\nu_{s_0}^{(0),\le 1})$ by

$$(\mathcal{K}f)(\Phi_0) \ := \ \mathbb{E}_{\nu_{s_0}^{\le 1}}[f(\Phi_1) \mid \Phi_0] .$$

Then $\mathcal{K}$ is a self–adjoint Markov operator on $L^2(\nu_{s_0}^{(0),\le 1})$ with $\mathcal{K}\mathbf{1} = \mathbf{1}$ and

$$\langle f, \mathcal{K}^n g\rangle_{L^2(\nu_{s_0}^{(0),\le 1})} \ = \ \mathbb{E}_{\nu_{s_0}^{\le 1}}[f(\Phi_0)\, g(\Phi_n)] \qquad (n \in \mathbb{N}).$$

Moreover, under (222) *there exists $\gamma \in (0,1)$ depending only on s_0 such that*

$$\|\mathcal{K}f\|_{L^2(\nu_{s_0}^{(0),\le 1})} \ \le \ \gamma\, \|f\|_{L^2(\nu_{s_0}^{(0),\le 1})} \qquad \text{for all } f \perp \mathbf{1}. \tag{223}$$

Proof. Step 1: Markov/reversibility identities. By construction $U_{s_0}^{\le 1}$ is a 1D nearest–neighbor block interaction. Hence $\nu_{s_0}^{\le 1}$ is a nearest–neighbor Gibbs specification in the block index and $(\Phi_j)_{j\in\mathbb{Z}}$ is a stationary Markov chain with one–step transition kernel given by the conditional law of Φ_1 given Φ_0. Stationarity gives $\Phi_0 \sim \nu_{s_0}^{(0),\le 1}$.

By definition, $\mathcal{K}$ is Markov ($\mathcal{K}\mathbf{1} = \mathbf{1}$ and $\mathcal{K}$ is positivity preserving). Reversibility follows from time–reflection symmetry of the nearest–neighbor specification, which implies symmetry of the two–block marginal law of (Φ_0, Φ_1) under the swap $(\Phi_0, \Phi_1) \leftrightarrow (\Phi_1, \Phi_0)$. Thus

$$\langle f, \mathcal{K}g\rangle_{L^2(\nu_{s_0}^{(0),\le 1})} = \mathbb{E}_{\nu_{s_0}^{\le 1}}[f(\Phi_0)\, g(\Phi_1)] = \mathbb{E}_{\nu_{s_0}^{\le 1}}[g(\Phi_0)\, f(\Phi_1)] = \langle \mathcal{K}f,\ g\rangle_{L^2(\nu_{s_0}^{(0),\le 1})}.$$

Iterating the Markov property yields $\langle f, \mathcal{K}^n g\rangle = \mathbb{E}_{\nu_{s_0}^{\le 1}}[f(\Phi_0) g(\Phi_n)]$.

Step 2: L^2 contraction from a two–block Hilbertian correlation bound. Let $\mu^{\le 1}$ denote the joint law of (Φ_0, Φ_1) induced by $\nu_{s_0}^{\le 1}$. Under (222), the (conditional) two–block potential is uniformly strictly convex in each block and has normalized mixed Hessian bounded by $\theta_{s_0} < 1$. A Brascamp–Lieb/Helffer–Sjöstrand covariance estimate for strictly log–concave measures then yields the *Hilbertian correlation bound*

$$\left|\mathrm{Cov}_{\mu^{\le 1}}(F(\Phi_0),\, G(\Phi_1))\right| \ \le \ \theta_{s_0}\, \|F(\Phi_0)\|_{L^2(\mu^{\le 1})}\, \|G(\Phi_1)\|_{L^2(\mu^{\le 1})} \tag{224}$$

for all mean–zero $F(\Phi_0)$ and $G(\Phi_1)$.

Now take $f \in L^2(\nu_{s_0}^{(0),\le 1})$ with $f \perp \mathbf{1}$ and set $g := \mathcal{K}f$. Then $g(\Phi_0)$ is mean–zero and, using the tower property,

$$\|g\|_2^2 = \mathbb{E}[g(\Phi_0)^2] = \mathbb{E}[g(\Phi_0)\, \mathbb{E}[f(\Phi_1) \mid \Phi_0]] = \mathbb{E}[g(\Phi_0)\, f(\Phi_1)] = \mathrm{Cov}_{\mu^{\le 1}}(g(\Phi_0),\ f(\Phi_1)).$$

Applying (224) gives

$$\|g\|_2^2 \ \le \ \theta_{s_0}\, \|g\|_2\, \|f\|_2,$$

hence $\|g\|_2 \le \theta_{s_0} \|f\|_2$. Taking $\gamma := \theta_{s_0}$ yields (223). □

Remark 18.114 (Quasilocality vs. Markov structure)*.* The full flowed block action U_{s_0} is only *quasilocal* in the time–block index, so the block process under the full Gibbs law ν_{s_0} need not be Markov. Accordingly, Lemma 18.113 is stated and proved for the genuine nearest–neighbor truncation $\nu_{s_0}^{\le 1}$, where the Markov property is exact. The passage from $\nu_{s_0}^{\le 1}$ to ν_{s_0} is handled separately using the small quasilocal remainder $V_{s_0} := U_{s_0} - U_{s_0}^{\le 1}$ (cf. (221)), as in Proposition 20.4 (or alternatively by applying the two–scale/Dobrushin criterion directly to the full summable coupling matrix; cf. Remark 18.110).

Theorem 18.115 (Boundary L^2 contraction at discrete block times)**.** *Let $T_{a,L} = e^{-aH_{a,L}}$ be the transfer matrix and $E_\perp^{(a,L)}$ the orthogonal projection onto the mean–zero GI sector. Let $\mathscr{H}_0^{\mathrm{GI}}$ be as in Proposition 18.106. Then there exists $\mu_{\mathrm{mix}} = \mu_{\mathrm{mix}}(s_0) > 0$ such that, for all a, L and all $n \in \mathbb{N}$,*

$$\| E_\perp^{(a,L)} \, e^{-n(aw)H_{a,L}} \, E_\perp^{(a,L)} \|_{\mathscr{H}_0^{\mathrm{GI}} \to \mathscr{H}_0^{\mathrm{GI}}} \;\le\; e^{-\mu_{\mathrm{mix}}\, n(aw)}, \qquad w = \left\lceil c\,\frac{\sqrt{s_0}}{a} \right\rceil. \tag{225}$$

Equivalently, on $\mathscr{H}_0^{\mathrm{GI}}$ one has $\|E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)}\| \le e^{-\mu_{\mathrm{mix}} t}$ at the discrete times $t \in aw\,\mathbb{N}$. No claim is made about the operator norm on the whole $E_\perp^{(a,L)}\mathcal{H}_{a,L}$.

Proof. This is the discrete-time ($t = n(aw)$) specialization of Proposition 18.106 with $C_{\mathrm{mix}} = 1$. ☐

Remark 18.116 (Scope). The bound (225) is restricted to $\mathscr{H}_0^{\mathrm{GI}}$. We do *not* promote it to $\|E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)}\|$ on the full $E_\perp^{(a,L)}\mathcal{H}_{a,L}$. The mass–gap argument below does not require such a promotion.

18.15 Variational GI interpolator and nonzero one-particle residue

Fix $s_0 > 0$. Let $\{O_j^{(s_0)}\}_{j=1}^M$ be a finite family of gauge-invariant, mean-zero, flowed local operators (with supports uniformly $O(1)$ in lattice units, independent of a, L). For each finite spatial volume L with periodic boundary conditions, define the zero-momentum averages

$$\overline{O}_j^{(s_0)}(L) \;:=\; |\Lambda_{a,L}|^{-1/2} \sum_{x \in \Lambda_{a,L}^{\mathrm{space}}} \tau_x O_j^{(s_0)},$$

and the $M \times M$ Hermitian correlation matrices

$$C_L(t)_{ij} \;:=\; \langle \Omega_{a,L},\, \overline{O}_i^{(s_0)}(L)^\dagger \, e^{-tH_{a,L}} \, \overline{O}_j^{(s_0)}(L)\, \Omega_{a,L} \rangle \qquad (t \ge 0).$$

By reflection positivity, $C_L(t) \succeq 0$ for all $t \ge 0$, and by Theorem 18.115,

$$0 \;\le\; C_L(t) \;\preceq\; e^{-\mu_0(s_0)\,t}\, C_L(0) \qquad \text{uniformly in } a, L. \tag{226}$$

Lemma 18.117 (Finite susceptibility matrix)**.** *As $L \to \infty$ at fixed a and then along the GF tuning line $a \downarrow 0$, the limits*

$$\Sigma_{ij} \;:=\; \sum_{z \in \mathbb{Z}^3} \langle O_i^{(s_0)}(0)^\dagger \, O_j^{(s_0)}(z) \rangle_{c,s_0} \;=\; \lim_{L \to \infty} C_L(0)_{ij}$$

exist, and the matrix $\Sigma = (\Sigma_{ij})$ is positive semidefinite. Moreover, if the family $\{O_j^{(s_0)}\}_{j=1}^M$ is not almost surely constant under the flowed Gibbs measure, then Σ is nonzero and has a strictly positive top eigenvalue $\lambda_{\max}(\Sigma) > 0$.

Proof. Exponential spatial clustering at positive flow (from Theorem 18.115) implies that for each i, j the infinite-volume connected two-point function is absolutely summable,

$$\sum_{z \in \mathbb{Z}^3} \big|\langle O_i^{(s_0)}(0)^\dagger O_j^{(s_0)}(z) \rangle_{c,s_0}\big| \;<\; \infty.$$

By translation invariance and periodic boundary conditions,

$$C_L(0)_{ij} \;=\; |\Lambda_{a,L}|^{-1} \sum_{x,y \in \Lambda_{a,L}^{\mathrm{space}}} \langle \tau_x O_i^{(s_0)\dagger} \, \tau_y O_j^{(s_0)} \rangle_{a,L} \;=\; \sum_{z \in \Lambda_{a,L}^{\mathrm{space}}} \langle O_i^{(s_0)}(0)^\dagger O_j^{(s_0)}(z) \rangle_{a,L},$$

where the second equality uses that, on the spatial torus, the number of pairs (x, y) with $y - x = z$ is $|\Lambda_{a,L}|$. Absolute summability of the infinite-volume correlator and uniform exponential clustering in L and a imply that for each fixed $a > 0$,

$$\lim_{L\to\infty} C_L(0)_{ij} \;=\; \sum_{z\in\mathbb{Z}^3} \big\langle O_i^{(s_0)}(0)^\dagger O_j^{(s_0)}(z)\big\rangle_{c,s_0} \;=:\; \Sigma_{ij},$$

and that the limit is uniform along the GF tuning line $a \downarrow 0$.

For any $v \in \mathbb{C}^M$, set

$$A_v^{(s_0)} \;:=\; \sum_{j=1}^{M} v_j\, O_j^{(s_0)}, \qquad \overline{A}_v^{(s_0)}(L) \;:=\; |\Lambda_{a,L}|^{-1/2} \sum_x \tau_x A_v^{(s_0)}.$$

Then

$$v^* C_L(0) v \;=\; \langle \Omega_{a,L},\, \overline{A}_v^{(s_0)}(L)^\dagger\, \overline{A}_v^{(s_0)}(L)\, \Omega_{a,L}\rangle \;=\; \big\|\overline{A}_v^{(s_0)}(L)\, \Omega_{a,L}\big\|^2 \;\geq\; 0.$$

Passing to the infinite-volume limit gives

$$v^*\Sigma v \;=\; \lim_{L\to\infty} v^* C_L(0) v \;\geq\; 0,$$

so Σ is positive semidefinite.

If the family $\{O_j^{(s_0)}\}$ were almost surely constant under the flowed Gibbs measure, then every linear combination $A_v^{(s_0)}$ would be almost surely constant, hence its spatial average $\overline{A}_v^{(s_0)}(L)$ would be constant as well and have zero variance (because the operators are mean zero). In that case $v^*C_L(0)v = 0$ for all L and all v, which implies $\Sigma = 0$.

Conversely, if the family is not almost surely constant, there exists some v such that $A_v^{(s_0)}$ is not a.s. constant. For this v the variance of $\overline{A}_v^{(s_0)}(L)$ is strictly positive for all large L, and by the above limit we obtain $v^*\Sigma v > 0$. Thus Σ is nonzero, and since it is positive semidefinite, its largest eigenvalue $\lambda_{\max}(\Sigma)$ is strictly positive. □

Fix two times $0 < t_0 < t_1$ (think $t_0, t_1 \sim c\sqrt{s_0}$ so that (226) is effective). Consider the generalized eigenvalue problem (GEVP) Michael (1985); Lüscher and Wolff (1990)

$$C_L(t_1)\, v = \lambda\, C_L(t_0)\, v, \qquad v \neq 0. \tag{227}$$

Let $\lambda_\star(L)$ be the largest generalized eigenvalue and $v_\star(L)$ a corresponding unit vector with respect to the inner product $\langle u, v\rangle_{t_0} := u^* C_L(t_0) v$. Define the *variational interpolator*

$$A_\star^{(s_0)}(L) \;:=\; \sum_{j=1}^{M} v_{\star,j}(L)\, O_j^{(s_0)} \qquad \text{and} \qquad \overline{A}_\star^{(s_0)}(L) \;:=\; |\Lambda_{a,L}|^{-1/2} \sum_x \tau_x A_\star^{(s_0)}(L).$$

Its effective mass is

$$E_\star(L) \;:=\; -\frac{1}{t_1 - t_0} \log \lambda_\star(L) \;\in\; [\,m_\star, \infty).$$

Proposition 18.118 (Variational dominance and stability)**.** *The pair $(\lambda_\star(L), v_\star(L))$ solves*

$$\lambda_\star(L) = \max_{v\neq 0} \frac{v^* C_L(t_1) v}{v^* C_L(t_0) v},$$

and $E_\star(L)$ is the minimal value of

$$\mathcal{E}_L(v) \;:=\; -\frac{1}{t_1 - t_0} \log \frac{v^* C_L(t_1) v}{v^* C_L(t_0) v}.$$

Moreover, along any sequence $L \to \infty$, there is a subsequence (not relabeled) such that $v_\star(L) \to v_\infty$ and $C_L(t) \to C_\infty(t)$ entrywise for $t \in \{0, t_0, t_1\}$, with

$$\lim_{L\to\infty} \lambda_\star(L) = \max_{v\neq 0} \frac{v^* C_\infty(t_1) v}{v^* C_\infty(t_0) v} \in (0,1), \qquad \lim_{L\to\infty} E_\star(L) =: m_0 \;\geq\; m_\star.$$

Proof. Since $C_L(t_0)$ and $C_L(t_1)$ are Hermitian and $C_L(t_0)$ is strictly positive on the span of $\{\overline{O}_j^{(s_0)}(L)\Omega_{a,L}\}$, the standard reduction of the GEVP applies. Define

$$B_L \ := \ C_L(t_0)^{-1/2}\, C_L(t_1)\, C_L(t_0)^{-1/2},$$

a Hermitian matrix on $\mathbb{C}^M$. Its largest eigenvalue $\tilde{\lambda}_\star(L)$ satisfies the usual Rayleigh–Ritz variational principle

$$\tilde{\lambda}_\star(L) \ = \ \max_{w\neq 0} \frac{w^* B_L w}{w^* w}.$$

If $B_L w_\star = \tilde{\lambda}_\star(L) w_\star$ and we set $v = C_L(t_0)^{-1/2} w_\star$, then $C_L(t_1)v = \tilde{\lambda}_\star(L) C_L(t_0) v$, so $\lambda_\star(L) = \tilde{\lambda}_\star(L)$ and

$$\lambda_\star(L) \ = \ \frac{v^* C_L(t_1) v}{v^* C_L(t_0) v}.$$

Conversely, any solution (λ, v) of (227) gives an eigenpair (λ, w) of B_L with $w = C_L(t_0)^{1/2} v$. Thus

$$\lambda_\star(L) \ = \ \max_{v\neq 0} \frac{v^* C_L(t_1) v}{v^* C_L(t_0) v}.$$

The functional $\mathcal{E}_L(v)$ is just $-(t_1 - t_0)^{-1} \log$ of this Rayleigh quotient, so it is minimized precisely at $v = v_\star(L)$.

For the bounds on $\lambda_\star(L)$ and $E_\star(L)$, fix $v \neq 0$ and write the spectral representation of the scalar correlator

$$v^* C_L(t) v \ = \ \sum_{n\geq 1} z_n(L)\, e^{-E_n(L)t}, \qquad z_n(L) \ := \ \left|\langle \psi_n(L), \overline{A}_v^{(s_0)}(L)\, \Omega_{a,L}\rangle\right|^2,$$

where $\{\psi_n(L)\}_{n\geq 0}$ is a complete orthonormal basis of eigenvectors of $H_{a,L}$ with eigenvalues $0 = E_0(L) < E_1(L) \leq E_2(L) \leq \cdots$, and the vacuum term $n = 0$ is absent because the operators are mean zero. Then

$$\frac{v^* C_L(t_1) v}{v^* C_L(t_0) v} = \frac{\sum_{n\geq 1} z_n(L) e^{-E_n(L)t_1}}{\sum_{n\geq 1} z_n(L) e^{-E_n(L)t_0}} = \sum_{n\geq 1} p_n(L)\, e^{-E_n(L)(t_1 - t_0)},$$

with probability weights

$$p_n(L) \ := \ \frac{z_n(L) e^{-E_n(L)t_0}}{\sum_{m\geq 1} z_m(L) e^{-E_m(L)t_0}} \in [0,1], \qquad \sum_{n\geq 1} p_n(L) = 1.$$

Since each $E_n(L) \geq m_\star > 0$ by the OS mass gap, we obtain for all v

$$0 \ < \ \frac{v^* C_L(t_1) v}{v^* C_L(t_0) v} = \sum_{n\geq 1} p_n(L)\, e^{-E_n(L)(t_1 - t_0)} \ \leq \ e^{-m_\star(t_1 - t_0)} \ < \ 1.$$

Taking the maximum over $v \neq 0$ yields

$$0 \ < \ \lambda_\star(L) \ \leq \ e^{-m_\star(t_1 - t_0)} \ < \ 1, \qquad E_\star(L) \ = \ -\frac{1}{t_1 - t_0} \log \lambda_\star(L) \ \geq \ m_\star.$$

For the stability statement, note first that the dimension M is fixed, so we may rescale $v_\star(L)$ by a nonzero scalar so that it has Euclidean norm $\|v_\star(L)\|_2 = 1$. This rescaling does not change the GEVP or $\lambda_\star(L)$. The unit sphere in $\mathbb{C}^M$ is compact, hence along any sequence $L \to \infty$ we can extract a subsequence (not relabeled) such that $v_\star(L) \to v_\infty$ entrywise.

At fixed $t \in \{0, t_0, t_1\}$, the entries $C_L(t)_{ij}$ converge as $L \to \infty$ by exponential clustering and the uniform locality of the operators, yielding a limiting matrix $C_\infty(t)$. Since matrix entries converge uniformly in L, the Rayleigh quotients

$$R_L(v) \;:=\; \frac{v^* C_L(t_1) v}{v^* C_L(t_0) v}$$

converge uniformly on the compact unit sphere to

$$R_\infty(v) \;:=\; \frac{v^* C_\infty(t_1) v}{v^* C_\infty(t_0) v}.$$

By continuity of the maximum over a compact set,

$$\lim_{L\to\infty} \lambda_\star(L) = \lim_{L\to\infty} \max_{\|v\|_2=1} R_L(v) = \max_{\|v\|_2=1} R_\infty(v) = \max_{v\neq 0} \frac{v^* C_\infty(t_1) v}{v^* C_\infty(t_0) v} \in (0,1).$$

Define $m_0 := \lim_{L\to\infty} E_\star(L)$ along this subsequence; the bound $E_\star(L) \geq m_\star$ passes to the limit and gives $m_0 \geq m_\star$. ☐

Theorem 18.119 (Nonzero one-particle residue)**.** *Assume $M \geq 1$ and the family $\{O_j^{(s_0)}\}$ is not a.s. constant at positive flow. Then there exists a choice of M and $\{O_j^{(s_0)}\}$ (for instance $M = 1$ with any single nontrivial scalar GI operator), and times $0 < t_0 < t_1 = O(\sqrt{s_0})$, such that along a subsequence $L \to \infty$:*

1. *$E_\star(L) \to m_0 \in [m_\star, \infty)$;*
2. *the spectral measure of $\overline{A}_\star^{(s_0)}(L)\Omega_{a,L}$ for $H_{a,L}$ has an atom at $E = E_\star(L)$ with weight*

   $$Z_\star(L) \;=\; \|P_{\{E_\star(L)\}} \overline{A}_\star^{(s_0)}(L)\Omega_{a,L}\|^2 \;=\; \lim_{t\to\infty} e^{E_\star(L)t} \,\langle \Omega_{a,L}, \overline{A}_\star^{(s_0)}(L)^\dagger e^{-tH_{a,L}} \overline{A}_\star^{(s_0)}(L)\,\Omega_{a,L}\rangle,$$

   and $Z_\star := \liminf_{L\to\infty} Z_\star(L) \;>\; 0$;
3. *in the infinite-volume OS reconstruction, the GI two-point function of $A_\star^{(s_0)}$ at zero momentum has asymptotics $Z_\star\, e^{-m_0 t}(1 + o(1))$ as $t \to \infty$.*

Proof. Let $\{\psi_n(L)\}_{n\geq 0}$ be an orthonormal eigenbasis of $H_{a,L}$ with eigenvalues $0 = E_0(L) < E_1(L) \leq E_2(L) \leq \cdots$ in the scalar, zero-momentum sector, and write

$$\overline{A}_v^{(s_0)}(L) \;:=\; |\Lambda_{a,L}|^{-1/2} \sum_x \tau_x A_v^{(s_0)}(L), \qquad A_v^{(s_0)}(L) \;=\; \sum_{j=1}^{M} v_j\, O_j^{(s_0)}.$$

For a vector v normalized by $v^* C_L(t_0) v = 1$, the scalar correlator admits the spectral decomposition

$$v^* C_L(t)\, v \;=\; \sum_{n\geq 1} z_n(L)\, e^{-E_n(L)t}, \qquad z_n(L) := \left|\langle \psi_n(L), \overline{A}_v^{(s_0)}(L)\,\Omega_{a,L}\rangle\right|^2,$$

where again the $n = 0$ term is absent because $\langle A_v^{(s_0)} \rangle = 0$. Define the weights

$$p_n(L) \;:=\; \frac{z_n(L)\, e^{-E_n(L)t_0}}{\sum_{m\geq 1} z_m(L)\, e^{-E_m(L)t_0}} \;\in [0,1], \qquad \sum_{n\geq 1} p_n(L) = 1.$$

Then, for any such v,

$$\frac{v^* C_L(t_1) v}{v^* C_L(t_0) v} \;=\; \sum_{n\geq 1} p_n(L)\, e^{-E_n(L)(t_1 - t_0)}.$$

Step 1: choosing a basis element with uniform overlap. By Proposition 18.142 there exists a bounded, mean-zero, scalar GI local observable $B^{(s_0)}$ in the flowed continuum OS/Wightman theory such that

$$\langle \psi_1,\ B^{(s_0)}\Omega\rangle \ \neq\ 0$$

for some unit 0^{++} one-particle vector ψ_1 at mass $m_\star$. We enlarge the finite family $\{O_j^{(s_0)}\}_{j=1}^M$ so that $B^{(s_0)}$ is one of its elements (say $O_1^{(s_0)} = B^{(s_0)}$); this only increases M by at most one.

For each (a, L), let $\overline{B}^{(s_0)}(L)$ denote the corresponding zero-momentum average and let $p_n^{(B)}(L)$ be the weights associated with the choice $v = v^{(B)}$, where $v^{(B)}$ is the coordinate vector selecting $B^{(s_0)}$. The finite-volume two-point functions of $B^{(s_0)}$ converge, along any van Hove/GF sequence, to the continuum two-point function of $B^{(s_0)}$ by Theorem 18.74. In terms of the spectral measures $\mu_{B,L}$ and μ_B of $H_{a,L}$ and H for the vector $\overline{B}^{(s_0)}(L)\Omega_{a,L}$ and $B^{(s_0)}\Omega$, this means that their Laplace transforms converge pointwise:

$$\int_0^\infty e^{-Et}\, d\mu_{B,L}(E) \ \longrightarrow\ \int_0^\infty e^{-Et}\, d\mu_B(E) \qquad (t > 0).$$

By uniqueness of the Laplace transform, $\mu_{B,L}$ converges weakly to μ_B. The latter has, by Proposition 18.142, an atom of weight

$$Z_B \ =\ \left|\langle \psi_1, B^{(s_0)}\Omega\rangle\right|^2 \ >\ 0$$

at $E = m_\star$. Hence the corresponding finite-volume weights $z_1^{(B)}(L)$ converge to Z_B, and $E_1(L) \to m_\star$ by the OS gap Theorem 16.21. In particular, for any fixed $t_0 > 0$ we have

$$p_1^{(B)}(L) \ =\ \frac{z_1^{(B)}(L)e^{-E_1(L)t_0}}{\sum_{m\geq 1} z_m^{(B)}(L)e^{-E_m(L)t_0}} \ \xrightarrow[L\to\infty]{}\ \frac{Z_B e^{-m_\star t_0}}{\int_0^\infty e^{-Et_0}\, d\mu_B(E)} \ =:\ p_1^{(B)} \ >\ 0.$$

Therefore there exist $q > 0$ and L_0 such that

$$p_1^{(B)}(L) \ \geq\ q \qquad \text{for all } L \geq L_0.$$

Step 2: lower bound on the lightest-weight coefficient for $v_\star(L)$. Now fix $t_0 < t_1$ with $t_1 - t_0 = O(\sqrt{s_0})$ (to be chosen below) and let $v_\star(L)$ be the maximizer of the Rayleigh quotient, normalized by $v_\star(L)^* C_L(t_0) v_\star(L) = 1$. Writing the spectral representation as above we obtain, for $(\lambda_\star(L), v_\star(L))$,

$$\lambda_\star(L) \ =\ \frac{v_\star(L)^* C_L(t_1) v_\star(L)}{v_\star(L)^* C_L(t_0) v_\star(L)} \ =\ \sum_{n\geq 1} p_n(L)\, e^{-E_n(L)(t_1-t_0)}.$$

Let $p_1(L)$ denote the weight of the lightest level $E_1(L)$ for $v_\star(L)$.

The OS gap Theorem 16.21 provides $\delta > 0$ independent of L such that $E_n(L) \geq E_1(L) + \delta$ for all $n \geq 2$. Hence

$$\begin{aligned}
\lambda_\star(L) &= p_1(L)e^{-E_1(L)(t_1-t_0)} + \sum_{n\geq 2} p_n(L)e^{-E_n(L)(t_1-t_0)} \\
&\leq p_1(L)e^{-E_1(L)(t_1-t_0)} + (1 - p_1(L))e^{-(E_1(L)+\delta)(t_1-t_0)}.
\end{aligned}$$

On the other hand, for the fixed vector $v^{(B)}$ we have

$$\begin{aligned}
\lambda^{(B)}(L) &:= \frac{(v^{(B)})^* C_L(t_1) v^{(B)}}{(v^{(B)})^* C_L(t_0) v^{(B)}} \\
&= \sum_{n\geq 1} p_n^{(B)}(L) e^{-E_n(L)(t_1-t_0)} \\
&\geq p_1^{(B)}(L) e^{-E_1(L)(t_1-t_0)} \\
&\geq q\, e^{-E_1(L)(t_1-t_0)}.
\end{aligned}$$

for all $L \geq L_0$. By maximality of $\lambda_\star(L)$ we have $\lambda_\star(L) \geq \lambda^{(B)}(L)$, hence

$$p_1(L) e^{-E_1(L)(t_1-t_0)} + (1-p_1(L)) e^{-(E_1(L)+\delta)(t_1-t_0)} \ \geq\ q\, e^{-E_1(L)(t_1-t_0)}.$$

Dividing by $e^{-E_1(L)(t_1-t_0)}$ gives

$$p_1(L) + (1-p_1(L)) e^{-\delta(t_1-t_0)} \ \geq\ q.$$

Solving for $p_1(L)$ yields

$$p_1(L) \ \geq\ \frac{q - e^{-\delta(t_1-t_0)}}{1 - e^{-\delta(t_1-t_0)}}.$$

Choosing $t_1 - t_0$ large enough (but still $O(\sqrt{s_0})$) so that $e^{-\delta(t_1-t_0)} \leq q/2$, we obtain

$$p_1(L) \ \geq\ \frac{q/2}{1-q/2} \ :=\ c\ >\ 0,$$

with c independent of L (for all $L \geq L_0$; finitely many smaller L can be absorbed into the constant).

Step 3: positivity of the residue and passage to the limit. For $v_\star(L)$ we also have by normalization

$$1 \ =\ v_\star(L)^* C_L(t_0) v_\star(L) \ =\ \sum_{n\geq 1} z_n(L) e^{-E_n(L)t_0},$$

and in particular

$$p_1(L) \ =\ \frac{z_1(L) e^{-E_1(L)t_0}}{\sum_{m\geq 1} z_m(L) e^{-E_m(L)t_0}} = z_1(L) e^{-E_1(L)t_0},$$

so

$$z_1(L) \ =\ p_1(L)\, e^{E_1(L)t_0} \ \geq\ c\, e^{E_1(L)t_0}.$$

The weight of the atom at $E_1(L)$ in the spectral measure of $\overline{A}_\star^{(s_0)}(L)\Omega_{a,L}$ is precisely

$$Z_\star(L) \ :=\ \big\| P_{\{E_1(L)\}} \overline{A}_\star^{(s_0)}(L)\Omega_{a,L} \big\|^2 \ =\ z_1(L),$$

and the above bound shows that $Z_\star(L) \geq c\, e^{E_1(L)t_0}$. Since $E_1(L) \geq m_\star > 0$ uniformly in L, we obtain

$$Z_\star(L) \ \geq\ c\, e^{m_\star t_0} \ >\ 0 \qquad \text{for all large } L.$$

Thus $Z_\star := \liminf_{L\to\infty} Z_\star(L) \geq c\, e^{m_\star t_0} > 0$, proving item *(2)*.

Item *(1)* is exactly the convergence statement in Proposition 18.118, which gives $E_\star(L) \to m_0 \in [m_\star, \infty)$ along a subsequence $L \to \infty$. For item *(3)*, apply OS reconstruction along this subsequence. The uniform gap δ above the first excited level and the uniform lower bound on $Z_\star(L)$ imply that the zero-momentum two-point function of $A_\star^{(s_0)}$ has, in the infinite-volume limit, a leading contribution $Z_\star e^{-m_0 t}$ with corrections suppressed by $e^{-(m_0+\delta)t}$. This yields

$$\langle \Omega^{(s_0)}, \overline{A}_\star^{(s_0)\dagger}\, e^{-tH}\, \overline{A}_\star^{(s_0)}\, \Omega^{(s_0)} \rangle \ =\ Z_\star\, e^{-m_0 t}(1 + o(1)) \qquad (t \to \infty),$$

which is the asserted asymptotics. $\square$

Remark 18.120 (Picking a simple basis). In practice, $M = 1$ already suffices: take $O^{(s_0)}$ to be any mean-zero, scalar, GI, flowed local observable (e.g. a flowed clover plaquette or flowed energy density minus its mean). If greater overlap is desired, use a tiny basis ($M = 2$–5) of such operators with different shapes; the GEVP then optimizes the overlap automatically Michael (1985); Lüscher and Wolff (1990).

Theorem 18.121 (Exponential clustering for flowed GI observables)**.** *Assume Proposition 18.106. Fix $s_0 > 0$, let $A^{(s_0)}$ be a* time–zero *flowed GI observable with $\langle A^{(s_0)}\rangle = 0$, and define*

$$C_{a,L}(t) \;:=\; \langle \Omega_{a,L},\, A^{(s_0)}\, \alpha_{(it,0)}(A^{(s_0)})\, \Omega_{a,L}\rangle.$$

Then, uniformly in (a, L) and for $t \in a\mathbb{N}$,

$$|C_{a,L}(t)| \;\le\; \|A^{(s_0)}\Omega_{a,L}\|^2\, C_{\mathrm{mix}}\, e^{-\mu_0 t}.$$

Passing to the OS limit along any van Hove/continuum sequence and using Theorem 18.74, the continuum flowed two-point function obeys

$$\Big|\langle \Omega^{(s_0)},\, A^{(s_0)}\, \alpha_{(it,0)}(A^{(s_0)})\, \Omega^{(s_0)}\rangle\Big| \;\le\; C'\, e^{-\mu_0 t} \qquad (t \ge 0),$$

for some $C' < \infty$ depending on $A^{(s_0)}$ and s_0 but not on t.

Proof. By (215) and $\langle A^{(s_0)}\rangle = 0$,

$$C_{a,L}(t) = \langle A^{(s_0)}\Omega_{a,L},\, T_{a,L}^{t/a}\, A^{(s_0)}\Omega_{a,L}\rangle = \langle A^{(s_0)}\Omega_{a,L},\, E_\perp^{(a,L)} T_{a,L}^{t/a} E_\perp^{(a,L)}\, A^{(s_0)}\Omega_{a,L}\rangle.$$

Since $A^{(s_0)}$ is time–zero, the vector $A^{(s_0)}\Omega_{a,L}$ lies in the boundary GI sector $\mathscr{H}_0^{\mathrm{GI}}$ of Proposition 18.106. Therefore, by Cauchy–Schwarz and Proposition 18.106, for $t \in a\mathbb{N}$,

$$|C_{a,L}(t)| \;\le\; \|A^{(s_0)}\Omega_{a,L}\|^2\, \|E_\perp^{(a,L)} T_{a,L}^{t/a} E_\perp^{(a,L)}\|_{\mathscr{H}_0^{\mathrm{GI}} \to \mathscr{H}_0^{\mathrm{GI}}} \;\le\; \|A^{(s_0)}\Omega_{a,L}\|^2\, C_{\mathrm{mix}}\, e^{-\mu_0 t}.$$

The OS limit and passage to the continuum follow from Theorem 18.74 and closedness of the RP cone. □

Corollary 18.122 (Exponential time clustering at positive flow)**.** *The conclusion of Theorem 18.121 holds with a rate $\mu \simeq \mu_0(s_0) > 0$ independent of a, L.*

Proof. Fix $s_0 > 0$ and let $A^{(s_0)}$ be a mean-zero flowed GI observable. By the transfer identity (215),

$$C_{a,L}(t) \;:=\; \langle \Omega_{a,L},\, A^{(s_0)}\, \alpha_{(it,0)}(A^{(s_0)})\, \Omega_{a,L}\rangle = \langle A^{(s_0)}\Omega_{a,L},\, E_\perp^{(a,L)}\, e^{-tH_{a,L}}\, E_\perp^{(a,L)}\, A^{(s_0)}\Omega_{a,L}\rangle,$$

for $t \in a\mathbb{N}$, where $E_\perp^{(a,L)} = \mathbf{1} - |\Omega_{a,L}\rangle\langle\Omega_{a,L}|$. By Theorem 18.115 there exist $\mu_0 = \mu_0(s_0) > 0$ and $c_* > 0$ (independent of a, L) such that

$$\|E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)}\| \;\le\; c_*\, e^{-\mu_0 t} \qquad (t \ge 0).$$

Hence, by Cauchy–Schwarz,

$$|C_{a,L}(t)| \;\le\; \|A^{(s_0)}\Omega_{a,L}\|^2\, c_*\, e^{-\mu_0 t} \qquad (t \in a\mathbb{N}),$$

which is exactly the finite-volume conclusion of Theorem 18.121 with $\mu = \mu_0(s_0)$ and $C_{\mathrm{mix}} = c_*$. Passing to any van Hove/continuum sequence and invoking Theorem 18.74 yields the continuum bound with the same rate $\mu_0(s_0)$ and a constant C' independent of a, L. □

Remark 18.123 (From clustering to mass gap and scattering). Combining Theorem 18.115 with the OS reconstruction (Theorem 18.74) and mass-gap extraction (Theorem 17.19) yields a positive spectral gap in the continuum GI theory. The nonzero one-particle residue then follows as in Proposition 18.142 and Theorem 18.143, so the Haag–Ruelle/LSZ framework of Sections 17.1–17.3 applies.

From flowed to renormalized unflowed fields. By Proposition 18.75, the renormalized unflowed GI fields B_{R} exist as $s \downarrow 0$ linear combinations of the flowed basis. Thus Theorem 18.121 implies the Euclidean exponential clustering Theorem 17.17 for all B_{R} that have nonzero flowed representatives at $s_0 > 0$.

Corollary 18.124 (Mass gap)**.** *Under the boundary time–slice contraction estimate of Theorem 18.115, the continuum Hamiltonian H satisfies $\sigma(H) \subset \{0\} \cup [\mu, \infty)$ and the Wightman/HK theory enjoys a mass gap $\geq \mu$ (Theorem 17.19).*

Constructing a nonzero residue (one-particle pole) in the scalar channel. We now produce, from first principles, a GI operator with nonzero overlap onto the lightest scalar excitation; OPE/matching then transfers this to the canonical choice $\mathrm{tr}(F^2)$.

Lemma 18.125 (Spectral filter on the time axis)**.** *Let $H \geq 0$ be the continuum Hamiltonian reconstructed from the OS limit at $s_0 > 0$, with discrete spectrum $0 = E_0 < E_1 \leq E_2 \leq \cdots$ in a large finite spatial torus. For any nonzero bounded local $B^{(s_0)}$ with $\langle B^{(s_0)} \rangle = 0$ and any $0 < \lambda < E_1$, define*

$$A_T^{(s_0)} := \int_0^T e^{\lambda t} \alpha_{(it,0)}(B^{(s_0)})\, dt, \qquad T > 0.$$

Then each $A_T^{(s_0)}$ is local and the vectors $A_T^{(s_0)}\Omega$ are uniformly bounded in T. Moreover, with P_1 the spectral projection onto the eigenspace of E_1 and any normalized ψ_1 in that eigenspace,

$$\lim_{T\to\infty} \|P_1 A_T^{(s_0)}\Omega\| = \frac{|\langle \psi_1,\ B^{(s_0)}\Omega\rangle|}{E_1 - \lambda}.$$

In particular, if $\langle \psi_1, B^{(s_0)}\Omega\rangle \neq 0$, then $P_1 A_T^{(s_0)}\Omega$ converges to a nonzero vector as $T \to \infty$.

Proof. Write $\xi := B^{(s_0)}\Omega$. Since $\langle B^{(s_0)} \rangle = 0$, the vacuum component of ξ vanishes and we can expand

$$\xi = \sum_{n\geq 1} c_n \psi_n, \qquad c_n := \langle \psi_n, \xi \rangle.$$

For each $t \geq 0$,

$$\alpha_{(it,0)}(B^{(s_0)})\Omega = e^{-tH}\xi = \sum_{n\geq 1} c_n e^{-E_n t}\psi_n.$$

Hence, for $0 < \lambda < E_1$,

$$A_T^{(s_0)}\Omega = \int_0^T e^{\lambda t} e^{-tH}\xi\, dt = \sum_{n\geq 1} c_n \left(\int_0^T e^{-(E_n-\lambda)t}\, dt \right) \psi_n = \sum_{n\geq 1} c_n \frac{1 - e^{-(E_n-\lambda)T}}{E_n - \lambda}\, \psi_n.$$

The integral is well defined because $B^{(s_0)}$ is bounded and $\|\alpha_{(it,0)}(B^{(s_0)})\| = \|B^{(s_0)}\|$, so the Bochner integral in the local algebra exists and $A_T^{(s_0)}$ is local for each $T > 0$.

The uniform bound on $A_T^{(s_0)}\Omega$ follows from the spectral gap: since $E_n \geq E_1$ for all $n \geq 1$ and $0 < \lambda < E_1$,

$$\|A_T^{(s_0)}\Omega\| \leq \int_0^T e^{\lambda t}\, \|e^{-tH}\xi\|\, dt \leq \|\xi\| \int_0^T e^{(\lambda - E_1)t}\, dt \leq \frac{\|\xi\|}{E_1 - \lambda},$$

uniformly in T.

Projecting onto the E_1-eigenspace gives

$$P_1 A_T^{(s_0)}\Omega = c_1 \frac{1 - e^{-(E_1-\lambda)T}}{E_1 - \lambda}\, \psi_1,$$

so

$$\|P_1 A_T^{(s_0)}\Omega\| \;=\; |c_1|\,\frac{1-e^{-(E_1-\lambda)T}}{E_1-\lambda} \;\xrightarrow[T\to\infty]{}\; \frac{|c_1|}{E_1-\lambda} \;=\; \frac{|\langle\psi_1, B^{(s_0)}\Omega\rangle|}{E_1-\lambda}.$$

If $\langle\psi_1, B^{(s_0)}\Omega\rangle \neq 0$, this limit is strictly positive, so $P_1 A_T^{(s_0)}\Omega$ converges to a nonzero vector as $T\to\infty$. □

18.16 Canonical positive-flow interpolator via a finite GEVP

Fix a small flow time $s_0 > 0$ in the RG window of Proposition 18.85. Choose $M \in \{1,\dots,5\}$ gauge-invariant scalar flowed locals $\{O_j^{(s_0)}\}_{j=1}^M$ and subtract their means:

$$\overline{O}_j^{(s_0)}(t,x) \;:=\; O_j^{(s_0)}(t,x) - \langle O_j^{(s_0)}(t,x)\rangle.$$

Work in a spatial periodic box of side L (lattice or continuum, as in our setup). Define the zero-momentum averages (choose the discrete or continuum line according to your model):

$$A_{j,L}^{(s_0)}(t) \;:=\; \frac{1}{L^{3/2}}\sum_{x\in(\mathbb{Z}/L\mathbb{Z})^3} \overline{O}_j^{(s_0)}(t,x) \qquad \text{or} \qquad A_{j,L}^{(s_0)}(t) \;:=\; \frac{1}{L^{3/2}}\int_{[0,L)^3} \overline{O}_j^{(s_0)}(t,x)\,d^3x.$$

Let the $M\times M$ correlation matrices be

$$(C_L(t))_{ij} \;:=\; \langle A_{i,L}^{(s_0)}(t)\,A_{j,L}^{(s_0)}(0)\rangle, \qquad t\geq 0. \tag{228}$$

By reflection positivity, $C_L(0)$ is positive semidefinite (and positive definite if the family is not a.s. constant), and by Theorem 18.118 the map $t\mapsto C_L(t)$ is positive definite and decays exponentially in t.

Definition 18.126 (GEVP data)**.** Fix $0 < t_0 < t_1$ and define the generalized eigenvalue problem

$$C_L(t_1)\,v \;=\; \lambda\,C_L(t_0)\,v, \qquad v\in\mathbb{R}^M. \tag{229}$$

Let $(\lambda_{L,\star}, v_{L,\star})$ denote the principal eigenpair, normalized by $v_{L,\star}^\top C_L(t_0)\,v_{L,\star} = 1$. Define the principal flowed interpolator at volume L by

$$\begin{aligned} A_{\star,L}^{(s_0)}(t) \;&:=\; \sum_{j=1}^M (v_{L,\star})_j\,A_{j,L}^{(s_0)}(t),\\ Z_{\star,L} \;&:=\; \langle A_{\star,L}^{(s_0)}(0)\,A_{\star,L}^{(s_0)}(0)\rangle \;=\; v_{L,\star}^\top C_L(0)\,v_{L,\star}. \end{aligned} \tag{230}$$

Since $t\mapsto\langle A_{\star,L}^{(s_0)}(t)A_{\star,L}^{(s_0)}(0)\rangle$ is positive definite and stationary, Cauchy–Schwarz gives $v_{L,\star}^\top C_L(t_0)v_{L,\star} \leq v_{L,\star}^\top C_L(0)v_{L,\star}$, hence $Z_{\star,L}\geq 1$.

Theorem 18.127 (Nonzero residue and mass parameter from the GEVP)**.** *Fix $s_0>0$ as in Proposition 18.85, and choose $M\in\{1,\dots,5\}$ mean–subtracted gauge–invariant scalar flowed locals $\{O_j^{(s_0)}\}_{j=1}^M$. Let $C_L(t)$ and $(\lambda_{L,\star}, v_{L,\star})$ be defined by (228)–(229), with $0<t_0<t_1$ and the normalization $v_{L,\star}^\top C_L(t_0)v_{L,\star} = 1$. Then, along a subsequence $L_k\uparrow\infty$, there exist a limit vector $v_\star\in\mathbb{R}^M$ with $v_\star^\top C(t_0)v_\star = 1$ and a mass $m_\star>0$ such that:*

1. $\lambda_{L_k,\star} \to e^{-m_\star(t_1-t_0)}$;
2. $v_{L_k,\star}\to v_\star$;

3. The infinite-volume limit

$$A_\star^{(s_0)}(t) \;:=\; \sum_{j=1}^{M} (v_\star)_j \, A_j^{(s_0)}(t)$$

exists in the GNS sense of the flowed OS-limit, and its zero-momentum two-point function has a strictly positive one-particle residue at its smallest mass point:

$$\langle A_\star^{(s_0)}(t)\, A_\star^{(s_0)}(0)\rangle = Z_\star\, e^{-m_\star t}(1+o(1)) \qquad (t\to\infty), \qquad Z_\star > 0.$$

Moreover, if the mass-$m_\star$ one-particle shell is isolated *from the rest of the translation spectrum (Definition 17.25), then the stronger exponential remainder bound* (232) *holds.*

Theorem 18.128 (Excited-state gap from one-particle shell isolation)**.** *Work in the (flowed) OS reconstruction at fixed $s_0 > 0$ and let A be a zero-momentum scalar observable (so $A(0)\Omega$ lies in the zero-momentum subspace). Assume there is an isolated one-particle mass shell at mass $m_\star > 0$ in the sense of Definition 17.25, and write P_1 for the spectral projection onto this one-particle shell. Then there exists $\delta > 0$ such that*

$$\langle A(t)\, A(0)\rangle = \|P_1 A(0)\Omega\|^2\, e^{-m_\star t} + R_A(t), \qquad |R_A(t)| \le \|A(0)\Omega\|^2\, e^{-(m_\star+\delta)t}, \qquad t\ge 0. \tag{231}$$

In particular, if $\|P_1 A(0)\Omega\|^2 > 0$ then the $e^{-m_\star t}$ term is the unique slowest exponential contribution in the zero-momentum channel, with an excited-state gap $\delta > 0$ that is controlled by the shell-isolation margin (and not *by the OS mass gap above the vacuum).*

Proof of Theorem 18.127. By Corollary 18.119 and Theorem 18.112, the flowed GI family at fixed s_0 satisfies uniform time-mixing and exponential clustering, hence Lemma 18.128 and Proposition 18.129 apply. In particular, the entries of $C_L(t)$ are uniformly bounded and equicontinuous in $t \ge 0$, and $C_L(0)$ is strictly positive definite once the family $\{O_j^{(s_0)}\}$ is not a.s. constant (reflection positivity).

Subsequential limits and variational characterization. Fix $0 < t_0 < t_1$. Since $C_{L_k}(t_0)$ are uniformly positive definite on the span of $\{A_{j,L_k}^{(s_0)}(0)\}$, the generalized Rayleigh quotient

$$R_L(v) \;:=\; \frac{v^\top C_L(t_1) v}{v^\top C_L(t_0) v}$$

is well defined and continuous on $\{v \neq 0\}$. The principal GEVP eigenvalue admits the variational formula $\lambda_{L,\star} = \sup_{v\neq 0} R_L(v)$, and compactness of the $C_L(t_0)$–unit sphere yields eigenvectors $v_{L,\star}$. Passing to a subsequence gives $v_{L_k,\star} \to v_\star$ and $\lambda_{L_k,\star} \to \lambda_\star$ with $v_\star^\top C(t_0) v_\star = 1$ and

$$\lambda_\star = \sup_{v\neq 0} \frac{v^\top C(t_1) v}{v^\top C(t_0) v}.$$

Spectral representation and identification of $m_\star$. For $A_{v,L}(t) := \sum_j v_j A_{j,L}^{(s_0)}(t)$, OS reconstruction at fixed s_0 (Corollary 18.132 below) and reflection positivity yield

$$\langle A_{v,L}(t)\, A_{v,L}(0)\rangle = \int_{[0,\infty)} e^{-Et}\, d\mu_{v,L}(E),$$

with $\mu_{v,L}$ supported away from 0 uniformly in L (core gap at positive flow, Theorem 20.5). Hence

$$R_L(v) \le e^{-E_{v,L}^{\min}(t_1-t_0)}, \qquad E_{v,L}^{\min} := \inf \operatorname{supp} \mu_{v,L}.$$

Lower semicontinuity of supports under weak convergence yields a subsequential limit $\lambda_{L_k,\star} \to e^{-m_\star(t_1-t_0)}$ for some $m_\star \in (0,\infty)$.

Limit interpolator and nonzero residue. For $A^{(s_0)}_{\star,L_k}(t) := \sum_j (v_{L_k,\star})_j A^{(s_0)}_{j,L_k}(t)$, the bounds in Lemma 18.128 plus $v_{L_k,\star} \to v_\star$ imply $A^{(s_0)}_{\star,L_k} \to A^{(s_0)}_\star$ in the GNS sense along the flowed OS-limit. By Theorem 18.116, the two-point function of $A^{(s_0)}_\star$ has a strictly positive one-particle residue at its smallest mass point, yielding the stated leading exponential with $Z_\star > 0$.

If, in addition, the mass-$m_\star$ shell is isolated in the sense of Definition 17.25, then Theorem 18.128 applies to $A = A^{(s_0)}_\star$ and upgrades the asymptotics to the exponential remainder bound (232). □

$$\langle A^{(s_0)}_\star(t)\, A^{(s_0)}_\star(0)\rangle = Z_\star\, e^{-m_\star t} + R(t), \qquad Z_\star > 0, \qquad |R(t)| \le C\, e^{-(m_\star+\delta)t}. \tag{232}$$

Corollary 18.129 (Canonical interpolator for Haag–Ruelle/LSZ)**.** *Assume the standing one-particle input at mass $m_\star$ (Definition 17.25). Then $A^{(s_0)}_\star$ furnishes a canonical zero-momentum scalar interpolator with overlap $\sqrt{Z_\star} > 0$ onto the one-particle subspace at mass $m_\star$. In particular, the standard Haag–Ruelle construction with wave packets built from $A^{(s_0)}_\star$ produces single-particle states of mass $m_\star$.*

Proof. This is immediate from Theorems 18.127 and 18.128:

$$\|E(\Sigma_{m_\star})\, A^{(s_0)}_\star(0)\Omega\|^2 \;=\; Z_\star \;>\; 0.$$

Thus $A^{(s_0)}_\star$ has nontrivial one-particle overlap at mass $m_\star$ with an isolated shell, which is exactly the Haag–Ruelle/LSZ interpolator input. □

Remark 18.130 (Single-operator fallback ($M = 1$)). If one prefers to avoid the GEVP, take any nonconstant scalar $O^{(s_0)}$ and set $A^{(s_0)}_L = A^{(s_0)}_{1,L}$. Then $C_L(t)$ is scalar, $\lambda_{L,\star} = C_L(t_1)/C_L(t_0)$, and the preceding argument reduces to the single-operator variational bound. The GEVP merely optimizes the overlap and removes the need to guess a good operator.

18.17 Flowed continuum limit (OS reconstruction) and persistence of the mass gap

Definition 18.131 (Flowed Schwinger distributions at fixed $s_0 > 0$)**.** Fix $s_0 > 0$. For each lattice spacing $a \in (0, a_0]$ and periodic box $\Lambda_{a,L}$, fix a finite family of gauge–invariant flowed locals $O^{(s_0)}_i$ (mean–subtracted). Define the pointwise n–point functions

$$S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1,\dots,x_n) \;:=\; \langle \tau_{x_1} O^{(s_0)}_{i_1} \cdots \tau_{x_n} O^{(s_0)}_{i_n} \rangle_{a,L}, \qquad x_j \in \Lambda_{a,L}.$$

Equivalently, define the associated Schwinger *distributions* $S^{(s_0)}_{i_1,\dots,i_n;a,L} \in \mathcal{S}'(\mathbb{R}^{4n})$ by

$$\langle S^{(s_0)}_{i_1,\dots,i_n;a,L}, \Phi\rangle \;:=\; a^{4n} \sum_{x_1,\dots,x_n \in \Lambda_{a,L}} \Phi(x_1,\dots,x_n)\, S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1,\dots,x_n), \qquad \Phi \in \mathcal{S}(\mathbb{R}^{4n}).$$

For $R \in a\mathbb{N}$, let $O^{(s_0),\le R}_i$ be the block-local truncations from (F3) and define $S^{(s_0),\le R}_{i_1,\dots,i_n;a,L}$ analogously by replacing $O^{(s_0)}_i$ with $O^{(s_0),\le R}_i$.

Lemma 18.132 (Uniform locality and moment bounds at fixed flow)**.** *Fix $s_0 > 0$. There exist $c, C < \infty$, independent of a and L, such that for all multi-indices and $n \ge 2$,*

$$\|S^{(a,L)}_{i_1,\dots,i_n;s_0}\|_{L^\infty} \le C,$$
$$|\, S^{(a,L)}_{i_1,\dots,i_n;s_0}(X \cup Y) - S^{(a,L)}_{i_1,\dots,i_{|X|};s_0}(X)\, S^{(a,L)}_{i_{|X|+1},\dots,i_n;s_0}(Y)\,| \le C\, e^{-c\,\mathrm{dist}(X,Y)/\sqrt{s_0}}.$$

for all finite sets $X, Y \subset \mathbb{Z}^4$ (embedded in $\mathbb{R}^4$ via lattice spacing a). Moreover, the dependence on the gauge links is GI-Lipschitz with constant decaying as $e^{-c\,\mathrm{dist}/\sqrt{s_0}}$ (by Lemma 18.80), and all polynomial moments are uniformly bounded (Proposition 13.2).

Proof of Lemma 18.132. Fix $s_0 > 0$ throughout and write $O_j := O_j^{(s_0)}$ for brevity. All constants below may depend on s_0 and on the choice of finitely many indices $\{i_1, \dots, i_n\}$ but are independent of $a \in (0, a_0]$ and L.

(1) Uniform L^∞ (moment) bounds. By the uniform moment bounds at positive flow (Proposition 13.2), for every $p \in [2, \infty)$ there exists $C_p < \infty$ such that

$$\sup_{a,L} \sup_{x} \|\tau_x O_j\|_{L^p(\mathbb{P}_{a,L})} \;\le\; C_p.$$

Hence, by Hölder/Cauchy–Schwarz,

$$|S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1, \dots, x_n)| = \big|\,\mathbb{E}_{a,L}\big[\prod_{k=1}^{n} \tau_{x_k} O_{i_k}\big]\big| \le \prod_{k=1}^{n} \|\tau_{x_k} O_{i_k}\|_{L^{2n}} \;\le\; C,$$

for a constant C depending only on n and $\{i_k\}$, proving the uniform L^∞ bound.

(2) Exponential decoupling across separated sets. Let $X = \{x_1, \dots, x_{|X|}\}$ and $Y = \{y_1, \dots, y_{|Y|}\}$ with $\mathrm{dist}(X, Y) =: R$. Set

$$F_X := \prod_{x \in X} \tau_x O_{i(x)}, \qquad G_Y := \prod_{y \in Y} \tau_y O_{i(y)},$$

so that

$$S^{(a,L)}_{i_1,\dots,i_n;s_0}(X \cup Y) - S^{(a,L)}_{i_1,\dots,i_{|X|};s_0}(X)\, S^{(a,L)}_{i_{|X|+1},\dots,i_n;s_0}(Y) = \mathrm{Cov}_{a,L}(F_X, G_Y).$$

By the positive–flow log–Sobolev inequality and its exponential clustering consequence (Corollary 18.88 and Theorem 18.121), there exist $c_0, C_0 > 0$ such that for any two gauge–invariant local functionals F, G with supports at distance at least R,

$$\big|\mathrm{Cov}_{a,L}(F, G)\big| \;\le\; C_0\, e^{-c_0 R/\sqrt{s_0}} \big(\mathrm{osc}_{\mathrm{supp}F}(F) + \|F\|_{L^2}\big) \big(\mathrm{osc}_{\mathrm{supp}G}(G) + \|G\|_{L^2}\big), \tag{233}$$

uniformly in a, L. (Here the input is the positive–flow LSI together with a *quasilocal* derivative/oscillation control for flowed observables: the relevant Fréchet/Gâteaux derivatives inherit Gaussian heat–kernel tails at scale $\sqrt{s_0}$ from Lemma 18.80 and its higher–derivative extensions; see Section 18.14 for the way these enter the Dobrushin/OR resolvent argument.)

We now bound the oscillations and L^2 norms of F_X, G_Y. By the uniform moment bounds already used in (1), $\|F_X\|_{L^2} \le C$ and $\|G_Y\|_{L^2} \le C$ with C independent of a, L.

For the oscillations we use the heat–kernel quasilocality of the flow (Lemma 18.80), which implies that the derivative of $O_j^{(s_0)}$ with respect to an underlying link variable at space–time distance r is suppressed by Gaussian tails $O\big(e^{-cr^2/s_0}\big)$ (and hence also by $O\big(e^{-c'r/\sqrt{s_0}}\big)$ after weakening). Therefore the oscillations of F_X and G_Y under changes of the field inside their supports are uniformly bounded in terms of the Lipschitz constants of the factors:

$$\mathrm{osc}_{\mathrm{supp}F_X}(F_X) \;\le\; C', \qquad \mathrm{osc}_{\mathrm{supp}G_Y}(G_Y) \;\le\; C',$$

with C' independent of a, L. Inserting these bounds into (233) gives

$$\big|\mathrm{Cov}_{a,L}(F_X, G_Y)\big| \;\le\; C\, e^{-c\,R/\sqrt{s_0}},$$

which is exactly the claimed decoupling estimate.

(3) GI–Lipschitz dependence. By Lemma 18.80 and the definition of $L_{\mathrm{ad}}^{\mathrm{GI}}$, the differential $D_\ell O_j^{(s_0)}$ with respect to any underlying link variable ℓ obeys a Gaussian off–diagonal bound in the space–time distance to $\mathrm{supp}(O_j)$, hence the same holds for products such as F_X by the Leibniz rule and the uniform moment bounds. This yields the stated GI–Lipschitz property.

Combining (1)–(3) proves the lemma. □

Proposition 18.133 (Equicontinuity and tightness)**.** *Fix $s_0 > 0$. For any sequence $a_k \downarrow 0$ and $L_k \uparrow \infty$, the family $\{S^{(a_k,L_k)}_{\cdot;s_0}\}_k$ is tight in the topology of tempered distributions on $\mathbb{R}^{4n}$ for each n. Hence there exists a subsequence (not relabeled) and limiting distributions*

$$S^{(s_0)}_{i_1,\dots,i_n} \in \mathcal{S}'(\mathbb{R}^{4n}) \quad \textit{such that} \quad S^{(a_k,L_k)}_{i_1,\dots,i_n;s_0} \implies S^{(s_0)}_{i_1,\dots,i_n} \quad \textit{for all } n.$$

Proof of Proposition 18.133. Fix $s_0 > 0$ and $n \geq 2$. For $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$ write the pairing

$$\langle S^{(a,L)}_{i_1,\dots,i_n;s_0}, \varphi \rangle = \int_{\mathbb{R}^{4n}} S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1,\dots,x_n)\, \varphi(x_1,\dots,x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

Equicontinuity. By Lemma 18.132 there are $C, c > 0$ with

$$|S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1,\dots,x_n)| \;\leq\; C,$$

$$|S^{(a,L)}_{i_1,\dots,i_n;s_0}(X) - S^{(a,L)}_{i_1,\dots,i_{|X|};s_0}(X)\, S^{(a,L)}_{i_{|X|+1},\dots,i_n;s_0}(Y)| \;\leq\; C \exp\Big(- c\, \frac{\mathrm{dist}(X,Y)}{\sqrt{s_0}} \Big).$$

A standard induction on n (tree–graph bound for truncated correlations) then yields

$$|S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1,\dots,x_n)| \;\leq\; C_n \sum_{T \in \mathfrak{T}_n} \prod_{(u,v) \in E(T)} e^{-c\,|x_u - x_v|/\sqrt{s_0}}, \tag{234}$$

where $\mathfrak{T}_n$ is the set of spanning trees on $\{1,\dots,n\}$ and $E(T)$ its edge set; the constants C_n are independent of a, L. Let $K(z) := e^{-c|z|/\sqrt{s_0}}$ and $|\varphi|$ denote the pointwise absolute value. Integrating (234) against $|\varphi|$ and applying iteratively Young's convolution inequality gives

$$|\langle S^{(a,L)}_{i_1,\dots,i_n;s_0}, \varphi \rangle| \;\leq\; C_n \sum_{T \in \mathfrak{T}_n} \big\|\, |\varphi| * \underbrace{K * \cdots * K}_{|E(T)| \text{ times}} \big\|_{L^1(\mathbb{R}^{4n})} \;\leq\; C_n' \sum_{|\alpha| \leq m} \|(1+|x|)^m\, \partial^\alpha \varphi\|_{L^1},$$

for some m and C_n' depending on n, s_0 but not on a, L (since $K \in L^1$ with norm independent of a, L). The right–hand side is a finite combination of the standard Schwartz seminorms, hence the family $\{S^{(a,L)}_{\cdot;s_0}\}_{a,L}$ is equicontinuous on $\mathcal{S}(\mathbb{R}^{4n})$.

Tightness and subsequential convergence. The Schwartz space is Montel; therefore bounded (equicontinuous) subsets of $\mathcal{S}'(\mathbb{R}^{4n})$ are relatively compact in the weak* topology. By the bound above, $\{S^{(a,L)}_{\cdot;s_0}\}_{a,L}$ is bounded in $\mathcal{S}'$; hence for any sequences $a_k \downarrow 0$ and $L_k \uparrow \infty$ there exists a subsequence (not relabeled) and distributions $S^{(s_0)}_{i_1,\dots,i_n} \in \mathcal{S}'(\mathbb{R}^{4n})$ such that

$$S^{(a_k,L_k)}_{i_1,\dots,i_n;s_0} \implies S^{(s_0)}_{i_1,\dots,i_n} \quad \text{in } \mathcal{S}'(\mathbb{R}^{4n}).$$

This proves tightness and the existence of subsequential limits claimed in the proposition. □

Lemma 18.134 (O(4) invariance from $O(a^2)$ improvement)**.** *Fix $s_0 > 0$. Let $S^{(a,L)}_{\cdot;s_0}$ be the finite–a, L flowed GI Schwinger functions and let $R \in O(4)$. There exist $C(s_0) < \infty$ and $a_0 > 0$ such that, uniformly in L and for all test functions $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$ with unit Schwartz seminorms,*

$$\Big|\; \langle S^{(a,L)}_{\cdot;s_0}, \varphi \rangle \;-\; \langle S^{(a,L)}_{\cdot;s_0}, \varphi \circ R \rangle \;\Big| \;\leq\; C(s_0)\, a^2 \qquad (0 < a \leq a_0).$$

Hence every subsequential continuum limit $S^{(s_0)}$ is $O(4)$–invariant.

Proof. By Theorem 15.9, each flowed local admits an $O(4)$–covariant $O(a^2)$ improvement uniformly in L. The uniform moment/locality bounds of Lemma 18.132 control the n-point remainders when paired with unit–seminorm φ, yielding the estimate and the $O(4)$ invariance of any limit. □

Lemma 18.135 (Two-regularization comparison at fixed flow)**.** *Assume Assumption 18.107. Fix $s_0 > 0$. Let $r_1, r_2 \in \mathfrak{R}$ and denote by $S^{(n)}_{a,L;s_0}[r]$ the finite-volume, flowed GI n-point Schwinger functional for regularization r at lattice spacing a. Then for every n and every Schwartz test F on $\mathcal{S}(\mathbb{R}^{4n})$ there exists $C = C(F, n, s_0)$ such that, uniformly in the volumes and for all $a_1, a_2 \le a_0$,*

$$\Big| \langle F,\ S^{(n)}_{a_1,L;s_0}[r_1]\rangle \ -\ \langle F,\ S^{(n)}_{a_2,L;s_0}[r_2]\rangle \Big| \ \le\ C(F,n,s_0)\,(a_1^2+a_2^2).$$

Proof. By (R1) and the Symanzik expansion used in the proof of Theorem 15.9, the difference of the two actions can be written (modulo TD/EOM) as $S^{(r_1)}_{a_1} - S^{(r_2)}_{a_2} = \sum_\ell \big(\kappa^{(1)}_\ell a_1^2 - \kappa^{(2)}_\ell a_2^2\big)\, Q_\ell + O(a_1^4 + a_2^4)$, with finitely many GI Q_ℓ of dimension ≥ 6 and coefficients uniformly bounded in r_j. Differentiating expectations with respect to these coefficients and summing the resulting connected insertions, the BKAR/tree representation together with the uniform moment/locality bounds at positive flow (Lemma 18.132) yields

$$\Big| \langle F, S^{(n)}_{a_1,L;s_0}[r_1] - S^{(n)}_{a_2,L;s_0}[r_2]\rangle \Big| \ \le\ C \sum_\ell \big(|\kappa^{(1)}_\ell| a_1^2 + |\kappa^{(2)}_\ell| a_2^2\big) \ \le\ C'(F,n,s_0)\,(a_1^2+a_2^2),$$

uniformly in the volume and in r_j. The $O(a^4)$ remainders are absorbed. Pairing with $\varphi \in \mathcal{S}$ uses the same seminorm control as in Theorem 15.9. □

Corollary 18.136 (Flowed OS limit and reconstruction)**.** *Each subsequential limit $S^{(s_0)}$ from Proposition 18.133 satisfies the OS axioms (OS0–OS4) (by Theorem 18.74). In particular, at fixed $s_0 > 0$ it obeys the cluster property.*

Consequently, the OS reconstruction theorem Osterwalder and Schrader (1973, 1975) produces a Hilbert space $\mathcal{H}_{s_0}$, a vacuum vector Ω_{s_0}, a local $*$–algebra generated by the limits of the fields $\tau_x O_i^{(s_0)}$, and a strongly continuous unitary representation of Euclidean translations whose time component is $e^{-tH_{s_0}}$ with $H_{s_0} \ge 0$ selfadjoint.

Proof of Corollary 18.136. Let $S^{(a_k,L_k)}_{\cdot;s_0} \Rightarrow S^{(s_0)}_{\cdot}$ be the subsequence from Proposition 18.133.

(OS0: Regularity). Equicontinuity of $\{S^{(a_k,L_k)}_{\cdot;s_0}\}_k$ on $\mathcal{S}$ (Proposition 18.133) implies that each limit $S^{(s_0)}_n$ is a tempered distribution and the family $\{S^{(s_0)}_n\}_{n\ge0}$ is jointly continuous on $\mathcal{S}(\mathbb{R}^{4n})$.

(OS1: Euclidean invariance and symmetry). Each finite–a, L Schwinger family is translation invariant and permutation symmetric by construction; these properties pass to the limit. Rotational invariance in the continuum follows from the $O(a^2)$ improvement at positive flow (Theorem 15.9) via Lemma 18.134; hence the limit is $O(4)$–invariant.

(OS2: Reflection positivity). Let ϑ_{s_0} denote the shifted reflection from Lemma 18.71 at flow time s_0. Reflection positivity for gauge–invariant observables at flow time s_0 holds with respect to ϑ_{s_0} (Lemma 18.71) and is stable under L^2 limits (Lemma 16.11). Therefore, for any finite linear combination $Z = \sum_j c_j\, \tau_{x_j} O^{(s_0)}_{i_j}$ supported in the positive half–space for ϑ_{s_0} (in particular, any Z supported in $\{x_0 \ge 0\}$),

$$\langle \vartheta_{s_0} Z,\ Z\rangle_{a_k,L_k} \ \ge\ 0 \quad \text{for all } k.$$

By the uniform bounds of Lemma 18.132, $\langle \vartheta_{s_0} Z, Z\rangle_{a_k,L_k} \to \langle \vartheta_{s_0} Z, Z\rangle_{s_0}$ along the convergent subsequence; hence $\langle \vartheta_{s_0} Z, Z\rangle_{s_0} \ge 0$, i.e. $S^{(s_0)}$ is OS–positive (equivalently, after recentering the reflection plane, OS2 holds in the standard form).

(OS3: Symmetry under permutations). Already addressed together with translation invariance.

(OS4: Cluster property). Lemma 18.132 yields, uniformly in a, L,

$$\left|\langle X\, \tau_{(t,x)} Y\rangle_{a,L} - \langle X\rangle_{a,L}\langle Y\rangle_{a,L}\right| \;\le\; C\, e^{-c\sqrt{t^2+|x|^2}/\sqrt{s_0}},$$

for any gauge–invariant locals X, Y with disjoint supports. Passing to the limit gives the cluster property for $S^{(s_0)}$.

Having verified OS0–OS4, the OS reconstruction theorem Osterwalder and Schrader (1973, 1975) produces a Hilbert space $\mathcal{H}_{s_0}$, a vacuum vector Ω_{s_0}, a local $*$–algebra generated by the limits of $\tau_x O_j^{(s_0)}$, and a unitary representation of Euclidean translations whose time component is $e^{-tH_{s_0}}$ with $H_{s_0} \ge 0$ selfadjoint. □

Lemma 18.137 (OS norm versus GI–adjoint Lipschitz seminorm)**.** *Fix $s_0 > 0$ and let $(\mathcal{H}_{s_0}, \Omega_{s_0})$ be the OS reconstruction from Corollary 18.136. Let Z be a mean–zero gauge–invariant cylinder functional supported in the positive half–space for ϑ_{s_0} and depending on finitely many blocks (so that $\|L^{\mathrm{GI}}_{\mathrm{ad}}(Z)\| < \infty$). Then there exists $C_{\mathrm{PI}}(s_0) < \infty$, independent of a, L and of Z, such that*

$$\|Z\Omega_{s_0}\|^2_{\mathcal{H}_{s_0}} \;=\; \langle \vartheta_{s_0} Z, Z\rangle_{s_0} \;\le\; C_{\mathrm{PI}}(s_0)\, \big\|L^{\mathrm{GI}}_{\mathrm{ad}}(Z)\big\|^2. \tag{235}$$

Equivalently,

$$\big\|L^{\mathrm{GI}}_{\mathrm{ad}}(Z)\big\| \;\ge\; C_{\mathrm{PI}}(s_0)^{-1/2}\, \|Z\Omega_{s_0}\|_{\mathcal{H}_{s_0}}.$$

Proof. Fix a convergent subsequence (a_k, L_k) from Proposition 18.133. For each k, by Cauchy–Schwarz and invariance of $\mathbb{P}_{a_k,L_k}$ under ϑ_{s_0},

$$\langle \vartheta_{s_0} Z, Z\rangle_{a_k,L_k} \;\le\; \|\vartheta_{s_0} Z\|_{L^2(\mathbb{P}_{a_k,L_k})}\, \|Z\|_{L^2(\mathbb{P}_{a_k,L_k})} \;=\; \|Z\|^2_{L^2(\mathbb{P}_{a_k,L_k})}.$$

Since Z is mean–zero, $\|Z\|^2_{L^2} = \mathrm{Var}_{a_k,L_k}(Z)$. By the (uniform) Poincaré inequality for gauge–invariant cylinder functionals (a consequence of the slice LSI/Poincaré control, cf. Theorem 18.115 and its Poincaré corollary),

$$\mathrm{Var}_{a_k,L_k}(Z) \;\le\; C_{\mathrm{PI}}(s_0)\, \big\|L^{\mathrm{GI}}_{\mathrm{ad}}(Z)\big\|^2,$$

with $C_{\mathrm{PI}}(s_0)$ independent of k and of Z. Combining the two estimates gives

$$\langle \vartheta_{s_0} Z, Z\rangle_{a_k,L_k} \;\le\; C_{\mathrm{PI}}(s_0)\, \big\|L^{\mathrm{GI}}_{\mathrm{ad}}(Z)\big\|^2.$$

Finally, by definition of the OS limit along the subsequence,

$$\|Z\Omega_{s_0}\|^2_{\mathcal{H}_{s_0}} = \langle \vartheta_{s_0} Z, Z\rangle_{s_0} = \lim_{k\to\infty} \langle \vartheta_{s_0} Z, Z\rangle_{a_k,L_k},$$

and (235) follows. □

Canonical choice of interpolator and LSZ normalization. Fix the positive flow time $s_0 > 0$ once and for all. We use the canonical flowed, gauge-invariant interpolator $A_\star^{(s_0)}$ constructed in Corollary 18.129, which satisfies the one-particle pole statement

$$\big\langle A_\star^{(s_0)}(t)\, A_\star^{(s_0)}(0)\big\rangle \;=\; Z_\star\, e^{-m_\star t} \;+\; R(t), \qquad Z_\star > 0, \quad |R(t)| \le C\, e^{-(m_\star+\delta)t}.$$

We work with the *LSZ-normalized* field

$$\widehat{A}_\star^{(s_0)} \;:=\; Z_\star^{-1/2}\, A_\star^{(s_0)}.$$

Then $\|E(\Sigma_{m_\star})\,\widehat{A}_\star^{(s_0)}(0)\Omega\| = 1$, and in particular

$$\langle \widehat{A}_\star^{(s_0)}(t)\,\widehat{A}_\star^{(s_0)}(0)\rangle \;=\; e^{-m_\star t} + O\big(e^{-(m_\star+\delta)t}\big).$$

All Haag–Ruelle and LSZ constructions below are performed with $\widehat{A}_\star^{(s_0)}$ and the mass parameter $m_\star > 0$. We denote by $\alpha_{(t,x)}$ the real-time space–time automorphism (Heisenberg evolution), so that

$$\widehat{A}_\star^{(s_0)}(t,x) \;:=\; \alpha_{(t,x)}\big(\widehat{A}_\star^{(s_0)}(0,0)\big).$$

Lemma 18.138 (Inherited quasi-locality/commutator bounds)**.** *Let $A_\star^{(s_0)} = \sum_j c_j\,\mathcal{A}_j^{(s_0)}$ be as in Corollary 18.129. Assume that for each j and for every local observable B disjoint from a radius-r neighborhood of* $\operatorname{supp}\mathcal{A}_j^{(s_0)}$ *one has, for all $N \in \mathbb{N}$,*

$$\big\|[\alpha_{(t,x)}(\mathcal{A}_j^{(s_0)}), B]\big\| \;\le\; C_N\,\big(1 + \operatorname{dist}(\operatorname{supp}\mathcal{A}_j^{(s_0)} + x, \operatorname{supp} B) - v|t|\big)^{-N},$$

(or the corresponding equal-time version). Then the same bound holds for $A_\star^{(s_0)}$, with a possibly different constant C'_N, uniformly in (t,x) (with s_0 fixed).

Proof. Write $A_\star^{(s_0)} = \sum_{j=1}^{M} c_j\,\mathcal{A}_j^{(s_0)}$ with $M < \infty$ as in Corollary 18.129. Fix a local observable B disjoint from a radius-r neighborhood of $\operatorname{supp} A_\star^{(s_0)} = \bigcup_j \operatorname{supp}\mathcal{A}_j^{(s_0)}$. By linearity of the commutator and the triangle inequality,

$$\big\|[\alpha_{(t,x)}(A_\star^{(s_0)}), B]\big\| \;\le\; \sum_{j=1}^{M} |c_j|\,\big\|[\alpha_{(t,x)}(\mathcal{A}_j^{(s_0)}), B]\big\|.$$

By hypothesis, for each j,

$$\big\|[\alpha_{(t,x)}(\mathcal{A}_j^{(s_0)}), B]\big\| \;\le\; C_N\,\big(1 + \operatorname{dist}(\operatorname{supp}\mathcal{A}_j^{(s_0)} + x, \operatorname{supp} B) - v|t|\big)^{-N}.$$

Since $\operatorname{dist}(\operatorname{supp}\mathcal{A}_j^{(s_0)} + x, \operatorname{supp} B) \ge \operatorname{dist}(\operatorname{supp} A_\star^{(s_0)} + x, \operatorname{supp} B)$ for all j and the map $d \mapsto (1 + d - v|t|)^{-N}$ is decreasing in d, we obtain

$$\big\|[\alpha_{(t,x)}(A_\star^{(s_0)}), B]\big\| \;\le\; \Big(\sum_{j=1}^{M} |c_j|\Big)\,C_N\,\big(1 + \operatorname{dist}(\operatorname{supp} A_\star^{(s_0)} + x, \operatorname{supp} B) - v|t|\big)^{-N}.$$

Thus the same quasi-local (or equal-time) commutator bound holds for $A_\star^{(s_0)}$ with $C'_N := C_N \sum_{j=1}^{M} |c_j|$, uniformly for fixed s_0. □

18.18 Haag–Ruelle wave packets at mass $m_\star$

Let $\omega_\star(p) := \sqrt{m_\star^2 + |p|^2}$ and choose $h \in \mathcal{S}(\mathbb{R}^3)$ with compact momentum support. Define the positive–energy Klein–Gordon solution

$$h_t(x) \;:=\; \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{\,ip\cdot x - i\,\omega_\star(p)\,t}\,\widehat{h}(p)\,\mathrm{d}^3p, \qquad t \in \mathbb{R},$$

and set the Haag–Ruelle operator (on the common polynomial core $\mathcal{D}_{\text{poly}}$)

$$B_t(h) \;:=\; \int_{\mathbb{R}^3} \Big(\dot{h}_t(x)\,\widehat{A}_\star^{(s_0)}(t,x) \;-\; h_t(x)\,\partial_t\widehat{A}_\star^{(s_0)}(t,x)\Big)\,\mathrm{d}^3x, \tag{236}$$

where $\partial_t\widehat{A}_\star^{(s_0)}(t,x) \;=\; i\,[H, \widehat{A}_\star^{(s_0)}(t,x)]$ is the Heisenberg derivative and $\widehat{A}_\star^{(s_0)}$ is the LSZ–normalized scalar interpolator with unit residue at mass $m_\star$ (cf. Corollary 18.129).

Proposition 18.139 (Haag–Ruelle one–particle limit at mass $m_\star$)**.** *Assume the reconstructed Wightman theory of §17 with positive energy and locality, and that the joint spectrum of* $(H, \mathbf{P})$ *contains an isolated mass shell* $\Sigma_{m_\star} = \{(p^0, \mathbf{p}) : p^0 = \omega_\star(\mathbf{p})\}$ *with spectral projection* $E_1 := E(\Sigma_{m_\star}) \neq 0$*. Then, for every* $h \in \mathcal{S}(\mathbb{R}^3)$ *with compact momentum support, the strong limit*

$$\Psi_\star(h) \ := \ \lim_{t\to+\infty} B_t(h)\,\Omega$$

exists, depends only on $\widehat{h}$ *through its restriction to* $\Sigma_{m_\star}$*, belongs to the one–particle space* $\mathcal{H}_1 := E_1\mathcal{H}$*, and*

$$\|\Psi_\star(h)\|^2 \ = \ \int_{\mathbb{R}^3} \frac{|\widehat{h}(p)|^2}{2\,\omega_\star(p)}\,\mathrm{d}^3p, \tag{237}$$

with the identification of $\Psi_\star(h)$ *as the standard one–particle wave packet at mass* $m_\star$*. The vector* $\Psi_\star(h)$ *is independent of the choice of the positive flow time* $s_0 > 0$ *used to define* $\widehat{A}_\star^{(s_0)}$*.*

Proof. 1) Four–dimensional smearing and its Fourier transform. Introduce $g_t \in \mathcal{S}'(\mathbb{R}^4)$ by

$$g_t(x^0, \mathbf{x}) := \dot{h}_t(\mathbf{x})\,\delta(x^0 - t) \ - \ h_t(\mathbf{x})\,\delta'(x^0 - t).$$

By definition of the smeared field, $B_t(h) = \widehat{A}_\star^{(s_0)}(g_t)$ on $\mathcal{D}_{\mathrm{poly}}$. Using the Fourier transform

$$\widetilde{f}(p^0, \mathbf{p}) \ = \ \int_{\mathbb{R}^4} e^{i(p^0x^0 - \mathbf{p}\cdot\mathbf{x})} f(x^0, \mathbf{x})\,\mathrm{d}x^0\,\mathrm{d}^3x,$$

and the explicit form of h_t, one computes

$$\int_{\mathbb{R}^3} e^{-i\mathbf{p}\cdot\mathbf{x}}\,h_t(\mathbf{x})\,\mathrm{d}^3x = (2\pi)^{3/2} e^{-i\omega_\star(\mathbf{p})t}\,\widehat{h}(\mathbf{p}),$$

and hence

$$\int_{\mathbb{R}^3} e^{-i\mathbf{p}\cdot\mathbf{x}}\,\dot{h}_t(\mathbf{x})\,\mathrm{d}^3x = -\,i\,(2\pi)^{3/2}\,\omega_\star(\mathbf{p})\,e^{-i\omega_\star(\mathbf{p})t}\,\widehat{h}(\mathbf{p}).$$

Using $\int e^{ip^0x^0}\delta(x^0 - t)\,\mathrm{d}x^0 = e^{ip^0t}$ and $\int e^{ip^0x^0}\delta'(x^0 - t)\,\mathrm{d}x^0 = i\,p^0\,e^{ip^0t}$, we obtain

$$\widetilde{g}_t(p^0, \mathbf{p}) \ = \ -\,i\,(2\pi)^{3/2}\,(p^0 + \omega_\star(\mathbf{p}))\,e^{\,i\,(p^0 - \omega_\star(\mathbf{p}))\,t}\,\widehat{h}(\mathbf{p}). \tag{238}$$

Thus $\widetilde{g}_t$ is a smooth function with compact $\mathbf{p}$–support and at most polynomial growth in p^0.

2) Spectral representation. Let $E(\cdot)$ be the joint spectral measure of $(H, \mathbf{P})$ and use translation covariance $\widehat{A}_\star^{(s_0)}(x) = U(x)\widehat{A}_\star^{(s_0)}(0)U(x)^*$ with $U(x) = e^{i(Hx^0 - \mathbf{P}\cdot\mathbf{x})}$. For $g \in \mathcal{S}(\mathbb{R}^4)$, the standard spectral calculus gives

$$\widehat{A}_\star^{(s_0)}(g)\,\Omega = \int_{\mathbb{R}^4} \widetilde{g}(p)\,E(\mathrm{d}p)\,\widehat{A}_\star^{(s_0)}(0)\Omega$$

(a vector–valued Bochner integral). By density this extends to our $g_t \in \mathcal{S}'$, and we obtain

$$B_t(h)\Omega \ = \ \int_{\mathbb{R}^4} \widetilde{g}_t(p)\,E(\mathrm{d}p)\,\widehat{A}_\star^{(s_0)}(0)\Omega.$$

Decompose with E_1 and $E_{\mathrm{c}} := \mathbf{1} - E_1$:

$$B_t(h)\Omega \ = \ E_1B_t(h)\Omega \ + \ E_{\mathrm{c}}B_t(h)\Omega.$$

3) Identification of the one–particle contribution. On the mass shell $\Sigma_{m_\star}$ we have $p^0 = \omega_\star(\mathbf{p})$, so by Equation (238),

$$\widetilde{g}_t(p^0, \mathbf{p})\big|_{\Sigma_{m_\star}} = -\,i\,(2\pi)^{3/2}\,2\,\omega_\star(\mathbf{p})\,\widehat{h}(\mathbf{p}),$$

which is independent of t. Hence

$$E_1 B_t(h)\Omega \;=\; -i\,(2\pi)^{3/2}\int_{\mathbb{R}^3} 2\,\omega_\star(\mathbf{p})\;\widehat{h}(\mathbf{p})\;E_1(\mathrm{d}^3\mathbf{p})\,\widehat{A}_\star^{(s_0)}(0)\Omega.$$

By the LSZ normalization of $\widehat{A}_\star^{(s_0)}$ (unit residue at $m_\star$), the vacuum–to–one–particle matrix elements of $\widehat{A}_\star^{(s_0)}$ coincide with those of a free scalar of mass $m_\star$. In particular, if $\{|\mathbf{p}\rangle\}$ denotes the Dirac momentum basis in $\mathcal{H}_1$ with

$$\langle\mathbf{p}|\mathbf{q}\rangle \;=\; 2\omega_\star(\mathbf{p})\,\delta(\mathbf{p}-\mathbf{q}),$$

then

$$E_1(\mathrm{d}^3\mathbf{p})\,\widehat{A}_\star^{(s_0)}(0)\Omega \;=\; \frac{1}{(2\pi)^{3/2}\,2\,\omega_\star(\mathbf{p})}\,|\mathbf{p}\rangle\,\mathrm{d}^3\mathbf{p}.$$

Substituting this into the previous display yields

$$E_1 B_t(h)\Omega \;=\; \int_{\mathbb{R}^3}\frac{\widehat{h}(\mathbf{p})}{2\,\omega_\star(\mathbf{p})}\,|\mathbf{p}\rangle\,\mathrm{d}^3\mathbf{p} =: \Psi_\star(h),$$

which is manifestly independent of t and belongs to $\mathcal{H}_1$. Moreover,

$$\left\|\Psi_\star(h)\right\|^2 = \int_{\mathbb{R}^3}\frac{|\widehat{h}(\mathbf{p})|^2}{2\,\omega_\star(\mathbf{p})}\,\mathrm{d}^3\mathbf{p},$$

which is exactly the right–hand side of Equation (237). In particular, $\Psi_\star(h)$ depends on $\widehat{h}$ only through its restriction to $\Sigma_{m_\star}$.

4) Vanishing of the continuum part as $t\to+\infty$. On $\operatorname{ran}E_{\mathrm{c}}$ the joint spectrum of $(H,\mathbf{P})$ is contained in the closed subset $\sigma_{\mathrm{c}}\subset\mathbb{R}^4$ obtained from the full spectrum by removing the isolated mass shell $\Sigma_{m_\star}$. Let $\widehat{h}$ have compact support contained in a fixed compact set $K\subset\mathbb{R}^3$. The energy bounds for $\widehat{A}_\star^{(s_0)}(0)\Omega$ (Nelson analyticity/subgaussian moments, see Lemma 17.2 and Proposition 17.3) imply that

$$(p^0,\mathbf{p})\;\longmapsto\;(p^0+\omega_\star(\mathbf{p}))\,\widehat{h}(\mathbf{p})\,E_{\mathrm{c}}(\mathrm{d}p)\,\widehat{A}_\star^{(s_0)}(0)\Omega$$

defines a Bochner–integrable $\mathcal{H}$–valued measure on $\mathbb{R}^4$. Using Equation (238), we can therefore write

$$E_{\mathrm{c}}B_t(h)\Omega = \;-i\,(2\pi)^{3/2}\int_{\sigma_{\mathrm{c}}\cap(\mathbb{R}\times K)} e^{\,i\,(p^0-\omega_\star(\mathbf{p}))\,t}\,(p^0+\omega_\star(\mathbf{p}))\,\widehat{h}(\mathbf{p})\,E_{\mathrm{c}}(\mathrm{d}p)\,\widehat{A}_\star^{(s_0)}(0)\Omega.$$

Set $\lambda(p):=p^0-\omega_\star(\mathbf{p})$. Since $\Sigma_{m_\star}$ is isolated, λ has no zeros on $\sigma_{\mathrm{c}}\cap(\mathbb{R}\times K)$, but this fact is not needed. Consider the pushforward $\mathcal{H}$–valued measure ν on $\mathbb{R}$ defined by

$$\nu(B)\;:=\;-i\,(2\pi)^{3/2}\int_{\{p\in\sigma_{\mathrm{c}}\cap(\mathbb{R}\times K):\ \lambda(p)\in B\}}(p^0+\omega_\star(\mathbf{p}))\,\widehat{h}(\mathbf{p})\,E_{\mathrm{c}}(\mathrm{d}p)\,\widehat{A}_\star^{(s_0)}(0)\Omega$$

for Borel sets $B\subset\mathbb{R}$. The total variation of ν is finite because of the energy bounds and the compact support of $\widehat{h}$. Then

$$E_{\mathrm{c}}B_t(h)\Omega \;=\; \int_{\mathbb{R}} e^{i\lambda t}\,\nu(\mathrm{d}\lambda).$$

By the vector–valued Riemann–Lebesgue lemma (Fourier transform of a finite $\mathcal{H}$–valued measure on $\mathbb{R}$), this Bochner integral converges to zero in norm as $t\to+\infty$:

$$\|E_{\mathrm{c}}B_t(h)\Omega\|\;\xrightarrow[t\to+\infty]{}\;0.$$

5) Existence of the limit and independence of s_0. Combining Steps 3 and 4 we obtain

$$\lim_{t\to+\infty} B_t(h)\Omega = \lim_{t\to+\infty} \big(E_1 B_t(h)\Omega + E_{\mathrm{c}} B_t(h)\Omega\big) = \Psi_\star(h) \in \mathcal{H}_1,$$

with norm given by Equation (237). The representation of $\Psi_\star(h)$ in Step 3 shows that it depends on h only through $\widehat{h}\restriction_{\Sigma_{m_\star}}$.

To see that $\Psi_\star(h)$ does not depend on the choice of $s_0 > 0$, let s_0' be another flow time and let $\widehat{A}_\star^{(s_0')}$ be the corresponding LSZ–normalized interpolator. By construction, both interpolators have unit residue at mass $m_\star$, so their one–particle projections coincide:

$$E_1\, \widehat{A}_\star^{(s_0)}(0)\Omega = E_1\, \widehat{A}_\star^{(s_0')}(0)\Omega.$$

Set $C := \widehat{A}_\star^{(s_0)} - \widehat{A}_\star^{(s_0')}$ and denote by $B_t^C(h)$ the operator obtained from Equation (236) with $\widehat{A}_\star^{(s_0)}$ replaced by C. Then $E_1 C(0)\Omega = 0$, so Step 3 yields $E_1 B_t^C(h)\Omega = 0$ for all t, while Step 4 applied to C in place of $\widehat{A}_\star^{(s_0)}$ gives

$$\lim_{t\to+\infty} \big\|E_{\mathrm{c}} B_t^C(h)\Omega\big\| = 0.$$

Thus $\lim_{t\to+\infty} B_t^C(h)\Omega = 0$, and the limits constructed with $\widehat{A}_\star^{(s_0)}$ and $\widehat{A}_\star^{(s_0')}$ coincide. This proves independence of s_0. ☐

Remark 18.140 (Isometry and domain)*.* The map $h \mapsto \Psi_\star(h)$ extends by density to an isometry from $L^2(\mathbb{R}^3, \mathrm{d}^3p/(2\omega_\star(p)))$ onto the one–particle space $\mathcal{H}_1$. Indeed, Equation (237) identifies the norm of $\Psi_\star(h)$ with the standard L^2 norm of $\widehat{h}$ on the mass shell, and the set of $\widehat{h}$ with compact support is dense in that L^2 space. The operators $B_t(h)$ are well defined on the common polynomial core $\mathcal{D}_{\mathrm{poly}}$, and the limit above is taken in the strong topology of $\mathcal{H}$.

Corollary 18.141 (Vacuum uniqueness at $T = 0$ for the flowed theory)**.** *Fix $s_0 > 0$ and consider the continuum OS limit from Corollary 18.136. Then the reconstructed Hamiltonian H_{s_0} has a unique (up to phase) translation-invariant ground state Ω_{s_0}.*

Proof. Exponential clustering for flowed gauge–invariant locals at fixed $s_0 > 0$ (Lemma 18.132) implies the OS cluster property for the corresponding Schwinger functions. In the OS reconstruction, clustering of Schwinger functions entails uniqueness of the translationally invariant vacuum vector; see, for instance, Glimm and Jaffe (1987, Thm. III.4.12). Applied to the continuum OS limit at flow time s_0, this yields uniqueness of the ground state Ω_{s_0} of H_{s_0} up to phase. ☐

Proposition 18.142 (Nonzero overlap in the lightest scalar channel)**.** *Let $s_0 > 0$ be fixed. Assume that the scalar one–particle subspace at mass $m_\star$ in the flowed OS/Wightman theory is nontrivial, i.e. there exists a unit vector $\psi_1 \in \mathcal{H}_1^{(0^{++})}$ at mass $m_\star$. Then there exists a bounded, gauge–invariant* scalar *local operator $A^{(s_0)}$ such that*

$$\langle \psi_1,\ A^{(s_0)}\,\Omega\rangle \ \neq\ 0.$$

Moreover, one may arrange $\langle \Omega, A^{(s_0)}\Omega\rangle = 0$.

Proof. Fix a unit one–particle vector $\psi_1 \in \mathcal{H}_1^{(0^{++})}$ at mass $m_\star$. Let $\mathcal{O} \subset \mathbb{R}^4$ be a nonempty open set (take $\mathcal{O}$ to be a ball to preserve rotational invariance). By the flowed gauge–invariant Reeh–Schlieder theorem Theorem 10.5, the set of vectors

$$\{\, B^{(s_0)}(f)\,\Omega : B^{(s_0)} \text{ a flowed GI local, } f \in C_c^\infty(\mathcal{O}) \,\}$$

is dense in $\mathcal{H}$. Hence there exist $B^{(s_0)}(f)$ with

$$\|B^{(s_0)}(f)\Omega - \psi_1\| < \tfrac{1}{2}.$$

Define the mean–zero operator

$$\widetilde{B}^{(s_0)} := B^{(s_0)}(f) - \langle \Omega, B^{(s_0)}(f)\Omega\rangle \, \mathbf{1},$$

so that $\langle \Omega, \widetilde{B}^{(s_0)}\Omega\rangle = 0$ and $\langle \psi_1, \widetilde{B}^{(s_0)}\Omega\rangle = \langle \psi_1, B^{(s_0)}(f)\Omega\rangle$ because $\psi_1 \perp \Omega$.

Now project to the 0^{++} sector by symmetrizing under spatial rotations (and, if desired, also parity and charge conjugation):

$$A^{(s_0)} := \int_{SO(3)} U(R)\, \widetilde{B}^{(s_0)}\, U(R)^*\, dR,$$

(where dR is Haar measure). Since $\mathcal{O}$ is a ball, each conjugate $U(R)\widetilde{B}^{(s_0)}U(R)^*$ is still localized in $\mathcal{O}$, hence $A^{(s_0)}$ is bounded, GI, and local; and by construction it is scalar. Because ψ_1 has spin 0 (and is P–even and C–even in the 0^{++} channel), ψ_1 is invariant under these symmetries, so

$$\langle \psi_1, A^{(s_0)}\Omega\rangle = \int_{SO(3)} \langle \psi_1, U(R)\widetilde{B}^{(s_0)}U(R)^*\Omega\rangle \, dR = \langle \psi_1, \widetilde{B}^{(s_0)}\Omega\rangle.$$

Finally,

$$|\langle \psi_1, B^{(s_0)}(f)\Omega\rangle - 1| = |\langle \psi_1, B^{(s_0)}(f)\Omega - \psi_1\rangle| \le \|B^{(s_0)}(f)\Omega - \psi_1\| < \tfrac{1}{2},$$

so $|\langle \psi_1, A^{(s_0)}\Omega\rangle| > \frac{1}{2}$ and in particular it is nonzero. □

From a filtered operator to $\mathrm{tr}(F^2)$ via quantitative flow removal. Let $\{\mathcal{O}_\Delta\}_{\Delta\le 4}$ be a renormalized gauge–invariant (GI) basis for the scalar 0^{++} channel, chosen so that $\mathcal{O}_4 \equiv \mathrm{tr}(F^2)_{\mathrm{R}}$. Fix $s_0 > 0$ and work on the common OS core at flow time s_0. Then the flow–removal theorem (Proposition 18.75, based on the nonperturbative SFTE with remainder bounds, Lemma 18.24) yields: for every $f \in C_c^\infty(\mathbb{R}^4)$ and every $s \in (0, s_0]$,

$$\langle v,\, A^{(s)}(f)\, w\rangle = \sum_{\Delta\le 4} c_{A,\Delta}(s)\, \langle v,\, \mathcal{O}_\Delta(f)\, w\rangle + \langle v,\, R_s(f)\, w\rangle,$$

with a quantitative remainder bound

$$\|R_s(f)\| \;\le\; C\, s^{\varepsilon}\, \|f\|_{C^N},$$

for some $\varepsilon > 0$, integer N, and constant C independent of $s \in (0, s_0]$. Any total–derivative term $\partial\cdot \mathcal{J}^{(s)}$ that appears before smearing is absorbed into $R_s(f)$ by integration by parts. Moreover, in the same matching scheme,

$$c_{A,\Delta}(s) = c^{(0)}_{A,\Delta} + O\big(s|\log s|\big) \qquad (s \downarrow 0),$$

in particular $c_{A,4}(s) \to c^{(0)}_{A,4}$.

Theorem 18.143 (Nonzero one-particle residue for $\mathrm{tr}(F^2)$)**.** *In the scalar 0^{++} channel, the renormalized composite $\mathrm{tr}(F^2)_{\mathrm{R}}$ has a strictly positive LSZ residue at the one-particle mass m_0:*

$$Z_{0^{++}} \;:=\; \big|\langle \psi,\; \mathrm{tr}(F^2)_{\mathrm{R}}(0)\,\Omega\rangle\big|^2 \;>\; 0 \quad \textit{for some unit one-particle } \psi \textit{ of mass } m_0.$$

Proof. Fix $s_0 > 0$ and let $A_\star^{(s_0)}$ be the principal interpolator in the scalar 0^{++} channel at flow time s_0, constructed in the GEVP/variational setup (so $A_\star^{(s_0)}$ is a finite linear combination of flowed basis fields $\Phi_\alpha^{(s_0)} := G_{s_0} * Q_\alpha^{\mathrm{ren}}$ with Q_α^{ren} renormalized point–local GI scalars of canonical dimension ≤ 4). Let ψ be a unit one–particle vector of mass m_0 such that

$$\langle \psi,\ A_\star^{(s_0)}(0)\Omega\rangle \ \neq\ 0,$$

as guaranteed by Proposition 18.142 (with $m_0 = m_\star$ in that notation).

Write

$$A_\star^{(s_0)} \ =\ \sum_{\alpha\in\mathcal{B}} v_\alpha\, \Phi_\alpha^{(s_0)} \ =\ \sum_{\alpha\in\mathcal{B}} v_\alpha\,(G_{s_0} * Q_\alpha^{\mathrm{ren}}) \ =\ G_{s_0} * A^{(0),\mathrm{ren}}, \qquad A^{(0),\mathrm{ren}} := \sum_{\alpha\in\mathcal{B}} v_\alpha\, Q_\alpha^{\mathrm{ren}}.$$

Equivalently, for every test function $f \in C_c^\infty(\mathbb{R}^4)$,

$$A_\star^{(s_0)}(f) \ =\ A^{(0),\mathrm{ren}}(G_{s_0} * f).$$

Choose f so that $\langle \psi, A_\star^{(s_0)}(f)\Omega\rangle \neq 0$ (possible since the distribution $f \mapsto \langle \psi, A_\star^{(s_0)}(f)\Omega\rangle$ is not identically zero). Setting $g := G_{s_0} * f$ then gives

$$\langle \psi,\ A^{(0),\mathrm{ren}}(g)\Omega\rangle \ =\ \langle \psi,\ A_\star^{(s_0)}(f)\Omega\rangle \ \neq\ 0,$$

so $A^{(0),\mathrm{ren}}$ has nonzero overlap with a one–particle vector of mass m_0.

In the scalar GI 0^{++} sector at canonical dimension ≤ 4, the renormalized quotient is spanned (modulo the identity and null fields in the physical sector) by $\mathrm{tr}(F^2)_{\mathrm{R}}$. Hence we may write

$$A^{(0),\mathrm{ren}} \ =\ c_0\,\mathbf{1} \ +\ c_4\ \mathrm{tr}(F^2)_{\mathrm{R}}.$$

Taking a vacuum–to–one–particle matrix element eliminates $\mathbf{1}$ (since $\psi \perp \Omega$), and therefore

$$\langle \psi,\ A^{(0),\mathrm{ren}}(0)\Omega\rangle \ =\ c_4\,\langle \psi,\ \mathrm{tr}(F^2)_{\mathrm{R}}(0)\Omega\rangle.$$

The left-hand side is nonzero, hence $c_4 \neq 0$ and $\langle \psi, \mathrm{tr}(F^2)_{\mathrm{R}}(0)\Omega\rangle \neq 0$. This implies $Z_{0^{++}} > 0$. ☐

18.19 RG window transport and explicit low–momentum coefficients

We now show that, in a robust renormalization–group (RG) window that survives the continuum/thermodynamic limit, the flowed GI two–point function admits a uniform small–momentum expansion whose inverse has strictly positive coefficients

$$(\widetilde{G}^{(s)}(p))^{-1} \ =\ c_0(s) \ +\ c_2(s)\,p^2 \ +\ O(p^4) \qquad \text{with } c_0(s), c_2(s) > 0,$$

and we identify $c_0(s), c_2(s)$ explicitly in terms of Euclidean correlator moments or, equivalently, the Källén–Lehmann spectral measure.

RG window. Fix a (physical) flow time $s > 0$ and define the *RG window of momenta*

$$\mathcal{W}(s,\kappa) \ :=\ \{\, p \in \mathbb{R}^4 :\ |p| \le \kappa/\sqrt{s}\,\},$$

with a *data–driven* $\kappa \equiv \kappa_{a,L}(s) \in (0,1)$ chosen as in Theorem 18.148. On the lattice with spacing a and linear size L (periodic b.c.), we restrict to the discrete momenta $p \in (2\pi/L)\,\mathbb{Z}^4 \cap \mathcal{W}(s, \kappa_{a,L}(s))$ and impose

$$a \ \ll\ \sqrt{s} \ \ll\ \ell \ \ll\ L, \tag{239}$$

where ℓ is a fixed coarse length (in physical units) used to separate UV and IR errors. We call (239) an *RG window schedule.* In the joint limit $a \downarrow 0$, $L \uparrow \infty$ with s, ℓ fixed (or slowly varying so that (239) holds), the window $\mathcal{W}(s, \kappa_{a,L}(s))$ remains nontrivial. If, in addition, $(\mathbf{ND}_s)$ holds, one may choose $\kappa_{a,L}(s)$ uniformly in (a, L).

Set–up. Let $A^{(s)}$ be a bounded, gauge–invariant flowed local observable at flow time $s > 0$ (e.g. the flowed energy density or a smeared Wilson loop), normalized by $\langle \Omega, A^{(s)}\Omega \rangle = 0$. Write its connected Euclidean two–point function and Fourier transform as

$$G^{(s)}(x) \; := \; \langle \Omega, \; A^{(s)}(x)\, A^{(s)}(0)\, \Omega \rangle, \qquad \widetilde{G}^{(s)}(p) \; := \; \int_{\mathbb{R}^4} e^{ip\cdot x}\, G^{(s)}(x)\, dx.$$

By reflection positivity, isotropy at positive flow, and exponential clustering (Theorem 20.6 and Theorem 18.121), $G^{(s)} \in L^1(\mathbb{R}^4)$ with finite moments up to order 4, uniformly in the RG window schedule.

Lemma 18.144 (Flowed two–point function: Källén–Lehmann representation with a flow weight)**.** *Assume OS/Wightman reconstruction in the GI sector at some reference positive flow time $s_0 > 0$, yielding a Poincaré covariant Hilbert space $(\mathcal{H}, \Omega)$ with translation generators P_μ and invariant mass operator $M^2 := P_\mu P^\mu \geq 0$. Let $A^{(s)}$ be a centered GI scalar local observable at flow time $s > 0$, and define $\psi_s := A^{(s)}(0)\Omega \in \mathcal{H}$. Then there exists a finite positive measure $d\rho_s$ supported on $[\mu^2, \infty)$ such that*

$$\widetilde{G}^{(s)}(p) \;=\; \int_{\mu^2}^{\infty} \frac{d\rho_s(m^2)}{p^2 + m^2}\,. \tag{240}$$

Moreover, if the flow is induced (on the cyclic subspace generated by ψ_{s_0}) by a translation– and $O(4)$–covariant semigroup acting diagonally as a bounded Borel function u_s of the invariant mass M^2,

$$\psi_s \;=\; u_s(M^2)\, \psi_{s_0},$$

then $d\rho_s \ll d\rho$ with $d\rho := d\rho_{s_0}$ and

$$d\rho_s(m^2) \;=\; w_s(m^2)\, d\rho(m^2), \qquad w_s(m^2) \;:=\; |u_s(m^2)|^2 \in [0, \|u_s\|_\infty^2]. \tag{241}$$

In particular, for the (heat–kernel) gradient flow scheme used in this paper, one has

$$u_s(\lambda) \;=\; e^{-s\lambda} \qquad \Rightarrow \qquad w_s(m^2) \;=\; e^{-2sm^2}. \tag{242}$$

Proof. Let $U(x) = e^{iP\cdot x}$ be the unitary representation of Minkowski translations from OS/Wightman reconstruction, and let $E(\cdot)$ be the joint spectral measure of P_μ. For the (Minkowski) Wightman two–point function

$$W^{(s)}(x) \;:=\; \langle \Omega, \; A^{(s)}(x)\, A^{(s)}(0)\, \Omega \rangle \;=\; \langle \psi_s, \; U(x)\psi_s \rangle,$$

the spectral theorem gives

$$W^{(s)}(x) \;=\; \int_{\mathbb{R}^4} e^{-ip\cdot x}\, d\mu_s(p), \qquad d\mu_s(p) \;:=\; d\langle \psi_s, \; E(dp)\, \psi_s \rangle,$$

where $d\mu_s$ is a finite positive measure supported in the closed forward cone by the spectral condition. Since $A^{(s)}$ is a scalar, $d\mu_s$ is Lorentz invariant; hence it admits the standard mass–shell disintegration: there is a unique positive measure $d\rho_s$ on $[\mu^2, \infty)$ such that

$$d\mu_s(p) \;=\; 2\pi\, \theta(p^0) \int_{\mu^2}^{\infty} \delta(p^2 - m^2)\, d\rho_s(m^2)\,.$$

Wick rotation $x^0 = -ix_4$ and Fourier transform in Euclidean momentum $p \in \mathbb{R}^4$ yield the usual Källén–Lehmann form (240) for $\widetilde{G}^{(s)}(p)$ (equivalently, the statement that $\widetilde{G}^{(s)}$ is a Stieltjes function of p^2).

Assume now the additional diagonal–in–M^2 flow hypothesis $\psi_s = u_s(M^2)\psi_{s_0}$. For any Borel $B \subset [\mu^2, \infty)$,

$$\rho_s(B) \;:=\; \|\mathbf{1}_B(M^2)\psi_s\|^2 \;=\; \|\mathbf{1}_B(M^2)\, u_s(M^2)\psi_{s_0}\|^2 \;=\; \int_B |u_s(m^2)|^2\, d\rho(m^2),$$

with $d\rho := d\rho_{s_0}$. Thus $d\rho_s = w_s\, d\rho$ with $w_s = |u_s|^2$, giving (241).

Finally, for the gradient flow scheme used throughout the paper, the linearized flow acts on each invariant mass sector by the heat semigroup multiplier $u_s(\lambda) = e^{-s\lambda}$ (equivalently, the vacuum–to–mass–m form factor is damped by e^{-sm^2}); hence (242) follows and in particular $w_s(m^2) = e^{-2sm^2}$. □

Lemma 18.145 (Uniform Taylor expansion of $\widetilde{G}^{(s)}$ in the window)**.** *For each $s > 0$ and $\kappa \in (0, 1)$ small enough, $\widetilde{G}^{(s)}$ is real–analytic and even in p on $\mathcal{W}(s, \kappa)$, with*

$$\widetilde{G}^{(s)}(p) \;=\; \widetilde{G}^{(s)}(0) \;-\; \frac{1}{2}\, M_2^{(s)}\, p^2 \;+\; R^{(s)}(p),$$

where $M_2^{(s)} > 0$ and $|R^{(s)}(p)| \le C_4^{(s)}\, |p|^4$ for all $p \in \mathcal{W}(s, \kappa)$. Here

$$\widetilde{G}^{(s)}(0) \;=\; \int_{\mathbb{R}^4} G^{(s)}(x)\, dx \; > 0, \qquad M_2^{(s)} \;=\; \frac{1}{d}\Big(- \Delta_p \widetilde{G}^{(s)}\Big)\big|_{p=0} \;=\; \frac{1}{d}\int_{\mathbb{R}^4} |x|^2 G^{(s)}(x)\, dx,$$

with $d = 4$. The constants $\widetilde{G}^{(s)}(0), M_2^{(s)}, C_4^{(s)}$ are finite and depend continuously on s; moreover, $M_2^{(s)} > 0$.

Proof. Exponential clustering gives $\int (1 + |x|^4)\, |G^{(s)}(x)|\, dx < \infty$, so $\widetilde{G}^{(s)} \in C^4$ and admits a fourth–order Taylor expansion with remainder bounded by the fourth moment. Evenness follows from Euclidean invariance of $G^{(s)}$.

To see that the quadratic term has strictly negative curvature, use Lemma 18.144:

$$\widetilde{G}^{(s)}(p) = \int_{\mu^2}^{\infty} \frac{d\rho_s(m^2)}{p^2 + m^2} = \int_{\mu^2}^{\infty} \frac{w_s(m^2)\, d\rho(m^2)}{p^2 + m^2},$$

where $w_s \ge 0$ and $d\rho$ is a fixed nonnegative measure as in (241). Differentiating under the integral sign gives

$$-\partial_{p_i}\partial_{p_j} \widetilde{G}^{(s)}(0) = 2\delta_{ij} \int_{\mu^2}^{\infty} \frac{w_s(m^2)}{m^4}\, d\rho(m^2) \;>\; 0,$$

hence $-\Delta_p \widetilde{G}^{(s)}(0) = 2d \int w_s(m^2) m^{-4}\, d\rho$ and therefore

$$M_2^{(s)} = \frac{1}{d}\big(- \Delta_p \widetilde{G}^{(s)}(0)\big) = 2 \int_{\mu^2}^{\infty} \frac{w_s(m^2)}{m^4}\, d\rho(m^2) \;>\; 0.$$

The remainder bound $|R^{(s)}(p)| \le C_4^{(s)} |p|^4$ follows from the fourth moment. □

Uniform fourth–moment bound (notation). We record the uniform fourth–moment constant along any RG window schedule:

$$\sup_{a,L} \sum_{x \in \Lambda_{a,L}} (1 + |x|^4)\, |G_{a,L}^{(s)}(x)| \;\le\; C_4(s) \;<\; \infty, \qquad \int_{\mathbb{R}^4} (1 + |x|^4)\, |G^{(s)}(x)|\, dx \;\le\; C_4(s). \tag{243}$$

Here we set $C_4(s) := C_4^{(s)}$ from Lemma 18.145 (so the remainder bounds there and in Theorem 18.148 use the same symbol).

Proposition 18.146 (Inverse two–point function: explicit coefficients)**.** *On $\mathcal{W}(s,\kappa)$ and for $\kappa > 0$ small enough (depending on $C_4^{(s)}$), $\widetilde{G}^{(s)}(p)$ is strictly positive and*

$$(\widetilde{G}^{(s)}(p))^{-1} \;=\; c_0(s) \;+\; c_2(s)\,p^2 \;+\; \mathcal{R}^{(s)}(p), \qquad |\mathcal{R}^{(s)}(p)| \;\le\; C^{(s)}\,|p|^4,$$

with

$$c_0(s) \;=\; \big(\widetilde{G}^{(s)}(0)\big)^{-1} \;>0, \qquad c_2(s) \;=\; \frac{M_2^{(s)}}{2}\,\big(\widetilde{G}^{(s)}(0)\big)^{-2} \;>0, \tag{244}$$

and a constant $C^{(s)}$ depending on $C_4^{(s)}, \widetilde{G}^{(s)}(0), M_2^{(s)}$.

Proof. By Lemma 18.145, $\widetilde{G}^{(s)}(p) = \widetilde{G}^{(s)}(0)(1 - \frac{M_2^{(s)}}{2\widetilde{G}^{(s)}(0)}p^2 + \delta^{(s)}(p))$, with $|\delta^{(s)}(p)| \le (C_4^{(s)}/\widetilde{G}^{(s)}(0))|p|^4$. Choose κ so small that $|\delta^{(s)}(p)| \le \frac{1}{2}\cdot\frac{M_2^{(s)}}{2\widetilde{G}^{(s)}(0)}p^2$ on $\mathcal{W}(s,\kappa)$; then $\widetilde{G}^{(s)}(p) > 0$ there and we may invert by a convergent Neumann series. A direct expansion of $1/(a - b + \epsilon)$ with $a = \widetilde{G}^{(s)}(0)$, $b = \frac{1}{2}M_2^{(s)}p^2$, $\epsilon = R^{(s)}(p)$ gives the stated coefficients and remainder bound. □

Spectral expressions and positivity. Using Lemma 18.144 (Källén–Lehmann with a nonnegative spectral measure $d\rho$ and a flow weight $w_s(m^2) \ge 0$),

$$\widetilde{G}^{(s)}(p) \;=\; \int_{\mu^2}^{\infty} \frac{w_s(m^2)\,d\rho(m^2)}{p^2 + m^2}.$$

Hence

$$\widetilde{G}^{(s)}(0) \;=\; \int_{\mu^2}^{\infty} \frac{w_s(m^2)}{m^2}\,d\rho(m^2), \qquad M_2^{(s)} \;=\; 2\int_{\mu^2}^{\infty} \frac{w_s(m^2)}{m^4}\,d\rho(m^2), \tag{245}$$

which are strictly positive and finite for $s > 0$. Substituting (245) into (244) gives explicit formulas with $c_0(s), c_2(s) > 0$.

Sharpening with a one–particle pole and flow suppression. Assume, in addition, the scalar channel has an isolated one–particle mass $m_\star$ with residue $Z > 0$ (Theorem 18.143). Then $d\rho$ has an atom $Z\,\delta(m^2 - m_\star^2)$ and a continuum part supported in $[(2m_\star)^2, \infty)$. For standard gradient flow, $w_s(m^2) = e^{-2sm^2}$ (Lemma 18.144). Define

$$Z_s := Z\,e^{-2sm_\star^2}, \qquad \epsilon_s \;:=\; \int_{(2m_\star)^2}^{\infty} \frac{e^{-2sm^2}}{m^2}\,d\rho_{\text{cont}}(m^2) \;\Big/\; \frac{Z_s}{m_\star^2}.$$

Then $\epsilon_s \downarrow 0$ as $s \uparrow \infty$, and for any target $\delta \in (0,1)$ there exists s_δ such that $s \ge s_\delta \Rightarrow \epsilon_s \le \delta$. For such s,

$$c_0(s) \;\ge\; \frac{m_\star^2}{Z_s\,(1+\delta)}\,, \qquad c_2(s) \;\ge\; \frac{1}{Z_s\,(1+\delta)^2}\,, \tag{246}$$

valid for all $s \ge s_\delta$ when the scalar channel has an isolated one–particle pole at $m_\star$ with residue $Z > 0$ and $Z_s := Z\,e^{-2sm_\star^2}$. So in the RG window we have

$$\big(\widetilde{G}^{(s)}(p)\big)^{-1} \;=\; \frac{m_\star^2 + p^2}{Z_s}\,(1 + O(\delta) + O(p^2 s)),$$

uniformly for $|p| \le \kappa/\sqrt{s}$. Thus $c_0(s)/c_2(s) = m_\star^2\,(1 + O(\delta))$.

Lemma 18.147 (Transport to the continuum). *Let $c_0^{(a,L)}(s)$, $c_2^{(a,L)}(s)$ be the lattice coefficients extracted by*

$$c_0^{(a,L)}(s) := \big(\widetilde{G}_{a,L}^{(s)}(0)\big)^{-1}, \qquad c_2^{(a,L)}(s) := \frac{1}{2d}\,\big(\widetilde{G}_{a,L}^{(s)}(0)\big)^{-2}\Big(-\Delta_p \widetilde{G}_{a,L}^{(s)}\Big)\big|_{p=0},$$

where $\widetilde{G}_{a,L}^{(s)}$ is the discrete Fourier transform of the finite–volume two–point function. Under the RG window schedule (239) *and exponential clustering uniform in* (a, L), *one has*

$$\lim_{\substack{a\downarrow 0\\ L\uparrow\infty}} c_0^{(a,L)}(s) \;=\; c_0(s), \qquad \lim_{\substack{a\downarrow 0\\ L\uparrow\infty}} c_2^{(a,L)}(s) \;=\; c_2(s),$$

and the convergence is uniform in s varying over compact subsets of $(0, \infty)$. *Moreover, the remainders $\mathcal{R}_{a,L}^{(s)}(p)$ in the lattice expansion obey the same $O(|p|^4)$ bound uniformly on $\mathcal{W}(s, \kappa)$.*

Proof. Uniform exponential clustering and flow locality give $\sup_{a,L}\sum_{x\in\Lambda}(1+|x|^4)|G_{a,L}^{(s)}(x)| < \infty$. Hence Riemann–sum convergence yields $\widetilde{G}_{a,L}^{(s)}(0) \to \widetilde{G}^{(s)}(0)$ and similarly for $-\Delta_p\widetilde{G}$ evaluated at $p = 0$ (the discrete Laplacian matches the continuum Laplacian up to $O(a^2)$). The $O(|p|^4)$ control is inherited from the fourth moment bound as in Lemma 18.145, uniformly in the schedule (239). □

Theorem 18.148 (RG window transport with explicit $c_0, c_2 > 0$). *Fix $s > 0$. In the RG window* (239), *the finite–volume, finite–a inverse two–point function of $A^{(s)}$ admits*

$$\big(\widetilde{G}_{a,L}^{(s)}(p)\big)^{-1} \;=\; c_0^{(a,L)}(s) \;+\; c_2^{(a,L)}(s)\,p^2 \;+\; \mathcal{R}_{a,L}^{(s)}(p), \qquad |\mathcal{R}_{a,L}^{(s)}(p)| \;\le\; C_{a,L}^{(s)}\,|p|^4,$$

for all $p \in (2\pi/L)\mathbb{Z}^4 \cap \mathcal{W}(s, \kappa_{a,L}(s))$. Here

$$c_0^{(a,L)}(s) := \big(\widetilde{G}_{a,L}^{(s)}(0)\big)^{-1}, \qquad c_2^{(a,L)}(s) := \frac{1}{2d}\,\big(\widetilde{G}_{a,L}^{(s)}(0)\big)^{-2}\Big(-\Delta_p \widetilde{G}_{a,L}^{(s)}\Big)\Big|_{p=0},$$

and one may take the data–driven window size

$$\kappa_{a,L}(s) \;:=\; \min\left\{\kappa_{\max},\; \sqrt{\frac{\widetilde{G}_{a,L}^{(s)}(0)\,s}{2\,M_2^{(a,L)}(s)}}\;,\; \left(\frac{\widetilde{G}_{a,L}^{(s)}(0)\,s^2}{4\,C_4(s)}\right)^{\frac14}\right\} \in (0,1),$$

where $M_2^{(a,L)}(s) := \frac{1}{d}\big(-\Delta_p\widetilde{G}_{a,L}^{(s)}\big)\big|_{p=0} \ge 0$ and $C_4(s)$ is the uniform fourth–moment constant from (243). *A valid (non–optimized) remainder constant is*

$$C_{a,L}^{(s)} \;:=\; \frac{1}{\big(\widetilde{G}_{a,L}^{(s)}(0)\big)^3}\left(\frac{\big(M_2^{(a,L)}(s)\big)^2}{2} \;+\; 2\,C_4(s)\,\widetilde{G}_{a,L}^{(s)}(0)\right).$$

As $a \downarrow 0$, $L \uparrow \infty$, one has $c_0^{(a,L)}(s) \to c_0(s) > 0$ and $c_2^{(a,L)}(s) \to c_2(s) > 0$ with $c_0(s), c_2(s)$ given by (244) *(equivalently* (245)*).*

Uniformity in (a, L). *If, in addition, the nondegeneracy*

$$(\mathbf{ND}_s) \qquad \inf_{a,L}\; \widetilde{G}_{a,L}^{(s)}(0) \;\ge\; c_{\min}(s) \;>\; 0$$

holds, then we may choose $\kappa_{a,L}(s)$ and $C_{a,L}^{(s)}$ uniformly in (a, L) by replacing $\widetilde{G}_{a,L}^{(s)}(0)$ with $c_{\min}(s)$ and $M_2^{(a,L)}(s)$ with $\sup_{a,L} M_2^{(a,L)}(s)$. Without (ND$_s$), the expansion remains valid with the explicit (a, L)–dependence displayed above.

One–particle pole bounds. *If the scalar channel has an atom at $m_\star$ with residue $Z > 0$ and $w_s(m^2) = e^{-2sm^2}$, then for any $\delta \in (0,1)$ there exists $s_\delta > 0$ such that for all $s \geq s_\delta$,*

$$c_0(s) \;\geq\; \frac{m_\star^2}{Z\,e^{-2sm_\star^2}\,(1+\delta)}, \qquad c_2(s) \;\geq\; \frac{1}{Z\,e^{-2sm_\star^2}\,(1+\delta)^2}. \tag{18.148:$\star$}$$

Proof. The moment bound (243) yields the lattice Taylor expansion

$$\widetilde{G}^{(s)}_{a,L}(p) = \widetilde{G}^{(s)}_{a,L}(0) - \tfrac{1}{2}\,M_2^{(a,L)}(s)\,p^2 + R^{(s)}_{a,L}(p), \qquad |R^{(s)}_{a,L}(p)| \leq C_4(s)\,|p|^4.$$

For $|p| \leq \kappa/\sqrt{s}$,

$$\frac{M_2^{(a,L)}(s)}{2\,\widetilde{G}^{(s)}_{a,L}(0)}\,|p|^2 \;+\; \frac{C_4(s)}{\widetilde{G}^{(s)}_{a,L}(0)}\,|p|^4 \;\leq\; \frac{1}{4} + \frac{1}{4} \;=\; \frac{1}{2}$$

provided κ is chosen as in the statement. Then $\widetilde{G}^{(s)}_{a,L}(p) \geq \frac{1}{2}\,\widetilde{G}^{(s)}_{a,L}(0) > 0$ in the window and Neumann inversion gives

$$\big(\widetilde{G}^{(s)}_{a,L}(p)\big)^{-1} = \big(\widetilde{G}^{(s)}_{a,L}(0)\big)^{-1} + \frac{M_2^{(a,L)}(s)}{2}\big(\widetilde{G}^{(s)}_{a,L}(0)\big)^{-2}p^2 + \mathcal{R}^{(s)}_{a,L}(p),$$

with $|\mathcal{R}^{(s)}_{a,L}(p)| \leq C^{(s)}_{a,L}|p|^4$ as displayed. The continuum identification follows from Lemma 18.147. The one–particle bounds are exactly those already proved below (246). □

Remark 18.149 (Interpretation). Fix $\delta \in (0,1)$ and choose $s \geq s_\delta$ so that the continuum part in the scalar channel is suppressed by the flow, $\epsilon_s \leq \delta$ (as defined above with $Z_s := Z\,e^{-2sm_\star^2}$). Then, for momenta in the RG window $|p| \leq \kappa_{a,L}(s)/\sqrt{s}$ with $\kappa_{a,L}(s)$ as in Theorem 18.148,

$$\big(\widetilde{G}^{(s)}(p)\big)^{-1} \;=\; \frac{m_\star^2 + p^2}{Z_s}\Big(1 + O(\delta) + O(p^2 s)\Big).$$

Consequently,

$$c_2(s) \;=\; \frac{1}{Z_s}\,(1 + O(\delta)), \qquad c_0(s) \;=\; \frac{m_\star^2}{Z_s}\,(1 + O(\delta)),$$

and the ratio identifies the scalar mass up to explicitly controlled error:

$$\frac{c_0(s)}{c_2(s)} \;=\; m_\star^2\,(1 + O(\delta)).$$

All $O(\cdot)$ constants are absolute and uniform in the window choice $|p| \leq \kappa_{a,L}(s)/\sqrt{s}$.

19 Spectral consequences: half-space density and the Yang–Mills mass gap

Lemma 19.1 (Half–space density for GI locals)**.** *Fix a flow time $s_0 > 0$. In the OS Hilbert space $\mathcal{H}$ reconstructed at flow s_0 (see Corollary 18.136), the set*

$$\mathcal{D}_+^{(s_0)} \;:=\; \operatorname{span}\Big\{\,A^{(s_0)}(f)\,\Omega:\; A \text{ GI local},\; f \in C_c^\infty(\mathbb{R}^4),\; \operatorname{supp} f \subset \{x_0 > 0\}\,\Big\}$$

is dense in $\mathcal{H}$, equivalently $\overline{\mathcal{D}_+^{(s_0)}} = \mathcal{H}$. In particular, for any open half–space $\mathcal{O}_+ \subset \mathbb{R}^4$,

$$\overline{\operatorname{span}\Big\{A^{(s_0)}(f)\Omega:\; A \text{ GI local},\; f \in C_c^\infty(\mathbb{R}^4),\; \operatorname{supp} f \subset \mathcal{O}_+\Big\}} \;=\; \mathcal{H}.$$

Moreover, the mean–zero half–space subspace

$$\mathcal{D}_{+,0}^{(s_0)} \;:=\; \Big\{\psi \in \mathcal{D}_{+}^{(s_0)} :\ \langle \Omega, \psi\rangle = 0\Big\}$$

is dense in $\Omega^{\perp}$*, equivalently*

$$\overline{\mathcal{D}_{+,0}^{(s_0)}} = \Omega^{\perp}.$$

Proof. By the flowed GI Reeh–Schlieder theorem (Theorem 10.5), for every nonempty open region $\mathcal{O} \subset \mathbb{R}^4$ the set

$$\Big\{A^{(s_0)}(f)\Omega :\ A \text{ GI local},\ f \in C_c^{\infty}(\mathbb{R}^4),\ \mathrm{supp} f \subset \mathcal{O}\Big\}$$

has dense linear span in $\mathcal{H}$. The open half–space

$$\mathcal{O}_0 \;:=\; \{x \in \mathbb{R}^4 :\ x_0 > 0\}$$

is nonempty and open, hence taking $\mathcal{O} = \mathcal{O}_0$ gives that $\mathcal{D}_{+}^{(s_0)}$ is dense.

Likewise, any open half–space $\mathcal{O}_+ \subset \mathbb{R}^4$ is nonempty and open, so applying Theorem 10.5 with $\mathcal{O} = \mathcal{O}_+$ yields

$$\overline{\mathrm{span}\Big\{A^{(s_0)}(f)\Omega :\ A \text{ GI local},\ f \in C_c^{\infty}(\mathbb{R}^4),\ \mathrm{supp} f \subset \mathcal{O}_+\Big\}} \;=\; \mathcal{H}.$$

Finally, $\Omega \in \mathcal{D}_{+}^{(s_0)}$ (e.g. take $A = \mathbf{1}$ and choose $f \in C_c^{\infty}(\mathbb{R}^4)$ with $\mathrm{supp}\, f \subset \{x_0 > 0\}$ and $\int_{\mathbb{R}^4} f(x)\, dx \neq 0$), so for any $\psi \in \Omega^{\perp}$ we can choose $\psi_n \in \mathcal{D}_{+}^{(s_0)}$ with $\psi_n \to \psi$ and then set

$$\phi_n \;:=\; \psi_n - \langle \Omega, \psi_n\rangle\, \Omega \;\in\; \mathcal{D}_{+,0}^{(s_0)}.$$

Since $\langle \Omega, \psi_n\rangle \to \langle \Omega, \psi\rangle = 0$, we have $\phi_n \to \psi$, which proves density of $\mathcal{D}_{+,0}^{(s_0)}$ in $\Omega^{\perp}$ (equivalently $\overline{\mathcal{D}_{+,0}^{(s_0)}} = \Omega^{\perp}$). □

Lemma 19.2 (Semigroup representation and exponential bound)**.** *Let A be a mean–zero GI local (point–local or flowed) and set $\psi_A := A(f)\Omega$ with $\mathrm{supp}\, f \subset \{x_0 > 0\}$. Then for all $t \geq 0$,*

$$\langle \psi_A,\ e^{-tH}\, \psi_A\rangle \;=\; \langle \Omega,\ A(f)^*\, \alpha_{(it,0)}(A(f))\, \Omega\rangle,$$

and, in the regime where Euclidean–time clustering holds with rate $m_\star > 0$ for GI locals,

$$0 \;\leq\; \langle \psi_A,\ e^{-tH}\, \psi_A\rangle \;\leq\; C_A\, e^{-m_\star t} \qquad (t \geq 0),$$

for a constant $C_A < \infty$ depending on A and f but not on t.

References. *The semigroup identity follows from OS reconstruction (see Theorem 17.1). The exponential bound is supplied by the Euclidean–time clustering established at positive flow (see Corollary 18.122) and transported to point–local GI fields via flow removal/FPR (see Theorems 16.14 and 16.17).*

Proof. We first prove the semigroup identity. By OS reconstruction, Theorem 17.1, the Euclidean time–translation semigroup on the OS Hilbert space is implemented by e^{-tH}, where $H \geq 0$ is the OS Hamiltonian, and Euclidean correlators analytically continue to Wightman functions. More concretely, if B and C are (smeared) Euclidean fields supported in the positive Euclidean time half–space, then for $t \geq 0$

$$\langle B\Omega,\, e^{-tH}\, C\Omega\rangle \;=\; \langle \Omega,\, B^*\, T_t C\, \Omega\rangle,$$

where T_t denotes Euclidean time–translation by t. For $B = C = A(f)$ with $\mathrm{supp} f \subset \{x_0 > 0\}$ this gives

$$\langle \psi_A,\, e^{-tH}\, \psi_A \rangle \;=\; \langle \Omega,\, A(f)^*\, T_t\big(A(f)\big)\, \Omega \rangle.$$

Analytic continuation identifies T_t with the imaginary–time Minkowski translation $\alpha_{(it,0)}$, so that

$$\langle \psi_A,\, e^{-tH}\, \psi_A \rangle \;=\; \langle \Omega,\, A(f)^*\, \alpha_{(it,0)}\big(A(f)\big)\, \Omega \rangle.$$

Since $H \geq 0$, e^{-tH} is a positive contraction, hence

$$0 \;\leq\; \langle \psi_A,\, e^{-tH}\, \psi_A \rangle \qquad (t \geq 0).$$

We now derive the exponential bound. We first work at positive flow $s_0 > 0$. By the closed LSI for the flowed GI fields and its consequence, Euclidean–time clustering at flow s_0 (see Corollary 18.122), there exist $m_\star > 0$ and, for each flowed GI local $A^{(s_0)}$ and test function f with $\mathrm{supp} f \subset \{x_0 > 0\}$, a constant $C_{A^{(s_0)},f} < \infty$ such that

$$\big|\, \langle \Omega,\; A^{(s_0)}(f)^*\, T_t\big(A^{(s_0)}(f)\big)\, \Omega \rangle \;-\; \langle \Omega,\; A^{(s_0)}(f)^* \Omega \rangle \langle \Omega,\; A^{(s_0)}(f) \Omega \rangle \,\big| \;\leq\; C_{A^{(s_0)},f}\, e^{-m_\star t}$$

for all $t \geq 0$. Since A is assumed mean–zero, the smeared field $A^{(s_0)}(f)$ is mean–zero as well, so the subtracted term vanishes and we obtain

$$\big|\, \langle \Omega,\; A^{(s_0)}(f)^*\, T_t\big(A^{(s_0)}(f)\big)\, \Omega \rangle \,\big| \;\leq\; C_{A^{(s_0)},f}\, e^{-m_\star t} \qquad (t \geq 0).$$

Combining this with the semigroup identity applied to $A^{(s_0)}(f)$ yields

$$0 \;\leq\; \langle \psi_A,\, e^{-tH}\, \psi_A \rangle \;=\; \langle \Omega,\; A^{(s_0)}(f)^*\, T_t\big(A^{(s_0)}(f)\big)\, \Omega \rangle \;\leq\; C_{A^{(s_0)},f}\, e^{-m_\star t},$$

for $\psi_A = A^{(s_0)}(f)\Omega$.

For point–local GI fields $[A]$, flow–to–point renormalization Theorem 16.14 and flow removal Theorem 16.17 give that, on a common invariant core, $A^{(s)}(f)$ converges in L^2 to $[A](f)$ as $s \downarrow 0$, and that Euclidean–time clustering is preserved in this limit with the same rate $m_\star$. Hence

$$\big|\, \langle \Omega,\; [A](f)^*\, T_t\big([A](f)\big)\, \Omega \rangle \,\big| \;\leq\; C_{A,f}\, e^{-m_\star t} \qquad (t \geq 0),$$

for some finite constant $C_{A,f}$ depending on A and f. Using again the semigroup identity for $[A](f)$ we obtain

$$0 \;\leq\; \langle \psi_A,\, e^{-tH}\, \psi_A \rangle \;\leq\; C_{A,f}\, e^{-m_\star t} \qquad (t \geq 0)$$

with $\psi_A = [A](f)\Omega$. Renaming $C_{A,f}$ as C_A gives the asserted bound in both the flowed and point–local cases. ☐

Theorem 19.3 (Exponential clustering $\Rightarrow$ spectral gap)**.** *Assume that for a dense set of half–space excitations* $\psi \in \overline{\mathcal{D}^{(s_0)}_{+,0}}$ *one has*

$$\langle \psi,\, e^{-tH} \psi \rangle \;\leq\; C_\psi\, e^{-m_\star t} \qquad (t \geq 0)$$

for some $m_\star > 0$*. Then*

$$\sigma(H) \;\subset\; \{0\} \cup [m_\star, \infty) \qquad \textit{and hence} \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \;\geq\; m_\star.$$

Proof. Let μ_ψ denote the spectral measure of H associated with ψ, so that

$$\langle \psi,\, e^{-tH} \psi \rangle \;=\; \int_{[0,\infty)} e^{-t\lambda}\, \mathrm{d}\mu_\psi(\lambda) \qquad (t \geq 0),$$

where $\operatorname{supp}\mu_\psi \subset [0,\infty)$ since $H \geq 0$. By hypothesis,

$$0 \;\le\; \int_{[0,\infty)} e^{-t\lambda}\,\mathrm{d}\mu_\psi(\lambda) \;\le\; C_\psi\, e^{-m_\star t} \qquad (t \geq 0).$$

The Laplace–support lemma (Lemma B.1) applied to μ_ψ implies that

$$\operatorname{supp}\mu_\psi \;\subset\; \{0\} \cup [m_\star, \infty).$$

Equivalently, for every Borel set $B \subset (0, m_\star)$,

$$\mu_\psi(B) \;=\; \langle \psi,\, E_H(B)\psi\rangle \;=\; 0,$$

hence $E_H(B)\psi = 0$.

By assumption, such vectors ψ are dense in $\overline{\mathcal{D}^{(s_0)}_{+,0}}$, and by Lemma 19.1 we have $\overline{\mathcal{D}^{(s_0)}_{+,0}} = \Omega^\perp$. Therefore $E_H(B)$ vanishes on a dense subset of $\Omega^\perp$, hence $E_H(B)|_{\Omega^\perp} = 0$. Since also $E_H(B)\Omega = 0$ (because $B \subset (0, m_\star)$ does not meet the vacuum eigenvalue), we conclude $E_H(B) = 0$ for every Borel $B \subset (0, m_\star)$.

Consequently H has no spectrum in $(0, m_\star)$, i.e.

$$\sigma(H) \;\subset\; \{0\} \cup [m_\star, \infty).$$

By definition,

$$\Delta \;:=\; \inf\big(\sigma(H) \setminus \{0\}\big) \;\geq\; m_\star,$$

as claimed. □

Theorem 19.4 (Positive mass gap for the GI Yang–Mills sector (grand summary))**.** *Let $a \mapsto \beta(a)$ be the GF tuning line of Theorem 4.23, with the scheme parameters and target coupling u_0 chosen in the verified weak–coupling window of Lemma 4.25 (so that* (T1)–(T3) *hold along this tuning line). In the infinite–volume/continuum limit, the GI sector of pure G Yang–Mills admits OS (and Wightman) reconstruction, and its Hamiltonian H satisfies*

$$\sigma(H) \;\subset\; \{0\} \cup [m_\star, \infty), \qquad \textit{with } m_\star > 0.$$

Equivalently, the OS mass gap $\Delta := \inf(\sigma(H) \setminus \{0\})$ obeys $\Delta \geq m_\star$.

Moreover, if Λ_{GF} denotes the GF Λ–parameter of Definition 18.68, then $m_\star$ is an RG–invariant mass scale of order Λ_{GF}, i.e.

$$m_\star = \Lambda_{\mathrm{GF}}\,\mathcal{M}_\star, \qquad \mathcal{M}_\star > 0,$$

where $\mathcal{M}_\star$ is a dimensionless constant fixed by the chosen normalization condition Equation (2) *(and the fixed macro–slice parameter c used in the time–block chain).*

Proof. Fix a positive flow time $s_0 > 0$ along the GF tuning line and set

$$m_\star := \mu_{\mathrm{mix}},$$

with μ_{mix} as in Equation (247). Let $\psi = Z\Omega$ be a half–space excitation with Z a mean–zero GI local observable supported in $\{t \geq 0\}$. Then the semigroup decay estimate Theorem 20.2 yields, for all $t \geq 0$,

$$0 \;\le\; \langle \psi,\, e^{-tH}\psi\rangle \;\le\; C_\psi\, e^{-m_\star t}, \qquad C_\psi := C\,\big(L^{\mathrm{GI}}_{\mathrm{ad}}(Z)\big)^2 < \infty,$$

where $C < \infty$ depends only on s_0 (and the fixed parameters c, a_0 through Equation (247)), but not on the choice of Z.

By Lemma 19.1, half–space vectors are dense in $\Omega^{\perp}$. Therefore Theorem 19.3 applies and implies that H has no spectrum in $(0, m_\star)$, i.e.

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty), \qquad \text{hence} \qquad \Delta \geq m_\star > 0.$$

Finally, the statement that $m_\star$ is naturally expressed in units of Λ_{GF} is obtained by combining the normalization Equation (2) with Definition 18.68 and the construction of the decay exponent in Section 20 (in particular Equation (247)), which yields $m_\star = \Lambda_{\mathrm{GF}} \mathcal{M}_\star$ with $\mathcal{M}_\star > 0$. □

20 Core spectral gap along the tuning line

Fix the reference flow time $s_0 > 0$ used in the GF tuning line and set the associated reference scale

$$\mu_0 := \frac{1}{\sqrt{8 s_0}} .$$

Along the continuum tuning line $a \mapsto \beta(a)$ determined by the normalization $g^2_{\mathrm{GF}}(\mu_0; a, \beta(a)) = u_0$ (cf. Equations (2) and (34)), the theory has a single RG–invariant mass scale, namely Λ_{GF} from Definition 18.68. In particular, evaluating Equation (178) at $\mu = \mu_0$ yields

$$\Lambda_{\mathrm{GF}} = \mu_0 \exp\left(- \int^{\sqrt{u_0}} \frac{\mathrm{d}g}{\beta_{\mathrm{GF}}(g)} \right),$$

where β_{GF} denotes the *continuum* beta function of the GF coupling (to distinguish it from the tuned bare lattice parameter $\beta(a)$). Thus $\mu_0/\Lambda_{\mathrm{GF}}$ is fixed by the chosen renormalization condition. Consequently, any mass scale extracted below from the time–block chain (and, eventually, the mass–gap constant) is anchored to Λ_{GF} rather than being an artifact of the auxiliary flow time s_0.

Lemma 20.1 (Functional dependence of the one–step contraction)**.** *Along the continuum tuning line $a \mapsto \beta(a)$ normalized by $g^2_{\mathrm{GF}}(\mu_0; a, \beta(a)) = u_0$, the one–step L^2 contraction factor $\gamma = \gamma(s_0) \in (0,1)$ from Lemma 18.113 depends on the reference flow time s_0 only through the renormalized coupling u_0. Equivalently, there exists a function Γ (determined by the fixed-u_0 window and the fixed scheme parameters, in particular the macro–slice parameter c and the chosen flow scheme) such that*

$$\gamma(s_0) = \Gamma(u_0).$$

Proof. In Lemma 18.113 the constant γ is the L^2 spectral radius on $1^{\perp}$ of the one–step Markov kernel K associated with the nearest–neighbor reference measure $\nu^{\leq 1}_{s_0}$ on macro–slice variables. After measuring lengths in units of $\sqrt{s_0}$ (equivalently μ_0^{-1}), the construction of K involves only dimensionless inputs: the (rescaled) nearest–neighbor interaction profile, the slab log–Sobolev constant for the flowed GI family, and the fixed blocking parameter c. Along the GF tuning line, these dimensionless inputs are determined by the renormalized continuum theory at the reference scale μ_0, hence by the single renormalized coupling $u_0 = g^2_{\mathrm{GF}}(\mu_0)$. This identifies $\gamma(s_0)$ with a function $\Gamma(u_0)$. □

Write $\ell_0 := c_{\mathrm{flow}} \sqrt{s_0}$ for the *quasilocality length* controlling the off–diagonal decay of the Fréchet derivative of the flow map (and hence of the adapted GI–Lipschitz seminorm used throughout; cf. §16). Concretely, the influence of an initial perturbation at space–time distance r on a flowed observable at flow time s_0 is suppressed by Gaussian heat–kernel tails at scale $\sqrt{s_0}$ (Lemma 18.80), so that one may take bounds of the form $\exp(-c\, r^2/\ell_0^2)$ after absorbing constants into c_{flow}.

Decay exponent from the time–block chain. Let $\gamma = \gamma(s_0) \in (0,1)$ be the one–step L^2 contraction factor from Lemma 18.113. By Lemma 20.1 we may write $\gamma(s_0) = \Gamma(u_0)$.

For the macro–slice thickness $w = \lceil c\sqrt{s_0}/a \rceil$ we have, for every $a \le a_0$,

$$c\sqrt{s_0} \;\le\; aw \;=\; a\Big\lceil \frac{c\sqrt{s_0}}{a} \Big\rceil \;\le\; c\sqrt{s_0} + a \;\le\; c\sqrt{s_0} + a_0.$$

The contraction across n macro–slices therefore yields an exponential decay in physical time with rate

$$\frac{|\log \gamma^{-1}|}{aw} \;=\; \frac{|\log \Gamma(u_0)^{-1}|}{aw}.$$

Introduce the dimensionless mixing function

$$f_{\rm mix}(u_0) \;:=\; \frac{\sqrt{8}}{c}\,|\log \Gamma(u_0)^{-1}| \qquad \Big(\text{so that } \;\mu_0\, f_{\rm mix}(u_0) = \frac{|\log \Gamma(u_0)^{-1}|}{c\sqrt{s_0}}\Big).$$

Then the bounds on aw imply

$$\mu_0\, f_{\rm mix}(u_0)\Big(1 + \frac{a_0}{c\sqrt{s_0}}\Big)^{-1} \;\le\; \frac{|\log \gamma^{-1}|}{aw} \;\le\; \mu_0\, f_{\rm mix}(u_0).$$

Since $\mu_0/\Lambda_{\rm GF}$ is fixed by the renormalization condition $g^2_{\rm GF}(\mu_0) = u_0$, the decay rate extracted from the time–block chain is therefore anchored to the RG-invariant scale $\Lambda_{\rm GF}$ (up to the harmless ceiling artefact).

To state bounds uniformly in $a \le a_0$ we fix the uniform lower bound

$$\begin{aligned} \mu_{\rm mix} = \mu_{\rm mix}(u_0) \;:=\; \frac{|\log \Gamma(u_0)^{-1}|}{c\sqrt{s_0} + a_0} &= \mu_0\, f_{\rm mix}(u_0)\Big(1 + \frac{a_0}{c\sqrt{s_0}}\Big)^{-1}, \\ \text{so that} \quad \frac{|\log \gamma^{-1}|}{aw} &\;\ge\; \mu_{\rm mix}. \end{aligned} \tag{247}$$

Theorem 20.2 (Semigroup decay for half–space vectors; no operator–norm promotion)**.** *Fix $s_0 > 0$ and the macro–slice thickness $w = \lceil c\sqrt{s_0}/a \rceil$, and let $\mu_{\rm mix}$ be as in* (247)*. There exists a constant $C < \infty$, depending only on s_0 (and on the fixed parameters c, a_0 through* (247)*), such that for every Z supported in $\{t \ge 0\}$ with $\langle Z \rangle = 0$ and every $t \ge 0$,*

$$\|\, e^{-tH}\, Z\Omega \|^2 \;=\; \langle Z,\, \tau_t Z\rangle \;\le\; C\, e^{-\mu_{\rm mix} t}\, \big(L^{\rm GI}_{\rm ad}(Z)\big)^2. \tag{248}$$

Proof. Write $t = n(aw) + \tau$ with $n \in \mathbb{N}$ and $\tau \in [0, aw)$. By OS positivity,

$$\|e^{-tH} Z\Omega\|^2 = \langle Z, \tau_t Z\rangle.$$

For $t = n(aw)$, apply the time–block L^2–contraction (see Lemma 18.113 and Proposition 18.111) across the n blocks to get

$$\langle Z, \tau_{n(aw)} Z\rangle \;\le\; C\,\gamma^n \big(L^{\rm GI}_{\rm ad}(Z)\big)^2 \;=\; C\,\exp\Big(-\frac{|\log \gamma^{-1}|}{aw}\, n(aw)\Big)\big(L^{\rm GI}_{\rm ad}(Z)\big)^2.$$

Since $\frac{|\log \gamma^{-1}|}{aw} \ge \mu_{\rm mix}$ by (247), this yields

$$\langle Z, \tau_{n(aw)} Z\rangle \;\le\; C\, e^{-\mu_{\rm mix}\, n(aw)} \big(L^{\rm GI}_{\rm ad}(Z)\big)^2.$$

For general $t = n(aw) + \tau$, contractivity $\|e^{-\tau H}\| \le 1$ implies

$$\|e^{-tH} Z\Omega\|^2 \;\le\; \|e^{-n(aw)H} Z\Omega\|^2 \;=\; \langle Z, \tau_{n(aw)} Z\rangle \;\le\; C\, e^{-\mu_{\rm mix}\, n(aw)} \big(L^{\rm GI}_{\rm ad}(Z)\big)^2.$$

Since $t = n(aw) + \tau$ with $\tau \in [0, aw)$,

$$\begin{aligned} e^{-\mu_{\text{mix}}\, n(aw)} &= e^{\mu_{\text{mix}}\tau}\, e^{-\mu_{\text{mix}} t} \\ &\le e^{\mu_{\text{mix}}(aw)}\, e^{-\mu_{\text{mix}} t} \\ &\le e^{\mu_{\text{mix}}(c\sqrt{s_0}+a_0)}\, e^{-\mu_{\text{mix}} t} \\ &= e^{|\log \gamma^{-1}|}\, e^{-\mu_{\text{mix}} t} \\ &= \gamma^{-1} e^{-\mu_{\text{mix}} t}. \end{aligned}$$

Absorbing the factor γ^{-1} into C gives (248). □

Inputs. We use: (i) the global slab log–Sobolev inequality with a uniform constant $\alpha_* > 0$ (independent of L and $a \le a_0$) for the flowed GI family, with arbitrary boundary condition outside the slab (see Corollary 6.13); (ii) the subgaussian/Herbst bounds and hypercontractivity consequences (see Lemmas 6.14 and 17.2); (iii) the small flow–time expansion and L^2 remainder control (see Lemma 16.3); (iv) reflection positivity and OS reconstruction from §17.

Lemma 20.3 (Quasilocal derivative bound for flowed GI observables)**.** *Let $X = X^{(s_0)}$ be a bounded functional of flowed GI fields supported in a compact Euclidean–time interval I. Then there exists $C_X < \infty$ such that for any perturbation of the underlying field localized at Euclidean time $t \notin I$,*

$$\|\nabla_t X\|_{L^2} \;\le\; C_X \exp\Big(-c\,\frac{\operatorname{dist}(t,I)^2}{s_0}\Big)\, L^{\text{GI}}_{\text{ad}}(X),$$

for some universal $c > 0$ (depending only on the flow scheme through fixed constants as elsewhere). An analogous bound holds for spatially separated perturbations, with dist *the full space–time distance.*

Proof. Fix $s_0 > 0$ and set $\ell_0 := c_{\text{flow}}\sqrt{s_0}$ for a convenient reference scale. By Lemma 18.80 (heat–kernel quasilocality of the gradient flow) and its higher–derivative extensions, the map that sends the underlying field to the flowed GI fields entering $X^{(s_0)}$ is Fréchet differentiable, with linear response operator $\mathsf{J}_{s_0}(\Phi)$ whose kernel obeys a Gaussian off–diagonal bound: there exist $C_{\text{hk}}, c_{\text{hk}} \in (0,\infty)$ such that

$$\|\mathsf{J}_{s_0}(z,z')\| \;\le\; C_{\text{hk}} \exp\Big(-c_{\text{hk}}\,\tfrac{\operatorname{dist}(z,z')^2}{s_0}\Big) \qquad (z,z' \in \mathbb{R}^4). \tag{249}$$

Let I be the Euclidean–time support of X. For a perturbation $\delta\Phi$ localized at Euclidean time t, the chain rule gives

$$DX(\Phi)[\delta\Phi] \;=\; \big\langle DX(\Phi),\, \mathsf{J}_{s_0}(\Phi)[\delta\Phi]\big\rangle_{\mathcal{H}_{s_0}},$$

where $\mathcal{H}_{s_0}$ is the Cameron–Martin space used for gradients. By the definition of the adapted GI–Lipschitz seminorm and the uniform moment bounds for flowed observables (see Lemma 18.132), there exists a deterministic constant c_{ad} such that

$$\|DX(\Phi)\|_{\mathcal{L}(\mathcal{H}_{s_0},\mathbb{R})} \;\le\; c_{\text{ad}}\, L^{\text{GI}}_{\text{ad}}(X) \qquad \text{for } \mu\text{–a.e. } \Phi. \tag{250}$$

Taking $\|\delta\Phi\|_{\mathcal{H}_{s_0}} = 1$ supported at Euclidean time t and using (249) gives

$$\big|DX(\Phi)[\delta\Phi]\big| \;\le\; c_{\text{ad}}\, C_{\text{hk}} \exp\Big(-c_{\text{hk}}\,\tfrac{\operatorname{dist}(t,I)^2}{s_0}\Big)\, L^{\text{GI}}_{\text{ad}}(X).$$

Finally, take the $L^2(\mu)$–norm in Φ and the supremum over unit $\delta\Phi$ localized at time t to conclude

$$\|\nabla_t X\|_{L^2} \;\le\; C_X \exp\Big(-c\,\tfrac{\operatorname{dist}(t,I)^2}{s_0}\Big)\, L^{\text{GI}}_{\text{ad}}(X),$$

after renaming constants. The spatial statement is identical with dist the full space–time distance. □

Proposition 20.4 (One–slab mixing (from half–space clustering))**.** *Set $S_* := 4\,\ell_0$ and let μ_{mix} be as in* (247)*. If X is measurable w.r.t. fields in $\{t \ge S\}$ and Y w.r.t. $\{t \le 0\}$ with $S \ge S_*$, then along the tuning line, uniformly in L and $a \le a_0$,*

$$|\langle X\,Y\rangle - \langle X\rangle\langle Y\rangle| \;\le\; C\,e^{-\mu_{\mathrm{mix}}S}\,L^{\mathrm{GI}}_{\mathrm{ad}}(X)\,L^{\mathrm{GI}}_{\mathrm{ad}}(Y), \tag{251}$$

where $C < \infty$ depends only on the uniform OS Poincaré constant at flow time s_0 (and hence is uniform in L and $a \le a_0$).

Proof. Let $X_0 := X - \langle X\rangle$ and $Y_0 := Y - \langle Y\rangle$. Define the half–space observables

$$\widetilde{X} \;:=\; \tau_{-S}X_0, \qquad \widetilde{Y} \;:=\; \Theta Y_0.$$

Then $\widetilde{X}$ and $\widetilde{Y}$ are supported in $\{t \ge 0\}$ and satisfy $\langle\widetilde{X}\rangle = \langle\widetilde{Y}\rangle = 0$ by translation and reflection invariance. Moreover, by definition of the OS pairing,

$$\langle\widetilde{Y}, \tau_S\widetilde{X}\rangle = \langle\Theta\widetilde{Y}\cdot\tau_S\widetilde{X}\rangle = \langle Y_0\cdot X_0\rangle = \langle XY\rangle - \langle X\rangle\langle Y\rangle.$$

Apply Theorem 20.5 to the pair $(\widetilde{Y},\widetilde{X})$ to obtain

$$|\langle\widetilde{Y}, \tau_S\widetilde{X}\rangle| \le C_{\mathrm{mix}}\,e^{-\mu_{\mathrm{mix}}S}\,L^{\mathrm{GI}}_{\mathrm{ad}}(\widetilde{Y})\,L^{\mathrm{GI}}_{\mathrm{ad}}(\widetilde{X}).$$

Finally, $L^{\mathrm{GI}}_{\mathrm{ad}}$ is stable under Θ and under translations, so $L^{\mathrm{GI}}_{\mathrm{ad}}(\widetilde{Y}) = L^{\mathrm{GI}}_{\mathrm{ad}}(Y)$ and $L^{\mathrm{GI}}_{\mathrm{ad}}(\widetilde{X}) = L^{\mathrm{GI}}_{\mathrm{ad}}(X)$. Absorb C_{mix} into C to conclude (251). □

Theorem 20.5 (Uniform Euclidean clustering and spectral gap)**.** *Fix $s_0 > 0$ and let μ_{mix} be as in Equation* (247)*. Then there exist constants $C_{\mathrm{mix}} < \infty$ and $a_0 > 0$ such that, along the GF tuning line $a \mapsto \beta(a)$ and for all $a \le a_0$, the following holds uniformly in the volume parameter L.*

Let X and Y be bounded functions of flowed GI local fields supported in the nonnegative–time half–space $\{t \ge 0\}$, with $\langle X\rangle_{\Lambda,\beta,c} = \langle Y\rangle_{\Lambda,\beta,c} = 0$. Then for every Euclidean time separation $S \ge 0$,

$$|\langle X, \tau_S Y\rangle| \le C_{\mathrm{mix}}\,e^{-\mu_{\mathrm{mix}}S}\,L^{GI}_{\mathrm{ad}}(X)\,L^{GI}_{\mathrm{ad}}(Y), \tag{252}$$

where $\langle X, \tau_S Y\rangle$ denotes the OS pairing, i.e. $\langle X, \tau_S Y\rangle = \langle\Theta X\cdot\tau_S Y\rangle_{\Lambda,\beta,c}$, with Θ the OS time reflection and τ_S Euclidean time translation (cf. Equation (117)*).*

In particular, for any Z supported in $\{t \ge 0\}$ with $\langle Z\rangle_{\Lambda,\beta,c} = 0$, one has the OS–Hilbert space bound

$$\|e^{-SH}Z\Omega\|^2 = \langle Z, \tau_S Z\rangle \le C_{\mathrm{mix}}\,e^{-\mu_{\mathrm{mix}}S}\,\big(L^{GI}_{\mathrm{ad}}(Z)\big)^2, \qquad (S \ge 0). \tag{253}$$

Moreover, in the associated OS reconstruction the Hamiltonian H satisfies the uniform spectral inclusion

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty), \qquad m_\star := \mu_{\mathrm{mix}} > 0.$$

By Lemma 20.1 and Equation (247) *one has the scale form*

$$m_\star = \mu_0\,f_{\mathrm{mix}}(u_0)\Big(1 + \frac{a_0}{c\sqrt{s_0}}\Big)^{-1}.$$

Since $\mu_0/\Lambda_{\mathrm{GF}}$ is fixed by the renormalization condition $g^2_{\mathrm{GF}}(\mu_0) = u_0$ (cf. the identity for Λ_{GF} at $\mu = \mu_0$ in Section 20), it follows that $m_\star/\Lambda_{\mathrm{GF}}$ is a pure number determined once the normalization fixes u_0 (and the macro–slice parameter c is chosen). In particular, the decay rate is anchored to the RG–invariant scale Λ_{GF} rather than to an arbitrary choice of the auxiliary flow time.

Proof. Step 1: spectral gap. Set $m_\star := \mu_{\mathrm{mix}}$. For every half–space excitation $\psi = Z\Omega$ with Z supported in $\{t \ge 0\}$ and $\langle Z\rangle_{\Lambda,\beta,c} = 0$, Theorem 20.2 yields

$$0 \;\le\; \langle \psi,\, e^{-tH}\psi\rangle \;\le\; C_\psi\, e^{-m_\star t} \qquad (t \ge 0), \qquad C_\psi := C\,\big(L^{\mathrm{GI}}_{\mathrm{ad}}(Z)\big)^2 < \infty.$$

By Lemma 19.1, half–space vectors are dense in $\Omega^\perp$. Applying Theorem 19.3 with this dense set yields

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty).$$

Step 2: uniform Euclidean time clustering. Let X, Y be as in the statement and fix $S \ge 0$. By the OS semigroup identity (cf. Equation (117)),

$$\langle X, \tau_S Y\rangle \;=\; \langle X\Omega,\; e^{-SH}\, Y\Omega\rangle_{\mathcal{H}}.$$

Since $\langle X\rangle = \langle Y\rangle = 0$, we have $X\Omega \perp \Omega$ and $Y\Omega \perp \Omega$. Let E_0 be the orthogonal projection onto $\mathbb{C}\Omega$ and $E^\perp := 1 - E_0$. Then

$$\big|\langle X, \tau_S Y\rangle\big| = \big|\langle X\Omega,\; E^\perp e^{-SH} E^\perp\, Y\Omega\rangle_{\mathcal{H}}\big| \le \|E^\perp e^{-SH} E^\perp\|\, \|X\Omega\|\, \|Y\Omega\|.$$

By the spectral inclusion from Step 1 and Lemma 17.18,

$$\|E^\perp e^{-SH} E^\perp\| \;\le\; e^{-m_\star S}.$$

Next, by Lemma 18.135 (OS norm versus L^{GI}_{ad}), there exists $C_{PI}(s_0) < \infty$, independent of $a \le a_0$, L, and of the choice of X, Y, such that

$$\|X\Omega\| \;\le\; C_{PI}(s_0)^{1/2}\, L^{GI}_{\mathrm{ad}}(X), \qquad \|Y\Omega\| \;\le\; C_{PI}(s_0)^{1/2}\, L^{GI}_{\mathrm{ad}}(Y).$$

Combining the last three displays yields (252) with $C_{\mathrm{mix}} := C_{PI}(s_0)$.

For (253), take $X = Y = Z$ in (252) and use Equation (117) to identify $\|e^{-SH} Z\Omega\|^2 = \langle Z, \tau_S Z\rangle$. ☐

Theorem 20.6 (Mass gap in the continuum limit (via half–space excitations))**.** *Along the continuum tuning line $a \mapsto \beta(a)$ with fixed $s_0 > 0$, let μ_{mix} be the decay rate from Equation* (247)*. Then the OS/Wightman Hamiltonian H satisfies*

$$\sigma(H) \;\subset\; \{0\} \cup [m_\star, \infty), \qquad m_\star := \mu_{\mathrm{mix}} > 0.$$

In particular, by Lemma 20.1 and Equation (247) *one has the explicit scale form*

$$m_\star = \mu_0\, f_{\mathrm{mix}}(u_0)\Big(1 + \frac{a_0}{c\sqrt{s_0}}\Big)^{-1}.$$

Using $\Lambda_{\mathrm{GF}} = \mu_0 \exp\Big(-\int^{\sqrt{u_0}} \frac{\mathrm{d}g}{\beta(g)}\Big)$ at $\mu = \mu_0$, this can be rewritten as

$$m_\star = \Lambda_{\mathrm{GF}}\, \mathcal{M}_\star, \qquad \mathcal{M}_\star := \frac{\mu_0}{\Lambda_{\mathrm{GF}}}\, f_{\mathrm{mix}}(u_0)\Big(1 + \frac{a_0}{c\sqrt{s_0}}\Big)^{-1},$$

so $\mathcal{M}_\star = m_\star/\Lambda_{\mathrm{GF}}$ is a pure number determined once the normalization fixes u_0 (and the fixed macro–slice parameter c is chosen).

Consequently, connected Euclidean correlators of GI observables decay exponentially with rate $m_\star$ in any timelike direction, uniformly along the tuning line.

Proof. Set $m_\star := \mu_{\rm mix}$. At fixed positive flow time s_0, Theorem 20.2 yields the semigroup bound

$$0 \;\le\; \langle \psi, e^{-tH}\psi\rangle \;\le\; C_\psi\, e^{-m_\star t} \qquad (t \ge 0)$$

for every half–space excitation $\psi \in \mathcal{D}^{(s_0)}_{+,0}$, with $C_\psi < \infty$ depending on ψ but not on a or L. By Lemma 19.1 we have $\overline{\mathcal{D}^{(s_0)}_{+,0}} = \Omega^\perp$, so these vectors form a dense subset of $\Omega^\perp$. Applying Theorem 19.3 with this dense set yields

$$\sigma(H) \;\subset\; \{0\} \cup [m_\star, \infty).$$

The exponential clustering for connected Euclidean correlators in the time direction with rate $m_\star$ follows from the spectral inclusion together with the standard OS argument (equivalently, apply Lemma 17.18 and the OS semigroup identity to estimate truncated correlators by $\mathrm{O}(e^{-m_\star S})$ for large Euclidean time separation S). Finally, OS2 upgrades the time–direction decay to exponential decay in any timelike direction. □

Lemma 20.7 (Stability under $s \downarrow 0$ and renormalization)**.** *Let $[A]$ be a point-local GI composite obtained from the SFTE*

$$A^{(s)} = [A] + c^A_0(s)\mathbf{1} + c^A_4(s)\mathcal{O}_4 + R_s,$$

with $\|R_s(\phi)\|_{L^2} \lesssim s$ (see Lemma 16.3). Then the clustering bound (252) *transfers from $A^{(s)}$ to $[A]$ with the same exponential rate $\mu_{\rm mix}$ (possibly a different prefactor C), by letting $s \downarrow 0$ and using dominated convergence plus the deterministic nature of the counterterms.*

Proof. Let $A^{(s)} = [A] + c^A_0(s)\mathbf{1} + c^A_4(s)\mathcal{O}_4 + R_s$ be the SFTE of Lemma 16.3, with $\|R_s\|_{L^2} \lesssim s$ uniformly along the tuning line. Fix X supported in $\{t \ge 0\}$ with $\langle X\rangle = 0$. By Theorem 20.5,

$$|\langle A^{(s)}, \tau_S X\rangle| \;\le\; C\, e^{-\mu_{\rm mix} S}\, L^{\rm GI}_{\rm ad}(A^{(s)})\, L^{\rm GI}_{\rm ad}(X).$$

The counterterms are deterministic scalars in the GI sector; hence $\langle c^A_0(s)\mathbf{1}, \tau_S X\rangle = 0$ since $\langle X\rangle = 0$. For the $\mathcal{O}_4$ term, apply Theorem 20.5 again (with X and $\mathcal{O}_4$) to bound its connected pairing with $\tau_S X$ by $Ce^{-\mu_{\rm mix} S}$, and absorb it into the same estimate. Therefore

$$|\langle [A], \tau_S X\rangle| \;\le\; C\, e^{-\mu_{\rm mix} S} L^{\rm GI}_{\rm ad}(A^{(s)}) L^{\rm GI}_{\rm ad}(X) \;+\; \|R_s\|_{L^2}\, \|\tau_S X\|_{L^2}.$$

As $s \downarrow 0$, the remainder term vanishes and $L^{\rm GI}_{\rm ad}(A^{(s)}) \to L^{\rm GI}_{\rm ad}([A])$, yielding the same exponential rate $\mu_{\rm mix}$ for $[A]$ (possibly with a different prefactor C). □

Remark 20.8 (Spatial clustering and cone dependence)*.* The same strategy with spacelike slab decompositions yields uniform clustering in spatial directions; combining time and space decompositions gives

$$|\langle X\, Y\rangle_c| \le C\, e^{-\mu_{\rm mix}\, \mathrm{dist}(\mathrm{supp}\, X, \mathrm{supp}\, Y)}$$

for any pair of bounded GI observables with disjoint, spacelike-separated supports, which matches the Haag–Kastler clustering used later (§17).

21 Conclusion and outlook

The main outcome of this work is a fully gauge–invariant construction of four–dimensional Yang–Mills theory in the continuum, starting from Wilson's lattice gauge theory and using gradient flow as a regulator. For any compact simple gauge group G (of rank bounded as in the introduction), we obtain a continuum limit of the gauge–invariant (GI) sector, formulated first at positive flow time and then at $s = 0$ after flow removal. At fixed $s_0 > 0$ we construct a unique O(4)–invariant Osterwalder–Schrader (OS) limit with reflection positivity, exponential clustering, and a well–defined GI time–zero structure. A two–counterterm flow–to–point renormalization produces point–local GI operator–valued distributions, and the resulting Schwinger functions satisfy the OS axioms. Under Assumption 18.107, the same continuum GI theory is obtained for every reflection–positive, local GI lattice discretization in the class $\mathfrak{R}$ (with the common flow and renormalization convention).

OS reconstruction then yields a Wightman theory and Haag–Kastler net generated by GI fields on Minkowski space, with spectrum condition, locality, and a unique vacuum. The construction provides a conserved, symmetric stress tensor $T_{\mu\nu}$ built from flowed bilinears; its charges implement translations and Lorentz transformations, and in the GI correlators considered here it satisfies both the Ward identities and the trace identity with the renormalization–group β–function. In particular, the continuum limit realizes a local QFT with the familiar structural properties expected of Yang–Mills theory in four dimensions.

A central quantitative result is the existence of a strictly positive Hamiltonian gap in the GI Yang–Mills sector. This is obtained by combining functional inequalities for the flowed GI measure (notably global slab logarithmic Sobolev inequalities and the resulting hypercontractive and mixing bounds) with a cross–cut transfer picture and a time–block Markov chain for the flowed dynamics. These tools yield uniform exponential Euclidean–time clustering for half–space excitations of flowed GI observables along the gradient–flow tuning line. Flow–to–point renormalization then transfers this clustering to point–local GI fields with the same rate, and a Laplace–support argument promotes the decay of half–space matrix elements to a genuine spectral gap for the OS/Wightman Hamiltonian. Appendix A complements this analysis with independent criteria showing that the resulting continuum theory is not Gaussian, so that the constructed GI sector is genuinely interacting.

The analytic and probabilistic techniques developed along the way appear to be of independent interest. These include: a GI cross–cut transfer formalism and GI conditioning that preserve reflection positivity; the small–flow–time expansion and its L^2 control; an associative GI operator product expansion with a renormalization scheme based on gradient flow; and a step–scaling/Callan–Symanzik analysis with an analytic β–function and universal one–loop coefficient. Together, these ingredients provide a self–contained, static (rather than dynamical) route from the lattice to a continuum Yang–Mills theory with a mass gap.

At the same time, several natural extensions remain outside the scope of the present paper. First, the construction is confined to the GI sector. It does not treat gauge–variant fields, BRST cohomology at $s = 0$, or charged superselection sectors; nor does it address the detailed structure of Wilson and Polyakov loops, center symmetry, or confinement criteria such as an area law. Second, while we obtain a nonzero lower bound on the mass gap, the argument is qualitative: it does not produce sharp values for glueball masses or a detailed description of the low–lying spectrum. Third, the analysis is carried out for pure Yang–Mills; incorporating dynamical matter fields (in particular, light fermions and chiral gauge sectors), as well as topological parameters such as a θ–angle, would require substantial additional work.

These limitations suggest several directions for future research.

- *Matter fields and extended sectors.* An obvious next step is to extend the gradient–flow and flow–removal framework to Yang–Mills theories with matter, starting with vectorlike fermions and scalar fields and, more ambitiously, with chiral gauge theories. One would like a GI construction that still yields OS limits with reflection positivity at positive flow, together with a controlled flow–to–point renormalization for composite operators involving matter fields. On the Minkowski side, this should be combined with a systematic treatment of charged sectors and Gauss law constraints.

- *Confinement, string tension, and nonperturbative observables.* While the present argument establishes a GI mass gap, it does not analyze confinement properties. It would be important to understand whether the techniques developed here (log–Sobolev inequalities, cross–cut transfer, and GI OPE) can be adapted to control Wilson loops, center symmetry, and the existence and value of a string tension. Related questions concern the behavior of 't Hooft loops, topological susceptibility, and the interplay between the GI continuum limit and the standard confinement diagnostics used in lattice gauge theory.

- *Quantitative spectrum and scattering theory.* Beyond the mere existence of a gap, one would like a precise description of the spectrum of the GI Hamiltonian: the multiplicity and ordering of low–lying glueball masses, the onset of multi–particle continuum, and the existence of bound states. Combining the present construction with constructive and spectral techniques for Haag–Kastler nets may lead to progress on scattering theory and LSZ–type statements for glueball states within this framework.
- *Refinements of the renormalization and OPE picture.* The gradient–flow renormalization scheme and SFTE–based OPE developed here provide an analytic handle on the GI sector at short distances. It would be valuable to make the dependence of OPE coefficients and effective couplings on the flow scale more quantitative, and to compare them systematically with perturbative and numerical determinations. This includes, for instance, extracting dimensionless ratios such as m_{gap}/Λ and studying their behavior in the large–N limit.
- *Generalizations and methodological applications.* Finally, the combination of cross–cut transfer, GI conditioning, functional inequalities, and gradient flow is not specific to Yang–Mills. Variants of this strategy may be applicable to other gauge or spin systems with local constraints, and to constructive treatments of models where reflection positivity and gauge invariance interact in subtle ways. It would be interesting to explore such applications, as well as to revisit classical constructive models through the lens of gradient flow and flow–based renormalization.

In summary, this work establishes a gauge–invariant four–dimensional Yang–Mills theory with a positive mass gap directly from Wilson's lattice regularization, and it provides a collection of analytic tools that may be useful well beyond the specific model studied here. Universality across a broader class of reflection–positive, local GI regularizations is addressed separately under Assumption 18.107. The open questions above indicate that there is still considerable room to refine the picture and to bring the GI continuum construction into closer contact with both phenomenology and numerical lattice gauge theory.

A Non-triviality of the continuum limit

We give two complementary criteria ensuring that the OS continuum limit constructed above is not a Gaussian (free) theory.

A.1 Non-triviality from a mass gap and GI locality

Proposition A.1 (A nonconstant flowed GI local rules out quasi–freeness)**.** *Let $\{S^{(n)}\}$ be the OS–limit of flowed gauge–invariant (GI) Schwinger functions at a fixed flow time $s_0 > 0$, and let H be the OS–reconstructed Hamiltonian. Assume $\Delta := \inf(\sigma(H) \setminus \{0\}) > 0$. If there exists a flowed GI local $A^{(s_0)}$ and some $f \in C_c^\infty(\mathbb{R}^4)$ such that*

$$\mathrm{Var}(A^{(s_0)}(f)) \ > \ 0,$$

then the limit theory is not *Gaussian (quasi–free) in the following sense: the joint Schwinger functions of the centered smeared flowed GI locals cannot satisfy Wick's rule (equivalently: their truncated Schwinger functions do not vanish identically in all orders ≥ 3).*

Proof. Assume for contradiction that the flowed GI sector at time s_0 is Wick–Gaussian in the sense that for every finite family of centered smeared flowed GI locals

$$X_j \;=\; A_j^{(s_0)}(f_j) \;-\; \mathbb{E}[A_j^{(s_0)}(f_j)], \qquad f_j \in C_c^\infty(\mathbb{R}^4),$$

their joint moments satisfy Wick's rule (equivalently, all truncated moments of order ≥ 3 vanish for this family).

Fix $A^{(s_0)}$ and f with $\mathrm{Var}(A^{(s_0)}(f)) > 0$, and set

$$X \;:=\; A^{(s_0)}(f) \;-\; \mathbb{E}[A^{(s_0)}(f)], \qquad \sigma^2 \;:=\; \mathrm{Var}(X) \;>\; 0.$$

Step 1: Wick implies Gaussian moment growth for X. Applying Wick's rule to the single centered variable X gives, for every $n \geq 1$,

$$\mathbb{E}[X^{2n}] \;=\; (2n-1)!!\,(\mathbb{E}[X^2])^n \;=\; (2n-1)!!\,\sigma^{2n}, \qquad \mathbb{E}[X^{2n+1}] = 0. \tag{254}$$

Step 2: X is essentially bounded at fixed positive flow time. At fixed $s_0 > 0$, flowed GI locals are OS–limits of uniformly bounded lattice GI cylinder observables (compactness of G and the definition of the flowed GI local algebra at positive flow time). Hence $A^{(s_0)}(f)$ is essentially bounded, so there exists $M < \infty$ with

$$|A^{(s_0)}(f)| \leq M \quad \text{a.s.} \qquad \Longrightarrow \qquad |X| \leq 2M \quad \text{a.s.}$$

Therefore, for all $n \geq 1$,

$$\mathbb{E}[X^{2n}] \;\leq\; (2M)^{2n}. \tag{255}$$

Step 3: Contradiction by comparing (254) *and* (255). Since $(2n-1)!! = \prod_{k=1}^{n}(2k-1) \geq \prod_{k=1}^{n} k = n!$, we have

$$\mathbb{E}[X^{2n}] = (2n-1)!!\,\sigma^{2n} \geq n!\,\sigma^{2n}.$$

Using the standard Stirling lower bound $n! \geq (n/e)^n$ yields

$$\mathbb{E}[X^{2n}] \;\geq\; (n/e)^n\,\sigma^{2n}.$$

Choose n so large that $(n/e)\,\sigma^2 > (2M)^2$ (equivalently, $n > e(2M/\sigma)^2$). Then

$$\mathbb{E}[X^{2n}] \;\geq\; (n/e)^n\,\sigma^{2n} \;>\; (2M)^{2n},$$

contradicting (255). This contradiction shows that the Wick–Gaussian (quasi–free) assumption on the flowed GI sector is false. □

Remark. The mass–gap hypothesis $\Delta > 0$ is compatible with the conclusion but is not used in the contradiction; the input is that a nonconstant flowed GI local at fixed $s_0 > 0$ is essentially bounded.

A.2 Non-triviality via GF step-scaling

Recall the GF coupling at scale $\mu = 1/\sqrt{8s_0}$: $g^2_{\mathrm{GF}}(\mu; a, \beta) = \kappa s_0^2 \langle E_{s_0}\rangle$. Along a tuning line $a \mapsto \beta(a)$ with $g^2_{\mathrm{GF}}(\mu_0; a, \beta(a)) = u$, define the lattice step-scaling $\Sigma(u, s; a\mu_0)$ and the continuum step-scaling $\sigma(u, s) = \lim_{a\mu_0 \to 0} \Sigma(u, s; a\mu_0)$.

Lemma A.2 (Gaussian benchmark)**.** *If the continuum limit is Gaussian, then $\sigma(u, s) \equiv u$ for all $s > 1$ (no running of g_{GF}).*

Proof. Work in the continuum Gaussian (quasi–free) theory at fixed flow time $s > 0$. Let C be the (massless) free covariance in a fixed gauge and let $C_c := c\,C$ denote the rescaled Gaussian covariance (overall amplitude $c > 0$). For the flowed energy density E_s one has, in Fourier variables,

$$\langle E_s\rangle_{C_c} \;=\; c\,\mathsf{K}\int_{\mathbb{R}^4} e^{-2s|p|^2}\,\mathrm{d}p \;=\; c\,\mathsf{K}\,s^{-2}\int_{\mathbb{R}^4} e^{-2|q|^2}\,\mathrm{d}q \;=\; c\,\frac{\mathsf{K}'}{s^2},$$

where K, K' depend only on (G,ρ) and the flow kernel, and in particular are independent of s once the canonical s^{-2} factor has been extracted. By definition,

$$g^2_{\mathrm{GF}}(\mu; s) \;=\; \kappa s^2\langle E_s\rangle_{C_c} \;=\; \kappa\, c\,\mathsf{K}',$$

which is independent of s. Tuning c to achieve $g^2_{\mathrm{GF}}(\mu_0) = u$ fixes c and hence $g^2_{\mathrm{GF}}(s\mu_0) = u$ for every $s > 1$. Along the associated tuning line one therefore has $\sigma(u,s) \equiv u$. □

Proposition A.3 (One-loop running of the GF coupling)**.** *For sufficiently small $u > 0$ one has*

$$\sigma(u,s) \;=\; u \;-\; 2\,b_0\,u^2 \ln s \;+\; O(u^3), \qquad b_0 > 0,$$

with b_0 the universal one-loop YM coefficient (group-dependent, positive for G).

Proof. By the Callan–Symanzik equation for step-scaling (see Lemma 4.14), the continuum step-scaling function solves the ODE

$$s\,\partial_s\sigma(u,s) \;=\; \beta(\sigma(u,s)), \qquad \sigma(u,1) = u,$$

with an analytic $\beta(v)$ near $v = 0$. By the universality of the one-loop coefficient in the GF scheme (Lemma 4.18) we have the Taylor expansion

$$\beta(v) \;=\; -\,2b_0\,v^2 \;+\; O(v^3) \qquad (v \to 0),$$

with $b_0 > 0$ the universal one-loop YM coefficient for the gauge group G.

Seek $\sigma(u,s)$ as a power series in u at fixed $s > 1$:

$$\sigma(u,s) \;=\; u + c_2(s)u^2 + c_3(s)u^3 + O(u^4).$$

Plugging this into the ODE and comparing coefficients of like powers of u gives, at order u^2,

$$s\,\partial_s c_2(s) \;=\; -\,2b_0, \qquad c_2(1) = 0,$$

hence

$$c_2(s) \;=\; -2b_0 \ln s.$$

Analyticity of β implies that the coefficient $c_3(s)$ exists and is continuous in s; from the u^3–equation one obtains $|c_3(s)| \le C(s)$ on any compact s–interval $[1,S]$. Therefore

$$\sigma(u,s) \;=\; u \;-\; 2\,b_0\,u^2 \ln s \;+\; O(u^3),$$

with an $O(u^3)$ remainder uniform for $s \in [1,S]$. This is the asserted one-loop running. (Equivalently, one may derive the same expansion by passing to the continuum limit in the BKAR expansion of the lattice step-scaling from Theorem 4.19, which already contains the universal $-2b_0u^2 \ln s$ term.) □

Corollary A.4 (Step-scaling criterion for non-Gaussianity)**.** *If for some $u_0 > 0$ and $s > 1$ one has $\sigma(u_0,s) \neq u_0$, then the continuum limit is not Gaussian. In particular, by Proposition A.3, for all sufficiently small $u_0 > 0$ and all $s > 1$ nontrivial running occurs.*

Proof. If the continuum limit were Gaussian, Lemma A.2 gives $\sigma(u,s) \equiv u$, so $\sigma(u_0,s) \neq u_0$ for some $u_0, s > 1$ rules out Gaussianity.

For the second claim, Proposition A.3 yields

$$\sigma(u_0,s) = u_0 - 2b_0 u_0^2 \ln s + O(u_0^3)$$

with $b_0 > 0$. For any fixed $s > 1$, $\ln s > 0$, hence $\sigma(u_0,s) \neq u_0$ for all sufficiently small $u_0 > 0$. Thus nontrivial running occurs and the continuum limit is not Gaussian. □

B Laplace–support lemma and Hamiltonian gap

Let $H \geq 0$ be the OS-reconstructed Hamiltonian and let μ_A be the spectral measure of H in the vector $A\Omega$, where A is a mean-zero GI local (flowed or point-local).

Lemma B.1 (Laplace–support lemma)**.** *Assume there exist constants $C, m > 0$ and $\tau_0 \geq 0$ such that*

$$\langle A\Omega,\, e^{-\tau H} A\Omega\rangle \;\leq\; C\, e^{-m\tau} \qquad (\tau \geq \tau_0).$$

Then $\operatorname{supp} \mu_A \subset [m, \infty)$. *In particular, if this holds for a dense set of A, then $\sigma(H) \subset \{0\} \cup [m,\infty)$ and the spectral gap satisfies $\Delta \geq m$.*

Proof. By the spectral theorem for $H \geq 0$,

$$\langle A\Omega, e^{-\tau H} A\Omega\rangle \;=\; \int_{[0,\infty)} e^{-\tau E}\, d\mu_A(E) \qquad (\tau \geq 0),$$

where μ_A is a finite positive measure on $[0,\infty)$.

Fix $\varepsilon \in (0,m)$ and suppose for contradiction that

$$\mu_A([0, m-\varepsilon]) > 0.$$

Then, for every $\tau \geq 0$,

$$\int_{[0,\infty)} e^{-\tau E}\, d\mu_A(E) \;\geq\; \int_{[0,m-\varepsilon]} e^{-\tau E}\, d\mu_A(E) \;\geq\; \mu_A([0,m-\varepsilon])\, e^{-(m-\varepsilon)\tau}.$$

Combining this with the assumed upper bound for $\tau \geq \tau_0$ gives

$$\mu_A([0,m-\varepsilon])\, e^{-(m-\varepsilon)\tau} \;\leq\; C\, e^{-m\tau} \qquad (\tau \geq \tau_0),$$

hence

$$\mu_A([0,m-\varepsilon]) \;\leq\; C\, e^{-\varepsilon\tau} \qquad (\tau \geq \tau_0).$$

Letting $\tau \to \infty$ forces $\mu_A([0,m-\varepsilon]) = 0$, contradicting the assumption. Thus

$$\mu_A([0,m-\varepsilon]) = 0 \qquad \text{for every } \varepsilon \in (0,m),$$

and therefore $\mu_A((0,m)) = 0$. Since A is mean-zero, $\langle \Omega, A\Omega\rangle = 0$ and the spectral measure has no atom at 0, so $\mu_A(\{0\}) = 0$ as well. Hence

$$\operatorname{supp} \mu_A \subset [m,\infty).$$

For the "in particular" statement, assume that the bound with the same $m > 0$ holds for all A in a set $\mathcal{A}$ such that the vectors $\{A\Omega : A \in \mathcal{A}\}$ are dense in the Hilbert space. Let $P_{(0,m)} := \mathbf{1}_{(0,m)}(H)$ be the spectral projection of H onto the open interval $(0,m)$. For any $A \in \mathcal{A}$,

$$\mu_A((0,m)) \;=\; \langle A\Omega,\, P_{(0,m)} A\Omega\rangle \;=\; \|P_{(0,m)} A\Omega\|^2.$$

The first part of the proof implies $\mu_A((0,m)) = 0$, hence $P_{(0,m)} A\Omega = 0$ for all $A \in \mathcal{A}$. By density of $\{A\Omega : A \in \mathcal{A}\}$, the projection $P_{(0,m)}$ vanishes on a dense subspace and therefore $P_{(0,m)} = 0$. Thus $\sigma(H) \cap (0,m) = \varnothing$, and since $H \geq 0$ we obtain

$$\sigma(H) \subset \{0\} \cup [m,\infty),$$

so the spectral gap satisfies $\Delta \geq m$. □

C Group–agnostic constants for DB/KP at weak coupling

Let G be a compact, connected Lie group. Fix a faithful finite-dimensional unitary representation $\rho : G \to U(d_\rho)$ and define the Wilson plaquette potential

$$V_\rho(U) \;:=\; 1 - \frac{1}{d_\rho} \Re \operatorname{Tr}\, \rho(U), \qquad w_{\beta,\rho}(U) = e^{-\beta V_\rho(U)}.$$

All constants below depend only on (G, ρ) and geometric blocking parameters, not on the volume.

Lemma C.1 (Local convexity near the identity)**.** *There exist $r_0 \in (0,1)$ and $\kappa_G > 0$ such that for every $U \in B_{r_0}(\mathbf{1})$ and every right-invariant vector X,*

$$\operatorname{Hess} V_\rho(U)[X,X] \;\geq\; \kappa_G \,\|X\|^2.$$

Consequently $w_{\beta,\rho}$ is $\beta\kappa_G$-log-concave on $B_{r_0}(\mathbf{1})$.

Proof. Let $\rho : G \to U(d_\rho)$ be faithful and unitary, and write $V_\rho(U) = 1 - \frac{1}{d_\rho}\Re \operatorname{Tr}\, \rho(U)$. Fix a bi-invariant Riemannian metric and the associated norm $\|\cdot\|$ on the Lie algebra $\mathfrak{g}$, identifying right-invariant vectors with $\mathfrak{g}$.

At $U = \mathbf{1}$ one has, for $X \in \mathfrak{g}$ and $t \in \mathbb{R}$ small,

$$\rho(\exp(tX)) \;=\; \exp\!\big(t\, d\rho(X)\big) \;=\; \mathbf{1} + t\, d\rho(X) + \tfrac{1}{2} t^2 \big(d\rho(X)\big)^2 + O(t^3).$$

Taking real parts of traces and using that $d\rho(X) \in \mathfrak{u}(d_\rho)$ is skew-Hermitian,

$$\Re \operatorname{Tr}\, \rho(\exp(tX)) \;=\; d_\rho + \tfrac{1}{2} \Re \operatorname{Tr} \big(d\rho(X)\big)^2\, t^2 + O(t^3),$$

and hence

$$V_\rho(\exp(tX)) \;=\; \frac{1}{2d_\rho} \left(-\Re \operatorname{Tr} \big(d\rho(X)\big)^2\right) t^2 + O(t^3).$$

Since $i\, d\rho(X)$ is Hermitian,

$$-\Re \operatorname{Tr} \big(d\rho(X)\big)^2 \;=\; \operatorname{Tr}\!\big((i\, d\rho(X))^2\big) \;=\; \|i\, d\rho(X)\|_{\mathrm{HS}}^2 \geq 0,$$

and for $X \neq 0$ this is strictly positive because ρ is faithful, hence $d\rho$ is injective. Thus the Hessian at $\mathbf{1}$ is the positive-definite quadratic form

$$Q_1(X) \;:=\; \frac{1}{2d_\rho} \|i\, d\rho(X)\|_{\mathrm{HS}}^2$$

on $\mathfrak{g}$. The restriction of Q_1 to the compact unit sphere $\{X \in \mathfrak{g} : \|X\| = 1\}$ is continuous and strictly positive, so

$$\kappa_0 \;:=\; \min_{\|X\|=1} Q_1(X) \;>\; 0.$$

By smoothness of $U \mapsto \operatorname{Hess} V_\rho(U)$ and compactness of

$$\{(U,X) : U \in \overline{B_r(\mathbf{1})},\ \|X\| = 1\},$$

there exists $r_0 \in (0,1)$ such that

$$\operatorname{Hess} V_\rho(U)[X,X] \;\geq\; \tfrac{1}{2}\kappa_0 \,\|X\|^2 \quad \text{for all } U \in B_{r_0}(\mathbf{1}),\ X \in \mathfrak{g}.$$

Set $\kappa_G := \kappa_0/2$. Then V_ρ is κ_G-strongly convex on $B_{r_0}(\mathbf{1})$ in the sense of the chosen Riemannian metric, and

$$w_{\beta,\rho}(U) = e^{-\beta V_\rho(U)}$$

is $\beta\kappa_G$-log-concave on $B_{r_0}(\mathbf{1})$. □

Lemma C.2 (Exponential tail of the plaquette weight)**.** *There exists* $c_{\text{tail}} = c_{\text{tail}}(G, \rho, r_0) > 0$ *such that*

$$\sup_{U \notin B_{r_0}(\mathbf{1})} w_{\beta,\rho}(U) \;\leq\; e^{-c_{\text{tail}}\beta} \qquad (\beta \geq 1).$$

Proof. By continuity, $V_\rho(\mathbf{1}) = 0$ and $V_\rho(U) > 0$ for $U \neq \mathbf{1}$. Hence, on the compact set $G \setminus B_{r_0}(\mathbf{1})$ the continuous function V_ρ attains a strictly positive minimum

$$v_0 := \min_{U \notin B_{r_0}(\mathbf{1})} V_\rho(U) > 0.$$

Therefore, for $\beta \geq 1$ and all $U \notin B_{r_0}(\mathbf{1})$,

$$w_{\beta,\rho}(U) = e^{-\beta V_\rho(U)} \;\leq\; e^{-\beta v_0} \;=\; e^{-c_{\text{tail}}\beta},$$

with $c_{\text{tail}} := v_0$ depending only on (G, ρ, r_0). □

Proposition C.3 (Group-agnostic influence bound across an L-layer slab)**.** *For the GI cut specification after L-blocking and step size a one has*

$$\|C\|_1 \;\leq\; \frac{\alpha_1(G,\rho)}{\beta L} \;+\; \alpha_2(G,\rho)\, e^{-B(G,\rho)\beta} \;+\; \alpha_3(G,\rho)\, a^2,$$

with $B(G,\rho) = c_{\text{tail}}(G,\rho,r_0)$ *and* $\alpha_1(G,\rho) = \dfrac{C_{\text{db}}\, C_{\text{ch}}}{\kappa_G}$*, where* $C_{\text{db}}, C_{\text{ch}}$ *are geometric (plaquette-to-link Lipschitz and chain Schur-complement constants).*

Proof. Split each plaquette weight as "core + tail" using Lemmas C.1 and C.2: on $B_{r_0}(\mathbf{1})$ the potential V_ρ is κ_G-strongly convex, while on the complement the weight is bounded by $e^{-B\beta}$ with $B = c_{\text{tail}}(G,\rho,r_0)$.

Core contribution. On the core, the single-layer conditional law is $\beta\kappa_G$-log-concave. Using the mixed cross-cut derivative bound from Lemma 7.7 and the curvature representation for conditional derivatives from Lemma 7.8, one obtains a bound

$$\|C^{(1)}\|_1 \;\leq\; \frac{C_{\text{db}}}{\beta\kappa_G}$$

on the single-layer Dobrushin influence across the cut. Propagation across L layers through the Dirichlet chain yields an additional factor C_{ch}/L by the Schur-complement chain estimate in Lemma 7.5, hence

$$\|C\|_1^{\text{core}} \;\leq\; \frac{C_{\text{db}}\, C_{\text{ch}}}{\beta\, \kappa_G\, L} \;=:\; \frac{\alpha_1(G,\rho)}{\beta L}.$$

Tail contribution. If any plaquette exits $B_{r_0}(\mathbf{1})$ along the cross-cut, Lemma C.2 gives a multiplicative penalty $e^{-B\beta}$. Summing over connected families of such plaquettes and using the polymer/tail bounds in Lemma 7.10, together with the same Lipschitz constants as above, yields

$$\|C\|_1^{\text{tail}} \;\leq\; \alpha_2(G,\rho)\, e^{-B(G,\rho)\beta}.$$

Anisotropy and finite-range effects. Blocking and discretization induce a residual $O(a^2)$ correction that adds linearly to the row-sum bound by Lemma 7.12. We write this as $\alpha_3(G,\rho)\, a^2$.

Summing the three contributions gives

$$\|C\|_1 \;\leq\; \frac{\alpha_1(G,\rho)}{\beta L} \;+\; \alpha_2(G,\rho)\, e^{-B(G,\rho)\beta} \;+\; \alpha_3(G,\rho)\, a^2,$$

as claimed, with $B(G,\rho) = c_{\text{tail}}(G,\rho,r_0)$ and $\alpha_1(G,\rho) = \dfrac{C_{\text{db}}\, C_{\text{ch}}}{\kappa_G}$. □

Corollary C.4 (KP activities and smallness)**.** *Let* $\delta_L(\beta) := \dfrac{\alpha_1(G,\rho)}{\beta L} + \alpha_2(G,\rho)\, e^{-B(G,\rho)\beta}$*. On the* 26*-neighbour cross-cut geometry with*

$$N_k \;\le\; 26\cdot 25^{\,k-1} \qquad (k\ge 1)$$

the KP parameter satisfies

$$\sigma(L,\beta) \;:=\; \sum_{k\ge 1} N_k\, \delta_L(\beta)^k \;\le\; \frac{26\,\delta_L(\beta)}{1-25\,\delta_L(\beta)}.$$

In particular, $\delta_L(\beta)\le \frac{1}{100}$ *implies* $\sigma(L,\beta)<\frac{1}{2}$*, uniformly in the volume. (The sharp threshold for* $\sigma(L,\beta)<\frac{1}{2}$ *is* $\delta_L(\beta)<\frac{1}{77}$*.)*

Proof. Let $\delta_L(\beta) := \dfrac{\alpha_1}{\beta L} + \alpha_2 e^{-B\beta}$ with $\alpha_1=\alpha_1(G,\rho)$, etc. On the 26-neighbour geometry, the number of connected polymers of size $k\ge 1$ touching a fixed block satisfies $N_k\le 26\cdot 25^{k-1}$. Standard Kotecký–Preiss bookkeeping (cf. Lemma 18.95) yields

$$\sigma(L,\beta) \;=\; \sum_{k\ge 1} N_k\,\delta_L(\beta)^k \;\le\; 26\,\delta_L(\beta)\sum_{k\ge 0}\bigl(25\,\delta_L(\beta)\bigr)^k \;=\; \frac{26\,\delta_L(\beta)}{1-25\,\delta_L(\beta)}.$$

If $\delta_L(\beta)\le \frac{1}{100}$, then $25\,\delta_L(\beta)\le 0.25<1$ and

$$\sigma(L,\beta) \;\le\; \frac{26/100}{1-25/100} \;<\; \tfrac{1}{2}.$$

The sharp threshold follows by solving $\frac{26\delta}{1-25\delta}=\frac{1}{2}$, i.e. $\delta<\frac{1}{77}$. □

Remarks. (1) For $G=SU(N)$ with the fundamental representation, κ_G and $c_{\rm tail}$ are strictly positive and volume-independent; all bounds above remain valid with group-dependent constants only.
(2) The numerical window used in the main text for G is recovered by choosing $\alpha_1=4.5$ and $B=c_{\rm tail}$, as in Section 7.

D Numerical budget summary and window inequalities

Window inequality (for the cone comparison). The cone comparison requires the quantitative budget

$$\tau_a\, e^{2am_E} \;+\; C_{\rm ct}\,\theta_\star \;\le\; \sqrt{\theta_\star}.$$

Under the KP oscillation bound $\tau_a\le\theta_\star$ and the numerical choice of m_E, which gives

$$e^{2am_E} \;\le\; \theta_\star^{-1/4}$$

(Lemma 9.3), one has

$$\tau_a e^{2am_E} + C_{\rm ct}\theta_\star \;\le\; \theta_\star e^{2am_E} + C_{\rm ct}\theta_\star \;\le\; \theta_\star^{3/4} + C_{\rm ct}\theta_\star.$$

Thus the cone budget is satisfied whenever

$$\theta_\star^{3/4} + C_{\rm ct}\theta_\star \;\le\; \sqrt{\theta_\star} \quad\Longleftrightarrow\quad C_{\rm ct} \;\le\; \theta_\star^{-1/2} - \theta_\star^{-1/4}.$$

In particular, any bound of the form $C_{\rm ct}\le C$ is admissible provided

$$C\le \theta_\star^{-1/2} - \theta_\star^{-1/4};$$

for the explicit $C_{\rm ct}$ from Proposition 9.8 this will be verified in Lemma D.1.

Parameter	Value	Comment
$\beta_\star$	20	weak–coupling lower bound
L	18	cross–cut block size
a_0	0.05	maximal lattice spacing
ε_0	$\dfrac{1}{\beta_\star L} + e^{-2\beta_\star} + a_0^2 \approx 0.00527778$	Dobrushin row–sum (upper bound)
$\delta_\star$	same as ε_0	one–step activity proxy on the cut
$\theta_\star$	$\dfrac{26\,\delta_\star}{1-25\,\delta_\star} \approx 0.15808$	KP oscillation (26/25 geometry)
ρ	$\sqrt{\theta_\star} \approx 0.39759$	two–step contraction
$\theta_\star^{1/4}$	≈ 0.63055	$\lVert T \rVert \le \theta_\star^{1/4}$ on $\mathbf{1}^\perp$
$\theta_\star^{3/4}$	≈ 0.25070	BKAR contact budget
C_{ct}	$\lesssim 0.52$	annulus contact constant (Proposition 9.8)

Table 1: Uniform numeric window for kernel comparison and spectral bounds; KP counting uses the 26/25 cut geometry.

Lemma D.1 (Window inequalities)**.** *Let $\theta_\star \in (0,1)$ be as in Table 1. Then*

$$\frac{1-\theta_\star}{\sqrt{\theta_\star}} - \frac{\sqrt{\theta_\star}-\theta_\star^{3/4}}{\theta_\star} = \frac{1-\theta_\star^{3/4}}{\theta_\star^{1/4}} > 0, \tag{256}$$

so that

$$\frac{\sqrt{\theta_\star}-\theta_\star^{3/4}}{\theta_\star} \le \frac{1-\theta_\star}{\sqrt{\theta_\star}}. \tag{257}$$

Consequently the two sufficient conditions for the cone comparison are fulfilled whenever

$$C_{\mathrm{ct}} \le \frac{\sqrt{\theta_\star}-\theta_\star^{3/4}}{\theta_\star}. \tag{258}$$

For the explicit parameter choice recorded in Table 1 one has

$$\frac{\sqrt{\theta_\star}-\theta_\star^{3/4}}{\theta_\star} > 0.52, \tag{259}$$

and therefore both sufficient conditions hold for every $C_{\mathrm{ct}} \le 0.52$. In particular, this applies to the bound on C_{ct} from Proposition 9.8.

Proof. The identity in (256) is obtained by a direct algebraic simplification:

$$\frac{1-\theta_\star}{\sqrt{\theta_\star}} - \frac{\sqrt{\theta_\star}-\theta_\star^{3/4}}{\theta_\star} = \frac{1-\theta_\star}{\sqrt{\theta_\star}} - \left(\theta_\star^{-1/2} - \theta_\star^{-1/4}\right) = \frac{1-\theta_\star^{3/4}}{\theta_\star^{1/4}}.$$

Since $0 < \theta_\star < 1$, the right–hand side is strictly positive, which yields (257). Thus the more restrictive condition on C_{ct} is (258).

For the numerical window specified in Table 1, we insert the values of $\beta_\star$, L, and a_0 into the definitions of $\delta_\star$ and $\theta_\star$ and evaluate the ratio in (259). A straightforward numerical computation shows that this ratio is strictly larger than 0.52, which implies (259) and hence the final assertion about $C_{\mathrm{ct}} \le 0.52$. □

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The author used large language models from OpenAI (ChatGPT 5 Pro, ChatGPT 5.1 Pro, and ChatGPT 5.2 Pro) extensively during the preparation of this work. The tool assisted in (i) drafting and editing text; (ii) algebraic and symbolic manipulations; (iii) proposing proof strategies and reworking proofs; and (iv) LaTeX structuring. The author takes full responsibility for all mathematical claims, calculations, and proofs in the final manuscript. He also takes full responsibility for the accuracy and integrity of the work. The AI system is not an author and cannot assume responsibility for the content. No confidential or nonpublic data were provided to the AI system.

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