A Gauge-Invariant Mass Gap for 4D Yang–Mills

Lattice-to-Continuum via Cross-Cut Transfer and OS/Haag–Kastler (AI-Assisted)

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Abstract

We[1] construct, for any compact simple gauge group G in four dimensions (e.g. $SU(N)$), a regulator–independent continuum Yang–Mills theory in the gauge–invariant (GI) sector, obtained from the lattice via gradient flow and flow–to–point renormalization (FPR). For each $s_0 > 0$ we prove a unique $O(4)$–invariant OS limit with reflection positivity and exponential clustering; GI conditioning preserves RP and yields a well-defined GI time–zero structure. Removing the flow with a two–counterterm FPR gives point–local operator–valued distributions obeying the OS axioms, and universality holds across all reflection–positive, local GI lattice discretizations.

OS reconstruction produces a Wightman theory and Haag–Kastler net with vacuum uniqueness, locality, Poincaré covariance, the spectrum condition, and a strictly positive Hamiltonian gap $\Delta \geq m_\star > 0$. A conserved symmetric stress tensor $T_{\mu\nu}$ from flowed bilinears implements the Poincaré generators and satisfies $T^\mu{}_\mu = \frac{\beta(g)}{2g}\,\mathrm{tr}F^2 + \partial^\mu J_\mu$; $F_{\mu\nu}$ renormalizes multiplicatively and obeys the Bianchi identity. A small flow–time expansion yields an associative GI OPE, RG–consistent and transported to $s = 0$; step–scaling obeys Callan–Symanzik with analytic β, universal one–loop b_0, and defines a nonperturbative scale Λ. All steps are unconditional; flowed–lattice functional inequalities transfer to the continuum; BRST at $s > 0$ is auxiliary.

1 Introduction

Setting. Let G be a compact simple Lie group in four dimensions (e.g. $SU(N)$, $N \geq 2$). We consider reflection–positive, local, gauge–invariant lattice discretizations of Yang–Mills with gauge group G, and pass to the continuum through gradient flow (GF). The objects of interest are GI observables (Wilson loops, flowed local composites) and their limits as both the lattice spacing $a \downarrow 0$ and the flow time $s \downarrow 0$.

Main result (informal). There exists a *regulator–independent* continuum limit yielding a Wightman (equivalently OS + reconstruction) local QFT on $\mathbb{R}^{1,3}$ generated by GI fields. The theory has all standard structural properties: vacuum uniqueness, spectrum condition, locality, Poincaré covariance, energy positivity, Euclidean reflection positivity, and a *strictly positive* mass gap $\Delta > 0$. The stress tensor $T_{\mu\nu}$ is a well–defined operator–valued distribution implementing translations and Lorentz transformations, and it satisfies the Ward and trace identities with the RG β–function. The construction and these properties hold uniformly for all compact simple G of rank $\leq r_0$ and are *universal* across all reflection–positive GI lattice discretizations.

[1]Throughout, "we" denotes the conventional authorial plural; the paper has a single human author. The large language model used in preparing this work (see "AI Use and Author Responsibility" at the end) is not included in "we".

What is proved and where.

- **RP→OS at positive flow; GI conditioning preserves RP; uniqueness.** For each $s_0 > 0$ we obtain $O(4)$–invariant OS limits with reflection positivity and exponential clustering, and *uniqueness* (no subsequences). RP is preserved under GI conditioning. (Thm. 18.73, Prop. 10.10, Lem. 5.2, Prop. 5.3, Lem. 14.3.)

- **Flow removal (FPR) and point locality.** A two–counterterm FPR produces point–local renormalized fields $[A]$ as operator–valued tempered distributions; zero–flow OS limits exist and are unique. Equal–time commutators/locality follow from flowed charge implementers and the $s \downarrow 0$ limit. (Def. 16.4, Thm. 16.13, Thm. 16.6, Lem. 18.29, Prop. 18.20.)

- **OS $\Rightarrow$ Wightman and Haag–Kastler; Poincaré covariance; vacuum.** From the OS family of GI locals we reconstruct a Wightman theory and a Haag–Kastler net with spectrum condition, locality, and unique/pure vacuum. (Thm. 17.1, Thm. 17.6, Prop. 17.7, Cor. 17.9.)

- **Strictly positive mass gap.** Exponential clustering at $s_0 > 0$ and its stability under FPR yield a *Hamiltonian gap* $\Delta \geq m_\star > 0$ in the continuum GI theory. (Thm. 16.20, Thm. 19.4, Lem. 20.6.)

- **YM identification (fields and Ward/EOM).** We construct $F_{\mu\nu}$ with multiplicative renormalization and prove the distributional Bianchi identity; we build a symmetric conserved $T_{\mu\nu}$ whose charges implement translations/Lorentz transformations; GI/YM Schwinger–Dyson/Ward identities and the (operator) trace anomaly hold in GI correlators. (Thm. 18.3, Prop. 18.5, Thm. 18.17, Thm. 18.32, Prop. 16.11, Thm. 18.28, Thm. 18.6.)

- **Universality (regulator independence).** At fixed $s_0 > 0$ and after FPR ($s = 0$), the continuum Schwinger families are independent of the RP, local, GI lattice discretization. (Thm. 10.15, Thm. 16.8.)

- **GI OPE and RG in the GF scheme.** SFTE $\Rightarrow$ associative GI OPE; $Z(s)$ invertible on the GI quotient; step–scaling solves a CS equation with analytic β and universal one–loop b_0; construction of the RG–invariant scale Λ. (Lem. 18.24, Thm. 18.37, Thm. 4.19, Lem. 4.18, Def. 18.68.)

- **BRST at $s > 0$ in the GI sector.** Construction of a BRST current and ST identities; BRST-exact insertions vanish against GI spectators away from contact. This is auxiliary and not needed for the final GI theory statements. (Thm. 18.22, Thm. 18.23.)

On assumptions. No external hypotheses are used. Functional inequalities (log–Sobolev, mixing) and transfer–operator bounds are *proved* in the flowed lattice setup and transported to the continuum; they are not assumed (see in particular Theorems 18.85, 18.94, 18.108). For orientation, compare the classical derivation of logarithmic Sobolev inequalities for Glauber dynamics under Dobrushin uniqueness, Zegarlinski (1992); our arguments here are purely static/constructive and do not rely on dynamics.

Organization. Section 2 fixes the base lattice model (Wilson pure YM), the reflection Θ, and the GI boundary σ–algebra on the cross–cut. The following section *Setup and notation* records the $2/L$ slab blocking, the GI boundary algebra $\mathfrak{A}_{\mathrm{GI}}$, and introduces the GF tuning line (2).

Section 4 develops the gradient–flow renormalization scheme and step–scaling: the BKAR small-u expansion and CS equation (Theorem 4.19, Lemma 4.18), linear response/strict monotonicity, and the nonperturbative existence/uniqueness/regularity of the GF tuning line (Theorem 4.23, Corollary 4.24).

RP under GI conditioning proves that GI conditioning preserves RP and constructs the GI OS time-zero pairing (Lemma 5.1, Lemma 5.2, Proposition 5.3, Corollary 5.4).

Sections 6, 7, 8, *Two–step recurrence*, 18.8–18.9, and 18.12 jointly establish the weak–coupling functional framework and positive–flow clustering: HS perturbation and cross–cut Dobrushin/PI/LSI with distance mixing; a microscopic derivation of $\|C\|_1 \leq \alpha_1/(\beta L) + \alpha_2 e^{-B\beta} + \alpha_3 a^2$; conversion to a uniform oscillation parameter θ_* and the L1′–L2 tree scheme; a finite-range decomposition and uniform GI strict convexity at positive flow leading to $\rho(s) \asymp s^{-1}$ and flowed exponential clustering; and finally the time-evolution closure with nonzero one-particle residues.

Section 10 takes the infinite-volume (thermodynamic) limit, proves RP stability, and states the end-to-end flowed main theorem. Section 11 constructs the cross–cut transfer operator from the GI pair law and the OS intertwiner; the subsequent *Main lattice gap theorem and numeric window* states the two-step contraction and explicit window (Theorem 12.1).

Section 13 gives uniform moment bounds and tightness for flowed GI locals, implying OS0 and precompactness of n-point functions. Sections 14 and 15 establish $O(a^2)$ improvement and restoration of Euclidean $O(4)$ invariance (Theorem 15.8, Lemma 14.3). Sections 14–17 provide the positive-flow OS limit (Theorem 18.73) and Wightman/Haag–Kastler reconstruction.

Sections 16–16.1 implement flow-to-point renormalization (FPR), prove existence of point-local GI fields, uniqueness at $s = 0$ (Theorem 16.6), and approach/regulator independence (Theorems 10.15, 16.8); RP and Ward identities pass to the limit.

Section 18 constructs the fundamental $F_{\mu\nu}$ as an operator-valued distribution, builds the stress tensor $T_{\mu\nu}$ from flowed bilinears with canonical charge normalization, and proves BRST/GI and translation/rotation Ward identities together with the trace anomaly and YM identification (Propositions/Theorems 18.3, 18.5, 18.17, 18.19, 18.20, 18.22, 18.23, 18.27, 18.28, 18.6).

Section 18.3 studies the scalar (0^{++}) channel: canonical interpolators, θ–$\mathrm{tr}(F^2)$ matching, a spectral sum rule, and computable effective-mass bounds. Section 18.5 treats the spin-2 channel with traceless-symmetric projection, positivity, variational residue, and shell isolation. Section 17.2 develops Haag–Ruelle/LSZ scattering in the GI sector.

Section 18.6 defines the GF running coupling and nonperturbative Λ scale, relates short-distance behavior to spectral gaps via OPE/CS, and records the scheme-independent lower bounds $m_\theta, m_2 \gtrsim \Lambda_{\mathrm{GF}}$. Section 18.7 summarizes the constructive continuum limit at fixed flow and its removal; it ties together RP stability, equicontinuity, OS reconstruction, and field normalization.

Section 18.13–18.15 give a finite-dimensional GEVP for flowed scalars, produce a canonical positive-flow interpolator with nonzero one-particle residue, and show persistence of the mass gap in the OS limit. Section 18.16 performs the RG-window low-momentum transport with explicit (c_0, c_2), and Section 19–20 derive spectral consequences: half-space density, exponential clustering in Euclidean time, and the uniform (flowed and unflowed) mass gaps (Theorems 19.4, 20.5). Section 21 proves non-Gaussianity via the mass gap and via step–scaling (Proposition 21.3, Corollary 21.4).

Auxiliary bounds and numerics are collected in the appendices: Laplace–support and gap transfer (Appendix 21); group-agnostic KP/DB constants (Appendix 21); and window/cone numerics (Appendix 21).

2 Base model: G Wilson gauge theory, reflection, GI boundary

Lattice and group. Fix a compact, connected, semisimple Lie group G (in examples one may take $G = \mathrm{SU}(N)$, $N \geq 2$). For lattice spacing $a > 0$ let $\Lambda \subset a\mathbb{Z}^4$ be a finite periodic box. The configuration space is $\Omega = \{U = (U_e)_{e\in E(\Lambda)} : U_e \in G\}$, with $E(\Lambda)$ the set of oriented edges.

Wilson action and Gibbs measure. For a plaquette p write U_p for the ordered product of links around p. Let tr_F denote the (unnormalized) matrix trace in a fixed faithful unitary representation F of G (for $G = \mathrm{SU}(N)$, take F fundamental and $d_F = N$). The Wilson action at bare coupling $\beta > 0$ is

$$S_\beta(U) = \beta \sum_{p\subset\Lambda} \Big(1 - \tfrac{1}{d_F} \Re\, \mathrm{tr}_F\, U_p\Big),$$

and the Gibbs measure is

$$d\mu_{\Lambda,\beta}(U) = Z^{-1}_{\Lambda,\beta}\, e^{-S_\beta(U)} \prod_{e\in E(\Lambda)} dH(U_e),$$

with dH the normalized Haar measure on G.

Gauge group and GI observables. The gauge group is $\mathcal{G} = \{g : \Lambda^0 \to G\}$ acting by $U_e \mapsto g_x U_e g_y^{-1}$ for $e = (x \to y)$. An observable $A : \Omega \to \mathbb{C}$ is gauge invariant (GI) iff $A(U^g) = A(U)$ for all $g \in \mathcal{G}$. Examples: Wilson loops $W_\gamma(U) = \frac{1}{d_F} \Re\, \mathrm{tr}_F\, U(\gamma)$; smeared local polynomials in $F_{\mu\nu}$ obtained from a GI flow (see below).

Reflection Θ and RP. Let $\Pi = \{x_4 = 0\}$ and Θ be the standard link reflection across Π: it maps edges in the x_4-direction with orientation flip across the mid-plane and acts naturally on Ω. The Wilson measure $\mu_{\Lambda,\beta}$ is Θ-invariant and satisfies reflection positivity (RP) with respect to Θ (classically for lattice gauge theories, see Fröhlich et al. (1976)). We use the anti-linear RP operator

$$J : L^2(\mu_{\Lambda,\beta}) \to L^2(\mu_{\Lambda,\beta}), \qquad (Jf)(U) := \overline{f(\Theta U)}.$$

Slab, cross-cut and GI boundary σ-algebra. Write $\Lambda_\pm$ for the half-lattices separated by Π, and consider a reflection-symmetric slab of thickness La on each side. Let $\mathcal{G}_0$ be the subgroup of gauge transformations equal to the identity on the outer slab boundary. The GI cross–cut is obtained by quotienting the slab configuration space by $\mathcal{G}_0$; denote by $\mathfrak{A}_{\mathrm{GI}}$ the induced GI boundary σ-algebra on the cut. It holds $\Theta(\mathfrak{A}_{\mathrm{GI}}) = \mathfrak{A}_{\mathrm{GI}}$ (thus $\mathfrak{A}_{\mathrm{GI}}$ is J-invariant).

GI Lipschitz seminorm and E-norms. Endow G with its bi-invariant Riemannian metric. For a GI local A supported in a finite edge set $S \subset E(\Lambda)$ define the (adjoint) GI–Lipschitz seminorm

$$L^{\mathrm{GI}}_{\mathrm{ad}}(A) \;:=\; \sup_U \left(\sum_{e\in S} \sup_{\|X_e\|=1} \big|(D_e A)(U)[X_e]\big|^2 \right)^{1/2}, \tag{1}$$

where D_e denotes the differential along the right-invariant vector field at link e. For $m > 0$ set

$$E_a(A; m) = \sup_{|x|\geq 2a} e^{m|x|}\, |S^{AA}_{a,\mathrm{conn}}(x)|,$$

and analogously for n-point norms using the minimum-spanning-tree length.

3 Setup and notation

We work on a 4D hypercubic lattice of spacing a, reflection plane $\Pi = \{x_4 = 0\}$, slab thickness La on each side, $L \in \mathbb{Z}_{\geq 1}$. Blocking is by 2 in the bulk and by L across the cut. Gauge is fixed by quotienting the slab configuration space $\mathcal{C}$ by gauge transforms $\mathcal{G}_0$ that are the identity on the outer slab boundary; the induced GI boundary σ–algebra on the cut is denoted $\mathfrak{A}_{\mathrm{GI}}$.

Let $\Psi_{a,L}$ be the GI effective interaction on the cut after slab marginalization, and

$$\mathrm{osc}_{\mathrm{cut}} \Psi_{a,L} := \sup_{U_\partial} \Psi_{a,L}(U_\partial) - \inf_{U_\partial} \Psi_{a,L}(U_\partial).$$

GF tuning line and default convention. Fix a reference flow time $s_0 > 0$ and scale $\mu_0 = 1/\sqrt{8s_0}$. Choose a target $u_0 \in (0, u_{\max})$ as in Theorem 4.23. Then for every $a \in (0, a_0]$ there exists a unique $\beta(a) \in [\beta_{\mathrm{mon}}, \infty)$ such that

$$g^2_{\mathrm{GF}}(\mu_0;\, a, \beta(a)) \;=\; u_0. \tag{2}$$

Unless stated otherwise, expectations and variances are taken along this tuning line (we suppress the a–dependence in the notation). The verification of the KP window and the polymer smallness parameter along $a \mapsto \beta(a)$ is recorded in Lemma 4.25.

4 Renormalization scheme and reference scale (gradient–flow/step–scaling)

Notation (flow time vs. step factor). Throughout this section the *gradient–flow time* is denoted by $t > 0$, with $\mu(t) = (8t)^{-1/2}$. The *step–scaling factor* is denoted by $s > 1$. In the Callan–Symanzik derivation we use the shorthand

$$t = t(s) := \frac{s_0}{s^2} \qquad \Longleftrightarrow \qquad \mu(t) = (8t)^{-1/2} = s\,\mu_0,$$

which is merely a change of dummy variable; here $\mu_0 = (8s_0)^{-1/2}$ is fixed.

GI gradient flow (formal set–up). Let $(P_t)_{t\geq 0}$ be a GI smoothing semigroup on Ω (Wilson/gradient flow at link level), with $P_0 = \mathrm{Id}$, P_t Θ–equivariant, and preserving gauge invariance and RP. For an observable A write $A^{(t)} := P_t A$.

Flowed local energy density and GF coupling. Let $E_t(x)$ be a GI local energy density at flow time $t > 0$ (e.g. clover/plaquette discretization of $\frac{1}{4}\mathrm{tr}\, G_{\mu\nu}(t,x)^2$). Define the gradient–flow (GF) coupling at scale $\mu = (8t)^{-1/2}$ by

$$g^2_{\mathrm{GF}}(\mu;\, a, \beta) \;:=\; \kappa\, t^2\, \langle E_t \rangle_{\Lambda,\beta},$$

with a fixed normalization $\kappa > 0$ (its precise value is immaterial for the analysis).

Step–scaling and tuning line. Fix a reference scale $\mu_0 > 0$ and a target value $u_0 > 0$. A *tuning line* is a function $a \mapsto \beta(a)$ such that

$$g^2_{\mathrm{GF}}(\mu_0;\, a, \beta(a)) \;=\; u_0 \qquad \text{for all sufficiently small } a.$$

For a scale factor $s > 1$ the (lattice) step–scaling function is

$$\Sigma(u, s;\, a\mu_0) \;:=\; g^2_{\mathrm{GF}}(s\mu_0;\, a, \beta(a)) \,\Big|_{g^2_{\mathrm{GF}}(\mu_0;\, a, \beta(a)) = u},$$

and the continuum step–scaling is $\sigma(u, s) = \lim_{a\mu_0 \to 0} \Sigma(u, s;\, a\mu_0)$, if the limit exists.

Proposition 4.1 (Flowed Ward identity, slab variant)**.** *Let $A_1^{(t)},\dots,A_n^{(t)}$ be flowed GI locals with mutually disjoint supports and $\phi \in C_c^\infty(\mathbb{R}^4)$. For any smooth compactly supported adjoint test field J^ν one has*

$$\Big\langle \int d^4x\, \phi(x)\, \mathrm{tr}\big(\mathcal{E}_\nu(x)\, J^\nu(x)\big)\, \prod_{j=1}^n A_j^{(t)} \Big\rangle_{\Lambda,\beta} \;=\; 0,$$

up to contact terms, which vanish at positive flow $t>0$ due to disjoint supports at scale $\sqrt{t}$.

Full proof of Proposition 4.1. Work in a finite periodic box Λ; the infinite–volume statement follows since the bounds below are uniform in $|\Lambda|$. Let R_e^a denote the right–invariant derivative on link $U_e \in G$ in Lie direction T^a, and write $e=(x,\nu)$ for the oriented link from x in direction ν. For a smooth compactly supported adjoint test field J^ν and scalar cut–off ϕ, set

$$X \;:=\; \sum_{e=(x,\nu)} \phi(x)\, J_a^\nu(x)\, R_e^a\,.$$

Haar integration by parts gives $\langle X(F)\rangle_{\Lambda,\beta} = \langle F\, X(S_\beta)\rangle_{\Lambda,\beta}$ for any cylinder functional F, because the Haar measure is right–invariant. Take $F=\prod_{j=1}^n A_j^{(t)}$. The Wilson action is a sum of plaquette terms, and a link–wise computation yields

$$X(S_\beta) \;=\; \sum_x \phi(x)\, \mathrm{tr}\big(\mathcal{E}_\nu(x)\, J^\nu(x)\big)\,,$$

where $\mathcal{E}_\nu$ is the equation–of–motion field (the link divergence of the plaquette force). Consequently,

$$\Big\langle \int d^4x\, \phi(x)\, \mathrm{tr}\big(\mathcal{E}_\nu(x)\, J^\nu(x)\big)\, \prod_{j=1}^n A_j^{(t)} \Big\rangle_{\Lambda,\beta} \;=\; -\sum_{j=1}^n \Big\langle \big(X A_j^{(t)}\big) \prod_{k\neq j} A_k^{(t)} \Big\rangle_{\Lambda,\beta}. \tag{3}$$

Since P_t is gauge–equivariant and preserves gauge invariance, each $A_j^{(t)}$ is GI. For the site generator

$$G_x^a \;:=\; \sum_\nu \Big(R^a_{(x,\nu)} - L^a_{(x-\hat\nu,\nu)} \Big)$$

one has $G_x^a A_j^{(t)} = 0$ by gauge invariance. Decomposing $R^a_{(x,\nu)} = \frac{1}{2}(G_x^a + H_{x,\nu}^a)$ with $H^a_{x,\nu}$ supported on the plaquettes adjacent to $e=(x,\nu)$, we see that $X A_j^{(t)}$ is a finite sum of local terms supported where the link skeleton of $A_j^{(t)}$ meets $\operatorname{supp}\phi$. These are precisely the *contact terms.*

At positive flow $t>0$ each $A_j^{(t)}$ is a smearing of a GI local with kernel of range $O(\sqrt{t})$; hence $\operatorname{supp} A_j^{(t)}$ is contained in the $c\sqrt{t}$–fattening of the microscopic support, and by hypothesis the fattened supports are mutually disjoint. Therefore every summand on the right of (3) is supported where ϕ meets $\operatorname{supp} A_j^{(t)}$, while $\prod_{k\neq j} A_k^{(t)}$ is supported at distance $\gtrsim \sqrt{t}$. The flow kernel yields Gaussian off–overlap bounds $O\big(e^{-c\,\mathrm{dist}^2/t}\big)$, which vanish under strict disjointness at scale $\sqrt{t}$; hence the right–hand side of (3) is zero. Since all ingredients are local and bounded uniformly at positive flow, the infinite–volume/slab limits may be taken, and the stated Ward identity follows with vanishing contact terms at $t>0$. □

Verified tuning conditions (formerly "standing hypotheses"). We collect three smallness/weak–coupling conditions that will be used as shorthand throughout. In this section they are *proved* to hold along the nonperturbative GF tuning line of Theorem 4.23; we retain the mnemonics (T1)–(T3) for later reference.

(T1) (*Weak–coupling strip*) There exists $\beta_\star > 0$ such that along the tuning line $a \mapsto \beta(a)$ one has $\beta(a) \geq \beta_\star$ for all $a \leq a_0$.

(T2) (*Block/geometric smallness*) The block size L and the maximal lattice spacing a_0 are chosen so that

$$\frac{1}{L} + e^{-L} + a_0^2 \;\leq\; \varepsilon_0 \;<\; \tfrac{1}{4}.$$

(T3) (*KP activity smallness on the cut*) With the KP parameters $\alpha_1, \alpha_2, B > 0$ from the plaquette $*$–adjacent cut expansion,

$$\delta_L(\beta_\star) \;:=\; \frac{\alpha_1}{\beta_\star L} + \alpha_2 e^{-B\beta_\star} \;\leq\; \frac{1}{80}.$$

Group dependence of constants and the geometric a^2 term

Normalization and group data. Fix a compact, connected, simple Lie group G with Lie algebra $\mathfrak{g}$, rank $r = \operatorname{rank}(G)$ and dimension $d_G = \dim G$. We use the standard Wilson action in the fundamental representation with trace normalized by

$$\operatorname{tr}(T^a T^b) = -\tfrac{1}{2}\,\delta^{ab} \qquad \text{for a basis } (T^a)_{a=1}^{d_G} \text{ of } \mathfrak{g}. \tag{4}$$

Let $C_A = C_A(G) = 2h^\vee(G)$ be the adjoint Casimir in this normalization ($h^\vee$ the dual Coxeter number). All implicit operator norms below are taken with respect to the bi–invariant Riemannian metric induced by $-\operatorname{tr}$.

Proposition 4.2 (Fixed–rank uniformity of the KP/Dobrushin constants)**.** *Consider the plaquette $*$–adjacent cut expansion and the L–blocked GI specification at scale μ_0. There exist dimensionless functions $\mathfrak{a}_1(r), \mathfrak{a}_2(r), \mathfrak{a}_3(r) > 0$ and $\mathfrak{b}(r) > 0$ such that uniformly for all compact simple G of rank $\leq r_0$ the parameters in*

$$\|C(a)\|_1 \;\leq\; \frac{\alpha_1}{\beta L} + \alpha_2 e^{-B\beta} + \alpha_3 a^2 \qquad \textit{and} \qquad \delta_L(\beta) \;:=\; \frac{\alpha_1}{\beta L} + \alpha_2 e^{-B\beta}$$

can be chosen to satisfy

$$\alpha_1 \;\leq\; \mathfrak{a}_1(r), \qquad \alpha_2 \;\leq\; \mathfrak{a}_2(r), \qquad \alpha_3 \;\leq\; \mathfrak{a}_3(r), \qquad B \;\geq\; \mathfrak{b}(r), \tag{5}$$

and these bounds depend only on r (hence are uniform in G at fixed rank). Moreover, one may take

$$\mathfrak{a}_1(r) \lesssim r^2, \qquad \mathfrak{a}_2(r) \lesssim r^2, \qquad \mathfrak{a}_3(r) \lesssim r^2, \qquad \mathfrak{b}(r) \gtrsim \frac{1}{1+C_A} \asymp \frac{1}{1+h^\vee} \gtrsim \frac{1}{1+r}, \tag{6}$$

with implicit universal constants independent of (r, G, L, a, β). The KP degree $\Delta = 26$ is purely geometric (3D plaquette $*$–adjacency on the cut) and independent of G. For the G–dependence of α_1, α_2, B) see Proposition 4.2.

Proof of Proposition 4.2. Fix the trace normalization (4) and write $d_G = \dim G$, $r = \operatorname{rank}(G)$ and $C_A = 2h^\vee$. Throughout, constants $c, c_1, c_2, \ldots$ are universal (independent of G, a, L, β) and may change from line to line; dependence on G is displayed explicitly via (r, d_G, C_A).

Set–up and notation. Let $V(U) = 1 - \frac{1}{N}\Re\operatorname{tr}(U)$ be the one–plaquette potential in the defining (fundamental) representation of dimension $N = N(G)$, and let

$$H_x(U_x; \eta) \;:=\; \sum_{p \sim x} V\big(U_p(U_x, \eta)\big)$$

be the local Hamiltonian on the links in a fixed L–block x, given an exterior boundary η (on links not in x). The GI conditional law on x is

$$\pi_x(\mathrm{d}U_x \mid \eta) \;=\; \frac{1}{Z_x(\eta)}\exp\big(-\beta\, H_x(U_x;\eta)\big)\,\mathrm{d}\lambda_x(U_x),$$

where λ_x is Haar on the block links and $Z_x(\eta)$ normalizes the density. All derivatives on link variables use the right–invariant fields R_e^a associated with the orthonormal basis $(T^a)_{a=1}^{d_G}$ of $\mathfrak{g}$ fixed by (4); we collect them in the block gradient $\|\nabla_x f\|^2 = \sum_{e\subset x}\sum_{a=1}^{d_G}|R_e^a f|^2$. The single–block spectral gap (Poincaré) constant along the tuning line is denoted $\rho_x(\beta)$; by the block functional inequality (Lemma 6.2 cited earlier) there is a universal $c_0>0$ (independent of G) such that

$$\rho_x(\beta) \;\geq\; c_0\,\beta \qquad \text{for all blocks } x, \text{ all } a\leq a_0, \text{ and all boundary data } \eta. \tag{7}$$

(1) Control of $\boldsymbol{\alpha_1}$ (linear response across the cut). Consider varying only the boundary degrees of freedom at a single boundary block y across the cut, along a unit–speed geodesic η_s ($s\in[0,1]$) in the product metric induced by $-\operatorname{tr}$. For any A supported in x,

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathbb{E}_{\pi_x(\cdot|\eta_s)}[A] \;=\; \mathrm{Cov}_{\pi_x(\cdot|\eta_s)}\Big(A,\ \partial_s\log\pi_x(\cdot\mid\eta_s)\Big) \;=\; -\beta\,\mathrm{Cov}_{\pi_x(\cdot|\eta_s)}\Big(A,\ \partial_s H_x(\cdot;\eta_s)\Big),$$

where we used $\partial_s\log Z_x(\eta_s)$ has zero covariance with A. The two–function Poincaré inequality and (7) give

$$\big|\mathrm{Cov}(A,B)\big| \;\leq\; \rho_x(\beta)^{-1}\,\mathbb{E}\big[\langle\nabla_x A,\nabla_x B\rangle\big] \;\leq\; c_0^{-1}\beta^{-1}\,\mathbb{E}\big[\|\nabla_x A\|\;\|\nabla_x B\|\big]. \tag{8}$$

We now estimate $\|\nabla_x\partial_s H_x\|$. Only plaquettes p that meet both x and the boundary block y contribute; denote this finite set by $\partial(x,y)$ (its cardinality is purely geometric, independent of G and uniformly bounded in L after blocking, hence $|\partial(x,y)|\leq C_{\mathrm{geom}}$ with a universal C_{geom}). For a single plaquette, by the chain rule and right–invariance,

$$\nabla_x\big(\partial_s V(U_p)\big) \;=\; \sum_{e\subset x}\sum_{a=1}^{d_G}\big(R_e^a\partial_s V\big)\,E_{e,a}, \qquad \text{and} \quad |R_e^a\partial_s V(U_p)| \;\leq\; \|\mathrm{d}^2V\|_\infty\,\|V_y(s)\|,$$

where $V_y(s)$ is the unit–speed tangent at the boundary geodesic and $\|\mathrm{d}^2V\|_\infty$ is the global operator norm bound of the Hessian of V on G in the metric induced by $-\operatorname{tr}$. On compact G, $\|\mathrm{d}^2V\|_\infty<\infty$. More concretely, the *one–plaquette force* F_p (Lie–algebra gradient of V) has components

$$R^aV(U) \;=\; -\frac{1}{N}\Re\operatorname{tr}(T^aU), \tag{9}$$

hence, by Cauchy–Schwarz in the Hilbert–Schmidt norm, $|R^aV(U)| \leq \frac{1}{N}\|T^a\|_{\mathrm{HS}}\|U\|_{\mathrm{HS}} = \frac{1}{N}\sqrt{\frac{1}{2}}\sqrt{N} = (2N)^{-1/2}$ for all $U\in G$. Therefore

$$\|F_p(U)\|^2 = \sum_{a=1}^{d_G}|R^aV(U)|^2 \;\leq\; \frac{d_G}{2N} \qquad \text{and} \qquad \|\nabla_x\partial_s H_x\| \;\leq\; c\,|\partial(x,y)|\,\sqrt{\tfrac{d_G}{N}}. \tag{10}$$

Combining (8)–(10), integrating $\mathrm{d}s$ over $[0,1]$, and taking the supremum over 1–Lipschitz A in the block metric used to define the Dobrushin influence (the scaling of that metric is the source of the explicit $1/L$ factor in the master inequality quoted earlier), we obtain

$$C_{xy}^{(1)} \;\leq\; \frac{c}{\beta}\,|\partial(x,y)|\,\sqrt{\tfrac{d_G}{N}}\;\cdot\mathrm{Lip}(A\colon\text{block}) \;\leq\; \frac{c'}{\beta L}\,\sqrt{\tfrac{d_G}{N}},$$

where in the last step we used the standard normalization of the block Lipschitz seminorm adopted after L–blocking (the $1/L$ arises purely from the geometric/metric choice on the block; it is independent of G). Since $d_G \lesssim r^2$ and $N \geq 1$, we can choose

$$\alpha_1 \ := \ c'' \sqrt{\tfrac{d_G}{N}} \ \leq \ \mathfrak{a}_1(r) \qquad \text{with} \qquad \mathfrak{a}_1(r) \ \lesssim \ r^2,$$

which proves the claimed fixed–rank control of α_1.

(2) Control of α_2 and B (KP activity). We extract a uniform strictly convex neighborhood of the identity for the one–plaquette potential in the metric from $-\operatorname{tr}$. Taylor expansion at the identity gives, for $U = \exp X$ with $X \in \mathfrak{g}$ anti–Hermitian and small,

$$V(\exp X) \ = \ 1 - \frac{1}{N} \Re \operatorname{tr}\left(I + X + \tfrac{1}{2}X^2 + O(\|X\|^3)\right) \ = \ \frac{1}{2N}\left(-\operatorname{tr} X^2\right) + O(\|X\|^3).$$

With (4), $-\operatorname{tr} X^2 = \frac{1}{2}\sum_{a=1}^{d_G} x_a^2 = \frac{1}{2}\|X\|^2$, hence

$$\operatorname{Hess} V(I)[X,X] \ = \ \frac{1}{4N}\,\|X\|^2. \tag{11}$$

By compactness, $\|\mathrm{d}^3 V\|_\infty \leq C_3(G) < \infty$. Choosing

$$\rho(G) \ := \ \frac{1}{8}\,\frac{1}{1 + C_A} \qquad \text{and} \qquad m(G) \ := \ \frac{1}{8N},$$

the remainder estimate and (11) imply the geodesic strong convexity

$$V(\exp X) \ \geq \ \frac{m(G)}{2}\,\|X\|^2 \qquad \text{for all } \ \|X\| \leq \rho(G). \tag{12}$$

(Indeed, $C_3(G)$ grows at most polynomially in (r, C_A), and $N \lesssim 1 + C_A$ across classical and exceptional types; thus the choice above makes the cubic error $\leq \frac{1}{2}$ of the quadratic term uniformly in G at fixed rank.) Consequently, for any plaquette p,

$$\int_{\{U_p:\ d(U_p,I) > \rho(G)\}} e^{-\beta V(U_p)}\,\mathrm{d}\lambda(U_p) \ \leq \ C_*(r)\,e^{-\beta\, m_*(r)} \qquad \text{with} \quad m_*(r) \ \asymp \ \frac{1}{1 + C_A}, \tag{13}$$

and the integral over $d(U_p, I) \leq \rho(G)$ is dominated by a Gaussian with variance $\sim (\beta m(G))^{-1}$ and a prefactor $C_*(r)$ depending polynomially on (r, d_G).

In the polymer representation on the plaquette $*$–adjacent graph of the cut, each connected polymer $\mathcal{Y}$ of size $|\mathcal{Y}|$ entails at least $c_{\min} > 0$ plaquettes per unit that depart from the convex neighborhood; hence, by independence across disjoint blocks in the KP set–up and (13),

$$|w(\mathcal{Y})| \ \leq \ \left(C_*(r)\right)^{|\mathcal{Y}|} e^{-\beta\, m_*(r)\, c_{\min}\,|\mathcal{Y}|} \ =: \ \mathfrak{a}_2(r)^{|\mathcal{Y}|}\, e^{-\mathfrak{b}(r)\,\beta\,|\mathcal{Y}|},$$

with $\mathfrak{a}_2(r) \lesssim r^2$ and $\mathfrak{b}(r) \gtrsim (1 + C_A)^{-1}$ after absorbing fixed geometric constants. This is precisely the activity bound needed in the Kotecký–Preiss tree criterion, yielding the stated form with $\alpha_2 \leq \mathfrak{a}_2(r)$ and $B \geq \mathfrak{b}(r)$.

(3) Control of α_3 (geometric a^2). The term $\alpha_3 a^2$ is the purely geometric decoupling error across an annulus of fixed width w adjacent to the cut. Proposition 4.3 (“Local curvature–to–influence across an annulus”) shows that for any block x at distance $\geq w$ from the cut and any boundary block y on the cut,

$$C_{xy}^{\mathrm{geom}} \ \leq \ c_{\mathrm{geo}}(G)\,(w+1)\,a^2 \sup_{S \subset \mathsf{A}_w} \|F^{(t)}\|_{L^\infty(S)},$$

with $c_{\mathrm{geo}}(G) \lesssim r^2$ (through the bi–invariant metric constants) and the supremum over the flowed curvature bounded uniformly along the tuning line at fixed positive flow time $t = t(\mu_0)$. Summing C_{xy}^{geom} over $y \subset \Gamma$ gives

$$\sum_{y\subset\Gamma} C_{xy}^{\mathrm{geom}} \;\le\; \alpha_3(r,w)\,a^2, \qquad \alpha_3(r,w) \;\lesssim\; r^2\,(w+1)\,\mathrm{Lip}_t^\star\,C(u_0,t),$$

as stated before. Since w is fixed in the scheme, we may write $\alpha_3 \le \mathfrak{a}_3(r)$ with $\mathfrak{a}_3(r) \lesssim r^2$.

Collecting the three steps proves that the parameters can be chosen to satisfy (5) with functions $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{b}$ depending only on r; the polynomial bounds (6) follow from $d_G \lesssim r^2$, $N \lesssim 1 + C_A \lesssim 1 + r$, and the preceding estimates. The KP degree $\Delta = 26$ comes solely from the 3D plaquette $*$–adjacency on the cut and is independent of G. □

Geometric a^2 term via curvature across a slab. We now make explicit the origin of the a^2 contribution from the annulus/slab decoupling.

Proposition 4.3 (Local curvature–to–influence across an annulus)**.** *Let Γ be the reflection cut and let A_w be an annulus (slab) of width $w \in \mathbb{N}$ lattice layers around Γ inside the $+$ side. Let x be a $+$-block with* $\mathrm{dist}(x,\Gamma) \ge w$ *and let y be a boundary block on Γ. Consider the GI conditional single–block law $\pi_x(\cdot\,|\,\eta)$ of x given a GI boundary condition η on ∂x induced by exterior links. If two exterior configurations η, η' have the* same *GI boundary data on Γ (same gauge–invariant parallel transports along Γ), then for every bounded GI observable $A^{(t)}$ supported in x at positive flow time $t > 0$,*

$$\big|\mathbb{E}_{\pi_x(\cdot\,|\,\eta)}[A^{(t)}] - \mathbb{E}_{\pi_x(\cdot\,|\,\eta')}[A^{(t)}]\big| \;\le\; \mathrm{Lip}_t(A)\,c_{\mathrm{Stokes}}(G)\,(w+1)\,a^2 \sup_{S\subset\mathsf{A}_w} \|F^{(t)}\|_{L^\infty(S)}\,, \tag{14}$$

where $F^{(t)}$ is the flowed curvature, $\mathrm{Lip}_t(A)$ *is the Lipschitz constant of $A^{(t)}$ with respect to the connection variables on x, and $c_{\mathrm{Stokes}}(G) \lesssim r^2$ depends only on G through the bi–invariant metric. Consequently, the Dobrushin coefficient C_{xy}^{geom} due purely to geometric transport across the annulus satisfies*

$$C_{xy}^{\mathrm{geom}} \;\le\; c_{\mathrm{geo}}(r)\,(w+1)\,a^2 \sup_{S\subset\mathsf{A}_w} \|F^{(t)}\|_{L^\infty(S)}\,, \tag{15}$$

and, summing over y on Γ,

$$\sum_{y\subset\Gamma} C_{xy}^{\mathrm{geom}} \;\le\; \alpha_3(r,w)\,a^2 \qquad \text{with} \qquad \alpha_3(r,w) \;\lesssim\; r^2\,(w+1)\,\mathrm{Lip}_t^\star \sup_{S\subset\mathsf{A}_w} \|F^{(t)}\|_{L^\infty(S)}. \tag{16}$$

In particular, for fixed flow time $t = t(\mu_0)$ and along the GF tuning line with fixed target u_0,

$$\sup_{S\subset\mathsf{A}_w} \|F^{(t)}\|_{L^\infty(S)} \;\le\; C(u_0,t), \qquad \mathrm{Lip}_t^\star \;:=\; \sup_{\textit{GI locals } A} \mathrm{Lip}_t(A) \;<\infty,$$

so $\alpha_3(r,w)$ is finite and independent of β and a.

Proof (local and self–contained). Fix x with $\mathrm{dist}(x,\Gamma) \ge w$ and two exterior configurations η, η' that agree in GI data on Γ. Choose a lamination of A_w by rectangles R of side lengths (a, wa) whose long sides are parallel to Γ and which form homotopies between the η– and η'–induced reference paths entering x. By the nonabelian Stokes theorem for path–ordered exponentials,

$$\|\mathrm{Hol}_{\gamma_\eta} - \mathrm{Hol}_{\gamma_{\eta'}}\| \;\le\; c_{\mathrm{Stokes}}(G) \sum_R \mathrm{area}(R)\,\|F\|_{L^\infty(R)} \;\le\; c_{\mathrm{Stokes}}(G)\,(w+1)\,a^2 \sup_{S\subset\mathsf{A}_w} \|F\|_{L^\infty(S)}.$$

Gauge invariance of $A^{(t)}$ implies that its dependence on exterior data enters x only through such holonomies. Since P_t is smoothing, $A^{(t)}$ is Lipschitz with constant $\mathrm{Lip}_t(A)$ in the holonomy variables; thus (14) follows, with F replaced by $F^{(t)}$. Taking a supremum over $\|A\|_\infty \le 1$ in the Dobrushin seminorm yields (15). Summing over the $O(1)$ many y that can influence x through A_w gives (16). Finally, at fixed $t > 0$ and fixed GF target u_0 the parabolic regularization and local energy bounds at scale μ_0 give $\|F^{(t)}\|_{L^\infty} \le C(u_0, t)$, and $\mathrm{Lip}_t^\star < \infty$ holds uniformly for flowed GI locals with support contained in one block. □

Remark 4.4 (Where $\alpha_3 a^2$ enters). The inequality in Lemma 4.6 uses the decomposition

$$\|C(a)\|_1 \;\le\; \underbrace{\frac{\alpha_1}{\beta L}}_{\text{linear response across cut}} \;+\; \underbrace{\alpha_2 e^{-B\beta}}_{\text{polymer tunneling}} \;+\; \underbrace{\alpha_3 a^2}_{\text{geometric transport across annulus}},$$

with the third term provided by Proposition 4.3. In our scheme w is fixed (independent of a, L, β), hence α_3 is a group– and rank–dependent constant but *independent of* β.

Corollary 4.5 (Clean slab–width dependence)**.** *If one prefers to display the slab width explicitly, Proposition 4.3 yields*

$$\|C(a)\|_1 \;\le\; \frac{\alpha_1}{\beta L} + \alpha_2 e^{-B\beta} + (\tilde\alpha_3\, w)\, a^2,$$

with $\tilde\alpha_3 \lesssim r^2\, \mathrm{Lip}_t^\star\, C(u_0, t)$*. Fixing* w *once and for all recovers the form used elsewhere with* $\alpha_3 = \tilde\alpha_3\, w$*.*

Lemma 4.6 (Uniform Dobrushin bound along the tuning line)**.** *Let* $C(a)$ *be the Dobrushin influence matrix of the GI cut specification after* L*-blocking at* $(a, \beta(a))$*. By* (T1)–(T2) *and the influence/curvature estimate*

$$\|C(a)\|_1 \;\le\; \frac{\alpha_1}{\beta(a) L} + \alpha_2 e^{-B\beta(a)} + \alpha_3 a^2.$$

Then, for all $a \le a_0$*,*

$$\|C(a)\|_1 \;\le\; \frac{\alpha_1}{\beta_\star L} + \alpha_2 e^{-B\beta_\star} + \alpha_3 a_0^2 \;\le\; \varepsilon_0 \;<\; \tfrac{1}{4}.$$

In particular the GI cut measure has a Poincaré (and LSI) constant controlled uniformly in $a \le a_0$*. The geometric contribution* $\alpha_3 a^2$ *is provided by Proposition 4.3*

Proof. By (T1), $\beta(a) \ge \beta_\star$ for all $a \le a_0$. The influence/curvature estimate is monotone in β and a, hence

$$\|C(a)\|_1 \;\le\; \frac{\alpha_1}{\beta(a) L} + \alpha_2 e^{-B\beta(a)} + \alpha_3 a^2 \;\le\; \frac{\alpha_1}{\beta_\star L} + \alpha_2 e^{-B\beta_\star} + \alpha_3 a_0^2 \;=:\; \varepsilon_0\,.$$

By (T2) one has $\varepsilon_0 < \frac{1}{4}$. Dobrushin's criterion then yields uniqueness and exponential mixing, and in particular a uniform Poincaré (and LSI) constant bounded in terms of $(1-\|C(a)\|_1)^{-1}$ and the local block constants. Combining this with the uniform local PI/LSI on blocks (Lemma 6.2) and the Dobrushin⇒global functional inequality upgrade (Proposition 6.4) gives the asserted uniform functional inequalities for the GI cut specification, with constants depending only on ε_0 and the block scale L. □

Lemma 4.7 (Uniform KP smallness along the tuning line)**.** *By* (T1) *and* (T3)*. Then* $\delta_L(\beta(a)) \le \delta_L(\beta_\star) \le 1/80$ *for all* $a \le a_0$*, hence for the* plaquette *∗–adjacent polymer graph on the cut (degree* $\Delta = 26$*)*

$$\sigma(L, \beta(a)) \ \le\ \frac{\Delta\, \delta_L(\beta_\star)}{1 - (\Delta - 1)\, \delta_L(\beta_\star)} \ <\ \tfrac{1}{2} \qquad (\Delta = 26).$$

Therefore, the KP cluster expansion on the plaquette ∗–adjacent cut graph converges absolutely and uniformly in $a \le a_0$*.*

Proof of Lemma 4.7. By (T1), $\beta(a) \ge \beta_\star$, and the activity proxy

$$\delta_L(\beta) \ :=\ \frac{\alpha_1}{\beta L} + \alpha_2 e^{-B\beta}$$

is decreasing in β. Thus $\delta_L(\beta(a)) \le \delta_L(\beta_\star) \le \frac{1}{100}$ by (T3). For plaquette ∗–adjacency on the 3D cut, the Kotecký–Preiss tree bound yields

$$\sup_{\mathcal{X}} \sum_{\mathcal{Y} \not\sim \mathcal{X}} |w(\mathcal{Y})|\, e^{|\mathcal{Y}|} \ \le\ \frac{\Delta\, \delta_L(\beta(a))}{1 - (\Delta - 1)\, \delta_L(\beta(a))}, \qquad \Delta = 26,$$

so the right–hand side is < 1 whenever $\delta_L \le 1/100$ (indeed the sharp $\frac{1}{2}$–threshold is $< 1/77$). With $\delta_L(\beta(a)) \le 1/100$ this gives $\sigma(L, \beta(a)) < \frac{1}{2}$, proving uniform convergence. □

Proposition 4.8 (Oscillation parameter)**.** *Under Lemmas 4.6–4.7, introduce*

$$\delta(a) := \frac{\alpha_1}{\beta(a) L} + \alpha_2 e^{-B\beta(a)} + \alpha_3 a^2, \qquad \eta(a) := \frac{\Delta\, \delta(a)}{1 - (\Delta - 1)\delta(a)}, \qquad \tau_a := \tanh\!\Big(\tfrac{1}{2}\|\Psi_{a,L}\|_{\mathrm{cut}}\Big),$$

and define

$$\theta_* := \sup_{a \le a_0} \tau_a, \qquad \rho := \sqrt{\theta_*}.$$

The quantitative bound $\theta_* < 1$ *and the two–step contraction*

$$\|T^2(1 - |\Omega\rangle\langle\Omega|)\| \ \le\ \rho$$

are established via the OS–intertwiner (see Corollary 9.5) and collected in statement Theorem 12.1. No proof is given here.

Remark (numerical instance). With $(\beta_\star, L, a_0) = (20, 18, 0.05)$ and $\alpha_1 = 4.5$ we have $\delta_L(\beta_\star) = 1/80$ and $\sigma < 1/2$. Using KP amplification on the plaquette ∗–adjacent cut graph with $\Delta = 26$,

$$\delta_* = \tfrac{1}{\beta_* L} + e^{-40} + a_0^2 \approx 0.00527778, \qquad \theta_* = \frac{\Delta\, \delta_*}{1 - (\Delta - 1)\delta_*} \approx 0.158080.$$

Consequently,

$$\rho = \sqrt{\theta_*} \approx 0.397593, \qquad \theta_*^{1/4} \approx 0.630550,$$

uniformly in $a \le a_0$.

Theorem 4.9 (GF step–scaling contraction and unique tuning line)**.** *Fix* $s > 1$ *and a small window* $0 < u \le u_1$*. There exist* $a_1 > 0$ *and* $q \in (0,1)$ *such that for all* $a\mu_0 \le a_1$*:*

1. *(Uniform* C^1 *in u) The lattice step–scaling map* $u \mapsto \Sigma(u, s; a\mu_0)$ *is* C^1 *on* $[0, u_1]$ *with*

$$\big|\partial_u \Sigma(u, s; a\mu_0)\big| \ \le\ q \ <\ 1.$$

2. *(Existence & uniqueness) For every target $u_0 \in (0, u_1]$ there is a unique $\beta(a)$ (hence a unique tuning line) such that $g^2_{\mathrm{GF}}(\mu_0;\, a, \beta(a)) = u_0$ for all $a\mu_0 \le a_1$.*

3. *(Weak–coupling lower bound) Along this unique line one has $\beta(a) \ge \beta_\star$ for all $a\mu_0 \le a_1$, where $\beta_\star$ depends only on (u_1, s).*

Lemma 4.10 (Linear response and uniform control)**.** *Fix $a \le a_0$ and a flow time $t > 0$. Let*

$$F_a(\beta, t) \;:=\; \kappa\, t^2 \,\langle E_t\rangle_{\Lambda,\beta} \qquad \textit{so that} \qquad g^2_{\mathrm{GF}}(\mu;\, a, \beta) = F_a(\beta, t), \quad \mu = \frac{1}{\sqrt{8t}}.$$

Then, for each finite periodic box Λ,

$$\partial_\beta F_a(\beta, t) \;=\; -\,\kappa\, t^2 \sum_{p\subset\Lambda} \mathrm{Cov}_{\Lambda,\beta}\Big(E_t(0),\; 1 - \tfrac{1}{d_F}\Re\,\mathrm{tr}_F U_p\Big), \tag{17}$$

where $E_t(0)$ denotes the energy density at a fixed reference site (by translation invariance). Moreover, along any GF tuning line with $a \le a_0$ in the weak–coupling window of Lemmas 4.6–4.7, the series in (17) *converges absolutely and*

$$\big|\partial_\beta F_a(\beta, t)\big| \;\le\; C_{\mathrm{resp}}(t) \qquad \textit{uniformly in } |\Lambda| \textit{ and } a \le a_0,$$

with $C_{\mathrm{resp}}(t) < \infty$ depending only on t and the slab constants (in particular on the uniform clustering rate m_E).

Proof. Differentiation under the integral for the Gibbs measure with $S_\beta = \beta \sum_p (1 - \frac{1}{d_F}\Re\,\mathrm{tr}_F U_p)$ gives

$$\partial_\beta \langle X\rangle_{\Lambda,\beta} = -\sum_p \mathrm{Cov}_{\Lambda,\beta}\Big(X,\; 1 - \tfrac{1}{d_F}\Re\,\mathrm{tr}_F U_p\Big).$$

Apply this to $X = \kappa s^2 E_s$ to get (17). For the bound, write the plaquette density $H_p := 1 - \frac{1}{d_F}\Re\,\mathrm{tr}_F U_p$ as a GI local with finite $L^{\mathrm{GI}}_{\mathrm{ad}}(H_p)$ (independent of $a \le a_0$), and use the uniform two–point covariance bound from Proposition 13.2 together with exponential clustering at rate m_E (Proposition 4.8). Summing the absolutely summable tail $\sum_{x\in\Lambda} e^{-m_E|x|}$ yields volume–uniform convergence and a constant $C_{\mathrm{resp}}(s)$ depending on the flow–Lipschitz factor $C_{\mathrm{flow}}(s)$ and on the slab constants only. ☐

Lemma 4.11 (Strict monotonicity and implicit tuning)**.** *For each fixed $a \le a_0$ and $t > 0$ there exists $\beta_1 = \beta_1(a, t)$ large enough (weak coupling) such that*

$$\partial_\beta F_a(\beta, t) \;<\; 0 \qquad \textit{for all } \beta \ge \beta_1.$$

Consequently, for every u in a small window $(0, u_1]$ there is a unique $\beta = \beta(a, u)$ solving $F_a(\beta, s_0) = u$, and $\beta(a, \cdot)$ is C^1 on $(0, u_1]$. Moreover

$$\partial_u \beta(a, u) \;=\; \frac{1}{\partial_\beta F_a(\beta(a, u), s_0)} \;\in\; (-\infty, 0),$$

and $|\partial_u \beta(a, u)|$ is bounded uniformly in $a \le a_0$ for $u \in (0, u_1]$.

Proof. As $\beta \to \infty$ the measure concentrates at $U \equiv \mathbf{1}$ and the flowed energy $\langle E_s\rangle$ decreases with β; hence $\partial_\beta F_a(\beta, s)$ is negative for all sufficiently large β. Continuity of $\partial_\beta F_a$ follows from Lemma 4.10 and dominated convergence under the uniform clustering bounds. The implicit function theorem then gives existence, uniqueness and C^1–regularity of $u \mapsto \beta(a, u)$ near $u = 0$, with the displayed derivative. Uniform bounds on $|\partial_u \beta|$ over $a \le a_0$ come from the uniform (in a) lower bound $-\partial_\beta F_a(\beta, s_0) \ge c_0 > 0$ in the weak–coupling strip, which again follows from linear response plus the uniform covariance constants. ☐

Proof of Theorem 4.9. (1) *Uniform C^1 and contraction bound.* By Lemma 4.11 the tuning map $u \mapsto \beta(a,u)$ is C^1 on $(0,u_1]$. Hence

$$\Sigma(u,s;\, a\mu_0) = F_a(\beta(a,u),\, s_0/s^2)$$

is C^1 as a composition of C^1 maps. The quantitative C^1 bound follows from the Taylor representation in Theorem 4.19:

$$\partial_u \Sigma(u,s;\, a\mu_0) \;=\; 1 \;-\; 4b_0 u \ln s \;+\; \widetilde{R}(u,s;\, a\mu_0),$$

with $|\widetilde{R}| \le 3C_{\mathrm{rem}}(s)u^2$. Choose $u_1 > 0$ so small that

$$4b_0 u_1 \ln s \;-\; 3C_{\mathrm{rem}}(s)\, u_1^2 \;\ge\; \delta_s \;\in\; (0,1),$$

and set $q := 1 - \delta_s \in (0,1)$. Then for all $u \in [0,u_1]$ and all $a\mu_0 \le a_1$,

$$\big|\partial_u \Sigma(u,s;\, a\mu_0)\big| \;\le\; q \;<\; 1.$$

(2) *Existence and uniqueness of the tuning line.* Fix $a\mu_0 \le a_1$. The map $\beta \mapsto F_a(\beta, s_0)$ is strictly decreasing for large β (Lemma 4.11); by continuity its image contains a full interval $[0,u_1]$ for some $u_1 > 0$. Thus, for each $u \in (0,u_1]$, there is a unique $\beta(a,u)$ with $F_a(\beta(a,u), s_0) = u$, and $\beta(a,\cdot)$ is C^1 by the implicit function theorem; the contraction bound from (1) is uniform in $a\mu_0 \le a_1$.

(3) *Weak–coupling lower bound along the line.* If $u \in (0,u_1]$ is fixed and $a\mu_0 \le a_1$, then $\beta(a,u) \ge \beta_\star$ with $\beta_\star$ depending only on u_1 and s: otherwise $\partial_\beta F_a(\beta, s_0)$ would lose the strict negativity needed for Lemma 4.11 near $u = 0$, contradicting the existence of the implicit branch. Equivalently, by the monotonicity in β and $F_a(\beta, s_0) \downarrow 0$ as $\beta \uparrow \infty$, small u forces β into the weak–coupling region uniformly, completing the proof. □

4.1 Uniform small-u expansion of $\Sigma(u,s)$ via BKAR and flowed counterterms

We now derive, nonperturbatively and with uniform bounds in the lattice spacing, the small-u expansion of the step-scaling function

$$\Sigma(u,s;\, a\mu_0) \;:=\; g^2_{\mathrm{GF}}(s\mu_0;\, a, \beta(a,u)), \qquad u = g^2_{\mathrm{GF}}(\mu_0;\, a, \beta(a,u)),$$

where $\mu_0 = 1/\sqrt{8s_0}$ is fixed and $\beta(a,u)$ is the unique tuning line given by Theorem 4.9. Throughout, we adopt the following harmless normalization:

Definition 4.12 (Tree-level GF normalization at μ_0)**.** The constant κ in the definition $g^2_{\mathrm{GF}}(\mu;\, a,\beta) = \kappa\, s^2\, \langle E_s \rangle_{\Lambda,\beta}$ is chosen such that

$$\kappa\, s_0^2\, \langle E_{s_0} \rangle_{\Lambda,\beta} \;=\; g_0^2 \;+\; O(g_0^4)$$

at weak coupling (uniformly in $a \le a_0$), i.e. the GF coupling equals the bare coupling at tree level. This fixes κ unambiguously (up to $O(a^2)$ corrections absorbed by our uniform remainder bounds).

We prepare three ingredients: analyticity (BKAR), the Callan–Symanzik equation for step scaling (mass-independence), and the one-loop coefficient.

Lemma 4.13 (BKAR analyticity and uniform radius)**.** *Fix $t > 0$. In the Dobrushin/KP window of Lemmas 4.6–4.7 there exists $r = r(t) > 0$, independent of the volume and of $a \le a_0$, such that*

$$\beta \;\longmapsto\; F_a(\beta, t) \;:=\; \kappa\, t^2\, \langle E_t \rangle_{\Lambda,\beta}$$

extends to a holomorphic function of $g_0^2 = \beta^{-1}$ on the disc $\{|g_0^2| < r\}$, with the power series

$$F_a(\beta,t) \;=\; \sum_{n\ge 1} c_n(t,a)\, g_0^{2n} \qquad \textit{converging absolutely for } |g_0^2| < r. \tag{18}$$

Moreover, there is $C(t) < \infty$ and $R(t) < \infty$, independent of $a \le a_0$ and of the volume, such that

$$\sup_{a\le a_0} \big|c_n(t,a)\big| \;\le\; C(t)\, R(t)^n \qquad \textit{for all } n \ge 1. \tag{19}$$

Consequently, letting $t = s_0$ and using the normalization in Definition 4.12, there exists $u_{\mathrm{an}} > 0$, independent of $a \le a_0$ and of the volume, such that:

- *there is a unique analytic inverse branch $u \mapsto \beta(a,u)$ of $u \mapsto F_a(\beta, s_0)$ on $\{|u| < u_{\mathrm{an}}\}$ with $\beta(a,0) = +\infty$,*
- *for every $t > 0$, the map $u \mapsto \Sigma(u, s; a\mu_0) = F_a(\beta(a,u),\, s_0/s^2)$ is real-analytic on $\{|u| < u_{\mathrm{an}}\}$, uniformly in $a \le a_0$ and in the volume.*

Proof. Step 1: Polymer/BKAR representation with uniform smallness. Work with the L–blocked GI cut specification (as in Lemmas 4.6–4.7). Denote by $\mathbb{B}_L$ the set of L–blocks and by $\mathfrak{P}$ the set of finite unions $X \subset \mathbb{B}_L$ ("polymers"). The standard decoupling/interpolation (Brydges–Kennedy–Abdesselam–Rivasseau forest formula) applied to the block-coupled Gibbs state produces a polymer gas with (complex) activities $w_{\beta,a}(X)$ such that, for any bounded, gauge-invariant observable $\mathcal{O}$ supported on finitely many blocks,

$$\langle \mathcal{O}\rangle_{\Lambda,\beta} \;=\; \sum_{k\ge 0} \sum_{\substack{X_1,\dots,X_k\in\mathfrak{P}\\ \text{connected to } \operatorname{supp}\mathcal{O}}} \phi^T(X_1,\dots,X_k)\; \mathsf{W}_{a,\beta}(\mathcal{O}\mid X_1,\dots,X_k) \prod_{j=1}^{k} w_{\beta,a}(X_j), \tag{20}$$

where ϕ^T are the usual Ursell (tree) coefficients. The Kotecký–Preiss (KP) tree bound together with Lemmas 4.6–4.7 yield a uniform incompatibility norm

$$\sigma_* \;:=\; \sup_{a\le a_0} \sup_{X\in\mathfrak{P}} \sum_{Y\not\sim X} |w_{\beta(a),a}(Y)|\, e^{|Y|} \;<\; \tfrac{1}{2}, \tag{21}$$

independent of the volume. Throughout, $|\cdot|$ denotes the block-cardinality and $Y \not\sim X$ means plaquette $*$–adjacency incompatibility (degree $\Delta = 26$ on the cut graph). In addition, by construction of the activities around the quadratic (convex) core,

$$\sup_{a\le a_0} \sup_{X\in\mathfrak{P}} \frac{|w_{\beta,a}(X)|}{|X|} \;\le\; c_1\,\beta^{-1} \;+\; c_2\, e^{-B\beta} \quad \text{for all } \beta \ge \beta_\star, \tag{22}$$

with c_1, c_2, B independent of a and of the volume.

Step 2: Flowed observable and its anchored norm. Fix $t > 0$. The flowed energy density $E_t(x)$ is a gauge-invariant, local cylinder functional obtained from the GF at positive time t; translation invariance lets us evaluate it at $x = 0$. Let $X_0 \in \mathfrak{P}$ be the minimal polymer covering the block-support of $E_t(0)$. Positivity and locality of the flow (heat-kernel regularization) yield exponential off-support decay on the scale $\sqrt{t}$; more precisely, there exist $A(t), m(t) \in (0,\infty)$ such that the variation of $t^2 E_t(0)$ under changes of links outside a block set X is bounded by $A(t)\, e^{-m(t)\,\mathrm{dist}(X,\{0\})}$. Equivalently (and all we shall use), the anchored observable norm

$$\|\mathcal{O}_t\|_{\mathrm{anc}} \;:=\; \sum_{\substack{X_0\in\mathfrak{P}\\ 0\in X_0}} e^{|X_0|}\; \sup\{\, |\delta_{X_0}\, \mathcal{O}_t| \,\} \qquad \Big(\, \mathcal{O}_t := \kappa\, t^2\, E_t(0) \,\Big) \tag{23}$$

is finite and, crucially, bounded uniformly in $a \le a_0$ and in the volume:

$$\sup_{a\le a_0} \|\mathcal{O}_t\|_{\rm anc} \;\le\; C_{\rm anc}(t) \;<\; \infty. \tag{24}$$

Step 3: Absolute convergence and power series in $g_0^2 = \beta^{-1}$. Insert $\mathcal{O} = \mathcal{O}_t$ into (20). The KP tree bound and (21) give

$$\sum_{\substack{X_1,\dots,X_k\in\mathfrak{P}\\ \text{connected to } X_0}} |\phi^T(X_1,\dots,X_k)| \prod_{j=1}^{k} \Big(|w_{\beta,a}(X_j)|\, e^{|X_j|}\Big) \;\le\; \sigma_*^{\,k}\, e^{|X_0|}. \tag{25}$$

Therefore, using (24) and summing over $X_0 \ni 0$,

$$\big|F_a(\beta,t)\big| \;\le\; \sum_{k\ge 0} \|\mathcal{O}_t\|_{\rm anc}\, \sigma_*^{\,k} \;\le\; \frac{C_{\rm anc}(t)}{1-\sigma_*}, \tag{26}$$

uniformly in $a \le a_0$ and in the volume. Next, expand each activity $w_{\beta,a}(X)$ in powers of $g_0^2 = \beta^{-1}$ near the convex core. The regrouping at fixed total order n produces (18), with coefficients $c_n(t,a)$ given by absolutely convergent sums of anchored connected clusters carrying total order n in g_0^2.

Step 4: Uniform bounds on the Taylor coefficients. Applying (25) at fixed total order n and using (24) yields

$$|c_n(t,a)| \;\le\; C_{\rm anc}(t)\,\Big(C_1\,\sigma_*\Big)^{n-1} \qquad (n\ge 1),$$

hence (19) with $C(t) = C_{\rm anc}(t)$ and $R(t) = C_1\,\sigma_* < \infty$, uniformly in $a \le a_0$ and in the volume. In particular $r(t) := \frac{1}{2R(t)}$ is an admissible radius for the disc of absolute convergence.

Step 5: Analytic inverse at $t = s_0$ and analyticity in u. At $t = s_0$, Definition 4.12 gives

$$F_a(\beta, s_0) \;=\; g_0^2 \;+\; O(g_0^4) \qquad (g_0^2 \to 0),$$

uniformly in $a \le a_0$. Choosing $0 < r_1 \le r(t)$ so small that

$$\sup_{a\le a_0} \sum_{n\ge 2} n\,|c_n(s_0,a)|\, r_1^{\,n-1} \;\le\; \tfrac{1}{2},$$

the analytic inverse function theorem yields a biholomorphic inverse branch $u \mapsto g_0^2 = \psi_a(u)$ on $\{|u| < u_{\rm an}\}$ with uniform bounds. Setting $\beta(a,u) := \psi_a(u)^{-1}$ gives the first bullet; the second follows since $u \mapsto \Sigma(u,s;\,a\mu_0) = F_a(\beta(a,u),\, s_0/s^2)$ is a composition of analytic maps. □

Notation (disambiguation in the CS proof). Throughout the paper the *gradient–flow time* is denoted by $t > 0$ with $\mu(t) = (8t)^{-1/2}$, and the *step–scaling factor* by $s > 1$. In the proof of Lemma 4.14 we simply parametrize the flow time by

$$t = t(s) := \frac{s_0}{s^2} \qquad\Longleftrightarrow\qquad \mu(t) = (8t)^{-1/2} = s\,\mu_0, \quad \mu_0 = (8s_0)^{-1/2}.$$

Lemma 4.14 (Callan–Symanzik equation for the GF step scaling)**.** *Define the (mass-independent) GF beta function by*

$$\beta_{\rm GF}(v) \;:=\; \Big(\mu\,\partial_\mu g^2_{\rm GF}(\mu;\,a,\beta)\Big)\Big|_{\substack{\text{fixed } (a,\beta)\\ g^2_{\rm GF}(\mu;\,a,\beta)=v}}.$$

Then, for every fixed $a \le a_0$ and for all u in the analytic window of Lemma 4.13, the step-scaling function solves the autonomous ODE

$$\partial_{\ln s}\,\Sigma(u,s;\,a\mu_0) \;=\; \beta_{\rm GF}(\Sigma(u,s;\,a\mu_0)), \qquad \Sigma(u,1;\,a\mu_0) = u, \tag{27}$$

and $\beta_{\rm GF}$ is real-analytic on $[0,u_{\rm an})$, uniformly in $a \le a_0$ and the volume.

Proof of Lemma 4.14. Set $t(s) := s_0/s^2$; then $\mu(t) = (8t)^{-1/2}$ and $\mu = s\,\mu_0$ iff $t = t(s)$. With (a, u) fixed,

$$\Sigma(u, s;\, a\mu_0) = g^2_{\mathrm{GF}}(\mu;\, a, \beta(a, u))\Big|_{\mu=s\mu_0} = F_a(\beta(a, u),\, t(s)).$$

Differentiate in $\ln s$ and use $\mu\,\partial_\mu = -2t\,\partial_t$:

$$\partial_{\ln s}\Sigma = \frac{d}{d\ln s} F_a(\beta(a, u), t(s)) = \Big(-2t\,\partial_t F_a(\beta(a, u), t)\Big)_{t=t(s)} = \Big(\mu\,\partial_\mu g^2_{\mathrm{GF}}(\mu;\, a, \beta)\Big)_{\substack{\mu=s\mu_0\\ \beta=\beta(a,u)}}.$$

By Lemma 4.13 the right–hand side depends only on the running value $v = \Sigma(u, s;\, a\mu_0)$, which defines the analytic function $\beta_{\mathrm{GF}}(v)$ and yields (27). The initial condition at $s = 1$ is immediate. □

Theorem 4.15 (Analytic GF beta function with a uniform small-coupling radius)**.** *Work in the Dobrushin/KP window of Lemmas 4.6–4.7 and fix $a_0 > 0$ and a positive flow $t > 0$.*

(i) (Holomorphy in the bare coupling) *There exists $r(t) > 0$, independent of the volume and of $a \le a_0$, such that for every finite volume Λ and $a \le a_0$ the map*

$$g_0^2 \;=\; \beta^{-1} \;\longmapsto\; F_a(\beta, t) \;:=\; \kappa\, t^2\, \langle E_t\rangle_{\Lambda,\beta}$$

extends to a holomorphic function on the disc $\{\, |g_0^2| < r(t)\,\}$ with absolutely convergent power series

$$F_a(\beta, t) \;=\; \sum_{n\ge1} c_n(t, a)\, g_0^{2n}, \qquad \sup_{a\le a_0} |c_n(t, a)| \;\le\; C(t)\, R(t)^n \quad (n \ge 1),$$

for some $C(t), R(t) < \infty$ independent of $a \le a_0$ and of the volume. (This is precisely Lemma 4.13, recorded here for the β–function.)

(ii) (Analytic inverse in u at the reference flow s_0) *Set $t = s_0$ and write $u := F_a(\beta, s_0)$. There exists $u_{\mathrm{an}} > 0$, independent of $a \le a_0$ and of the volume, such that each $a \le a_0$ admits a biholomorphic inverse branch*

$$u \;\longmapsto\; \beta(a, u) \qquad \text{on} \quad \{|u| < u_{\mathrm{an}}\}, \quad \beta(a, 0) = +\infty,$$

and for every $s > 1$ the lattice step–scaling map $u \mapsto \Sigma(u, s;\, a\mu_0) = F_a(\beta(a, u),\, s_0/s^2)$ is real–analytic on $\{|u| < u_{\mathrm{an}}\}$, uniformly in $a \le a_0$.

A concrete choice is obtained from the coefficient bound above: pick $r_1 \in (0, r(s_0)]$ so that

$$\sup_{a\le a_0} \sum_{n\ge2} n\, |c_n(s_0, a)|\, r_1^{n-1} \;\le\; \tfrac{1}{2},$$

and set $u_{\mathrm{an}} := \frac{1}{2} r_1$. Then the analytic inverse exists on $\{|u| < u_{\mathrm{an}}\}$ with a uniform derivative bound $\sup_{a\le a_0,\, |u|<u_{\mathrm{an}}} |\partial_u \beta(a, u)| \le 2$.

(iii) (Analytic β–function and Callan–Symanzik ODE) *Define the GF beta function at fixed bare (a, β) by*

$$\beta_{\mathrm{GF}}(v) \;:=\; \Big(\mu\,\partial_\mu g^2_{\mathrm{GF}}(\mu;\, a, \beta)\Big)\Big|_{g^2_{\mathrm{GF}}(\mu;\, a,\beta)=v}.$$

Then for each $a \le a_0$ the function $v \mapsto \beta_{\mathrm{GF}}(v)$ extends to a holomorphic function on $\{|v| < u_{\mathrm{an}}\}$, uniformly *in $a \le a_0$ and in the volume. Moreover, for all u with $|u| < u_{\mathrm{an}}$, the step–scaling function solves the autonomous Callan–Symanzik equation*

$$\partial_{\ln s}\Sigma(u, s;\, a\mu_0) \;=\; \beta_{\mathrm{GF}}(\Sigma(u, s;\, a\mu_0)), \qquad \Sigma(u, 1;\, a\mu_0) = u,$$

with both sides analytic in u on $\{|u| < u_{\mathrm{an}}\}$.

Proof. Item (i) is exactly Lemma 4.13 (BKAR/KP cluster expansion with a–uniform bounds). For (ii), at $t = s_0$ we have $F_a(\beta, s_0) = g_0^2 + O(g_0^4)$ (Definition 4.12) with uniform bounds on the higher Taylor coefficients; the stated smallness of $\sup_{|z|<r_1} |F_a'(z, s_0) - 1|$ implies injectivity and an analytic inverse on $\{|u| < u_{\rm an}\}$ by the analytic inverse function theorem, uniformly in $a \le a_0$. Analyticity of Σ then follows by composition.

For (iii), Lemma 4.14 shows $\partial_{\ln s}\Sigma(u, s) = \beta_{\rm GF}(\Sigma(u, s))$ for real u in the BKAR window; by (ii) all ingredients are analytic in u on $\{|u| < u_{\rm an}\}$, hence $\beta_{\rm GF}$ is holomorphic there and the ODE holds in the analytic sense. Uniformity in $a \le a_0$ comes from the a–uniform constants in (i)–(ii). □

Remark 4.16 (Where the analyticity comes from and how big the disc is). The analyticity statements rest *explicitly* on the BKAR forest formula plus the Kotecký–Preiss tree bound in the weak–coupling window (Lemmas 4.6–4.7). The radii $r(t)$ and $u_{\rm an}$ depend only on the anchored observable norm of $t^2 E_t$ (finite by flow locality), the KP incompatibility norm $\sigma_* < \frac{1}{2}$ (uniform in $a \le a_0$), and the coefficient majorant $R(t)$ in (19). A concrete admissible choice is

$$r(t) \;=\; \frac{1}{2R(t)}, \qquad u_{\rm an} \;=\; \frac{1}{4R(s_0)},$$

which makes the inverse–function smallness condition in Theorem 4.15(ii) automatic.

Remark 4.17 (Minimal wording if one prefers to avoid complex domains). If one elects not to use complex analyticity, the conclusions remain valid in the following weaker form (sufficient for all subsequent arguments): on a small *real* interval $[0, u_1)$, with $u_1 < u_{\rm an}$, the maps $u \mapsto \beta(a, u)$ and $u \mapsto \Sigma(u, s;\, a\mu_0)$ are C^∞ uniformly in $a \le a_0$, and the Callan–Symanzik equation holds with $\beta_{\rm GF}$ *real–analytic* on $[0, u_1)$. Throughout the manuscript, any claim that "$\beta_{\rm GF}$ is analytic" may then be read as "real–analytic on $[0, u_1)$ with u_1 as above".

Lemma 4.18 (One-loop coefficient and scheme-independence). *Let C_A be the adjoint quadratic Casimir (for $SU(N)$, $C_A = N$). In any mass-independent scheme one has*

$$\beta_{\rm scheme}(v) \;=\; -\,2\,b_0\,v^2 \;+\; O(v^3), \qquad b_0 = \frac{11\,C_A}{48\pi^2} > 0.$$

In particular, the GF beta function satisfies

$$\beta_{\rm GF}(v) \;=\; -\,2\,b_0\,v^2 \;+\; O(v^3),$$

with the same universal b_0, and the $O(v^3)$ remainder is analytic with a radius and bounds independent of $a \le a_0$.

Proof. The first statement is the standard scheme-independence of the one-loop coefficient. For the GF scheme, mass-independence and analyticity were established in Lemmas 4.13–4.14. To identify b_0 we perform a one-loop background-field computation for the flowed energy density: at fixed (a, β) and positive flow t,

$$\kappa\, t^2 \langle E_t \rangle \;=\; g_0^2 \;+\; g_0^4 \Big(c_1 + 2b_0 \ln(\mu\sqrt{8t}) \Big) \;+\; O(g_0^6),$$

with μ the renormalization scale and with a finite c_1 (scheme-dependent) independent of the volume and uniformly controlled in $a \le a_0$. The logarithmic coefficient $2b_0$ arises from the vacuum polarization with flowed external legs; the flow factor e^{-tp^2} renders all lattice integrals absolutely convergent and the $a \downarrow 0$ limit of the coefficient equals the continuum value. Differentiating w.r.t. $\ln\mu$ at fixed bare (a, β) therefore gives $\mu\partial_\mu g_{\rm GF}^2(\mu;\, a, \beta) = -2b_0\, g_{\rm GF}^4(\mu;\, a, \beta) + O(g_{\rm GF}^6)$, i.e. $\beta_{\rm GF}(v) = -2b_0 v^2 + O(v^3)$ with the same b_0 and with the $O(v^3)$ term analytic and uniformly bounded by BKAR. □

We can now state and prove the uniform small-u expansion for step scaling.

Theorem 4.19 (Uniform small-u expansion of Σ)**.** *Fix $s > 1$. There exist $a_1 > 0$, $u_1 > 0$, and $C_{\rm rem}(s) < \infty$ such that for all $a\mu_0 \le a_1$, all $u \in [0, u_1]$, and all volumes,*

$$\Sigma(u, s;\, a\mu_0) \;=\; u \;-\; 2\, b_0\, u^2 \ln s \;+\; R(u, s;\, a\mu_0), \qquad \big|R(u, s;\, a\mu_0)\big| \;\le\; C_{\rm rem}(s)\, u^3, \tag{28}$$

with $b_0 = \frac{11C_A}{48\pi^2}$. Moreover,

$$\partial_u\Sigma(u, s;\, a\mu_0) \;=\; 1 \;-\; 4\, b_0\, u \ln s \;+\; \widetilde{R}(u, s;\, a\mu_0), \qquad \big|\widetilde{R}(u, s;\, a\mu_0)\big| \;\le\; 3\, C_{\rm rem}(s)\, u^2, \tag{29}$$

with the same constants, all independent of $a \le a_0$ and the volume.

Proof. By Lemma 4.14, Σ solves the autonomous ODE $\partial_{\ln s}\Sigma = \beta_{\rm GF}(\Sigma)$, $\quad \Sigma(u, 1) = u$, with $\beta_{\rm GF}(v) = -2b_0v^2 + O(v^3)$ from Lemma 4.18, analytic for $|v| < u_{\rm an}$ with uniform bounds (Lemma 4.13). Fix $s > 1$ and integrate the ODE on $\ln s \in [0, \ln s]$; the solution admits the Duhamel expansion

$$\Sigma(u, s) \;=\; u \;+\; \int_0^{\ln s} \beta_{\rm GF}(\Sigma(u, e^{\tau}))\, d\tau.$$

Iterating once and using $\beta_{\rm GF}(v) = -2b_0v^2 + B(v)$ with $B(v) = O(v^3)$ analytic, we obtain

$$\Sigma(u, s) \;=\; u \;-\; 2b_0\, u^2 \ln s \;+\; \int_0^{\ln s}\Big(-\, 4b_0 u \int_0^{\tau} (-2b_0u^2)\, d\tau' \;+\; B(\Sigma(u, e^{\tau}))\Big)\, d\tau.$$

The double integral of the b_0^2 term is $O(u^3)(\ln s)^2$; the B-term is bounded by

$$C_B \sup_{0\le\tau\le\ln s} \Sigma(u, e^{\tau})^3 \ln s.$$

For $u \le u_1$ small enough, Grönwall's inequality with the analytic bound on $\beta_{\rm GF}$ implies $\sup_{0\le\tau\le\ln s} \Sigma(u, e^{\tau}) \le 2u$, hence both contributions are $\le C_{\rm rem}(s)u^3$ for some finite $C_{\rm rem}(s)$ independent of $a \le a_0$. This proves (28). Differentiating the ODE w.r.t. u and repeating the same argument yields (29) with the displayed bound (the factor 3 is a harmless majorant for the quadratic remainder coming from differentiating B). ☐

Remark 4.20 (Recovery of Proposition 21.3 *(one–loop universality of* b_0*)*). Equation (28) implies, in particular, $\sigma(u, s) = \lim_{a\mu_0\to 0} \Sigma(u, s;\, a\mu_0) = u - 2b_0u^2 \ln s + O(u^3)$ with the universal $b_0 > 0$. This recovers Proposition 21.3 (Prop. 20.3).

4.2 Nonperturbative existence and regularity of the GF tuning line

We now *prove* the existence (and regularity) of a gauge–invariant gradient–flow (GF) tuning line $a \mapsto \beta(a)$ that fixes the renormalized GF coupling at a reference scale $\mu_0 = 1/\sqrt{8s_0}$:

$$g^2_{\rm GF}(\mu_0;\, a, \beta(a)) \;=\; u_0.$$

This removes the only remaining hypothesis in §4 and makes the continuum statements unconditional within our weak–coupling window.

Lemma 4.21 (Uniform weak–coupling analyticity and expansion of the flowed energy)**.** *Fix $s_0 > 0$ and $a_0 > 0$. There exists $\beta_\sharp \ge \beta_\star$ and constants $c_1(s_0) > 0$, $C_2(s_0) < \infty$ (independent of $a \le a_0$ and of the volume) such that, for all $\beta \ge \beta_\sharp$:*

(i) The map $\beta \mapsto \langle E_{s_0}\rangle_{\Lambda,\beta}$ (and its infinite–volume limit) is real–analytic on $(\beta_\sharp, \infty)$.

(ii) One has the uniform expansion

$$\Big|\ \langle E_{s_0}\rangle_\beta\ -\ \frac{c_1(s_0)}{\beta}\ \Big|\ \le\ \frac{C_2(s_0)}{\beta^2},\qquad \Big|\ \partial_\beta\langle E_{s_0}\rangle_\beta\ +\ \frac{c_1(s_0)}{\beta^2}\ \Big|\ \le\ \frac{2C_2(s_0)}{\beta^3}. \tag{30}$$

Proof. We work at fixed positive flow $s_0 > 0$. By the KP/Dobrushin smallness in our window (Lemmas 4.6–4.7) the high–β cluster (BKAR/KP) expansion is absolutely convergent and uniform in $a \le a_0$ for all β beyond some $\beta_\sharp \ge \beta_\star$. As a consequence, $\beta \mapsto \langle E_{s_0}\rangle_\beta$ is represented by a locally absolutely convergent power series in $1/\beta$, hence (i).

For (ii), expand the Wilson weight near the identity (convex core) and write the interacting measure as a perturbation of a strictly log–concave Gaussian–type reference measure obtained from the quadratic approximation of the plaquette potential (Lemma 7.1). Flow positivity and locality ensure that E_{s_0} is a bounded cylinder quadratic form in the small–field coordinates, hence its Gaussian expectation is of order $1/\beta$ with a strictly positive coefficient

$$c_1(s_0)\ =\ \tfrac{1}{4}\,\mathrm{Tr}\Big(\mathsf{K}_{s_0}\,\mathsf{C}\,\mathsf{K}_{s_0}^*\Big)\ >\ 0,$$

where C is the covariance of the quadratic core, and K_{s_0} the (gauge–invariant) linear map implementing the flow and local field tensor at time s_0. The interacting corrections are given by absolutely convergent connected cluster integrals whose absolute value is $O(\beta^{-2})$ uniformly in $a \le a_0$ due to the KP activity bound $\delta_L(\beta) = O(1/(\beta L) + e^{-B\beta})$ and the finite support of E_{s_0} (in lattice units $\sim \sqrt{s_0}/a$). This gives the first estimate in (30). Differentiation in β acts by insertion of the centered energy density $\sum_p V(U_p)$; the same BKAR/KP bounds (termwise differentiation in an absolutely convergent series) yield the second estimate. All constants are uniform in $a \le a_0$ by the a–uniform Dobrushin/KP bounds and the fixed flow range s_0. □

Proposition 4.22 (Strict monotonicity at large β)**.** *With s_0 and $\beta_\sharp$ as in Lemma 4.21, there exists $\beta_{\rm mon} \ge \beta_\sharp$ such that, for all $\beta \ge \beta_{\rm mon}$ and all $a \le a_0$,*

$$\partial_\beta\ \langle E_{s_0}\rangle_\beta\ \le\ -\frac{c_1(s_0)}{2\,\beta^2}\ <\ 0.$$

Proof. By the second estimate in (30),

$$\partial_\beta\langle E_{s_0}\rangle_\beta\ =\ -\frac{c_1(s_0)}{\beta^2}\ +\ R(\beta),\qquad |R(\beta)|\ \le\ \frac{2C_2(s_0)}{\beta^3}.$$

Choose $\beta_{\rm mon} \ge \beta_\sharp$ so large that $\frac{2C_2(s_0)}{\beta_{\rm mon}} \le \frac{1}{2}c_1(s_0)$. Then for all $\beta \ge \beta_{\rm mon}$, $\partial_\beta\langle E_{s_0}\rangle_\beta \le -\frac{c_1(s_0)}{2\beta^2} < 0$, uniformly in $a \le a_0$. □

Theorem 4.23 (Existence, uniqueness, and regularity of the GF tuning line)**.** *Fix $s_0 > 0$ and pick any target $u_0 \in (0, u_{\rm max})$ with*

$$u_{\rm max}\ :=\ \frac{\kappa\,s_0^2}{2}\,\frac{c_1(s_0)}{\beta_{\rm mon}}\,.$$

Then there exists a unique function $\beta(\cdot)$ defined on $(0, a_0]$ with values in $[\beta_{\rm mon}, \infty)$ such that

$$g^2_{\rm GF}(\mu_0;\,a,\beta(a))\ =\ \kappa\,s_0^2\,\langle E_{s_0}\rangle_{\beta(a)}\ =\ u_0\qquad \text{for all } a \in (0, a_0]. \tag{31}$$

Moreover, $\beta(a)$ is continuous on $(0, a_0]$ and locally Lipschitz; in particular it is bounded below by $\beta_{\rm mon}$ and satisfies the weak–coupling window assumed in §4.

Proof. Fix $a \in (0, a_0]$. By Lemma 4.21, $\beta \mapsto \langle E_{s_0} \rangle_\beta$ is continuous on $[\beta_{\text{mon}}, \infty)$, tends to 0 as $\beta \to \infty$, and is strictly decreasing there by Proposition 4.22. At $\beta = \beta_{\text{mon}}$ we have

$$\kappa s_0^2 \langle E_{s_0} \rangle_{\beta_{\text{mon}}} \;\geq\; \kappa s_0^2 \Big(\frac{c_1(s_0)}{\beta_{\text{mon}}} - \frac{C_2(s_0)}{\beta_{\text{mon}}^2} \Big) \;\geq\; \frac{\kappa s_0^2}{2} \frac{c_1(s_0)}{\beta_{\text{mon}}} \;=\; u_{\max},$$

after increasing β_{mon} if needed to ensure $C_2(s_0)/(\beta_{\text{mon}} c_1(s_0)) \leq \frac{1}{2}$. Hence the range of $g_{\text{GF}}^2(\mu_0; a, \beta)$ on $[\beta_{\text{mon}}, \infty)$ contains the whole interval $(0, u_{\max})$. By the intermediate value theorem and strict monotonicity, there is a unique $\beta(a) \in [\beta_{\text{mon}}, \infty)$ solving (31).

To see that $a \mapsto \beta(a)$ is continuous (indeed locally Lipschitz), note that E_{s_0} is a finite–range flowed local and its expectation is jointly continuous in (a, β) under our uniform Dobrushin/KP bounds (uniform L^p controls and dominated convergence; see Proposition 13.2). Furthermore, on $[\beta_{\text{mon}}, \infty)$, $\partial_\beta g_{\text{GF}}^2(\mu_0; a, \beta) = \kappa s_0^2 \, \partial_\beta \langle E_{s_0} \rangle_\beta$ is uniformly bounded away from 0 by Proposition 4.22 (indeed $\leq -\kappa s_0^2 c_1(s_0)/(2\beta_{\text{mon}}^2)$). The implicit function theorem (or quantitative monotone–inverse bound) then yields local Lipschitz continuity of $\beta(a)$. □

Corollary 4.24 (Removal of the tuning hypothesis)**.** *All results in §4 that were stated "along a tuning line" now hold with the tuning line $a \mapsto \beta(a)$ supplied by Theorem 4.23, with $\beta(a) \geq \beta_{\text{mon}} \geq \beta_\star$ for all $a \leq a_0$. In particular, Lemmas 4.6–4.7 and Proposition 4.8 apply uniformly along this nonperturbative tuning line.*

Proof of Corollary 4.24. By Theorem 4.23 there exists a unique tuning line $a \mapsto \beta(a) \in [\beta_{\text{mon}}, \infty)$ with $g_{\text{GF}}^2(\mu_0; a, \beta(a)) = u_0$ for all $a \in (0, a_0]$. In particular $\beta(a) \geq \beta_{\text{mon}} \geq \beta_\star$, so (T1) holds along this line. The choices of L and a_0 already ensure (T2), and (T3) concerns fixed KP parameters, independent of a. Therefore Lemmas 4.6–4.7 apply uniformly along $a \mapsto \beta(a)$, and Proposition 4.8 follows uniformly as well. All statements in §4 that were conditional on the existence of a tuning line therefore hold *along* the line produced by Theorem 4.23. □

Lemma 4.25 (Verification of (T1)–(T3) along the GF tuning line)**.** *Fix $s_0 > 0$ and let $\beta(\cdot)$ be the unique GF tuning line from Theorem 4.23 at target $u_0 \in (0, u_{\max})$. Then, after fixing $L \in \mathbb{Z}_{\geq 1}$ and $a_0 > 0$ with* (T2)*, there exists a choice of $u_0 \in (0, u_{\max})$ (depending only on s_0, L, a_0 and the KP parameters) for which* (T1)–(T3) *all hold. In particular, the constants $\beta_\star$, L, a_0 and ε_0 may be fixed once and for all, and every statement below that cites* (T1)–(T3) *can be read as invoking this lemma rather than an external hypothesis.*

Proof. (T1). By Theorem 4.23 there is a unique $\beta(a) \in [\beta_{\text{mon}}, \infty)$ solving $g_{\text{GF}}^2(\mu_0; a, \beta(a)) = u_0$ for every $a \in (0, a_0]$. Hence (T1) holds with $\beta_\star := \beta_{\text{mon}}$ (independent of a).

(T2). This is a choice, not an assumption: pick any L and a_0 satisfying $L^{-1} + e^{-L} + a_0^2 \leq \varepsilon_0 < 1/4$. For concreteness, $L = 18$ and $a_0 = 0.05$ give $L^{-1} + e^{-L} + a_0^2 \approx 0.0580556 < 1/4$.

(T3). The map $\beta \mapsto \delta_L(\beta) = \alpha_1/(\beta L) + \alpha_2 e^{-B\beta}$ is strictly decreasing. Let $\beta_{\text{KP}} = \beta_{\text{KP}}(L)$ be any value with $\delta_L(\beta_{\text{KP}}) \leq 1/80$ (existence follows by monotonicity). Set

$$\beta_\star \;:=\; \max\{\beta_{\text{mon}},\, \beta_{\text{KP}}\}.$$

By Lemma 4.21 there exist $c_1(s_0) > 0$ and $C_2(s_0) < \infty$ such that

$$\Big|\, \langle E_{s_0} \rangle_\beta - \frac{c_1(s_0)}{\beta} \,\Big| \;\leq\; \frac{C_2(s_0)}{\beta^2} \qquad \text{for all } \beta \geq \beta_\sharp,$$

uniformly in $a \leq a_0$ and the volume. Choose $\beta_{\text{req}} \geq \max\{\beta_\star, \beta_\sharp\}$ and define

$$u_{\text{crit}} \;:=\; \kappa\, s_0^2 \left(\frac{c_1(s_0)}{\beta_{\text{req}}} - \frac{C_2(s_0)}{\beta_{\text{req}}^2} \right) \;>\; 0.$$

If we now fix the target coupling to satisfy $0 < u_0 \le u_{\rm crit}$, then the monotonicity $\partial_\beta \langle E_{s_0}\rangle_\beta < 0$ (Proposition 4.22) implies

$$g^2_{\rm GF}(\mu_0; a, \beta_{\rm req}) \;\ge\; u_0 \qquad \Longrightarrow \qquad \beta(a) \;\ge\; \beta_{\rm req} \;\ge\; \beta_\star$$

for all $a \le a_0$. Consequently $\delta_L(\beta(a)) \le \delta_L(\beta_\star) \le 1/80$ uniformly in a, i.e. (T3) holds. This completes the verification. □

Remark 4.26 (Explicit admissible window). With the constants entering the cut–KP bound from Appendix A (plaquette ∗–adjacency, degree $\Delta = 26$), one admissible choice is

$$(\beta_\star, L, a_0) = (20, 18, 0.05), \qquad \delta_L(\beta_\star) \le \tfrac{1}{80}, \qquad \varepsilon_0 = \tfrac{1}{L} + e^{-L} + a_0^2 \approx 0.0580556.$$

In the a–uniform polymer budget used later we also employ

$$\delta_*(a) \;:=\; \frac{1}{\beta_\star L} + e^{-B\beta_\star} + a_0^2,$$

and, with $\beta_\star = 20$ and any $B \ge 2$ (as in Appendix A), we have $e^{-B\beta_\star} \le e^{-40}$, so numerically $\delta_*(a) \lesssim \frac{1}{\beta_\star L} + e^{-40} + a_0^2 \approx 0.0052778$. These numerics are recorded for orientation; the proof of Lemma 4.25 does not rely on any particular values.

5 RP under GI conditioning (anti–linear J)

Let $(\Omega, \mathfrak{A}, \mu)$ be a probability space, $\Theta : \Omega \to \Omega$ an involutive reflection with $\mu \circ \Theta^{-1} = \mu$, and let $\mathfrak{A}_\pm, \mathfrak{A}_0 \subset \mathfrak{A}$ be the σ–algebras of observables localized in $\{x_0 \gtrless 0\}$ and on the reflection hyperplane, respectively, with $\Theta(\mathfrak{A}_+) = \mathfrak{A}_-$, $\Theta(\mathfrak{A}_0) = \mathfrak{A}_0$. We assume *reflection positivity (RP)* in the standard Osterwalder–Schrader form:

$$\langle JF, F\rangle_{L^2(\mu)} \;=\; \int \overline{F \circ \Theta}\, F \, \mathrm{d}\mu \;\ge 0 \qquad \text{for all } F \in L^2(\mu) \text{ with } F\ \mathfrak{A}_+\text{–measurable}, \tag{32}$$

where $J : L^2(\mu) \to L^2(\mu)$ is the anti–linear isometry

$$(Jf)(\omega) \;:=\; \overline{f(\Theta\omega)} \qquad (J^2 = \mathrm{id},\ \langle Jf, Jg\rangle = \langle g, f\rangle). \tag{33}$$

Gauge–invariant boundary algebra. Let $\mathfrak{A}_{\rm GI} \subset \mathfrak{A}_0$ be a reflection–invariant σ–subalgebra encoding the *gauge–invariant* (GI) boundary data at time 0, i.e. $\Theta(\mathfrak{A}_{\rm GI}) = \mathfrak{A}_{\rm GI}$. Denote by

$$P \;:=\; \mathbb{E}[\,\cdot\,|\,\mathfrak{A}_{\rm GI}] \,:\, L^2(\mu) \longrightarrow L^2(\mu) \tag{34}$$

the orthogonal projection (conditional expectation) onto $L^2(\mathfrak{A}_{\rm GI}, \mu)$.

Lemma 5.1 (Compatibility: J preserves $L^2(\mathfrak{A}_{\rm GI})$ and commutes with P)**.** *If*

$$\Theta(\mathfrak{A}_{\rm GI}) = \mathfrak{A}_{\rm GI} \quad \textit{and} \quad \mu \textit{ is } \Theta\textit{-invariant}$$
$$\Longrightarrow\; J\big(L^2(\mathfrak{A}_{\rm GI})\big) \;\subset\; L^2(\mathfrak{A}_{\rm GI})\,.$$

and

$$JP \;=\; PJ \qquad \textit{on } L^2(\mu). \tag{35}$$

Proof. If g is $\mathfrak{A}_{\rm GI}$–measurable then $g \circ \Theta$ is also $\mathfrak{A}_{\rm GI}$–measurable, hence $Jg = \overline{g \circ \Theta} \in L^2(\mathfrak{A}_{\rm GI})$. Thus J preserves $L^2(\mathfrak{A}_{\rm GI})$. The orthogonal projection P is characterized by $\langle Pf, h\rangle = \langle f, h\rangle$ for all $h \in L^2(\mathfrak{A}_{\rm GI})$. Using that J is anti–unitary with $J^2 = \mathrm{id}$ and that $J(L^2(\mathfrak{A}_{\rm GI})) = L^2(\mathfrak{A}_{\rm GI})$, for any $f \in L^2(\mu)$ and any $h \in L^2(\mathfrak{A}_{\rm GI})$,

$$\langle JPf, h\rangle = \langle Pf, Jh\rangle = \langle f, Jh\rangle = \langle PJf, h\rangle.$$

Since h ranges over a dense set in the range of P, we conclude $JPf = PJf$. □

Lemma 5.2 (RP preserved by GI conditioning). *If* (32) *holds,* μ *is* Θ*–invariant and* $\Theta(\mathfrak{A}_{\mathrm{GI}}) = \mathfrak{A}_{\mathrm{GI}}$*, then for every* $\mathfrak{A}_+$*–measurable* F*,*

$$\big\langle J\,\mathbb{E}[F\,|\,\mathfrak{A}_{\mathrm{GI}}],\ \mathbb{E}[F\,|\,\mathfrak{A}_{\mathrm{GI}}]\big\rangle \ \geq\ 0. \tag{36}$$

Proof. *By Lemma 5.1,* $JP = PJ$*. Therefore* $\langle JPF, PF\rangle = \langle PJF, F\rangle \geq 0$ *by* (32)*.*

The previous lemma has the following standard matrix (Gram)–positivity consequence.

Proposition 5.3 (Matrix RP after GI conditioning). *Let* $F_1, \dots, F_n$ *be* $\mathfrak{A}_+$*–measurable. Then the* $n \times n$ *matrix*

$$M_{ij} \ :=\ \langle JPF_i,\ PF_j\rangle$$

is Hermitian positive semidefinite. Equivalently,

$$\sum_{i,j=1}^{n} \overline{c_i}\, c_j\, \langle JPF_i,\ PF_j\rangle \ \geq\ 0 \qquad \text{for all } (c_1, \dots, c_n) \in \mathbb{C}^n.$$

Proof. Apply Lemma 5.2 to $F = \sum_j c_j F_j$ and use polarization. □

Corollary 5.4 (GI RP seminorm and OS pre–Hilbert space). *Define, for* $\mathfrak{A}_+$*–measurable* F, G*,*

$$\langle F, G\rangle_{\mathrm{GI}} \ :=\ \langle JPF,\ PG\rangle, \qquad \|F\|_{\mathrm{GI}}^2 := \langle F, F\rangle_{\mathrm{GI}}.$$

Then $\langle\cdot,\cdot\rangle_{\mathrm{GI}}$ *is a positive semidefinite Hermitian form on* $\{F :\ F\ \mathfrak{A}_+\text{–measurable}\}$*. Modding out the null space* $\mathcal{N}_{\mathrm{GI}} = \{F : \|F\|_{\mathrm{GI}} = 0\}$ *and completing yields a Hilbert space* $\mathcal{H}_+^{(\mathrm{GI})}$*, canonically isometric to the RP time–zero Hilbert space built from the GI boundary algebra. Moreover,*

$$\big|\langle F, G\rangle_{\mathrm{GI}}\big| \ \leq\ \|PF\|_2\, \|PG\|_2 \ \leq\ \|F\|_2\, \|G\|_2. \tag{37}$$

Proof. Positivity follows from Proposition 5.3. The Cauchy–Schwarz bound (37) is the L^2 Cauchy–Schwarz inequality together with $\|Jh\|_2 = \|h\|_2$ and the contractivity $\|P\|_{2\to 2} = 1$. □

Remark 5.5 (Monotonicity under enlarging the boundary σ–algebra). If $\mathfrak{A}_{\mathrm{GI}} \subset \mathfrak{B} \subset \mathfrak{A}_0$ are reflection–invariant σ–algebras with projections $P_{\mathrm{GI}}, P_{\mathfrak{B}}$, then

$$\|F\|_{\mathrm{GI}}^2 = \langle JP_{\mathrm{GI}}F, P_{\mathrm{GI}}F\rangle \ \leq\ \langle JP_{\mathfrak{B}}F, P_{\mathfrak{B}}F\rangle$$

for all $\mathfrak{A}_+$–measurable F. Thus refining the boundary information can only *increase* the RP seminorm.

GI sufficiency and descent of Markov factorization. We write $\mathfrak{A}_\pm^{\mathrm{GI}} \subset \mathfrak{A}_\pm$ for the σ–algebras of *gauge–invariant* observables localized in the halves $\{x_0 \gtrless 0\}$ (so $\Theta(\mathfrak{A}_+^{\mathrm{GI}}) = \mathfrak{A}_-^{\mathrm{GI}}$).

Assumption 5.6 (GI sufficiency of the time–zero boundary). For every $F \in L^\infty(\mathfrak{A}_\pm^{\mathrm{GI}})$ one has

$$\mathbb{E}[F \mid \mathfrak{A}_0] \ \text{ is } \mathfrak{A}_{\mathrm{GI}}\text{–measurable}, \qquad \text{equivalently } \ \mathbb{E}[F \mid \mathfrak{A}_0] = \mathbb{E}[F \mid \mathfrak{A}_{\mathrm{GI}}]. \tag{38}$$

Remark 5.7 (Verification for Wilson–type specifications). In standard lattice Yang–Mills (Wilson) measures with compact structure group, the time–zero gauge group $\mathcal{G}_0$ acts measurably on $\mathfrak{A}_0$, the measure μ and the half–space specifications are $\mathcal{G}_0$–equivariant, and $\mathfrak{A}_{\mathrm{GI}}$ is the fixed–point σ–algebra $\mathfrak{A}_0^{\mathcal{G}_0}$. Hence, if $F \in \mathfrak{A}_+^{\mathrm{GI}}$ then $F \circ g_+ = F$ for all $g \in \mathcal{G}_0$, which forces $\mathbb{E}[F \mid \mathfrak{A}_0]$ to be $\mathcal{G}_0$–invariant and therefore $\mathfrak{A}_{\mathrm{GI}}$–measurable. The same argument applies with $+$ replaced by $-$. This is the usual "sufficiency of the GI boundary variables."

Lemma 5.8 (Descent of conditional independence along a sufficient boundary)**.** *Assume* $\mathfrak{A}_+$ *and* $\mathfrak{A}_-$ *are conditionally independent given* $\mathfrak{A}_0$*. Let* $\mathfrak{B} \subset \mathfrak{A}_0$ *be a sub–*σ*–algebra such that for every bounded* $\mathfrak{A}_\pm$*–measurable* H *one has* $\mathbb{E}[H \mid \mathfrak{A}_0]$ $\mathfrak{B}$*–measurable. Then* $\mathfrak{A}_+$ *and* $\mathfrak{A}_-$ *are conditionally independent given* $\mathfrak{B}$ *and, for all* $F \in L^\infty(\mathfrak{A}_+)$ *and* $G \in L^\infty(\mathfrak{A}_-)$*,*

$$\mathbb{E}[FG \mid \mathfrak{B}] \;=\; \mathbb{E}[F \mid \mathfrak{B}]\, \mathbb{E}[G \mid \mathfrak{B}]. \tag{39}$$

Proof. By the tower property and the Markov property across $\mathfrak{A}_0$,

$$\mathbb{E}[FG \mid \mathfrak{B}] \;=\; \mathbb{E}[\mathbb{E}[F \mid \mathfrak{A}_0]\, \mathbb{E}[G \mid \mathfrak{A}_0] \mid \mathfrak{B}]\,.$$

If both inner factors are $\mathfrak{B}$–measurable, the right–hand side equals $\mathbb{E}[F \mid \mathfrak{B}]\, \mathbb{E}[G \mid \mathfrak{B}]$, proving (39). □

Proposition 5.9 (GI Markov property on the GI sector)**.** *Assume the Markov property across the reflection hyperplane (*$\mathfrak{A}_+ \perp\!\!\!\perp \mathfrak{A}_- \mid \mathfrak{A}_0$*) and Assumption 5.6. Then* $\mathfrak{A}_+^{\mathrm{GI}}$ *and* $\mathfrak{A}_-^{\mathrm{GI}}$ *are conditionally independent given* $\mathfrak{A}_{\mathrm{GI}}$ *and, for all* $F \in L^2(\mathfrak{A}_+^{\mathrm{GI}})$*,* $G \in L^2(\mathfrak{A}_-^{\mathrm{GI}})$*,*

$$\begin{aligned} \mathbb{E}[FG \mid \mathfrak{A}_{\mathrm{GI}}] &= \mathbb{E}[F \mid \mathfrak{A}_{\mathrm{GI}}]\, \mathbb{E}[G \mid \mathfrak{A}_{\mathrm{GI}}], \\ \langle JG,\, F\rangle &= \big\langle J\, \mathbb{E}[F \mid \mathfrak{A}_{\mathrm{GI}}],\ \mathbb{E}[G \mid \mathfrak{A}_{\mathrm{GI}}]\big\rangle = \langle JPF,\, PG\rangle. \end{aligned} \tag{40}$$

Proof. Apply Lemma 5.8 with $\mathfrak{B} = \mathfrak{A}_{\mathrm{GI}}$ and use (38). The identity for the OS pairing follows from $\langle JG, F\rangle = \mathbb{E}(\overline{G \circ \Theta}\, F)$ and the first equality in (40). The last equality is the definition of $P = \mathbb{E}[\cdot \mid \mathfrak{A}_{\mathrm{GI}}]$. □

Lemma 5.10 (Factorization and conditional independence)**.** *Assume, in addition, the (standard) Markov property across the reflection hyperplane:* $\mathfrak{A}_+$ *and* $\mathfrak{A}_-$ *are conditionally independent given* $\mathfrak{A}_0$*. Then for* F $\mathfrak{A}_+$*–measurable and* G $\mathfrak{A}_-$*–measurable,*

$$\langle JG, F\rangle \;=\; \big\langle J\, \mathbb{E}[F \mid \mathfrak{A}_0],\ \mathbb{E}[G \mid \mathfrak{A}_0]\big\rangle. \tag{41}$$

Amendment to Lemma 5.10. *The statement and proof up to* (41) *are unchanged.* For the GI pairing, use Proposition 5.9: if $F \in L^2(\mathfrak{A}_+^{\mathrm{GI}})$ and $G \in L^2(\mathfrak{A}_-^{\mathrm{GI}})$, then $\langle JG, F\rangle = \langle JPF,\ PG\rangle$.

Proof. By conditional independence, $\mathbb{E}\big[\overline{G \circ \Theta}\, F\big] = \mathbb{E}\big[\mathbb{E}(\overline{G \circ \Theta} \mid \mathfrak{A}_0)\, \mathbb{E}(F \mid \mathfrak{A}_0)\big]$. Since Θ fixes $\mathfrak{A}_0$, $\mathbb{E}(\overline{G \circ \Theta} \mid \mathfrak{A}_0) = \overline{\mathbb{E}(G \mid \mathfrak{A}_0) \circ \Theta}$, which yields (41). □

Remark 5.11 (Bridge to the cross–cut transfer operator). To avoid duplication with Section 11, we refrain here from introducing the pair law on the GI boundary and the associated correlation/transfer operators. Section 11 realizes the bounded positive form $(f, g) \mapsto \langle Jf, g\rangle$ on $L^2(\mathfrak{A}_{\mathrm{GI}}, \mu)$ as a symmetric integral operator induced by the GI pair law across the cut and proves the full OS–intertwiner identity there. The results of the present section provide the input (RP under GI conditioning and the Markov factorization) used in that construction.

6 Dobrushin/Holley–Stroock and the slab constants

We index the GI cut blocks by a finite set $\mathcal{I}$ (face/edge/vertex adjacency on $L\mathbb{Z}^3$). For $x \in \mathcal{I}$ let $\mathfrak{F}_{x^c}$ be the σ–algebra generated by all blocks $\neq x$ and write

$$\mathbb{E}_x[f] \;:=\; \mathbb{E}[f \,|\, \mathfrak{F}_{x^c}], \qquad \mathrm{Var}_x(f) \;:=\; \mathbb{E}\big[\,(f - \mathbb{E}_x f)^2 \,\big|\, \mathfrak{F}_{x^c}\big].$$

We also use the block GI-adjoint Lipschitz seminorm (right-invariant gradient restricted to block x):

$$L^{\mathrm{GI}}_{\mathrm{ad},x}(f) \;:=\; \sup_U \left(\sum_{e \subset \text{block } x} \ \sup_{\|X_e\|=1} \big|(D_e f)(U)[X_e]\big|^2 \right)^{1/2},$$

so that $L^{\mathrm{GI}}_{\mathrm{ad}}(f)^2 = \sum_{x \in \mathcal{I}} L^{\mathrm{GI}}_{\mathrm{ad},x}(f)^2$ whenever f is supported on $\bigcup_x$.

A. Holley–Stroock (HS) perturbation and local Poincaré constant

Lemma 6.1 (Holley–Stroock Perturbation). *Let μ_0 and μ be probability measures on a smooth manifold with $d\mu = Z^{-1}e^h\,d\mu_0$. If $\operatorname{osc}(h) := \sup h - \inf h \le \delta$ and μ_0 satisfies a Poincaré inequality*

$$\operatorname{Var}_{\mu_0}(f) \;\le\; C_0 \int \|\nabla f\|^2\,d\mu_0 \qquad (\forall f \in C^1),$$

then μ satisfies

$$\operatorname{Var}_{\mu}(f) \;\le\; e^{2\delta}\,C_0 \int \|\nabla f\|^2\,d\mu \qquad (\forall f \in C^1).$$

Proof. Since $e^{-\delta} \le e^h \le e^{\delta}$, we have $e^{-\delta}\,d\mu_0 \le Z\,d\mu \le e^{\delta}\,d\mu_0$, hence $\|g\|^2_{L^2(\mu)} \le e^{\delta}Z^{-1}\|g\|^2_{L^2(\mu_0)}$ and $\int \|\nabla f\|^2 d\mu_0 \le e^{\delta}Z \int \|\nabla f\|^2 d\mu$. Apply the Poincaré inequality for μ_0 to $f - \mathbb{E}_\mu f$ and use the two-sided L^2 comparison. □

Lemma 6.2 (Block–HS: uniforme lokale Poincaré-Konstante). *Uniformly in the boundary condition on $\mathfrak{F}_{x^c}$ there exists a constant*

$$C_{\mathrm{PI,loc}} \;=\; \frac{C_{\mathrm{db}}}{\beta\,\kappa_G}\,e^{2\delta_{\mathrm{loc}}}$$

(depending only on geometry, not on the volume) such that, for every $x \in \mathcal{I}$ and C^1 functional f,

$$\operatorname{Var}_x(f) \;\le\; C_{\mathrm{PI,loc}}\,\mathbb{E}_x[\,\|\nabla_x f\|^2\,] \;\le\; C_{\mathrm{PI,loc}}\,\big(L^{\mathrm{GI}}_{\mathrm{ad},x}(f)\big)^2,$$

with $\|\nabla_x f\|^2 = \sum_{e\subset x}\sup_{\|X_e\|=1}|(D_e f)[X_e]|^2$. Here κ_G is from Lemma 7.1, $C_{\mathrm{db}} \ge 1$ collects the deterministic plaquette-to-link/GI-quotient Lipschitz factors, and $\delta_{\mathrm{loc}} = O(a^2) + O(e^{-B\beta})$ bounds the oscillation of the block tail potential (uniform in $a \le a_0$).

Proof. The conditional law on block x has density $e^{-\Phi_x}$ w.r.t. the product of Haar measures on the links in x. On the convex core Lemma 7.1 gives $\operatorname{Hess}\Phi_x \succeq \beta\kappa_G\,\mathrm{Id}$ along right-invariant directions; the deterministic projection from plaquettes to link variables and the GI quotient cost a factor C_{db}. The non-core/tail contribution has bounded oscillation δ_{loc} (weak coupling and finite block), hence Lemma 6.1 yields the bound with constant $(C_{\mathrm{db}}/(\beta\kappa_G))e^{2\delta_{\mathrm{loc}}}$. □

B. Dobrushin-Matrix und globale Poincaré-Ungleichung

Definition 6.3 (Dobrushin-Einflussmatrix). Let $C = (c_{xy})_{x,y\in\mathcal{I}}$ with

$$c_{xy} \;:=\; \sup_{\substack{f \text{ meas. w.r.t. block } y\\ L^{\mathrm{GI}}_{\mathrm{ad},y}(f)\le 1}}\;\sup_{U}\;L^{\mathrm{GI}}_{\mathrm{ad},x}(\mathbb{E}_y f)(U)\,.$$

We set $\|C\|_1 := \sup_x \sum_y c_{xy}$.

Proposition 6.4 (Dobrushin–Poincaré). *Assume $\|C\|_1 \le \varepsilon < 1$ and Lemma 6.2. Then for every $f \in L^2(\mu)$,*

$$\operatorname{Var}(f) \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon}\sum_{x\in\mathcal{I}}\mathbb{E}[\|\nabla_x f\|^2] \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon}\sum_{x\in\mathcal{I}}\mathbb{E}\big[\,\big(L^{\mathrm{GI}}_{\mathrm{ad},x}(f)\big)^2\big].$$

Consequently, the GI cut measure satisfies a Poincaré inequality with constant

$$C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{db}}}{(1-\varepsilon)\,\beta\,\kappa_G}\,e^{2\delta_{\mathrm{loc}}}\,.$$

Proof. Let $P_x := \mathbb{E}_x$ denote conditional expectation on block x (given the complement), and $\mathrm{Var}_x(f) := \mathbb{E}_x[(f - \mathbb{E}_x f)^2]$. Assume $\|C\|_1 \le \varepsilon < 1$, where C is the Dobrushin influence matrix (Definition 6.3).

Step 1: Dobrushin covariance/variance bound. Set $R := (I - C)^{-1} = \sum_{n\ge 0} C^n$. By $\|C\|_1 < 1$, R is well defined and $\|R\|_1 \le (1 - \|C\|_1)^{-1}$. The Dobrushin resolvent inequality (Lemma 9.6) applied to $g = f$ gives

$$\mathrm{Var}(f) = \mathrm{Cov}(f, f) \ \le \ \sum_{x,y\in\mathcal{I}} R_{xy} \sqrt{\mathbb{E}\mathrm{Var}_x(f)}\,\sqrt{\mathbb{E}\mathrm{Var}_y(f)} \ \le \ \|R\|_1 \sum_{x\in\mathcal{I}} \mathbb{E}\mathrm{Var}_x(f),$$

whence

$$\mathrm{Var}(f) \ \le \ \frac{1}{1 - \|C\|_1} \sum_{x\in\mathcal{I}} \mathbb{E}\mathrm{Var}_x(f). \tag{42}$$

Step 2: Local PI on blocks. By Lemma 6.2, for each block x, $\mathbb{E}\mathrm{Var}_x(f) \ \le \ C_{\mathrm{PI,loc}}\, \mathbb{E}[\|\nabla_x f\|^2]$. Summing over x and inserting into (42) yields

$$\mathrm{Var}(f) \ \le \ \frac{C_{\mathrm{PI,loc}}}{1 - \varepsilon} \sum_{x\in\mathcal{I}} \mathbb{E}[\|\nabla_x f\|^2].$$

Step 3: GI Lipschitz domination. By definition of the GI Lipschitz seminorm, $\|\nabla_x f\| \le L^{\mathrm{GI}}_{\mathrm{ad},x}(f)$ pointwise. Therefore,

$$\sum_x \mathbb{E}[\|\nabla_x f\|^2] \ \le \ \sum_x \mathbb{E}[\,(L^{\mathrm{GI}}_{\mathrm{ad},x}(f))^2],$$

which proves the second inequality in the display.

Step 4: Global PI constant. Combining the above with the quantitative GI gradient/Lipschitz comparison (uniform block coercivity $\beta\kappa_G$ and bounded local oscillation δ_{loc}, as used throughout §6) gives

$$\sum_x \mathbb{E}[\|\nabla_x f\|^2] \ \le \ \frac{C_{\mathrm{db}}}{\beta\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}} \sum_x \mathbb{E}[\,(L^{\mathrm{GI}}_{\mathrm{ad},x}(f))^2].$$

Inserting this into Step 2 yields the global Poincaré inequality with

$$C_{\mathrm{PI}} \ \le \ \frac{C_{\mathrm{db}}}{(1-\varepsilon)\,\beta\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}}\,.$$

□

Corollary 6.5 (Slab constants)**.** *If the influence/curvature estimate of Proposition 7.11 holds, then for all $a \le a_0$*

$$\|C(a)\|_1 \ \le \ \frac{\alpha_1}{\beta(a)L} + \alpha_2 e^{-B\beta(a)} + \alpha_3 a^2 \ =: \ \varepsilon(L, a)\,.$$

Under (T1)–(T2) one has $\varepsilon(L, a) \le \varepsilon_0 < \frac{1}{4}$ uniformly, hence

$$C_{\mathrm{PI}} \ \le \ \frac{C_{\mathrm{db}}}{(1-\varepsilon_0)\,\beta_\star\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}} \ \le \ \frac{4\,C_{\mathrm{db}}}{\beta_\star\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}}.$$

In particular, the GI cut measure has a Poincaré (and, by the same argument with logarithmic Sobolev constants, an LSI) with constants uniform in $a \le a_0$.

Proof. By Proposition 7.11 the Dobrushin row sum satisfies, for all $a \le a_0$,

$$\|C(a)\|_1 \;\le\; \varepsilon(L,a) \;:=\; \frac{\alpha_1}{\beta(a)L} \;+\; \alpha_2 e^{-B\beta(a)} \;+\; \alpha_3 a^2 \,.$$

Fix a window (T1)–(T2) with $\sup_{a\le a_0} \varepsilon(L,a) \;\le\; \varepsilon_0 \;<\; \frac{1}{4}$. Applying Proposition 6.4 and Lemma 6.2 gives the global Poincaré constant

$$C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon_0} \;=\; \frac{1}{1-\varepsilon_0}\,\frac{C_{\mathrm{db}}}{\beta\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}} \;\le\; \frac{4\,C_{\mathrm{db}}}{\beta_\star\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}},$$

uniformly in $a \le a_0$ and along the tuning line, where $\beta_\star = \inf \beta(a)$ in the window. The last sentence follows because the same argument applies with the block LSI input (Bakry–Émery on the core plus Holley–Stroock perturbation) in place of the block PI; see also Lemma 6.10 below. □

C. Distance mixing on the cut graph

We work on the coarse cut graph G_{2a} whose vertices are the $2a$–blocks; two vertices are adjacent if their blocks meet by face/edge/vertex ($\Delta = 26$–neighbor geometry; no–backtracking 25). For sets of blocks $X, Y \subset \mathcal{I}$ we define the coarse graph distance

$$\mathrm{dist}_{2a}(X,Y) \;:=\; \min\{\, \mathrm{dist}_{G_{2a}}(x,y) :\; x\in X,\; y \in Y\,\}.$$

Lemma 6.6 (One–step L^2 propagation)**.** *Let $C = (c_{xy})$ be the Dobrushin matrix from Definition 6.3. For all $H \in L^2(\mu)$ and all $x, z \in \mathcal{I}$,*

$$\|\Delta_x P_z H\|_{L^2} \;\le\; c_{xz}\, \|\Delta_z H\|_{L^2}.$$

Consequently, for every $n \ge 1$ and blocks x, y,

$$\big\|\Delta_x T^n \Delta_y H\big\|_{L^2} \;\le\; (C^n)_{xy}\; \|\Delta_y H\|_{L^2}.$$

Proof. Fix z and decompose $H = \mathbb{E}_z H + \Delta_z H$. Since $P_z \Delta_z H = 0$, we have $\Delta_x P_z H = \Delta_x P_z(\mathbb{E}_z H) = \Delta_x(\mathbb{E}_z H)$. By the block Poincaré inequality (Lemma 6.2),

$$\|\Delta_x(\mathbb{E}_z H)\|_{L^2}^2 \;=\; \mathbb{E}\,\mathrm{Var}_x(\mathbb{E}_z H) \;\le\; C_{\mathrm{PI,loc}}\, \sup_U \big\|\nabla_x(\mathbb{E}_z H)\big\|^2.$$

By Definition 6.3, $\sup_U \|\nabla_x(\mathbb{E}_z h)\| \;\le\; c_{xz} \sup_U \|\nabla_z h\|$ for any z–measurable h with unit GI–Lipschitz constant on block z. Apply this with $h = (\Delta_z H)/L^{\mathrm{GI}}_{\mathrm{ad},z}(\Delta_z H)$ and combine with Lemma 6.2 again (now on block z) to bound $\sup_U \|\nabla_z h\| \le C_{\mathrm{PI,loc}}^{-1/2} \|\Delta_z H\|_{L^2}$. Altogether,

$$\|\Delta_x P_z H\|_{L^2} \;\le\; c_{xz}\, \|\Delta_z H\|_{L^2}.$$

For the iterated bound,

$$\Delta_x T H \;=\; \frac{1}{|\mathcal{I}|} \sum_{z\in\mathcal{I}} \Delta_x P_z H,$$

hence $\|\Delta_x T H\|_{L^2} \le \sum_z c_{xz}\, \|\Delta_z H\|_{L^2}$. Iterating gives $\|\Delta_x T^n \Delta_y H\|_{L^2} \le (C^n)_{xy}\, \|\Delta_y H\|_{L^2}$. □

Lemma 6.7 (Dobrushin distance mixing)**.** *Assume $\|C\|_1 \le \varepsilon < 1$, with C the Dobrushin influence matrix of Definition 6.3. Let F and G be mean–zero functionals measurable w.r.t. the blocks in finite sets $X, Y \subset \mathcal{I}$. Then*

$$|\mathrm{Cov}(F,G)| \;\le\; \frac{\varepsilon^{\,\mathrm{dist}_{2a}(X,Y)}}{1-\varepsilon} \sum_{x\in X}\sum_{y\in Y} \Big(\mathbb{E}\,\mathrm{Var}_x(F)\Big)^{1/2} \Big(\mathbb{E}\,\mathrm{Var}_y(G)\Big)^{1/2}. \tag{43}$$

By the norm-convergent resolvent identity on $L^2_0(\mu)$,

$$I \;=\; \sum_{n\ge 0}(T^n - T^{n+1}) \;=\; \frac{1}{|\mathcal{I}|}\sum_{n\ge 0}\sum_{y\in\mathcal{I}} T^n\,\Delta_y, \qquad \Delta_y := I - P_y.$$

Hence, for mean–zero $F, G \in L^2_0(\mu)$,

$$\mathrm{Cov}(F,G) \;=\; \langle F,\, G\rangle \;=\; \frac{1}{|\mathcal{I}|}\sum_{n\ge 0}\sum_{y\in\mathcal{I}}\langle F,\; T^n\Delta_y G\rangle \;=\; \frac{1}{|\mathcal{I}|^2}\sum_{n\ge 0}\sum_{x,y\in\mathcal{I}}\langle \Delta_x F,\; T^n\Delta_y G\rangle. \quad (44)$$

Proof of Lemma 6.7. Let F, G be mean–zero and supported in finite $X, Y \subset \mathcal{I}$. Set $T := |\mathcal{I}|^{-1}\sum_z P_z$ and note the norm–convergent resolvent identity on L^2_0:

$$I \;=\; \sum_{n\ge 0}(T^n - T^{n+1}) \;=\; \frac{1}{|\mathcal{I}|}\sum_{n\ge 0}\sum_{y\in\mathcal{I}} T^n\Delta_y.$$

Therefore

$$\mathrm{Cov}(F,G) \;=\; \langle F, G\rangle \;=\; \frac{1}{|\mathcal{I}|}\sum_{n\ge 0}\sum_{y\in\mathcal{I}}\;\langle F,\; T^n\Delta_y G\rangle.$$

By Cauchy–Schwarz and the Efron–Stein inequality $\|H\|^2_{L^2} \le \sum_x \|\Delta_x H\|^2_{L^2}$,

$$|\langle F,\; T^n\Delta_y G\rangle| \;\le\; \Big(\sum_{x\in\mathcal{I}}\|\Delta_x F\|^2_{L^2}\Big)^{1/2}\Big(\sum_{x\in\mathcal{I}}\|\Delta_x T^n\Delta_y G\|^2_{L^2}\Big)^{1/2}.$$

Apply Lemma 6.6 and then Cauchy–Schwarz in the x–sum:

$$\sum_{x\in\mathcal{I}}\|\Delta_x T^n \Delta_y G\|_{L^2} \;\le\; \sum_{x\in\mathcal{I}}(C^n)_{xy}\;\|\Delta_y G\|_{L^2}.$$

Since $c_{xy} = 0$ unless x and y are 26–neighbors, $(C^n)_{xy} = 0$ whenever $n < \mathrm{dist}_{2a}(x,y)$, and $\sum_{n\ge 0}(C^n)_{xy} \le \varepsilon^{\,\mathrm{dist}_{2a}(x,y)}/(1-\varepsilon)$ with $\varepsilon = \|C\|_1$. Restricting to $x \in X$, $y \in Y$ (otherwise $\Delta_x F$ or $\Delta_y G$ vanishes) gives

$$|\mathrm{Cov}(F,G)| \;\le\; \frac{1}{1-\varepsilon}\;\sum_{x\in X}\sum_{y\in Y}\;\varepsilon^{\,\mathrm{dist}_{2a}(x,y)}\;\|\Delta_x F\|_{L^2}\;\|\Delta_y G\|_{L^2},$$

which is (43). □

Lemma 6.8 (Fluctuation covariance bound (used in L2))**.** *Let A be a GI local and P_{2a} the coarse conditional expectation onto the $2a$–block σ–algebra. Set $h := (I - P_{2a})A$. Then h is supported on a single coarse block (up to a fixed finite collar), thus $|X|, |Y| \le C_{\mathrm{geom}}$ when $F = h(x)$ and $G = h(y)$ are placed at two distinct blocks x, y. Under Lemma 6.2 and $\|C\|_1 \le \varepsilon < 1$,*

$$\big|\mathrm{Cov}(h(x), h(y))\big| \;\le\; \frac{C_{\mathrm{geom}}\;C_{\mathrm{PI,loc}}}{1-\varepsilon}\;\varepsilon^{\,\mathrm{dist}_{2a}(\{x\},\{y\})}\;\big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2,$$

uniformly in the boundary condition and in $a \le a_0$.

Proof. Apply Lemma 6.7 with $X = \mathrm{supp}(h(x))$, $Y = \mathrm{supp}(h(y))$ and $\mathbb{E}\,\mathrm{Var}_x(h) \le C_{\mathrm{PI,loc}}\,\mathbb{E}\|\nabla_x h\|^2 \le C_{\mathrm{PI,loc}}\,\big(L^{\mathrm{GI}}_{\mathrm{ad},x}(A)\big)^2$. Sum over the $O(1)$ many x in the support of h to get the displayed bound. □

Lemma 6.9 (Geometric summability for the fluctuation tail)**.** *Let $r = |x - y|$ be the Euclidean separation on the fine grid, so that $d := \mathrm{dist}_{2a}(\{x\}, \{y\}) \geq \lfloor r/(2a) \rfloor - 1$. If $\varepsilon \leq \varepsilon_0 < \frac{1}{4}$ and m_E is such that $e^{2am_E} \leq \theta_*^{-1/4}$* (here θ_* is the KP–amplified two–step supremum on the cut with $\Delta = 26$, and $\|T\| \leq \theta_*^{1/4}$)*, then*

$$\sup_{r \geq 2a} \; e^{m_E r} \, \varepsilon^{\lfloor r/(2a) \rfloor - 1} \; \leq \; \frac{e^{2am_E}}{1 - \varepsilon_0 \, e^{2am_E}} \; < \; \infty.$$

In particular this supremum is bounded uniformly in $a \leq a_0$ by our window where $\varepsilon_0 \, \theta_*^{-1/4} < 1$.

Proof. Write $r \in [2a(d+1), 2a(d+2))$. Then $e^{m_E r} \varepsilon^d \leq e^{2am_E} \big(\varepsilon \, e^{2am_E}\big)^d$ and sum the geometric series in d. □

D. Global PI/LSI constants and L^p bounds

We work with the block conditional structure of the cut specification. For a block index $x \in \mathcal{I}$, let $\mathbb{E}_x[\cdot]$ denote conditional expectation given all blocks except x, and

$$\mathrm{Var}_x(F) \; := \; \mathbb{E}_x\big[(F - \mathbb{E}_x F)^2\big], \qquad \mathrm{Ent}_x(F^2) \; := \; \mathbb{E}_x\big[F^2 \log F^2\big] - \mathbb{E}_x[F^2] \log \mathbb{E}_x[F^2].$$

Let ∇_x denote the right–invariant differential along the links of block x, and set the local carré–du–champ $\Gamma_x(F) := \|\nabla_x F\|_2^2$ (sum of squared right–invariant derivatives over the links in block x).

Lemma 6.10 (Block Poincaré and LSI)**.** *There exist block–level constants $C_{\mathrm{PI,loc}}, C_{\mathrm{LSI,loc}} < \infty$ (independent of $a \leq a_0$ along the tuning line) such that for all smooth cylinder F,*

$$\mathrm{Var}_x(F) \; \leq \; C_{\mathrm{PI,loc}} \, \mathbb{E}_x \Gamma_x(F), \qquad \mathrm{Ent}_x(F^2) \; \leq \; 2\, C_{\mathrm{LSI,loc}} \, \mathbb{E}_x \Gamma_x(F).$$

Moreover, in the weak–coupling slab regime,

$$C_{\mathrm{PI,loc}} + C_{\mathrm{LSI,loc}} \; \leq \; C_{\mathrm{core}} \Big(\frac{1}{\beta \kappa_G} + e^{-B\beta} + a^2 \Big),$$

with C_{core} geometric and κ_G, B as in Lemmas 7.1–7.2.

Proof. Fix $x \in \mathcal{I}$ and condition on $\mathfrak{F}_{x^c}$. The conditional density on the links in block x is $d\mu_x = Z_x^{-1} \exp(-\Phi_x) \, d\lambda_x$, with $d\lambda_x$ the product of Haar measures. On the convex core (Lemma 7.1) the right–invariant Hessian satisfies $\mathrm{Hess}\, \Phi_x \succeq \beta \kappa_G \, \mathrm{Id}$, hence the Bakry–Émery Γ_2 criterion yields, for all smooth F,

$$\mathrm{Var}_x(F) \; \leq \; (\beta \kappa_G)^{-1} \, \mathbb{E}_x \Gamma_x(F), \qquad \mathrm{Ent}_x(F^2) \; \leq \; 2 (\beta \kappa_G)^{-1} \, \mathbb{E}_x \Gamma_x(F).$$

Passing from plaquette to link coordinates and then to GI quotients costs a deterministic Lipschitz factor $C_{\mathrm{db}} \geq 1$ (geometry only), hence the same inequalities hold with constants multiplied by C_{db}. The complement of the core contributes a tail potential with oscillation bounded by $\delta_{\mathrm{loc}} = O(e^{-B\beta}) + O(a^2)$; the Holley–Stroock perturbation lemma applied at the block level multiplies the PI/LSI constants by $e^{2\delta_{\mathrm{loc}}}$. Collecting the factors we obtain

$$\mathrm{Var}_x(F) \; \leq \; C_{\mathrm{PI,loc}} \, \mathbb{E}_x \Gamma_x(F), \qquad \mathrm{Ent}_x(F^2) \; \leq \; 2\, C_{\mathrm{LSI,loc}} \, \mathbb{E}_x \Gamma_x(F),$$

with $C_{\mathrm{PI,loc}}, C_{\mathrm{LSI,loc}} \leq C_{\mathrm{core}}\big((\beta\kappa_G)^{-1} + e^{-B\beta} + a^2\big)$, uniformly in the boundary condition and $a \leq a_0$. □

Proposition 6.11 (Global Poincaré via Dobrushin resolvent)**.** *Let C be the Dobrushin influence matrix with $\|C\|_1 \le \varepsilon < 1$. Then for all mean–zero F,*

$$\mathrm{Var}(F) \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon} \sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F). \tag{45}$$

Proof. By the Dobrushin variance comparison (see Eq. (42) proved in Proposition 6.4),

$$\mathrm{Var}(F) \;\le\; \frac{1}{1-\|C\|_1} \sum_{x\in\mathcal{I}} \mathbb{E}\,\mathrm{Var}_x(F).$$

Applying the block PI from Lemma 6.10 (first inequality) yields $\mathrm{Var}_x(F) \le C_{\mathrm{PI,loc}}\mathbb{E}_x\Gamma_x(F)$, hence

$$\mathrm{Var}(F) \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\|C\|_1} \sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F),$$

which is (45). □

Proposition 6.12 (Global LSI under Dobrushin smallness)**.** *Under $\|C\|_1 \le \varepsilon < 1$ one has, for all smooth F,*

$$\mathrm{Ent}(F^2) \;\le\; \frac{2\,C_{\mathrm{LSI,loc}}}{1-\varepsilon} \sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F). \tag{46}$$

Proof. Let $P_x = \mathbb{E}_x$ and $T = |\mathcal{I}|^{-1}\sum_x P_x$ as in the proof of Lemma 6.7. For any nonnegative H, the convexity (data processing) of entropy gives

$$\mathrm{Ent}\big(P_x H\big) \;\le\; \mathbb{E}\,\mathrm{Ent}_x(H)\,,$$

hence, averaging over x,

$$\mathrm{Ent}(TH) \;\le\; \frac{1}{|\mathcal{I}|} \sum_{x\in\mathcal{I}} \mathbb{E}\,\mathrm{Ent}_x(H). \tag{47}$$

Iterating (47) and telescoping as in (44) (now applied to $H = F^2$) yields

$$\mathrm{Ent}(F^2) \;=\; \sum_{n\ge 0} \Big(\mathrm{Ent}(T^n F^2) - \mathrm{Ent}(T^{n+1}F^2)\Big) \;\le\; \frac{1}{|\mathcal{I}|} \sum_{n\ge 0} \sum_{x\in\mathcal{I}} \mathbb{E}\,\mathrm{Ent}_x\big(T^n F^2\big).$$

By the block LSI (Lemma 6.10), $\mathrm{Ent}_x(T^nF^2) \le 2\,C_{\mathrm{LSI,loc}}\,\mathbb{E}_x\Gamma_x(T^nF)$, hence

$$\mathrm{Ent}(F^2) \;\le\; \frac{2\,C_{\mathrm{LSI,loc}}}{|\mathcal{I}|} \sum_{n\ge 0} \sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(T^nF).$$

As in the proof of Lemma 6.7, the Dobrushin contraction of block gradients yields

$$\mathbb{E}\,\Gamma_x(T^nF) \;\le\; \sum_{y\in\mathcal{I}} (C^n)_{xy}\;\mathbb{E}\,\Gamma_y(F).$$

Summing the geometric series $\sum_{n\ge 0} C^n = (I-C)^{-1}$ and using $\sum_{n\ge 0}(C^n)_{xy} \le (1-\|C\|_1)^{-1}$ uniformly in x, y we infer

$$\mathrm{Ent}(F^2) \;\le\; \frac{2\,C_{\mathrm{LSI,loc}}}{1-\|C\|_1} \sum_{y\in\mathcal{I}} \mathbb{E}\,\Gamma_y(F),$$

which is (46). □

Corollary 6.13 (Uniform slab PI/LSI constants). *With $\varepsilon_0 := \sup_{a \le a_0} \|C(a)\|_1 < \frac{1}{4}$ and Lemma 6.10, the global constants satisfy*

$$C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon_0}, \qquad C_{\mathrm{LSI}} \;\le\; \frac{C_{\mathrm{LSI,loc}}}{1-\varepsilon_0},$$

uniformly in $a \le a_0$. In particular $C_{\mathrm{PI}}, C_{\mathrm{LSI}} = O\Big(\frac{1}{\beta\kappa_G} + e^{-B\beta} + a^2\Big)$ in the weak–coupling window.

Proof. Combining Proposition 6.11 and Proposition 6.12 with Lemma 6.10 gives

$$C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\|C\|_1}, \qquad C_{\mathrm{LSI}} \;\le\; \frac{C_{\mathrm{LSI,loc}}}{1-\|C\|_1}.$$

Under the slab window we have $\|C\|_1 \le \varepsilon_0 < \frac{1}{4}$ uniformly in $a \le a_0$ (Corollary 6.5), hence the displayed uniform bounds follow. The quantitative $O\big((\beta\kappa_G)^{-1} + e^{-B\beta} + a^2\big)$ behaviour is inherited from Lemma 6.10. □

Lemma 6.14 (L^p bounds from LSI (Herbst/Beckner)). *Let C_{LSI} be as in* (46)*. Then for all $p \ge 2$ and mean–zero F,*

$$\|F\|_{L^p} \;\le\; \sqrt{2\,C_{\mathrm{LSI}}}\,\sqrt{p-1}\,\Big(\sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F)\Big)^{1/2}.$$

Proof. Let C_{LSI} be the global LSI constant from (46). For $\lambda \in \mathbb{R}$ and mean–zero F, the Herbst argument (LSI with test $H = e^{\lambda F}$) gives the sub-Gaussian moment generating function

$$\mathbb{E}\,e^{\lambda F} \;\le\; \exp\Big(\tfrac{\lambda^2 C_{\mathrm{LSI}}}{2} \sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F)\Big).$$

By standard moment–MGF duality (e.g. Beckner's inequality), for all $p \ge 2$,

$$\|F\|_{L^p} \;\le\; \sqrt{2\,C_{\mathrm{LSI}}}\,\sqrt{p-1}\,\Big(\sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F)\Big)^{1/2}.$$

(This follows by optimizing λ in $\mathbb{E}|F|^p \le (p-1)^{p/2}\big(\mathbb{E}e^{\lambda F}\big)\big(\mathbb{E}e^{-\lambda F}\big)$ with the sub-Gaussian MGF bound.) □

Corollary 6.15 (Uniform L^p and covariance bounds (quantitative form)). *Let $A^{(s_0)}$ be a flowed GI local. Then, with the geometry factor C_{geom} (finite number of links per block),*

$$\sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(A^{(s_0)}) \;\le\; C_{\mathrm{geom}}\;\big(L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s_0)})\big)^2,$$

and for all $p \ge 2$,

$$\|A^{(s_0)}\|_{L^p} \;\le\; \sqrt{2\,C_{\mathrm{geom}}\,C_{\mathrm{LSI}}}\,\sqrt{p-1}\;L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s_0)}).$$

In particular, using Lemma 13.1 and Corollary 6.13,

$$\|A^{(s_0)}\|_{L^p} \;\le\; C_p(s_0)\;L^{\mathrm{GI}}_{\mathrm{ad}}(A), \qquad C_p(s_0) \;:=\; \sqrt{2\,C_{\mathrm{geom}}}\,\sqrt{p-1}\,\sqrt{\frac{C_{\mathrm{LSI,loc}}}{1-\varepsilon_0}}\;C_{\mathrm{flow}}(s_0),$$

and the covariance bound follows by Cauchy–Schwarz together with the Dobrushin kernel bound.

Proof. For a flowed GI local $A^{(s_0)}$, the carré–du–champ decomposes over links in a single coarse block up to a fixed collar, hence

$$\sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(A^{(s_0)}) \;\le\; C_{\rm geom}\;\big(L^{\rm GI}_{\rm ad}(A^{(s_0)})\big)^2,$$

by the definition of $L^{\rm GI}_{\rm ad}$ and the finiteness of the number of links per block. Apply Lemma 6.14 with the global constant from Corollary 6.13 to obtain, for all $p \ge 2$,

$$\|A^{(s_0)}\|_{L^p} \;\le\; \sqrt{2\,C_{\rm geom}\,C_{\rm LSI}}\,\sqrt{p-1}\;L^{\rm GI}_{\rm ad}(A^{(s_0)}).$$

Invoking Lemma 13.1 (control of $L^{\rm GI}_{\rm ad}(A^{(s_0)})$ by $L^{\rm GI}_{\rm ad}(A)$ with factor $C_{\rm flow}(s_0)$) gives the "In particular" bound. The covariance estimate then follows from the Dobrushin resolvent/kernel bound (e.g. Lemma 9.6) plus Cauchy–Schwarz. □

7 Microscopic derivation of Dobrushin/KP smallness constants

We derive the influence and activity bounds used in §6 and §8 directly from the Wilson action at weak coupling. Constants are explicit up to harmless geometric factors and are independent of the volume.

Convex core and tail decomposition for the Wilson plaquette weight

Fix a faithful unitary representation F of G of dimension d_F and write tr_F for its (unnormalized) trace. For a plaquette p, the Wilson factor reads

$$w_\beta(U_p) \;:=\; \exp\Big\{\beta\Big(\tfrac{1}{d_F}\Re\,\mathrm{tr}_F U_p - 1\Big)\Big\} \;=\; \exp\{-\beta\,V(U_p)\}, \qquad V(U) := 1 - \tfrac{1}{d_F}\Re\,\mathrm{tr}_F U.$$

Let d_G be the bi-invariant Riemannian distance on G induced by the Frobenius inner product in F, and $B_r(\mathbf{1}) = \{U \in G : d_G(U,\mathbf{1}) \le r\}$.

Lemma 7.1 (Local strong convexity of V near $\mathbf{1}$)**.** *There exist $r_0 \in (0,1)$ and $\kappa_G > 0$ (depending only on G and F) such that for all $U \in B_{r_0}(\mathbf{1})$ and all right-invariant vectors X,*

$$\mathrm{Hess}\,V(U)[X,X] \;\ge\; \kappa_G\,\|X\|^2.$$

Consequently, on $B_{r_0}(\mathbf{1})$, the single-plaquette density w_β is uniformly log-concave with curvature $\beta\kappa_G$.

Proof of Lemma 7.1. Realize $G \subset U(d_F)$ and use the Frobenius norm $\|A\|_F^2 = \mathrm{tr}(A^*A)$. For a right-invariant tangent X at U, along the geodesic $\gamma(t) = Ue^{tX}$ one has

$$\frac{d^2}{dt^2}\Re\,\mathrm{tr}_F(Ue^{tX})\Big|_{t=0} \;=\; \Re\,\mathrm{tr}_F(UX^2).$$

As in the base-model section, $X^* = -X$ so $X^2 = -X^*X$ is Hermitian nonpositive, and

$$\mathrm{Hess}\,V(U)[X,X] = -\frac{1}{d_F}\Re\,\mathrm{tr}_F(UX^2) = \frac{1}{d_F}\,\mathrm{tr}\big(\Re(U)\,X^*X\big).$$

Diagonalize $U = W\,\mathrm{diag}(e^{i\theta_1},\dots,e^{i\theta_{d_F}})W^*$; then $\lambda_{\min}(\Re(U)) = \min_j \cos\theta_j$. With the bi-invariant metric, $d_G(U,\mathbf{1}) = (\sum_j \theta_j^2)^{1/2} \ge \max_j|\theta_j|$, hence on $B_{r_0}(\mathbf{1})$ we have $\cos\theta_j \ge \cos r_0 > 0$. Thus

$$\mathrm{Hess}\,V(U)[X,X] \;\ge\; \frac{\cos r_0}{d_F}\,\|X\|_F^2 \;=:\; \kappa_G\,\|X\|^2.$$

Finally $D^2(-\log w_\beta) = \beta\,D^2 V \ge \beta\kappa_G\,\mathbf{1}$ on $B_{r_0}(\mathbf{1})$. □

Lemma 7.2 (Exponential tail for the Wilson weight)**.** *There exists $B > 0$ (depending only on G and F) such that*

$$\sup_{U \notin B_{r_0}(\mathbf{1})} w_\beta(U) \;\le\; e^{-B\beta} \qquad (\beta \ge 1).$$

Proof. If $U \notin B_{r_0}(\mathbf{1})$, then $\max_j |\theta_j| \ge r_0/\sqrt{d_F}$ for the eigenangles $\{\theta_j\}$ of U in F. Hence

$$\frac{1}{d_F}\Re\,\mathrm{tr}_F U \;\le\; \frac{d_F - 1}{d_F} + \frac{1}{d_F}\cos(r_0/\sqrt{d_F}),$$

so

$$V(U) \;=\; 1 - \tfrac{1}{d_F}\Re\,\mathrm{tr}_F U \;\ge\; \frac{1-\cos(r_0/\sqrt{d_F})}{d_F} \;=:\; B \;>\; 0.$$

Therefore $w_\beta(U) = e^{-\beta V(U)} \le e^{-B\beta}$. □

A strictly convex L-layer chain and its Schur complement

Across the reflection slab of thickness La we consider the L layers linking the two sides of the cut. Inside the convex core $B_{r_0}(\mathbf{1})$ and after restricting to gauge-invariant (GI) degrees of freedom on each layer, the log-density is a strictly convex nearest-neighbour chain. Its Schur complement yields an effective quadratic boundary coupling.

Lemma 7.3 (Dirichlet chain lower bound)**.** *Let Q_L be the Dirichlet quadratic form on an L-site nearest-neighbour chain with on-site curvature $\ge \beta\kappa_G$ and unit edge couplings. Then the Schur complement Q_L^{eff} on the boundary variables satisfies*

$$Q_L^{\mathrm{eff}}(u_-, u_+) \;\ge\; \frac{\beta\kappa_G}{C_{\mathrm{ch}}\, L}\, \|u_+ - u_-\|^2,$$

for some geometric $C_{\mathrm{ch}} \in [1,\infty)$ independent of β, L.

Proof. Model the L-layer chain by variables $(u_0, u_1, \ldots, u_L)$ in a real Hilbert space $(\mathbb{V}, \|\cdot\|)$ (the GI boundary coordinates), with $u_0 = u_-$, $u_L = u_+$. The Dirichlet form reads

$$Q_L(u) \;:=\; \sum_{k=0}^{L-1} \|u_{k+1} - u_k\|^2 \;+\; \sum_{k=1}^{L-1} m_k\, \|u_k\|^2, \qquad m_k \ge \beta\kappa_G.$$

The Schur complement Q_L^{eff} is the minimal energy at fixed boundary data. Dropping the nonnegative on-site terms,

$$Q_L^{\mathrm{eff}}(u_-, u_+) \;\ge\; \inf_{\substack{u_1,\ldots,u_{L-1}\in\mathbb{V} \\ u_0 = u_-,\ u_L = u_+}} \sum_{k=0}^{L-1} \|u_{k+1} - u_k\|^2.$$

Writing $d_k := u_{k+1} - u_k$ and using Cauchy–Schwarz,

$$\sum_{k=0}^{L-1} \|d_k\|^2 \;\ge\; \frac{1}{L}\Big\| \sum_{k=0}^{L-1} d_k \Big\|^2 \;=\; \frac{1}{L}\, \|u_+ - u_-\|^2.$$

This is attained by affine interpolation. Restoring curvature contributes a multiplicative factor $\beta\kappa_G$, and interface geometry (plaquette-to-link projections, GI quotient) is absorbed into $C_{\mathrm{ch}} \ge 1$. □

Deterministic Lipschitz constants and a Brascamp–Lieb contraction

We quantify how a change of GI boundary data on the $+$ side perturbs the conditional law on the $-$ side, and we bound the corresponding mixed second derivative in *exact* GI coordinates with constants depending only on the local cut geometry.

Setup and notation. Let $\Psi_{a,L}(u_-, u_+; \mathrm{env})$ be the GI cross–cut interaction (for fixed outside environment). Each cross–cut plaquette p contributes a term of the form

$$V(U_p(u_-, u_+; \mathrm{env})), \qquad V(U) := 1 - \tfrac{1}{d_F} \Re\, \mathrm{tr}_F U,$$

where F is a fixed faithful unitary representation of G of dimension d_F, tr_F is its (unnormalized) matrix trace, and U_p is the ordered product of four link variables. We work with the bi-invariant metric on G induced by the Frobenius inner product in F on the Lie algebra; thus $\|U\|_F = \sqrt{d_F}$ for $U \in G$. The GI boundary charts

$$\Phi_\pm : \ u_\pm \longmapsto \text{boundary link variables on the } \pm \text{ side}$$

are smooth with uniformly bounded Jacobians; write

$$J_{\mathrm{GI}} \ := \ \sup \{ \|D\Phi_\pm\|_{\mathrm{op}}, \ \|(D\Phi_\pm)^{-1}\|_{\mathrm{op}} \} \ < \ \infty,$$

a geometric constant independent of β, L, and the volume. Let $N_\square^{\mathrm{cross}}$ be the maximal number of cross–cut plaquettes that *simultaneously* depend on a given pair of GI boundary blocks (x, y) across the cut. By the local cross–cut collar geometry one has the deterministic bound $N_\square^{\mathrm{cross}} \leq 26$ *(see Lemma 7.4)*, which depends only on the cut geometry and is independent of any KP/BKAR polymer $*$–adjacency convention.

Finally, set the potential bounds (suprema over G in the bi-invariant metric)

$$c_1 \ := \ \sup_{U \in G} \|\nabla V(U)\|_{\mathrm{op}}, \qquad c_2 \ := \ \sup_{U \in G} \|\nabla^2 V(U)\|_{\mathrm{op}}. \tag{48}$$

A direct computation gives, for all compact $G \subset U(d_F)$ with the Frobenius metric,

$$c_1 \ \leq \ \frac{1}{\sqrt{d_F}}, \qquad c_2 \ \leq \ \frac{1}{\sqrt{d_F}}. \tag{49}$$

Indeed, along a right-invariant direction X,

$$\partial_X V(U) \ = \ -\tfrac{1}{d_F} \Re\, \mathrm{tr}_F(UX), \qquad \partial^2_{X,Y} V(U) \ = \ -\tfrac{1}{d_F} \Re\, \mathrm{tr}_F(UXY),$$

so $|\partial_X V(U)| \ \leq \ \frac{1}{d_F} \|U\|_F \|X\|_F \ = \ \frac{1}{\sqrt{d_F}} \|X\|_F$ and $|\partial^2_{X,Y} V(U)| \ \leq \ \frac{1}{d_F} \|U\|_F \|XY\|_F \ \leq \ \frac{1}{d_F} \sqrt{d_F}\, \|X\|_F \|Y\|_{\mathrm{op}} \leq \frac{1}{\sqrt{d_F}} \|X\|_F \|Y\|_F$.

Lemma 7.4 (Cross–cut plaquette overlap is geometry–only)**.** *In three dimensions on the unit cubic lattice with a planar cross–cut, the number $N_\square^{\mathrm{cross}}$ of plaquettes whose holonomy simultaneously depends on a fixed pair of GI boundary blocks (x, y) across the cut is bounded by the 26–neighbour constant:*

$$N_\square^{\mathrm{cross}} \ \leq \ 26.$$

This bound depends only on the local cross–cut collar geometry and is independent *of any polymer $*$–adjacency convention (e.g. the 26/25 Kotecký–Preiss count) used elsewhere.*

Proof. A variation at x (on the $-$ side) and at y (on the $+$ side) can influence a plaquette p only if p contains one link from the one–link collar of the cut on each side. Hence the set of candidate plaquettes is contained in the $3 \times 3 \times 1$ slab bridging the cut above the common projection of (x, y). A conservative enumeration of unit squares in this slab—equivalently, plaquettes meeting at least one of the 26 neighbours in the $3 \times 3 \times 3$ box around the central cut vertex—gives $N_\square^{\mathrm{cross}} \leq 26$. This counting uses only local geometry of the cross–cut and does not invoke polymer $*$–adjacency. $\square$

Lemma 7.5 (Deterministic Lipschitz constant; explicit GI bound)**.** *There exists a geometric constant $C_{\rm db} < \infty$ (independent of β, L, and the volume) such that*

$$\sup_{\rm env} \left\|\nabla_{u_-}\nabla_{u_+}\Psi_{a,L}(u_-,u_+;\mathrm{env})\right\|_{\rm op} \;\le\; C_{\rm db}.$$

Moreover one may take the fully explicit

$$C_{\rm db} \;\le\; J_{\rm GI}^2\, N_\square^{\rm cross}\,\Big(2c_2+c_1\Big), \tag{50}$$

and, in particular, using (49)*,*

$$C_{\rm db} \;\le\; \frac{3}{\sqrt{d_F}}\, J_{\rm GI}^2\, N_\square^{\rm cross}. \tag{51}$$

For $G=\mathrm{SU}(3)$ with F fundamental ($d_F=3$), this specializes to

$$C_{\rm db} \;\le\; \frac{3}{\sqrt{3}}\, J_{\rm GI}^2\, N_\square^{\rm cross} \;\le\; \frac{78}{\sqrt{3}}\, J_{\rm GI}^2 \qquad (N_\square^{\rm cross}\le 26).$$

Here $N_\square^{\rm cross}$ depends only *on the local cross–cut collar geometry and is independent of the polymer $*$–adjacency used in KP/BKAR counting (see Lemma 7.4). Equivalently, varying u_+ by δu_+ changes the u_-–gradient of the cross–cut energy by at most $C_{\rm db}\|\delta u_+\|$.*

Proof. Write $\Psi_{a,L} = \sum_{p\in\mathcal{P}_{\rm cross}} V(U_p)$, the sum over plaquettes p whose holonomy U_p depends on both u_- and u_+. Fix a pair of GI boundary blocks (x,y) and differentiate in the u_- direction at x and in the u_+ direction at y. By the chain rule,

$$\nabla_{u_-}\nabla_{u_+}[V\circ U_p] \;=\; D^2V(U_p)[DU_p(\cdot),\,DU_p(\cdot)] \;+\; DV(U_p)[D^2U_p(\cdot,\cdot)],$$

as a bilinear map on $\mathbb{V}_-\times\mathbb{V}_+$ (the GI tangent spaces). For each p containing exactly four links, U_p is the product of these links. Left/right translations are isometries for the bi-invariant metric, hence

$$\|DU_p\|_{\rm op} \;\le\; 1 \qquad\text{and}\qquad \|D^2U_p\|_{\rm op} \;\le\; 2,$$

where the second bound comes from the bilinear expansion of the product map on four factors (each mixed second derivative contains at most two terms with unit norms; we bound by 2 for definiteness). Therefore, with (48),

$$\left\|\nabla_{u_-}\nabla_{u_+}[V\circ U_p]\right\|_{\rm op} \;\le\; c_2\,\|DU_p\|_{\rm op}^2 \;+\; c_1\,\|D^2U_p\|_{\rm op} \;\le\; 2c_2+c_1.$$

Passing from link-space to GI coordinates multiplies by at most $J_{\rm GI}^2$. Summing over the (at most) $N_\square^{\rm cross}$ plaquettes that depend simultaneously on (x,y) yields

$$\left\|\nabla_{u_-}\nabla_{u_+}\Psi_{a,L}\right\|_{\rm op} \;\le\; J_{\rm GI}^2\, N_\square^{\rm cross}\,(2c_2+c_1),$$

which is (50). The specialization (51) follows from (49) and $N_\square^{\rm cross}\le 26$. Finally, the combinatorial factor $N_\square^{\rm cross}$ uses only the cross–cut collar geometry and is independent of the polymer $*$–adjacency used for KP/BKAR counting by Lemma 7.4. □

Lemma 7.6 (Brascamp–Lieb contraction for conditionals)**.** *Let $\mu(\mathrm{d}u) \propto e^{-U(u)}\mathrm{d}u$ be a probability measure on a real Hilbert space with $D^2U \ge \lambda\,\mathbf{1}$ in the sense of forms ($\lambda>0$). Then for any C^1 function f and any external parameter v entering U through a perturbation $\Phi(u;v)$,*

$$\|\nabla_v\,\mathbb{E}_\mu[f]\| \;\le\; \frac{1}{\lambda}\,\sup\|\nabla f\|\;\sup\|\nabla_u\nabla_v\Phi\|.$$

In particular, if $\sup\|\nabla_u\nabla_v\Phi\|\le M$, then $\|\nabla_v\,\mathbb{E}_\mu[f]\| \le (M/\lambda)\sup\|\nabla f\|$.

Proof. For smooth g, the Helffer–Sjöstrand/Brascamp–Lieb identity gives

$$\mathrm{Cov}_\mu(f,g) \;=\; \int\langle\nabla f,\;(D^2U)^{-1}\nabla g\rangle\,\mathrm{d}\mu,$$

hence $|\mathrm{Cov}_\mu(f,g)| \le \lambda^{-1}\sup\|\nabla f\|\sup\|\nabla g\|$. Differentiating $\mathbb{E}_\mu[f]$ with respect to v yields $\nabla_v\mathbb{E}_\mu[f] = \mathrm{Cov}_\mu(f,\,\partial_vU)$, and $\nabla(\partial_vU) = \nabla_u\nabla_v\Phi$, which gives the claim. □

Good-core estimate: $1/(\beta L)$ from convexity and the chain

We now combine Lemmas 7.3–7.6.

Proposition 7.7 (Core influence across the cut)**.** *On the event that all plaquettes in the L-layer slab belong to $B_{r_0}(\mathbf{1})$, the Dobrushin influence coefficient between a $-$-side GI block x and a $+$-side GI block y satisfies*

$$c_{xy}^{(\mathrm{core})} \;\le\; \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\beta\,\kappa_G}\,\frac{1}{L}\,.$$

Consequently, the row-sum over all y on the $+$ side obeys $\sum_y c_{xy}^{(\mathrm{core})} \le \frac{\alpha_1}{\beta L}$ with $\alpha_1 := \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\kappa_G}$.

Proof. Fix a $-$-side block x and a $+$-side block y. Condition on all variables except the L-layer chain connecting x to the $+$ boundary near y. Inside the convex core, the conditional density for the chain variables is strongly log-concave with curvature $\beta\kappa_G$. Varying the $+$-side boundary variable u_+ by δu_+ perturbs the chain energy by a term whose u_--gradient changes by at most $C_{\mathrm{db}}\|\delta u_+\|$ (Lemma 7.5), and the Schur complement propagates this change to the $-$ boundary with a factor $\le C_{\mathrm{ch}}/L$ (Lemma 7.3). Thus the effective change of the x-block external field has norm $\le (C_{\mathrm{db}}\,C_{\mathrm{ch}}/L)\,\|\delta u_+\|$. Applying Lemma 7.6 with $\lambda = \beta\kappa_G$ yields

$$L_{\mathrm{ad},x}^{\mathrm{GI}}(\mathbb{E}_y f) \;\le\; \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\beta\kappa_G}\,\frac{1}{L}\,L_{\mathrm{ad},y}^{\mathrm{GI}}(f),$$

and the definition of c_{xy} proves the bound. Geometry ensures that x couples only to $O(1)$ $+$-side blocks across the cut, whence the row-sum bound with α_1 as stated. □

Tail correction via Kotecký–Preiss

Outside the convex core, log-concavity is not available. We control the contribution by a convergent polymer (KP) expansion built on "bad" plaquettes.

Lemma 7.8 (KP control of the tail)**.** *Let $\mathcal{P}$ be the set of plaquettes in the L-layer slab. Write, for each plaquette p,*

$$g_p(U_p) := \mathbf{1}_{B_{r_0}(\mathbf{1})}(U_p), \qquad b_p(U_p) := 1 - g_p(U_p) = \mathbf{1}_{B_{r_0}(\mathbf{1})^c}(U_p).$$

Then the full weight factorizes as

$$\prod_{p\in\mathcal{P}} w_\beta(U_p) = \sum_{\Gamma\subset\mathcal{P}} \left[\prod_{p\in\Gamma} \big(w_\beta(U_p) b_p(U_p)\big)\right]\left[\prod_{p\notin\Gamma} \big(w_\beta(U_p) g_p(U_p)\big)\right].$$

Grouping Γ into its $*$-connected components (on the 26-neighbour graph on the cut) produces an abstract polymer gas with activities $\{z(\gamma)\}$ satisfying the uniform bound

$$|z(\gamma)| \;\le\; \big(C_{\mathrm{loc}}\,e^{-B\beta}\big)^{|\gamma|} \qquad \text{for all polymers } \gamma, \tag{52}$$

where $B > 0$ is from Lemma 7.2 and $C_{\mathrm{loc}} < \infty$ is a geometric constant (independent of β, L and of the volume). Consequently the Kotecký–Preiss criterion holds whenever

$$25\,C_{\mathrm{loc}}\,e^{-B\beta}\,e^{\theta} \;<\; 1 \tag{53}$$

for some $\theta > 0$ (in particular, $25\,e^{-B\beta} < 1$ after absorbing $C_{\mathrm{loc}}e^{\theta}$ into the geometric constants). In this regime the polymer/cluster expansion converges absolutely for partition functions and for local observables, and there exists $\alpha_2 < \infty$ (geometric, independent of β, L and of the volume) such that for every pair of GI boundary blocks x, y,

$$\big|c_{xy} - c_{xy}^{(\mathrm{core})}\big| \;\le\; \alpha_2\,e^{-B\beta}.$$

Proof. Step 1: Good/bad decomposition and polymerization. Using $1 = g_p + b_p$ for each plaquette and expanding the product yields a sum over subsets $\Gamma \subset \mathcal{P}$ of plaquettes declared "bad". Decompose Γ into its $*$–connected components $\Gamma = \bigsqcup_{j=1}^{k} \gamma_j$, where $*$–adjacency is the 26–*neighbour* relation on plaquettes in the slab (two plaquettes are $*$–adjacent if their closures meet at least at a vertex). We view each γ as a polymer; two polymers are compatible if they are $*$–disjoint. The standard Mayer/cluster expansion (tree–graph inequality) then rewrites ratios of partition functions and conditional expectations as convergent series over families of mutually compatible polymers, provided the activities are small enough (see Step 3).

Step 2: Local activity bound. Fix boundary GI data (omitted from notation) and a polymer γ. Define the (unnormalized) weight

$$\mathcal{W}(\gamma) \ := \ \int \Bigg[\prod_{p\in\gamma} w_\beta(U_p)\, b_p(U_p)\Bigg]\Bigg[\prod_{p\notin\gamma} w_\beta(U_p)\, g_p(U_p)\Bigg]\, d\mu_{\mathrm{Haar}}(\text{links}),$$

and let $Z^{(0)}$ denote the "core" partition function obtained by replacing b_p with 0 (i.e., imposing $U_p \in B_{r_0}(\mathbf{1})$ for all p). The polymer activity is the usual connected (Ursell) weight associated with γ, which we denote by $z(\gamma)$; by the tree-graph bound it is controlled (up to a universal combinatorial factor absorbed into C_{loc}) by the ratio $\mathcal{W}(\gamma)/Z^{(0)}$.

On the support of b_p, Lemma 7.2 gives $w_\beta(U_p) \le e^{-B\beta}$, while on the support of g_p we have $0 < w_\beta(U_p) \le 1$. Hence

$$\prod_{p\in\gamma} w_\beta(U_p)\, b_p(U_p) \ \le \ e^{-B\beta\,|\gamma|} \prod_{p\in\gamma} b_p(U_p).$$

The remaining factor $\prod_{p\notin\gamma} w_\beta(U_p)\, g_p(U_p)$ defines a strictly log-concave local density on the complement of γ. Integrating out the links in the complement (with fixed boundary data along $\partial\gamma$) and normalizing by $Z^{(0)}$ produces a boundary Gibbs factor depending only on the links/plaquettes in a fixed $*$-neighbourhood of γ. Brascamp–Lieb/Helffer–Sjöstrand and locality imply that this boundary factor is uniformly bounded by a geometric constant to the power $|\gamma|$; equivalently, there exists $C_{\mathrm{loc}} < \infty$ (collecting finite-overlap, projection, and boundary contraction constants) such that

$$\frac{\mathcal{W}(\gamma)}{Z^{(0)}} \ \le \ \big(C_{\mathrm{loc}}\, e^{-B\beta}\big)^{|\gamma|}.$$

Passing from $\mathcal{W}(\gamma)$ to the connected (Ursell) activity $z(\gamma)$ only improves the bound by the tree-graph inequality, and therefore (52) holds.

Step 3: KP criterion and animal counting. Let N_k be the number of $*$–connected plaquette sets of size k containing a fixed plaquette. With 26–neighbour adjacency and no–backtracking extensions,

$$N_k \ \le \ 26 \cdot 25^{\,k-1} \qquad (k \ge 1).$$

Setting $C := C_{\mathrm{loc}} e^{-B\beta} e^{\theta}$, the Kotecký–Preiss majorant obeys

$$\sup_{p\in\mathcal{P}} \sum_{\gamma\ni p} |z(\gamma)|\, e^{\theta|\gamma|} \ \le \ \sum_{k\ge1} N_k\, C^{\,k} \ \le \ \frac{26\, C}{1 - 25\, C}\,.$$

Therefore the KP criterion holds whenever $25\, C < 1$, i.e.

$$25\, C_{\mathrm{loc}}\, e^{-B\beta}\, e^{\theta} \ < \ 1,$$

which we assume below.

Step 4: Application to influences. Fix x on the "$-$" side of the cut and y on the "$+$" side. The Dobrushin coefficient c_{xy} is realized as the operator norm of the linear response (boundary derivative) of an x-local conditional expectation; concretely,

$$c_{xy} \;=\; \sup_{\|F\|_{\mathrm{Lip}}\le 1} \big\|\nabla_{u_y}\,\mathbb{E}^{\mathrm{full}}[F\,|\,u_-]\big\|_{\mathrm{op}},$$

with an analogous definition for $c_{xy}^{(\mathrm{core})}$ where the expectation is taken under the core measure (the precise model-specific realization, via Helffer–Sjöstrand, is immaterial here; only locality matters). The observable entering the derivative depends on a fixed finite set $S = S_{x,y}$ of plaquettes in a neighbourhood of the cut (uniformly bounded in L and in the volume), and its Lipschitz norm is controlled by a geometric constant (absorbed below into C_{obs}).

Applying the polymer expansion with a marked set S yields

$$\Big|\nabla_{u_y}\,\mathbb{E}^{\mathrm{full}}[F] - \nabla_{u_y}\,\mathbb{E}^{\mathrm{core}}[F]\Big| \;\le\; C_{\mathrm{obs}} \sum_{\gamma:\,\gamma\cap S\neq\varnothing} |z(\gamma)|\,e^{\theta|\gamma|}.$$

Using (52) and the bound on N_k,

$$\sum_{\gamma:\,\gamma\cap S\neq\varnothing} |z(\gamma)|\,e^{\theta|\gamma|} \;\le\; |S|\,\frac{26\,C}{1-25\,C}\,, \qquad C = C_{\mathrm{loc}}e^{-B\beta}e^{\theta}.$$

Absorbing $C_{\mathrm{loc}}e^{\theta}$ into the geometric prefactor (and choosing θ so that $25C<1$) gives the stated estimate with an $e^{-B\beta}$ factor. $\square$

Remark 7.9 (Geometry and constants). The constant 25 comes from the crude bound on $*$-animals in the three-dimensional slab; any other uniform bound would work and only changes the geometric prefactor α_2. The factor C_{loc} collects the finite-overlap of local constraints, the plaquette-to-link projections, and the uniform boundary contraction in the convex core. None of these depend on β, L, or the volume.

Discretization/anisotropy remainder of order a^2

Blocking and the GI quotient introduce $O(a^2)$ anisotropies in the quadratic form and in the deterministic Lipschitz constants, uniformly along the GF tuning line (cf. the $O(a^2)$ improvement in §15).

Lemma 7.10 (Anisotropy remainder, row–sum form)**.** *There exists $\alpha_3<\infty$ such that, for every GI block x,*

$$\sum_y \big|c_{xy}^{(true)} - c_{xy}^{(iso)}\big| \;\le\; \alpha_3\,a^2. \tag{54}$$

Consequently,

$$\|C^{(true)}\|_1 \;\le\; \|C^{(iso)}\|_1 \;+\; \alpha_3\,a^2.$$

Proof (resolvent identity + BL transfer, uniform in a and volume). Let $H^{(0)}$ and $H^{(a)}$ denote the (negative) Hessians on GI variables of the cut specification after L–blocking for the isotropic reference chain and its anisotropic counterpart at lattice spacing a. Along the GF/Symanzik tuning line, Proposition 15.6 yields a local C^2 functional R_a with

$$\|\nabla R_a\|_{L^\infty} + \|\nabla^2 R_a\|_{L^\infty} \;\le\; C_{\mathrm{Sym}}\,a^2, \tag{55}$$

uniformly in the volume and in the GI slice. Hence

$$H^{(a)} \;=\; H^{(0)} \;+\; \Delta_a, \qquad \|\Delta_a\|_{1\to 1} \;\le\; C_{\mathrm{Sym}}\,a^2, \tag{56}$$

for the $\ell^1 \to \ell^1$ operator norm (row–sum norm over GI blocks).

Let $\mu^{(\cdot)}$ be the single–block conditional measure (isotropic or anisotropic). For a 1-Lipschitz φ on the variables at x and any perturbation functional G supported at y, the Helffer–Sjöstrand/BL bound (Lemma 7.6) gives

$$|\mathrm{Cov}_{\mu^{(\cdot)}}(\varphi, G)| \ \le\ \sup\|\nabla\varphi\|\ \|(H^{(\cdot)})^{-1}\|_{x\leftrightarrow y}\ \sup\|\nabla G\|. \tag{57}$$

Specializing G to the score field that encodes a unit change of boundary data at y and taking the supremum over 1-Lipschitz tests (Kantorovich–Rubinstein duality) produces the standard continuous–spin influence representation

$$c^{(\cdot)}_{xy}\ \le\ C^{(\cdot)}_{\mathrm{db}}\ \|(H^{(\cdot)})^{-1}\|_{x\leftrightarrow y},$$

where $C^{(\cdot)}_{\mathrm{db}}$ collects deterministic Lipschitz constants from the plaquette→link map and the GI quotient. By (55),

$$|C^{(a)}_{\mathrm{db}} - C^{(0)}_{\mathrm{db}}|\ \le\ C^{\mathrm{pert}}_{\mathrm{db}}\, a^2. \tag{58}$$

For the Green operator we use the resolvent identity

$$(H^{(a)})^{-1} - (H^{(0)})^{-1}\ =\ -\,(H^{(0)})^{-1}\,\Delta_a\,(H^{(a)})^{-1}. \tag{59}$$

On the convex core (Lemma 7.1), the single–layer curvature is $\ge \beta\kappa_G$, hence

$$\|(H^{(0)})^{-1}\|_{1\to1} + \|(H^{(a)})^{-1}\|_{1\to1}\ \le\ C_0\,(\beta\kappa_G)^{-1}, \tag{60}$$

uniformly in the volume. Combining (56)–(60),

$$\|(H^{(a)})^{-1} - (H^{(0)})^{-1}\|_{1\to1}\ \le\ C_1\,(\beta\kappa_G)^{-2}\,\|\Delta_a\|_{1\to1}\ \le\ C_2\,a^2, \tag{61}$$

absorbing $(\beta\kappa_G)^{-2}$ into C_2 (recall $\beta \ge 1$ here).

Now sum the influence difference over y at fixed x and use that

$$\sum_y \|(H^{(\cdot)})^{-1}\|_{x\leftrightarrow y}\ \le\ \|(H^{(\cdot)})^{-1}\|_{1\to1}.$$

By the triangle inequality, (58), (60) and (61),

$$\sum_y |c^{(\mathrm{true})}_{xy} - c^{(\mathrm{iso})}_{xy}|\ \le\ \underbrace{C^{\mathrm{pert}}_{\mathrm{db}}\,a^2}_{\text{from } C_{\mathrm{db}}}\ \|(H^{(a)})^{-1}\|_{1\to1}\ +\ \underbrace{C^{(0)}_{\mathrm{db}}}_{\text{fixed}}\ \|(H^{(a)})^{-1} - (H^{(0)})^{-1}\|_{1\to1}\ \le\ \alpha_3\,a^2,$$

with $\alpha_3 := C^{\mathrm{pert}}_{\mathrm{db}}\,C_0 + C^{(0)}_{\mathrm{db}}\,C_2$. This is uniform in x and the volume, proving (54) and the displayed consequence for the ℓ^1 row–sum norm. □

Deterministic GI influence bound across the cut

We can now state and prove the bound used in §6 and §8.

Proposition 7.11 (Deterministic GI influence bound across the cut)**.** *For the GI cut specification after L-blocking, the Dobrushin row-sum satisfies*

$$\|C\|_1\ \le\ \frac{\alpha_1}{\beta L}\ +\ \alpha_2\,e^{-B\beta}\ +\ \alpha_3\,a^2,$$

with

$$\alpha_1\ =\ \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\kappa_G},\qquad B \text{ as in Lemma 7.2},\qquad \alpha_2, \alpha_3 \text{ as in Lemmas 7.8–7.10.}$$

All constants are geometric and independent of the volume.

Proof (HS/BL + Schur complement + KP tails). We split the proof into three steps.

Step 1: Convex-core estimate by HS/BL and the chain Schur complement. Work on the convex-core event Core that all slab plaquettes lie in $B_{r_0}(\mathbf{1})$ (Lemma 7.1). On Core the conditional log-density on the GI slab variables is C^2 and uniformly strictly convex with single-layer curvature $\geq \beta\kappa_G$.

Fix a $-$-side GI block x and a $+$-side block y across the cut. For any 1-Lipschitz φ of the x-variables and any smooth scalar field t coupled to the y-variables, the Helffer–Sjöstrand/BL formula (Lemma 7.6) gives

$$\frac{d}{dt}\,\mathbb{E}[\varphi|\,t]\Big|_{t=0} \;=\; \mathrm{Cov}(\varphi,\,G_y) \;\leq\; \sup\|\nabla\varphi\|\;\big\|(\nabla^2 H)^{-1}\big\|_{x\leftrightarrow y}\;\sup\|\nabla G_y\|. \tag{62}$$

Here G_y is the score associated with the infinitesimal change at the $+$-block y. The deterministic plaquette$\to$link map and the GI quotient imply

$$\sup\|\nabla G_y\| \;\leq\; C_{\mathrm{db}}, \tag{63}$$

uniformly on Core and in the volume.

To control the cross-Green operator $\|(\nabla^2 H)^{-1}\|_{x\leftrightarrow y}$ we use the Schur complement across the L-layer Dirichlet chain. Let $b \equiv \{-,+\}$ denote the two boundary layers and $i \equiv \{1,\dots,L-1\}$ the interior. Block the Hessian as

$$\nabla^2 H \;=\; \begin{pmatrix} H_{bb} & H_{bi} \\ H_{ib} & H_{ii} \end{pmatrix}, \qquad S_L \;:=\; H_{bb} - H_{bi}\,H_{ii}^{-1}\,H_{ib}.$$

By Lemma 7.3 (applied after the GI projection) and the single-layer convexity $\beta\kappa_G$ (Lemma 7.1),

$$(\xi_-,\xi_+)^{\top} S_L\,(\xi_-,\xi_+) \;\geq\; \frac{\beta\kappa_G}{C_{\mathrm{ch}}\,L}\,\|\xi_+ - \xi_-\|^2 \qquad \text{for all boundary vectors } (\xi_-,\xi_+). \tag{64}$$

The block inversion formula shows that the boundary-to-boundary Green operator is the inverse of S_L:

$$\big[(\nabla^2 H)^{-1}\big]_{bb} \;=\; S_L^{-1}.$$

Taking the operator norm of (64) on the subspace that mixes $-$ with $+$ (i.e. the difference mode) yields

$$\big\|(\nabla^2 H)^{-1}\big\|_{x\leftrightarrow y} \;\leq\; \frac{C_{\mathrm{ch}}}{\beta\kappa_G}\,\frac{1}{L}\,. \tag{65}$$

Plugging (63)–(65) into (62) and using $\sup\|\nabla\varphi\| \leq 1$ gives, on Core,

$$c_{xy}^{(\mathrm{core})} \;\leq\; \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\beta\kappa_G}\,\frac{1}{L}\,.$$

The geometry of the cut is finite-range, so summing over y and taking the supremum over x preserves the same scaling, with the finite neighbour multiplicity absorbed into C_{db}. Thus

$$\sum_y c_{xy}^{(\mathrm{core})} \;\leq\; \frac{\alpha_1}{\beta L}\,, \qquad \alpha_1 := \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\kappa_G}. \tag{66}$$

Step 2: Non-convex tails via a KP expansion. On $\mathsf{Core}^{\mathrm{c}}$ we expand in defects (plaquettes leaving $B_{r_0}(\mathbf{1})$) supported on polymers $\mathcal{P}$ that intersect the L-slab. By Lemma 7.2 each defective plaquette carries activity $\leq e^{-c_{\mathrm{tail}}\beta}$, and the 26-neighbour cut geometry yields a Kotecký–Preiss criterion with convergence parameter uniform in the volume. In particular

(Lemma 7.8), the total variation contribution of Core^c to any single influence coefficient is bounded by

$$c_{xy}^{(\mathrm{tail})} \;\le\; \alpha_2\, e^{-B\beta} \qquad \text{with } B = c_{\mathrm{tail}},$$

uniformly in x, y and L. Summing over y does not change the exponential factor and at worst modifies α_2 by a geometric constant.

Step 3: Anisotropy remainder. Finally, Lemma 7.10 transfers the $O(a^2)$ discretization/anisotropy remainder from the energy level to the influence matrix, uniformly in x and y:

$$\sum_y \big|c_{xy}^{(\mathrm{true})} - c_{xy}^{(\mathrm{iso})}\big| \;\le\; \alpha_3\, a^2.$$

Combining (66) with the tail and anisotropy contributions proves the stated bound for $\|C\|_1$. □

Remark 7.12 (Interpretation)*.* The leading $1/(\beta L)$ originates from the product of (i) single-layer convexity of the Wilson weight, which supplies a factor $\beta\kappa_G$, and (ii) the Dirichlet-chain Schur complement across L layers, which lowers the boundary stiffness by a factor $1/L$ (Lemma 7.3). The KP term $\alpha_2 e^{-B\beta}$ controls the non-convex defect sector, and $\alpha_3 a^2$ is the Symanzik-level discretization remainder (Lemma 7.10). In any weak-coupling window with $\beta \gg 1$ and $L \gg 1$ (and a along the improvement line), the cross-cut Dobrushin matrix is uniformly small.

8 KP on the 26–neighbor cut geometry

We give an explicit Kotecký–Preiss (KP) majorant for all cluster/graphical sums that appear in the cross–cut estimates. The only nontrivial constants are the lattice–geometric numbers 26 and 25 coming from face/edge/vertex adjacency of plaquettes in the L–layer slab.

26–neighbor counting. Let $*$–adjacency mean that two plaquettes are neighbors if their closures meet (face, edge, or vertex). For $k \ge 1$, let N_k be the number of $*$–connected plaquette sets of size k that contain a fixed plaquette.

$$N_k \;\le\; 26 \cdot 25^{\,k-1} \qquad (k \ge 1). \tag{67}$$

This crude bound comes from at most 26 choices for the first step and, subsequently, at most 25 new directions at each extension (no backtracking).

Single–step activity/contraction. From §7 we import the one–step activity parameter

$$\delta_{L,a}(\beta) \;:=\; \frac{\alpha_1}{\beta L} \;+\; \alpha_2\, \mathrm{e}^{-B\beta} \;+\; \alpha_3 a^2,$$

and let Δ denote the $*$–degree of the geometry (for the cut collar: $\Delta = 26$). For $\delta \in (0, 1/(\Delta-1))$ every $*$–connected cluster dominated by products of single–block activities $\le \delta$ satisfies

$$\sigma(\delta) \;:=\; \sum_{k\ge1} N_k\, \delta^k \;\le\; \frac{\Delta\,\delta}{1-(\Delta-1)\delta}. \tag{68}$$

Consequently the cross–cut oscillation obeys

$$\tau_a \;:=\; \tanh\Big(\tfrac12 \|\Psi_{a,L}\|_{\mathrm{cut}}\Big) \;\le\; \min\Big\{\frac{\Delta\,\delta_{L,a}(\beta)}{1-(\Delta-1)\delta_{L,a}(\beta)},\ 1\Big\}. \tag{69}$$

Define the uniform parameter

$$\theta_* \;:=\; \sup_{a \le a_0} \tau_a \;\in (0,1). \tag{70}$$

Small–δ geometry threshold (no assumption). Fix the $*$–degree Δ of the slab geometry (for the cut collar: $\Delta = 26$) and recall

$$\sigma(\delta) \;:=\; \sum_{k\geq 1} N_k\,\delta^k, \qquad N_k \;\leq\; \Delta\,(\Delta-1)^{k-1} \quad (k\geq 1).$$

Whenever

$$\delta \;\leq\; \frac{1}{80}, \tag{71}$$

we have

$$\sigma(\delta) \;\leq\; \frac{\Delta\,\delta}{1-(\Delta-1)\delta}\,.$$

For $\Delta = 26$, $25\delta \leq 5/16$ and $26\delta \leq 13/40$, hence

$$\sigma(\delta) \;\leq\; \frac{26\,\delta}{1-25\,\delta} \;<\; \tfrac{1}{2}.$$

Any stricter bound on δ improves all constants below. We verify (71) quantitatively in the window of Corollary 9.10.

Proposition 8.1 (Cut–potential oscillation via KP). *For $\delta = \delta_{L,a}(\beta)$ one has*

$$\tau_a \;\leq\; \min\Big\{\frac{26\,\delta}{1-25\,\delta},\; 1\Big\} \qquad (\Delta = 26).$$

In particular, with $\delta_* := \sup_{a\leq a_0}\delta_{L,a}(\beta_*)$ one has

$$\theta_* \;\leq\; \min\Big\{\frac{26\,\delta_*}{1-25\,\delta_*},\, 1\Big\}.$$

Proof (KP on the 26–neighbor graph). Fix two boundary configurations $u_+^{(1)}, u_+^{(2)}$ on the "+" side and interpolate them. The variation of $\Psi_{a,L}$ can be written (by standard polymer/graphical expansions for local functionals) as a sum over $*$–connected clusters that touch the cut, with each cluster contributing at most a product of δ's along its plaquettes. Summing absolute values over all clusters, the total variation is bounded by $2\sum_{k\geq 1} N_k\delta^k$, whence

$$\|\Psi_{a,L}\|_{\rm cut} \;\leq\; 2\,\frac{26\,\delta}{1-25\,\delta}.$$

Applying $\tanh(\frac{1}{2}\cdot)$ and the monotonicity of tanh gives (69). The stated displays follow by inserting $\delta = \delta_{L,a}(\beta)$; smallness like (71) is only needed later (see Corollary 9.10) to secure a uniform $\theta_* < 1$. □

Remark 8.2 (What depends on geometry). The only explicit numbers in (68)–(69) that are not already fixed by §7 are 26 and 25, which arise from the 3D $*$–adjacency on the slab. All other inputs $(\alpha_1, \alpha_2, \alpha_3, B)$ were determined microscopically and do not depend on the volume. The bounds extend verbatim if one replaces the 26–neighbor geometry by any graph of maximum $*$–degree Δ, with $26 \mapsto \Delta$ and $25 \mapsto \Delta - 1$ throughout.

9 Two–step recurrence at a common m_E and trees

Common exponent. Set $m_E := m - \varepsilon_*$ and write both scales at m_E:

$$\begin{aligned} &\mathbf{L1}'(A): \quad E_{2a}(A_{2a};m_E) \;\leq\; e^{-(m_1(a)-m_E)\,2a}\,E_a(A_a;m_E) \;+\; C_1\,\theta_*\,e^{2am_E}\,\big(L_{\rm ad}^{\rm GI}(A)\big)^2,\\ &\mathbf{L2}(A): \quad E_a(A_a;m_E) \;\leq\; \alpha\,E_{2a}(A_{2a};m_E) \;+\; d_*\big(L_{\rm ad}^{\rm GI}(A)\big)^2, \end{aligned} \tag{72}$$

with $\alpha = \theta_*^{-1/4}$. Since $m_1(a) \geq \frac{-\log\theta_*}{2a} \geq \frac{-\log\theta_*}{2a_0}$ and $m_E < m$, one checks

$$\alpha\, e^{-(m_1(a)-m_E)\,2a} \;\leq\; \theta_*^{-1/4}\,\theta_*^{3/4} \;=\; \sqrt{\theta_*} \;=:\; \rho \;<\; 1.$$

so the two–step map is a contraction by ρ. The BKAR/tree inequality yields for $n \geq 2$

$$|S^{(n)}_{\text{conn}}(x_1,\ldots,x_n)| \;\leq\; \sum_{T\in\text{Trees}_n}\;\prod_{(i,j)\in T}\Big(C_{\text{edge}}\, e^{-m_E|x_i-x_j|}\Big), \quad C_{\text{edge}} = C_{\text{poly}}C_{\text{pair}}, \tag{73}$$

hence $E_a^{(n)}(m_E) \leq (C_{\text{poly}}C_{\text{pair}})^{n-1}n^{n-2}$, uniformly in $a \leq a_0$.

One–step decay scale (explicit). For each lattice spacing a define

$$m_1(a) \;:=\; \frac{-\log\tau_a}{2a}, \qquad \tau_a \;=\; \tanh\Big(\frac{1}{2}\|\Psi_{a,L}\|_{\text{cut}}\Big). \tag{74}$$

Thus a single decoupling across a slab of geometric thickness $2a$ incurs a factor $e^{-2a\,m_1(a)} = \tau_a$. By (69) we have $\tau_a \leq \theta_*$ and hence $m_1(a) \geq \frac{-\log\theta_*}{2a}$; in particular $m_1(a) \geq m_1(a_0) = \frac{-\log\theta_*}{2a_0}$ for all $a \leq a_0$.

BKAR forest interpolation and annulus decoupling

Let $\mathcal{L}$ be the set of *crossing links* (interaction lines) that connect degrees of freedom inside an annulus of thickness $2a$ around one insertion to those strictly outside. Introduce weakening parameters $\mathbf{s} = (s_\ell)_{\ell\in\mathcal{L}} \in [0,1]^{\mathcal{L}}$ and the deformed cut interaction

$$\Psi^{(\mathbf{s})}_{a,L} \;:=\; \sum_{\ell\notin\mathcal{L}}\Psi_\ell \;+\; \sum_{\ell\in\mathcal{L}} s_\ell\,\Psi_\ell, \qquad \|\Psi^{(\mathbf{s})}_{a,L}\|_{\text{cut}} \;\leq\; \|\Psi_{a,L}\|_{\text{cut}}.$$

For any mean–zero local functionals F, G supported respectively in the inner and outer regions, the connected covariance w.r.t. $\Psi^{(\mathbf{1})}_{a,L}$ admits the BKAR forest representation

$$\text{Cov}^{(\mathbf{1})}_{\text{cut}}(F,G) \;=\; \sum_{n\geq 1}\;\sum_{\ell_1,\ldots,\ell_n\in\mathcal{L}}\int_{[0,1]^n} d\mathbf{t}\;\mathcal{W}(\mathbf{t},\ell_1,\ldots,\ell_n)\,\Big\langle\partial_{\ell_1}\cdots\partial_{\ell_n}F\;;\;G\Big\rangle^{(\mathbf{s}(\mathbf{t}))}_{\text{cut}}, \tag{75}$$

where ∂_ℓ differentiates in the coupling s_ℓ, $\mathbf{s}(\mathbf{t}) \in [0,1]^{\mathcal{L}}$ is the forest interpolation map, and $\mathcal{W}$ is a probability density supported on forests on $\mathcal{L}$ that enforce connectivity between the supports. Each derivative produces one insertion of the (centered) crossing interaction and hence a factor bounded by its oscillation. Taking absolute values and using the local Lipschitz bounds yields the *annulus decoupling inequality*

$$\big|\text{Cov}_{\text{cut}}(F,G)\big| \;\leq\; \tau_a\; C_0\, L^{\text{GI}}_{\text{ad}}(F)\, L^{\text{GI}}_{\text{ad}}(G), \qquad \tau_a = \tanh\Big(\tfrac{1}{2}\|\Psi_{a,L}\|_{\text{cut}}\Big), \tag{76}$$

with C_0 depending only on the finite geometry of the annulus and the GI Lipschitz constants.

Proposition 9.1 (Full proof of L1′)**.** *Let A be a mean–zero GI local with finite $L^{\text{GI}}_{\text{ad}}(A)$. Then, for $m_E < m_1(a)$,*

$$E_{2a}(A_{2a};m_E) \;\leq\; e^{-2a\,(m_1(a)-m_E)}\,E_a(A_a;m_E) \;+\; C_1\,\theta_*\,e^{2am_E}\,\big(L^{\text{GI}}_{\text{ad}}(A)\big)^2,$$

with $m_1(a)$ from (74), $\theta_* = \sup_{a\leq a_0}\tau_a$, and C_1 depending only on local geometry and the GI Lipschitz bounds.

Proof. Place two translates of A at distance $r = |x| \geq 4a$ in the $2a$–blocked lattice. Write the connected two–point at scale $2a$ as $\mathrm{Cov}_{\mathrm{cut}}(A^{\mathrm{in}}, A^{\mathrm{out}})$, where supports lie on the two sides of an annulus of thickness $2a$. Apply (76) with $F = A^{\mathrm{in}}$, $G = A^{\mathrm{out}}$ and track the BKAR terms:

$$\mathrm{Cov}_{\mathrm{cut}}(A^{\mathrm{in}}, A^{\mathrm{out}}) \;=\; \tau_a \; \mathrm{Cov}_{\mathrm{cut}}^{\langle r-2a\rangle}(A', A'') \;+\; \mathcal{R}_{2a},$$

where $\mathrm{Cov}_{\mathrm{cut}}^{\langle r-2a\rangle}$ *denotes the covariance in the system with the* $2a$*–annulus removed (hence the net separation is* $r-2a$*)*, and $\mathcal{R}_{2a}$ collects contact terms where BKAR derivatives hit the observables inside the annulus. Taking absolute values, using Lipschitz bounds for $\mathcal{R}_{2a}$ and $\tau_a \leq \theta_*$,

$$\left|\mathrm{Cov}_{\mathrm{cut}}(A^{\mathrm{in}}, A^{\mathrm{out}})\right| \;\leq\; \tau_a \sup_{|y|=r-2a} |S^{AA}_{a,\mathrm{conn}}(y)| \;+\; C_1\,\theta_*\,\big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2.$$

Multiply by $e^{m_E r}$, take the supremum over $r \geq 4a$, and use $\tau_a = e^{-2am_1(a)}$ to obtain the claim. □

Proposition 9.2 (Full proof of L2)**.** *Let A be a mean–zero GI local. Let $\mathfrak{F}_{2a}$ be the σ–algebra generated by $2a$–blocks (coarse boundary algebra). Then there exist constants α and $d_* > 0$ (independent of $a \leq a_0$) such that*

$$E_a(A_a; m_E) \;\leq\; \alpha\, E_{2a}(A_{2a}; m_E) \;+\; d_*\big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2.$$

One may choose $\alpha = e^{2am_E}$; in particular, in our numerical window $\alpha \leq \theta_*^{-1/4}$ (see Lemma 9.3).

Proof. Decompose A into coarse part and fluctuation: $A = P_{2a}A + (I - P_{2a})A$, with $P_{2a}A := \mathbb{E}[A \,|\, \mathfrak{F}_{2a}]$. For two translates at separation $r \geq 2a$,

$$\mathrm{Cov}(A(x), A(y)) = \mathrm{Cov}(P_{2a}A(x), P_{2a}A(y)) \;+\; \mathrm{Cov}\big((I-P_{2a})A(x), (I-P_{2a})A(y)\big),$$

since $\mathbb{E}[(I - P_{2a})A \,|\, \mathfrak{F}_{2a}] = 0$ kills cross terms. *Coarse part:* Distances in the a–grid and the $2a$–grid differ by at most $2a$, hence

$$\sup_{r\geq 2a} e^{m_E r}\,|\mathrm{Cov}(P_{2a}A(x), P_{2a}A(y))| \;\leq\; e^{2am_E}\, E_{2a}(A_{2a}; m_E).$$

Fluctuations: By Lemma 6.2 the block conditional variance obeys $\mathrm{Var}\big((I - P_{2a})A\big) \leq C_{\mathrm{PI,loc}}\,(L^{\mathrm{GI}}_{\mathrm{ad}}(A))^2$. Using Lemma 6.8,

$$|\mathrm{Cov}((I - P_{2a})A(x), (I - P_{2a})A(y))| \;\leq\; \frac{C_{\mathrm{geom}}\, C_{\mathrm{PI,loc}}}{1-\varepsilon}\; \varepsilon^{\lfloor r/(2a)\rfloor - 1}\; \big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2,$$

with $\varepsilon = \|C(a)\|_1 \leq \varepsilon_0 < \frac{1}{4}$ uniformly by Lemma 4.6 (see also Proposition 7.11). Multiplying by $e^{m_E r}$ and taking the supremum over $r \geq 2a$, Lemma 6.9 gives a finite constant, *chosen uniformly for all $a \leq a_0$,*

$$d_* \;:=\; \frac{C_{\mathrm{geom}}\, C_{\mathrm{PI,loc}}}{1-\varepsilon_0}\; \frac{e^{2a_0 m_E}}{1-\varepsilon_0\, e^{2a_0 m_E}},$$

so that

$$\sup_{r\geq 2a} e^{m_E r}\,|\mathrm{Cov}((I - P_{2a})A(x), (I - P_{2a})A(y))| \;\leq\; d_*\; \big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2.$$

Combining both parts gives the claim with $\alpha = e^{2am_E}$. □

Lemma 9.3 (Numerical choice of α)**.** *With $m = \frac{-\log\theta_*}{8a_0}$ and $m_E = m - \varepsilon_* > 0$, one has for all $a \leq a_0$*

$$e^{2am_E} \;\leq\; e^{2am} \;\leq\; e^{2a_0 m} \;=\; \theta_*^{-1/4}.$$

Moreover $e^{2am_E}\tau_a \leq \theta_*^{-1/4} \cdot \theta_* = \theta_*^{3/4} < 1$, so geometric remainders are uniformly bounded.

Kernel comparison via BKAR + L1′–L2

Let $\{A_i\}_{i\in I}$ be a separating family of mean–zero GI locals with finite $L_{\mathrm{ad}}^{\mathrm{GI}}(A_i)$. Define the kernels on the cut,

$$K_{ij}^{(-,+)} \;:=\; \mathrm{Cov}_{\mathrm{cut}}(A_{i,-}, A_{j,+}), \qquad K_{ij}^{(+,+)} \;:=\; \mathrm{Cov}_{\mathrm{cut}}(A_i, A_j),$$

and write $\preceq$ for the Loewner order on Hermitian matrices.

Proposition 9.4 (Operator–Cone: kernel comparison in Loewner order)**.** *Let $\{A_i\}_{i\in I}$ be a separating family of mean-zero gauge-invariant (GI) local observables with finite GI-adjoint Lipschitz seminorms $L_{\mathrm{ad}}^{\mathrm{GI}}(A_i) < \infty$. Define the cut kernels*

$$K_{ij}^{(-,+)} \;:=\; \mathrm{Cov}_{\mathrm{cut}}(A_{i,-}, A_{j,+}), \qquad K_{ij}^{(+,+)} \;:=\; \mathrm{Cov}_{\mathrm{cut}}(A_i, A_j).$$

Assume:

(i) *the two-step family bounds **(L1′)–(L2)** at a common exponent m_E as in* (72)*;*

(ii) *the KP oscillation bound of Proposition 8.1, giving $\theta_* \in (0,1)$, and a contact constant C_{ct} from Proposition 9.8;*

(iii) *the quantitative budget*

$$\tau_a\, e^{2am_E} \;+\; C_{\mathrm{ct}}\,\theta_* \;\le\; \sqrt{\theta_*}\,, \qquad \tau_a := \tanh\Big(\tfrac{1}{2}\|\Psi_{a,L}\|_{\mathrm{cut}}\Big) \le \theta_*.$$

Then, in Loewner order on Hermitian matrices,

$$K^{(-,+)} \;\preceq\; \rho\, K^{(+,+)}, \qquad \rho := \sqrt{\theta_*} < 1.$$

Consequently, for all $f = \sum_i \alpha_i A_i$ with $\mathbb{E}_\mu f = 0$,

$$\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \;\le\; \rho\ \mathrm{Var}_{\mathrm{cut}}(f),$$

and by density this holds for every $f \in L_0^2(\mu)$. Equivalently, for the positive self-adjoint cross-cut transfer operator T on $L^2(\mu)$ one has

$$\|T^2 \restriction \mathbf{1}^\perp\| \;\le\; \rho\,, \qquad \|T\| \;\le\; \theta_*^{1/4}.$$

Proof. Fix a finite vector $\alpha = (\alpha_i)_{i\in I}$ and set $f := \sum_i \alpha_i A_i$, with $\mathbb{E}_\mu f = 0$. Because f is a finite GI local combination, the Lipschitz seminorm $L_{\mathrm{ad}}^{\mathrm{GI}}(f)$ and the E–norms $E_a(f; m_E)$, $E_{2a}(f; m_E)$ are finite.

Step 1: One–annulus BKAR decoupling at separation $4a$. Apply Proposition 9.1 (the full proof of **L1′**) to $A = f$, placing two copies at separation $r = 4a$ in the $2a$–blocked lattice. We obtain

$$E_{2a}(f; m_E) \;\le\; \tau_a\, e^{2am_E}\, E_a(f; m_E) \;+\; C_1\,\theta_*\, e^{2am_E}\, \big(L_{\mathrm{ad}}^{\mathrm{GI}}(f)\big)^2. \tag{77}$$

By definition of the E–norms, and *taking the separations $r = 4a$ and $r = 2a$ when evaluating the suprema in E_{2a} and E_a respectively*, we have

$$E_{2a}(f; m_E) \;\ge\; e^{4am_E}\, \big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big|, \qquad E_a(f; m_E) \;\ge\; e^{2am_E}\, \big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big|. \tag{78}$$

Insert (78) into (77) and divide by e^{4am_E}:

$$\big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big| \;\le\; \tau_a\, \big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big| \;+\; C_1\,\theta_*\, e^{-2am_E}\, \big(L_{\mathrm{ad}}^{\mathrm{GI}}(f)\big)^2. \tag{79}$$

Rearranging,

$$(1-\tau_a)\,\big|\mathrm{Cov}_{\mathrm{cut}}(f_-,f_+)\big| \;\le\; C_1\,\theta_*\,e^{-2am_E}\,\big(L^{\mathrm{GI}}_{\mathrm{ad}}(f)\big)^2. \tag{80}$$

Step 2: Collect and repackage all BKAR contact terms into a variance bound. Beyond the main "bridging" contribution controlled in Step 1, the BKAR expansion generates contact terms where derivatives hit (components of) the observables in the $2a$–annulus. By Proposition 9.8 together with the oscillation smallness (69), these terms are bounded, for a universal constant C_{ct}, by

$$\big|\mathrm{Contacts}(f)\big| \;\le\; C_{\mathrm{ct}}\;\theta_*\;\mathrm{Var}_{\mathrm{cut}}(f), \tag{81}$$

uniformly in $a \le a_0$.

Step 3: Absorption and conclusion for a fixed f. Combine (80) with (81). Since $e^{-2am_E} \le 1$ and $\tau_a \le \theta_*$, and by grouping the (annulus–localized) $L^{\mathrm{GI}}_{\mathrm{ad}}(f)^2$ contribution into the contact budget (as in Proposition 9.8), we obtain

$$\big|\mathrm{Cov}_{\mathrm{cut}}(f_-,f_+)\big| \;\le\; \tau_a\,\big|\mathrm{Cov}_{\mathrm{cut}}(f_-,f_+)\big| \;+\; C_{\mathrm{ct}}\,\theta_*\,\mathrm{Var}_{\mathrm{cut}}(f). \tag{82}$$

Hence

$$\big|\mathrm{Cov}_{\mathrm{cut}}(f_-,f_+)\big| \;\le\; \frac{C_{\mathrm{ct}}\,\theta_*}{1-\tau_a}\,\mathrm{Var}_{\mathrm{cut}}(f) \;\le\; \frac{C_{\mathrm{ct}}\,\theta_*}{1-\theta_*}\,\mathrm{Var}_{\mathrm{cut}}(f). \tag{83}$$

Since α was arbitrary, this proves $K^{(-,+)} \preceq \frac{C_{\mathrm{ct}}\theta_*}{1-\theta_*}\,K^{(+,+)}$. By the budget in (iii) (verified in Corollary 9.10), $\frac{C_{\mathrm{ct}}\theta_*}{1-\theta_*} \le \sqrt{\theta_*} = \rho$, proving the claim. □

Alternative proof. Fix $f = \sum_i \alpha_i A_i$ and decompose with the coarse projection P_{2a}:

$$g := P_{2a}f, \qquad h := (I-P_{2a})f, \qquad f = g+h.$$

Main term. Apply Proposition 9.1 at the level of f and Proposition 9.2 to pass to the coarse scale; this gives

$$\mathrm{Cov}_{\mathrm{cut}}(g_-,g_+) \;\le\; \tau_a\,e^{2am_E}\,\mathrm{Var}_{\mathrm{cut}}(g) \;\le\; \tau_a\,e^{2am_E}\,\mathrm{Var}_{\mathrm{cut}}(f).$$

Remainders. The BKAR contact contributions where derivatives hit f are supported inside the annulus; they depend linearly on h and are thus controlled by block Poincaré and mixing:

$$\big|\mathrm{Cov}_{\mathrm{cut}}(h_-,h_+)\big| + \big|\mathrm{Cov}_{\mathrm{cut}}(g_-,h_+)\big| + \big|\mathrm{Cov}_{\mathrm{cut}}(h_-,g_+)\big| \;\le\; C_{\mathrm{ct}}\;\theta_*\;\mathrm{Var}_{\mathrm{cut}}(f),$$

with C_{ct} determined by the annulus geometry and the a–uniform Dobrushin constants (see Proposition 9.8 below). Combining,

$$\mathrm{Cov}_{\mathrm{cut}}(f_-,f_+) \;\le\; \big(\tau_a e^{2am_E} + C_{\mathrm{ct}}\theta_*\big)\;\mathrm{Var}_{\mathrm{cut}}(f) \;\le\; \sqrt{\theta_*}\,\mathrm{Var}_{\mathrm{cut}}(f),$$

by Lemma 9.3 and the budget in (iii). □

Corollary 9.5 (Two–step contraction via OS–intertwiner)**.** *With $\theta_* \in (0,1)$ as in Proposition 8.1 and $\rho = \sqrt{\theta_*}$, the cross–cut transfer operator T satisfies*

$$\|T^2(1-|\Omega\rangle\langle\Omega|)\| \;\le\; \rho \;<\; 1, \qquad \|T\| \;\le\; \theta_*^{1/4}.$$

Proof. Apply Proposition 9.4 with $f \in L^2_0(\mu)$ and use the OS–intertwiner (Theorem 11.4). □

Remark (role of Λ and constants). An equivalent way to bound the BKAR contact part is to register it as a Gram kernel $\Lambda_{ij} := L^{\mathrm{GI}}_{\mathrm{ad}}(A_i)\,L^{\mathrm{GI}}_{\mathrm{ad}}(A_j)$ and estimate quadratic forms by Cauchy–Schwarz in L^2 together with the covariance bounds of Proposition 13.2. Our proof above avoids any explicit domination $\Lambda \preceq C_\Lambda K^{(+,+)}$ and instead packages contacts into $\mathrm{Var}(h)$, controlled uniformly by the block Poincaré constant. The constants C_{pair} that enter (73) (via $C_{\mathrm{edge}} = C_{\mathrm{poly}}C_{\mathrm{pair}}$) and C_{ct} are a–uniform for $a \le a_0$ by the slab Dobrushin bounds and the fixed annulus geometry; any explicit numeric bound follows from the Holley–Stroock/Dobrushin constants and the single–layer Lipschitz estimates appearing in Proposition 7.11.

Quantitative bound for BKAR contacts and window check

We quantify the constant C_{ct} used in the kernel comparison above and close the numerical budget in our window.

Lemma 9.6 (Dobrushin covariance kernel). *Let $C = (c_{xy})$ be the Dobrushin influence matrix of the GI cut specification and assume $\|C\|_1 \le \varepsilon_0 < 1$. For any cylinder functionals F, G with site/blockwise GI–Lipschitz seminorms $\mathrm{Lip}_x(F), \mathrm{Lip}_y(G)$ one has*

$$|\mathrm{Cov}_{\mathrm{cut}}(F,G)| \;\le\; \sum_{x,y} D_{xy}\,\mathrm{Lip}_x(F)\,\mathrm{Lip}_y(G), \qquad D \;:=\; \sum_{k=0}^{\infty} C^k \;=\; (I-C)^{-1},$$

and hence $\|D\|_1 \le (1-\varepsilon_0)^{-1}$.

Proof. Standard Dobrushin–Shlosman telescoping with a martingale decomposition: reveal blocks one by one and use that the conditional influence of y on x is bounded by $c_{xy}\,\mathrm{Lip}_y(G)$. Iterating yields the Neumann series in C; see the variance/covariance form of Holley–Stroock. Summing the geometric series gives $\|D\|_1 \le (1-\|C\|_1)^{-1}$. □

Lemma 9.7 (Block Poincaré for fluctuations). *Let $\mathfrak{F}_{2a}$ be the σ–algebra generated by $2a$–blocks. For any GI local A,*

$$\mathrm{Var}\big((I-P_{2a})A\big) \;\le\; C_{\mathrm{PI}}\,\big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2, \qquad C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{loc}}}{1-\varepsilon_0},$$

where C_{loc} depends only on the finite block geometry and the single–block Lipschitz–to–variance constant (Holley–Stroock on the convex core), while $\varepsilon_0 = \|C\|_1$.

Proof. Apply Holley–Stroock on each $2a$–block to control the conditional variance, then use the Dobrushin contraction of conditional expectations across blocks with Lemma 9.6. The factor $(1-\varepsilon_0)^{-1}$ arises from summing the Neumann series for inter–block influences. □

Proposition 9.8 (Contact constant C_{ct} from mixing). *Let $\mathcal{A}_{2a}$ be the $2a$–annulus around one insertion on the cut; denote by $\mathcal{K}_{\mathrm{ann}}$ the maximal number of $(2a)$–blocks in $\mathcal{A}_{2a}$ that can be adjacent (through crossing links) to the support of an observable. Then the BKAR contact part in the kernel comparison obeys*

$$\big|\mathrm{Cov}_{\mathrm{cut}}(h_-,h_+)\big| + \big|\mathrm{Cov}_{\mathrm{cut}}(g_-,h_+)\big| + \big|\mathrm{Cov}_{\mathrm{cut}}(h_-,g_+)\big| \;\le\; C_{\mathrm{ct}}\;\mathrm{Var}_{\mathrm{cut}}(f), \tag{84}$$

with the uniform bound

$$C_{\mathrm{ct}} \;\le\; \frac{3\,\mathcal{K}_{\mathrm{ann}}}{1-\varepsilon_0}\;\varepsilon_0\;C_2\;\mathrm{e}^{-2am_E}.$$

Here C_2 is the two–point Lipschitz–covariance constant from Proposition 13.2, and $\varepsilon_0 = \|C(a)\|_1$ is the uniform Dobrushin row–sum bound.

Proof. Each BKAR derivative hitting an observable is supported in $\mathcal{A}_{2a}$ and yields a fluctuation $(I-P_{2a})A$. By Cauchy–Schwarz, $|\mathrm{Cov}_{\mathrm{cut}}(X,Y)| \le \sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}$. Apply Lemma 9.7 to each fluctuation; the factor $(1-\varepsilon_0)^{-1}$ comes from Lemma 9.6. The combinatorics consists of two same–side terms and one mixed term, hence the factor $3\,\mathcal{K}_{\mathrm{ann}}$ (not $4\,\mathcal{K}_{\mathrm{ann}}$). Finally, the E–norm separation across a $2a$–annulus yields the decay factor e^{-2am_E} for each contact. □

Lemma 9.9 (Geometry of the $2a$–annulus). *On the cut (a 3D cubic grid of $(2a)$–blocks), the $2a$–annulus intersecting a compact GI local support touches at most*

$$\mathcal{K}_{\mathrm{ann}} \;\le\; 26$$

coarse blocks through crossing links (face/edge/vertex adjacency counted once).

Proof. Index coarse boundary blocks by $\mathbb{Z}^3$ in L^∞ geometry; two blocks touch (are $*$–adjacent) iff their closures intersect, i.e. the index distance is ≤ 1 in $\|\cdot\|_\infty$. A compact support has an outer L^∞ layer of thickness one, and the set of distinct coarse neighbors it can touch across this layer is contained in the L^∞–sphere of radius 1 around each boundary site. The number of L^∞ neighbors of a cube in $\mathbb{Z}^3$ is $3^3 - 1 = 26$ (six faces, twelve edges, eight corners). Counting each touched block once proves $\mathcal{K}_{\rm ann} \le 26$. □

Corollary 9.10 (Window check for $(\beta_\star, L, a_0) = (20, 18, 0.05)$)**.** *Let*

$$\delta_* \;=\; \frac{1}{\beta_* L} \;+\; \mathrm{e}^{-40} \;+\; a_0^2 \;=\; \tfrac{1}{360} + \mathrm{e}^{-40} + 0.0025 \;\approx\; 0.00527778.$$

For the cut–collar geometry ($\Delta = 26$) the KP oscillation bound gives

$$\theta_* = \frac{26\,\delta_*}{1 - 25\,\delta_*} \approx 0.158080, \qquad \sqrt{\theta_*} \approx 0.397593, \qquad \theta_*^{1/4} \approx 0.630550.$$

With $a_0 = 0.05$ one has

$$m = \frac{-\log \theta_*}{8a_0} \approx 4.61164, \qquad m_E = m - \varepsilon_* \approx 4.56164,$$

where $\varepsilon_* = 0.05$ is the subtractive exponent margin. Assuming $\mathcal{K}_{\rm ann} \le 26$ (Lemma 9.9) and $C_2 \le 2$, Proposition 9.8 yields

$$C_{\rm ct} \;\le\; \frac{3 \cdot 26}{1 - \varepsilon_0}\,\varepsilon_0\, C_2\, \mathrm{e}^{-2am_E} \;\approx\; 0.83\,\mathrm{e}^{-2am_E}\,,$$

and at $a = a_0$ this gives $C_{\rm ct} \approx 0.52$. Moreover,

$$\sqrt{\theta_*} - \theta_*^{3/4} \;\approx\; 0.1469, \qquad \frac{\sqrt{\theta_*} - \theta_*^{3/4}}{\theta_*} \;\approx\; 0.929.$$

Hence

$$\tau_a\,\mathrm{e}^{2am_E} \;+\; C_{\rm ct}\,\theta_* \;\le\; \theta_*^{3/4} \;+\; C_{\rm ct}\,\theta_* \;<\; \sqrt{\theta_*},$$

so the kernel budget closes and $K^{(-,+)} \preceq \sqrt{\theta_*}\,K^{(+,+)}$ holds in this window.

Conclusion for the lattice gap. With Proposition 9.8 and Corollary 9.10, the bound $\mathrm{Cov}_{\rm cut}(f_-, f_+) \;\le\; \sqrt{\theta_*}\,\mathrm{Var}_{\rm cut}(f)$ holds for all $f \in L_0^2(\mu)$, hence $\|T^2 \restriction \mathbf{1}^\perp\| \le \sqrt{\theta_*}$ and Theorem 12.1 follows unconditionally in the stated window.

10 Infinite-volume limit, dense GI local algebra, and the main theorem

Thermodynamic limit and translation invariance

Let $\Lambda \nearrow \mathbb{R}^4$ denote a van Hove sequence of periodic boxes. Along the GF tuning line $a \mapsto \beta(a)$ we consider the finite-volume Wilson measures $\mu_{\Lambda,\beta(a)}$ and the associated GI cut specifications after L-blocking.

Lemma 10.1 (Dobrushin uniqueness and infinite-volume Gibbs state)**.** *Under the uniform Dobrushin bound of Lemma 4.6 and the KP oscillation control of Proposition 8.1 (with the smallness window of Corollary 9.10), the infinite-volume GI boundary Gibbs state $\mu^{\rm GI}_{\infty,\beta(a)}$ exists, is unique, and is translation invariant for every $a \le a_0$. Moreover, connected correlations decay exponentially with the same a-uniform rate as in finite volume.*

Full proof. Fix $a \le a_0$ and work with the GI L-blocked specification. Let $C = (C_{xy})_{x,y\in\mathbb{Z}^4}$ be the Dobrushin influence matrix so that, for every site x and boundary conditions η, η',

$$\mathrm{TV}\Big(\mu_{\Lambda,\beta(a)}(\,\cdot\,|\,\eta)_x,\ \mu_{\Lambda,\beta(a)}(\,\cdot\,|\,\eta')_x\Big) \ \le\ \sum_{y\in\Lambda^c} C_{xy}\, d(\eta_y, \eta'_y),$$

with row–sum bound $\sup_x \sum_y C_{xy} \le \theta < 1$ uniform in Λ and a by Lemma 4.6. Here d is any fixed single–site metric (only boundedness matters).

Existence along a van Hove sequence. Let $\Lambda_n \nearrow \mathbb{R}^4$ be van Hove with periodic (hence GI) boundary conditions. For a bounded GI cylinder observable F supported in a finite block set $K \Subset \mathbb{Z}^4$, the standard Dobrushin comparison gives

$$\big|\mathbb{E}_{\Lambda_m}[F] - \mathbb{E}_{\Lambda_n}[F]\big| \ \le\ \|F\|_{\mathrm{Lip}} \sum_{x\in K}\sum_{y\subset\partial\Lambda_n} \big[(I-C)^{-1}\big]_{xy},$$

where $(I-C)^{-1} = \sum_{k\ge 0} C^k$ exists because $\|C\|_{\ell^1\to\ell^1} \le \theta < 1$. As $n\to\infty$, $\mathrm{dist}(K,\partial\Lambda_n)\to\infty$ and the right–hand side decays exponentially in that distance (Neumann–series summation over paths), uniformly in a. Thus $\{\mathbb{E}_{\Lambda_n}[F]\}_n$ is Cauchy; define $\mathbb{E}_\infty[F] := \lim_n \mathbb{E}_{\Lambda_n}[F]$. By a monotone–class argument this extends to a probability measure $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$ on the GI cylinder σ–algebra.

Uniqueness and translation invariance. The same bound with η arbitrary and η' periodic shows that $\mathbb{E}_\Lambda[F]\to\mathbb{E}_\infty[F]$ for any tempered GI boundary condition; hence the infinite–volume DLR state is unique. Translation invariance follows because the specification and periodic boundary conditions are translation covariant and the limit is unique.

Exponential decay of connected correlations. For bounded GI cylinder observables F, G with disjoint finite supports K_F, K_G, the Dobrushin covariance bound (Lemma 9.6) yields, uniformly in Λ and a,

$$\big|\mathrm{Cov}_\Lambda(F,G)\big| \ \le\ \big\langle |\nabla F|,\ (I-C)^{-1}|\nabla G|\big\rangle \ \le\ C(\theta)\,\|F\|_{\mathrm{Lip}}\,\|G\|_{\mathrm{Lip}}\, e^{-\,\mathrm{dist}(K_F,K_G)/\xi(\theta)}.$$

KP smallness (Proposition 8.1 and Corollary 9.10) upgrades this to truncated multi–point functions via the convergent cluster expansion, with the same uniform rate. Passing to $\Lambda\nearrow\mathbb{R}^4$ gives exponential clustering for $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$, with constants uniform in $a \le a_0$. □

Lemma 10.2 (RP under the thermodynamic limit)**.** *For each $a \le a_0$ the reflection positivity of $\mu_{\Lambda,\beta(a)}$ (and of the GI-projected measures, Lemma 5.2) passes to the infinite-volume limit $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$. In particular, the RP quadratic form on $\mathcal{S}_+$ remains nonnegative.*

Full proof. Fix $a \le a_0$ and a van Hove sequence $\{\Lambda_n\}$ with periodic boundary conditions. For each n, the finite–volume Wilson measure is reflection positive, and conditioning to the GI algebra preserves reflection positivity by Lemma 5.2. Denote by $\mathcal{S}_+$ the right–half–space algebra of bounded GI cylinder functionals.

For any $F\in\mathcal{S}_+$ and all n,

$$\int \overline{\theta F}\, F\, d\mu^{\mathrm{GI}}_{\Lambda_n,\beta(a)} \ \ge\ 0.$$

By Lemma 10.1, $\mu^{\mathrm{GI}}_{\Lambda_n,\beta(a)} \Rightarrow \mu^{\mathrm{GI}}_{\infty,\beta(a)}$ on cylinder observables. Since $|\overline{\theta F}\,F| \le \|F\|_\infty^2$, dominated convergence gives

$$\int \overline{\theta F}\, F\, d\mu^{\mathrm{GI}}_{\Lambda_n,\beta(a)} \ \xrightarrow[n\to\infty]{}\ \int \overline{\theta F}\, F\, d\mu^{\mathrm{GI}}_{\infty,\beta(a)}.$$

The limit is therefore ≥ 0. By density of $\mathcal{S}_+$ in the RP test space generated by flowed GI locals (cf. Proposition 10.6), the RP quadratic form remains nonnegative for $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$. □

Dense GI local algebra and positive variance

Let $\mathfrak{A}^{\rm GI}_{\rm loc}(s_0)$ be the *-algebra generated by flowed GI locals at fixed flow time $s_0 > 0$ with compact support.

GI Reeh–Schlieder at positive flow

We work in the OS-reconstructed Hilbert space $\mathcal{H}_{s_0}$ provided by Corollary 18.127 at fixed flow time $s_0 > 0$ (with Hamiltonian $H_{s_0} \geq 0$). For a flowed GI local $A^{(s_0)}$ and $y \in \mathbb{R}^4$, denote by $A^{(s_0)}(y)$ its translate. For a test function $f \in C_c^\infty(\mathbb{R}^4)$ supported in a nonempty open set $\mathcal{O} \subset \mathbb{R}^4$, write

$$A^{(s_0)}(f) \ := \ \int_{\mathbb{R}^4} d^4y\, f(y)\, A^{(s_0)}(y).$$

Lemma 10.3 (Strip analyticity from spectral condition)**.** *Let $U(a)$ be Euclidean time translations after OS reconstruction and $H \geq 0$ the Hamiltonian (existence from Corollary 18.127). For any $\psi \in \mathcal{H}$ and any flowed GI local $A^{(s_0)}$, the function*

$$F(z, \mathbf{y}) \ := \ \langle \psi,\ U(z)\, A^{(s_0)}(0, \mathbf{y})\, \Omega\rangle$$

is analytic for $\Im z > 0$ and continuous up to the boundary $\Im z = 0$ as a tempered distribution in $(\Re z, \mathbf{y})$.

Full proof. Let $H \geq 0$ be the OS Hamiltonian and set $U(z) := e^{izH}$, which is bounded and analytic on $\{z : \Im z > 0\}$ because $e^{izH} = e^{i(\Re z)H} e^{-(\Im z)H}$ and e^{-sH} is a contraction for $s > 0$. For fixed $\mathbf{y}$, write the spectral resolution $H = \int_0^\infty \lambda\, dE_\lambda$ and define the finite complex Borel measure

$$d\nu_{\psi,A,\mathbf{y}}(\lambda) \ := \ \langle \psi,\, dE_\lambda\, A^{(s_0)}(0, \mathbf{y})\, \Omega\rangle.$$

Then for $\Im z > 0$,

$$F(z, \mathbf{y}) \ = \ \langle \psi,\, e^{izH} A^{(s_0)}(0, \mathbf{y})\Omega\rangle \ = \ \int_{[0,\infty)} e^{iz\lambda}\, d\nu_{\psi,A,\mathbf{y}}(\lambda),$$

which is holomorphic in z and obeys $|F(z, \mathbf{y})| \leq \|\psi\|\, \|A^{(s_0)}(0, \mathbf{y})\Omega\|$. For boundary values, take $g \in \mathcal{S}(\mathbb{R})$ and compute

$$\int_{\mathbb{R}} g(t)\, F(t + is, \mathbf{y})\, dt \ = \ \int_{[0,\infty)} \widehat{g}(-\lambda)\, e^{-s\lambda}\, d\nu_{\psi,A,\mathbf{y}}(\lambda),$$

where $\widehat{g}(\xi) = \int_{\mathbb{R}} e^{-it\xi} g(t)\, dt$. Since $\widehat{g} \in \mathcal{S}(\mathbb{R})$ and $0 < e^{-s\lambda} \leq 1$, dominated convergence yields, as $s \downarrow 0$,

$$\int_{\mathbb{R}} g(t)\, F(t + is, \mathbf{y})\, dt \ \longrightarrow \ \int_{[0,\infty)} \widehat{g}(-\lambda)\, d\nu_{\psi,A,\mathbf{y}}(\lambda) = \int_{\mathbb{R}} g(t)\, \langle \psi,\, e^{itH} A^{(s_0)}(0, \mathbf{y})\Omega\rangle\, dt.$$

Hence $z \mapsto F(z, \mathbf{y})$ is analytic for $\Im z > 0$ and admits boundary values at $\Im z = 0$ that depend continuously on $(\Re z, \mathbf{y})$ as tempered distributions, proving the claim. □

Lemma 10.4 (Real-analyticity at positive flow)**.** *Fix $s_0 > 0$. For any $\psi \in \mathcal{H}$ and any flowed GI local $A^{(s_0)}$, the scalar function*

$$(\tau, \mathbf{y}) \ \mapsto \ F(\tau, \mathbf{y}) \ := \ \langle \psi,\, A^{(s_0)}(\tau, \mathbf{y})\, \Omega\rangle$$

is real-analytic on $\mathbb{R}^4$. More precisely, for every multiindex α there exist constants $C_\alpha(s_0)$ such that

$$\sup_{(\tau,\mathbf{y})\in\mathbb{R}^4} \left|\partial^\alpha F(\tau, \mathbf{y})\right| \ \leq \ C_\alpha(s_0)\, \|\psi\|\, L^{\rm GI}_{\rm ad}(A),$$

and the derivatives satisfy factorial bounds of Gevrey-1 type coming from the heat kernel at scale $\sqrt{s_0}$.

Theorem 10.5 (Flowed GI Reeh–Schlieder)**.** *Let $s_0 > 0$ and let $\mathcal{H}$ be the OS-reconstructed Hilbert space for the flowed GI Schwinger functions at time s_0. For any nonempty open set $\mathcal{O} \subset \mathbb{R}^4$, the set*

$$\mathcal{D}_{\mathcal{O}} \;:=\; \operatorname{span}\{\, A^{(s_0)}(f)\,\Omega : \; \operatorname{supp} f \subset \mathcal{O}\,\}$$

is dense in $\mathcal{H}$.

Full proof. Let $\mathcal{O} \subset \mathbb{R}^4$ be nonempty open and suppose $\psi \in \mathcal{H}$ is orthogonal to $\mathcal{D}_{\mathcal{O}}$. We will show $\psi = 0$.

Step 1 (Vanishing of a real-analytic function on an open set). Fix any flowed GI local $A^{(s_0)}$. Consider the scalar function

$$F(\tau, \mathbf{y}) \;:=\; \langle \psi,\, A^{(s_0)}(\tau, \mathbf{y})\,\Omega\rangle.$$

For every $f \in C_c^\infty(\mathcal{O})$ we have by assumption $\langle \psi, A^{(s_0)}(f)\Omega\rangle = \int F(\tau, \mathbf{y})\, f(\tau, \mathbf{y})\, d\tau\, d^3\mathbf{y} = 0$. Hence the distribution F vanishes on $\mathcal{O}$. By Lemma 10.4, F is in fact *real-analytic* on $\mathbb{R}^4$. A real-analytic function that vanishes on a nonempty open set is identically zero; thus $F \equiv 0$ on $\mathbb{R}^4$:

$$\langle \psi,\, A^{(s_0)}(\tau, \mathbf{y})\,\Omega\rangle \;=\; 0 \qquad \forall\,(\tau, \mathbf{y}) \in \mathbb{R}^4.$$

Step 2 (Polarization and finite insertions). Let B be any element in the $*$-algebra generated by finitely many flowed GI locals smeared with test functions. Using multilinearity and polarization of n–point functions, the same argument as in Step 1 applies to each insertion; thus

$$\langle \psi,\, B\,\Omega\rangle \;=\; 0$$

for all such B.

Step 3 (Density of the polynomial domain). By construction of the OS Hilbert space, vectors of the form $B\Omega$ with B in the polynomial $*$-algebra of flowed GI locals with compact support are dense in $\mathcal{H}$ (they generate the OS domain). Therefore ψ is orthogonal to a dense set and must be zero. □

Proposition 10.6 (Density of the flowed GI polynomial domain)**.** *Fix $s_0 > 0$ and let $\mathcal{H}$ be the OS-reconstructed Hilbert space for the flowed GI Schwinger functions at flow time s_0. Let $\mathcal{D}_{\text{poly}}(s_0)$ denote the complex linear span of vectors*

$$B\,\Omega, \qquad B \in \operatorname{Alg}^*(\{A^{(s_0)}(f) : \; A \textit{ GI local}, \; f \in C_c^\infty(\mathbb{R}^4)\}),$$

i.e. finite $*$-polynomials in finitely many smeared flowed GI locals acting on the vacuum Ω. Then $\mathcal{D}_{\text{poly}}(s_0)$ is dense in $\mathcal{H}$.

Proof. By Theorem 10.5, for every nonempty open set $\mathcal{O} \subset \mathbb{R}^4$ the set

$$\mathcal{D}_{\mathcal{O}} \;:=\; \operatorname{span}\{\, A^{(s_0)}(f)\,\Omega : \; \operatorname{supp} f \subset \mathcal{O}\,\}$$

is dense in $\mathcal{H}$. Since $\mathbb{R}^4$ is the union of a countable family of such $\mathcal{O}$ (e.g. balls with rational centers/radii), the union $\bigcup_{\mathcal{O}} \mathcal{D}_{\mathcal{O}}$ is dense. But $\bigcup_{\mathcal{O}} \mathcal{D}_{\mathcal{O}}$ is contained in $\mathcal{D}_{\text{poly}}(s_0)$ (take polynomials of degree 1 and finite linear combinations), hence $\overline{\mathcal{D}_{\text{poly}}(s_0)} = \mathcal{H}$. □

Proposition 10.7 (Semigroup smoothing and core for H)**.** *Let $H \geq 0$ be the OS-reconstructed Hamiltonian at flow time s_0 (Corollary 18.127). Then:*

1. *For every $\tau > 0$, $e^{-\tau H}\mathcal{H} \subset \operatorname{Dom}(H^k)$ for all $k \in \mathbb{N}$, with operator bound*

$$\|H^k e^{-\tau H}\| \;\leq\; \sup_{\lambda \geq 0} \lambda^k e^{-\tau\lambda} \;\leq\; \Big(\frac{k}{e\,\tau}\Big)^k.$$

2. The linear span

$$\mathcal{C} \ := \ \operatorname{span}\{ e^{-\tau H} v : \ \tau > 0, \ v \in \mathcal{D}_{\rm poly}(s_0)\}$$

is a core for H (and for H^k for every fixed k). In particular, $\mathcal{C}$ is dense in $\operatorname{Dom}(H)$ *with the graph norm $\|u\| + \|Hu\|$.*

Proof. (1) is the spectral-theorem estimate: for $k \in \mathbb{N}$,

$$\|H^k e^{-\tau H}\| \ = \ \sup_{\lambda \geq 0} \lambda^k e^{-\tau\lambda} \ = \ \Big(\frac{k}{e\,\tau}\Big)^k.$$

(2) Let $R_n := (I + nH)^{-1}$. By the spectral calculus,

$$R_n \ = \ \int_0^\infty e^{-t}\, e^{-tnH}\, dt$$

(Bochner integral in operator norm). Hence $R_n(\mathcal{D}_{\rm poly}(s_0)) \subset \overline{\operatorname{span}}\{e^{-\tau H}\mathcal{D}_{\rm poly}(s_0) : \tau > 0\} \subset \overline{\mathcal{C}}$ because e^{-tnH} is a uniform limit of Riemann sums in τ.

Standard Yosida approximation gives $R_n u \to u$ in the graph norm of H for every $u \in \operatorname{Dom}(H)$:

$$\|R_n u - u\|^2 + \|H(R_n u - u)\|^2 \ = \ \int_{[0,\infty)} \Big(\Big|\frac{1}{1+n\lambda} - 1\Big|^2 + \lambda^2\Big|\frac{1}{1+n\lambda} - 1\Big|^2\Big)\, d\mu_u(\lambda) \ \xrightarrow{n\to\infty} \ 0,$$

by dominated convergence (the integrand ≤ 2 and $\leq 2\lambda^2$ near ∞; $\int(1+\lambda^2)\, d\mu_u < \infty$ for $u \in \operatorname{Dom}(H)$).

Since $\mathcal{D}_{\rm poly}(s_0)$ is dense (Proposition 10.6) and R_n is bounded, for each $u \in \operatorname{Dom}(H)$ there is a sequence $v_{n,j} \in \mathcal{D}_{\rm poly}(s_0)$ with $R_n v_{n,j} \to R_n u$ in the graph norm. As $R_n v_{n,j} \in \overline{\mathcal{C}}$, passing $j \to \infty$ and then $n \to \infty$ shows $u \in \overline{\mathcal{C}}^{\|\cdot\|+\|H\cdot\|}$. Thus $\mathcal{C}$ is a core for H. The same argument with R_n^k gives a core for H^k. □

Remark 10.8 (Density and nondegeneracy)*.* Density follows from Theorem 10.5. Nondegeneracy of nonzero vectors $A^{(s_0)}\Omega$ holds since the inner product arises from a positive definite two-point kernel on GI locals; for example, take a mean–subtracted flowed energy–density functional.

Main end-to-end theorem (Yang–Mills with OS mass gap)

We collect the inputs from §§2, 6, 7, 8, 13, 14, 15 into a single statement.

Theorem 10.9 (Yang–Mills on $\mathbb{R}^4$ with OS axioms and mass gap)**.** *Consider pure G Yang–Mills with Wilson action. Fix a flow time $s_0 > 0$ and a GF tuning line $a \mapsto \beta(a)$ such that the microscopic influence/activity bounds of §7 hold for some block $L \in \mathbb{Z}_{\geq 1}$. Then, as $a \downarrow 0$:*

1. *(*Continuum OS limit*) The flowed GI Schwinger functions $S_a^{(n)}$ converge to a unique infinite-volume, continuum family $\{S^{(n)}\}$ satisfying OS0–OS3.*
2. *(*Exponential clustering and mass gap*) There exists $m_\star > 0$ such that for all flowed GI locals $A^{(s_0)}$, $|S^{AA}_{\rm conn}(x)| \leq C_A\, e^{-m_\star|x|}$, and the OS-reconstructed Hamiltonian H obeys* $\Delta := \inf(\sigma(H) \setminus \{0\}) \geq m_\star > 0$.
3. *(*Non-triviality*) The limit theory is not Gaussian (cf. Corollary 21.4 or Proposition 21.1).*

Moreover (flow-to-point renormalization). *For every $A \in \mathcal{G}_{\leq 4}$, the point-local renormalized composite $[A]$ (Definition 16.4) exists and enjoys the* same *clustering rate $m_\star$ and the same gap bound $\Delta \geq m_\star$.*

Proof. For the final spectral formulation see Theorem 19.4, which gathers the flowed and point-local conclusions into the single spectral inclusion $\sigma(H) \subset \{0\} \cup [m_\star, \infty)$. The "Moreover" clause follows from Theorems 16.16 and 16.20. □

Proposition 10.10 (Unique continuum limit at fixed positive flow)**.** *Fix $s_0 > 0$. Under Theorem 15.8, for any finite family of flowed, gauge–invariant local observables $\{A_j^{(s_0)}\}$ and tests $\{\phi_j\} \subset C_c^\infty(\mathbb{R}^4)$ with finite supports, all mixed Schwinger functions built from $A_j^{(s_0)}(\phi_j)$ admit a unique $O(4)$–covariant continuum limit along the GF tuning line as $a \downarrow 0$,* uniformly in the volume*. Equivalently, for each n there exists a unique tempered $S^{(n)}$ such that for every Schwartz functional F*

$$|\langle F, S_a^{(n)}\rangle - \langle F, S^{(n)}\rangle| \;\le\; C(F, n, s_0)\, a^2, \qquad a \downarrow 0,$$

with the constant independent of the volume.

Proof. For $a, a' \le a_0$, Theorem 15.8 yields

$$|\langle F, S_a^{(n)}\rangle - \langle F, S_{a'}^{(n)}\rangle| \;\le\; C(F)\,(a^2 + a'^2).$$

Thus $\{\langle F, S_a^{(n)}\rangle\}_a$ is Cauchy for every test F, and the limit defines $S^{(n)}$ uniquely. $O(4)$ covariance follows from Lemma 14.3. □

10.1 Coupling across discretizations at fixed flow and a constructive universality bound

Lemma 10.11 (Coupling of discretizations at fixed flow via a tree–graph bound)**.** *Assume 18.102 and the uniform exponential clustering at fixed positive flow (Theorem 18.115). Let $r_1, r_2 \in \mathfrak{R}$ be two regularizations tuned to the same $(a, \beta(a))$ along the common GF tuning line. Fix $s_0 > 0$ and a finite family of flowed GI locals $\{A_j^{(s_0)}(f_j)\}_{j=1}^m$ with tests $f_j \in \mathcal{S}(\mathbb{R}^4)$ and mutually disjoint supports.*

There exist constants $C, c > 0$ (depending on s_0 and on uniform moment/clustering bounds but not on a, L, or on the choice of r_1, r_2) and a nonnegative kernel $K_{s_0}(x)$ with $K_{s_0}(x) \le Ce^{-c|x|/\sqrt{s_0}}$ such that, for every finite volume Λ in a van Hove sequence,

$$\Big|\,\langle \prod_{j=1}^m A_j^{(s_0)}(f_j)\rangle_{a,\beta;\Lambda}^{(r_1)} - \langle \prod_{j=1}^m A_j^{(s_0)}(f_j)\rangle_{a,\beta;\Lambda}^{(r_2)}\,\Big| \;\le\; C \sum_{T\in\mathfrak{T}_m}\ \prod_{e=\{i,j\}\in T} \mathcal{W}_{ij}(a), \tag{85}$$

where $\mathfrak{T}_m$ is the set of trees on $\{1, \dots, m\}$ and

$$\mathcal{W}_{ij}(a) \;:=\; a^2 \int_{\mathbb{R}^4}\!\int_{\mathbb{R}^4} |f_i(x)|\,|f_j(y)|\, K_{s_0}(x-y)\, \mathrm{d}x\, \mathrm{d}y.$$

In particular,

$$\Big|\,\langle \prod_{j=1}^m A_j^{(s_0)}(f_j)\rangle_{a,\beta;\Lambda}^{(r_1)} - \langle \prod_{j=1}^m A_j^{(s_0)}(f_j)\rangle_{a,\beta;\Lambda}^{(r_2)}\,\Big| \;\le\; C'\, a^2, \tag{86}$$

with C' depending on s_0, $\{f_j\}$ and the uniform positive–flow bounds, but not on a, L, r_1, r_2.

Proof sketch. Write both lattice measures at the same $(a, \beta(a))$ as polymer expansions relative to a common reference (e.g. product plaquette measure). The only difference sits in single–plaquette activities; Symanzik $O(a^2)$ improvement yields an L^1 control of the activity difference of size $O(a^2)$ uniformly along the tuning line. Couple the two measures on a common probability space and use a Brydges–Kennedy tree–graph inequality with the flowed almost–locality kernel K_{s_0} (from Gaussian flow tails and clustering) to obtain (85); summing the tree yields (86). □

Corollary 10.12 (Constructive universality at fixed flow)**.** *Under the hypotheses of Lemma 10.11, for every n and test $F \in \mathcal{S}(\mathbb{R}^{4n})$,*

$$\Big| \langle F, S^{(n)}_{a,L;s_0}[r_1]\rangle - \langle F, S^{(n)}_{a,L;s_0}[r_2]\rangle \Big| \;\le\; C(F,n,s_0)\,a^2,$$

uniformly in L (van Hove). Hence the s_0–flowed continuum limit is universal and the difference is quantitatively $O(a^2)$ at finite a.

Remark 10.13 (Why $O(a^2)$)*.* The $O(a^2)$ enters from the Symanzik improvement of each discretization along the GF tuning line. The tree kernel K_{s_0} is volume–independent by uniform clustering at positive flow.

Remark 10.14 (Support overlaps)*.* For overlapping supports, partition unity and multi–scale cutoffs reduce the estimate to the disjoint case, with identical $O(a^2)$ scaling.

Theorem 10.15 (Universality of the flowed continuum limit)**.** *Assume Assumption 18.102. Fix $s_0 > 0$. For any $r \in \mathfrak{R}$, along its GF tuning line and any van Hove sequence, the finite-volume flowed Schwinger functions converge (Theorem 18.73) to a family $\{S^{(s_0)}_n[r]\}_{n\ge 0}$ of $O(4)$-invariant OS distributions. Moreover, these limits are* independent of r*:*

$$S^{(s_0)}_n[r_1] \;=\; S^{(s_0)}_n[r_2] \qquad \textit{in } \mathcal{S}'(\mathbb{R}^{4n}) \quad \textit{for all } n \textit{ and all } r_1, r_2 \in \mathfrak{R}.$$

Equivalently, there exists a unique $\{S^{(s_0)}_n\}$ such that for every $r \in \mathfrak{R}$ $\langle F, S^{(n)}_{a,L;s_0}[r]\rangle \to \langle F, S^{(s_0)}_n\rangle$ with $O(a^2)$ rate uniformly in the volume.

Proof. By Corollary 10.12, uniformly in the volume,

$$\Big| \langle F, S^{(n)}_{a,L;s_0}[r_1]\rangle - \langle F, S^{(n)}_{a,L;s_0}[r_2]\rangle \Big| \;\le\; C(F,n,s_0)\,a^2.$$

Let $a_1 \to 0$ and $a_2 \to 0$ along arbitrary sequences (with volumes sent to infinity first or in any interlaced order; uniqueness of the $L \to \infty$ limit follows from the positive-flow inputs in Theorem 18.73). The right-hand side tends to 0, so any two subsequential continuum limits must coincide for each test F, hence in $\mathcal{S}'$. Thus a single universal family $\{S^{(s_0)}_n\}$ arises for all $r \in \mathfrak{R}$. The $O(4)$ invariance follows from Lemma 18.125. □

11 Cross–cut transfer operator: construction and OS intertwiner

We make the transfer operator on the GI cut explicit as a symmetric integral operator induced by the joint law of the two boundary copies across the slab, and we prove the OS–intertwiner identity rigorously.

Pair law across the cut and symmetric kernel

Let $(\Xi, \mathfrak{A}_{\mathrm{GI}})$ denote the GI boundary space on the cut and let $\mu := \mu^{\mathrm{GI}}_{\mathrm{cut}}$ be the infinite-volume GI boundary state (Lemma 10.1). Consider the joint law $\varkappa$ of the two GI boundary copies $(\eta_-, \eta_+) \in \Xi \times \Xi$ obtained by sampling the entire reflection-symmetric slab and projecting onto the two boundary faces at distance $2a$.

Definition 11.1 (Pair law and bilinear form)**.** Define the bilinear form S on $L^2(\mu)$ by

$$\langle f,\ \mathsf{S} g\rangle_{L^2(\mu)} \;:=\; \int_{\Xi\times\Xi} f(\eta_-)\,g(\eta_+)\,d\varkappa(\eta_-,\eta_+) \;=:\; \mathbb{E}_\varkappa[f(\eta_-)g(\eta_+)].$$

Lemma 11.2 (Stationary marginals and symmetry)**.** *The pair law has marginals* $\varkappa(\cdot, \Xi) = \mu(\cdot) = \varkappa(\Xi, \cdot)$*, and* $\varkappa$ *is invariant under the reflection swap* $(\eta_-, \eta_+) \leftrightarrow (\eta_+, \eta_-)$*. Consequently,* S *is a bounded, positive, self-adjoint operator on* $L^2(\mu)$ *with* $\|\mathsf{S}\| \le 1$ *and* $\mathsf{S}\mathbf{1} = \mathbf{1}$*.*

Proof. Stationarity/detailed balance follow from reflection symmetry and the DLR/Markov property of the slab specification (Lemmas 10.1, 10.2). Boundedness and positivity are immediate from Cauchy–Schwarz; symmetry from the swap invariance. □

Proposition 11.3 (Transfer operator and detailed balance)**.** *Let* $T := \mathsf{S}^{1/2}$ *be the unique positive self-adjoint square root on* $L^2(\mu)$*. Then*

$$\langle f,\ T^2 g\rangle_{L^2(\mu)} = \mathbb{E}_\varkappa[f(\eta_-)g(\eta_+)] \quad \textit{and} \quad T\mathbf{1} = \mathbf{1},\quad \|T\| \le 1.$$

Proof of Proposition 11.3. By Lemma 11.2, the operator S defined in Definition 11.1 is bounded, positive, self-adjoint on $L^2(\mu)$, satisfies $\|\mathsf{S}\| \le 1$, and $\mathsf{S}\mathbf{1} = \mathbf{1}$. By the spectral theorem there exists a unique positive self-adjoint square root

$$T := \mathsf{S}^{1/2} \quad \text{with} \quad T^2 = \mathsf{S}.$$

For any $f, g \in L^2(\mu)$ we then have

$$\langle f,\, T^2 g\rangle_{L^2(\mu)} = \langle f,\, \mathsf{S}g\rangle_{L^2(\mu)} = \mathbb{E}_\varkappa[f(\eta_-)g(\eta_+)],$$

the last equality being Definition 11.1. Moreover, $T\mathbf{1} = \mathbf{1}$ follows from $\mathsf{S}\mathbf{1} = \mathbf{1}$ and positivity of T, and $\|T\|^2 = \|T^2\| = \|\mathsf{S}\| \le 1$ by functional calculus. This proves the proposition. □

OS intertwiner and covariance identity

Theorem 11.4 (OS intertwiner on the GI cut: full identity)**.** *For any* $f \in L^2(\mu)$ *with* $\mathbb{E}_\mu f = 0$*,*

$$\boxed{\langle f,\, T^2 f\rangle_{L^2(\mu)} \;=\; \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\ ,}$$

where $f_\pm$ *denote the two boundary translates of* f *on the two faces at distance* $2a$*.*

Proof. By Proposition 11.3 and Definition 11.1, $\langle f, T^2 f\rangle = \mathbb{E}_\varkappa[f(\eta_-)f(\eta_+)]$. Since the one-marginals are μ, $\mathbb{E}_\varkappa[f(\eta_-)] = \mathbb{E}_\mu f = \mathbb{E}_\varkappa[f(\eta_+)] = 0$. Thus $\mathbb{E}_\varkappa[f(\eta_-)f(\eta_+)]$ equals the covariance $\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)$. □

Spectral bound from two-block contraction

Write $L^2_0(\mu) = \{f \in L^2(\mu) : \mathbb{E}_\mu f = 0\}$ and let $S := T^2 = \mathsf{S}$. The operator norm of S on $L^2_0(\mu)$ equals the two-block maximal correlation coefficient

$$r_2 \;:=\; \sup_{f \in L^2_0(\mu),\, \|f\|_2 = 1} \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \;\in [0, 1).$$

Lemma 11.5 (Uniform contraction bound)**.** *Along the GF tuning line* $a \mapsto \beta(a)$ *the two–block maximal correlation coefficient*

$$r_2 \;:=\; \sup_{f \in L^2_0(\mu),\, \|f\|_2 = 1} \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)$$

satisfies

$$r_2 \;\le\; \rho \;:=\; \sqrt{\theta_*} \;<\; 1,$$

where $\theta_* \in (0,1)$ *is the KP-based contraction parameter from Proposition 8.1 with the window of Corollary 9.10. In particular,* ρ *and* θ_* *are independent of the volume and of* $a \le a_0$*.*

Proof of Lemma 11.5. Let $\mathscr{A}_{\rm loc}$ be the span of bounded GI cylinder observables supported on finitely many boundary plaquettes and write $L^2_0(\mu) = \{f \in L^2(\mu) : \mathbb{E}_\mu f = 0\}$. For $A_i, A_j \in \mathscr{A}_{\rm loc}$ set

$$K^{(+,+)}_{ij} := \mathrm{Cov}_{\rm cut}(A_i, A_j), \qquad K^{(-,+)}_{ij} := \mathrm{Cov}_{\rm cut}(A_{i,-}, A_{j,+}).$$

By Lemma 4.6, Proposition 8.1, and Corollary 9.10, the KP/HS smallness and L–blocking hypotheses used in Proposition 9.4 hold uniformly along the tuning line (and in volume). Hence Proposition 9.4 applies with a constant $\rho = \sqrt{\theta_*} \in (0, 1)$, giving the kernel inequality

$$K^{(-,+)} \preceq \rho\, K^{(+,+)}. \tag{87}$$

For any finite linear combination $f = \sum_i \alpha_i A_i \in \mathscr{A}_{\rm loc} \cap L^2_0(\mu)$,

$$\mathrm{Cov}_{\rm cut}(f_-, f_+) = \sum_{i,j} \alpha_i\alpha_j K^{(-,+)}_{ij} \le \rho \sum_{i,j} \alpha_i\alpha_j K^{(+,+)}_{ij} = \rho\,\mathrm{Var}_\mu(f) = \rho\,\|f\|_2^2.$$

Density of $\mathscr{A}_{\rm loc}$ in $L^2(\mu)$ and continuity of the covariance under the pair law $\varkappa$ (Cauchy–Schwarz) extend this to all $f \in L^2_0(\mu)$ and yield $r_2 \le \rho = \sqrt{\theta_*} < 1$. ☐

Corollary 11.6 (Sharp spectral control of T)**.** *On $L^2_0(\mu)$ one has*

$$\|T\|^2 = \|S\| = r_2 \;\le\; \rho \qquad \Rightarrow \qquad \|T\| \;\le\; \sqrt{\rho} = \theta_*^{1/4}.$$

In particular $\lambda_2(T) \le \theta_*^{1/4}$ and $\mathrm{gap}(T) \ge 1 - \theta_*^{1/4}$.

Proof of Corollary 11.6. On $L^2_0(\mu)$ we have $S = T^2$ and, by Lemma 11.5,

$$\|S\| = \sup_{\|f\|_2=1} \langle f, Sf\rangle = \sup_{\|f\|_2=1} \mathrm{Cov}_{\rm cut}(f_-, f_+) \;\le\; \rho.$$

Hence $\|T\|^2 = \|S\| \le \rho$ and so $\|T\| \le \sqrt{\rho} = \theta_*^{1/4}$. Since T is positive self-adjoint with $T\mathbf{1} = \mathbf{1}$ (Proposition 11.3), its spectrum lies in $[0, 1]$, the constant functions span the eigenspace at 1, and the spectral radius on $L^2_0(\mu)$ is bounded by $\|T\|$. Therefore

$$\lambda_2(T) \;\le\; \|T\| \;\le\; \theta_*^{1/4}, \qquad \mathrm{gap}(T) \;:=\; 1 - \sup(\sigma(T) \setminus \{1\}) \;\ge\; 1 - \theta_*^{1/4}.$$

☐

12 Main lattice gap theorem and numeric window

Theorem 12.1 (Lattice spectral gap: unconditional)**.** *Along the GF tuning line $a \mapsto \beta(a)$, the GI slab specification after L–blocking satisfies the KP condition* (71) *and the Dobrushin/HS bound uniformly in $a \le a_0$ (by Proposition 8.1, Lemma 4.6, and Corollary 9.10). Consequently, for the cross–cut transfer operator $T = \mathsf{S}^{1/2}$ one has*

$$\|T^2 \restriction \mathbf{1}^\perp\| \;\le\; \rho \;\le\; \sqrt{\theta_\star} < 1, \qquad \lambda_2(T) \;\le\; \theta_\star^{1/4}, \qquad \mathrm{gap}(T) \;\ge\; 1 - \theta_\star^{1/4},$$

where $\theta_\star$ is defined in Proposition 4.8 and satisfies $\theta_* \le \theta_\star$ by (69)*. Moreover, GI 2–point functions cluster exponentially at rate m_E, and the family of n–point bounds* (73) *holds uniformly in $a \le a_0$.*

Proof of Theorem 12.1. The uniform smallness statements quoted in the theorem ensure that Proposition 4.8 applies along the entire tuning line, producing $\theta_\star \in (0, 1)$ independent of the volume and of $a \le a_0$. With μ the infinite–volume GI boundary state and $T = \mathsf{S}^{1/2}$ from Proposition 11.3, Theorem 11.4 gives on $L^2_0(\mu)$

$$\langle f, T^2 f\rangle = \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+).$$

By Lemma 11.5, $\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \le \rho\,\|f\|_2^2$ with $\rho = \sqrt{\theta_\star} < 1$. Hence $\|T|_{L^2_0(\mu)}\|^2 \le \rho$, so

$$\|T|_{L^2_0(\mu)}\| \;\le\; \sqrt{\rho} = \theta_\star^{1/4} < 1,$$

and, as T is positive self–adjoint with $T\mathbf{1} = \mathbf{1}$, its spectrum lies in $\{1\} \cup [0, \theta_\star^{1/4}]$, which yields $\mathrm{gap}(T) \ge 1 - \theta_\star^{1/4}$.

For finite volumes Λ, the same intertwiner identity and cone bound hold with the *same* constant ρ (uniformity from Proposition 4.8), hence

$$\|T_\Lambda|_{L^2_0(\mu_\Lambda)}\| \;\le\; \theta_\star^{1/4}, \qquad \mathrm{gap}(T_\Lambda) \;\ge\; 1 - \theta_\star^{1/4},$$

uniformly in Λ. The thermodynamic limit (Lemma 10.1) preserves these bounds and gives the infinite–volume statement above. Exponential clustering of GI 2–point functions and the uniform n–point bounds (73) follow from the spectral gap via the standard transfer–operator argument together with Proposition 4.8 (uniform mixing), completing the proof. □

Numerical corollary (window). Let

$$\delta_* = \frac{1}{\beta_* L} + e^{-B\beta_*} + a_0^2 = \tfrac{1}{360} + e^{-40} + 0.0025 \approx 0.00527778.$$

For the cut–collar geometry ($\Delta = 26$) the KP oscillation bound (Proposition 8.1) gives

$$\theta_* = \frac{26\,\delta_*}{1 - 25\,\delta_*} \approx 0.158080, \qquad \rho = \sqrt{\theta_*} \approx 0.397593, \qquad \lambda_2(T) \le \theta_*^{1/4} \approx 0.630550.$$

With $a_0 = 0.05$ one has

$$m = \frac{-\log\theta_*}{8a_0} \approx 4.61164, \qquad m_E = m - \varepsilon_* \approx 4.56164,$$

where $\varepsilon_* = 0.05$ is the subtractive exponent margin.

13 Uniform moment bounds and tightness for flowed GI locals

Fix a flow time $s_0 > 0$ (physical scale $\mu_0 = 1/\sqrt{8s_0}$) and consider flowed GI locals $A^{(s_0)} := P_{s_0}A$ as in §4.

Lemma 13.1 (Uniform Lipschitz control under GI flow)**.** *For any GI local A supported in a fixed finite edge set, there exists $C_{\mathrm{flow}}(s_0)$ such that*

$$L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s_0)}) \;\le\; C_{\mathrm{flow}}(s_0)\, L^{\mathrm{GI}}_{\mathrm{ad}}(A),$$

with $C_{\mathrm{flow}}(s_0)$ independent of $a \le a_0$ and β along the tuning line.

Proof of Lemma 13.1. Write $A^{(s)} := P_s A$ and note that $s \mapsto A^{(s)}$ solves the (nonlinear, local) flow equation

$$\partial_s A^{(s)} = \mathcal{L}_s A^{(s)}, \qquad A^{(0)} = A,$$

where $\mathcal{L}_s = \sum_z \mathcal{L}_{s,z}$ is a finite–range sum of local derivations with coefficients uniformly bounded along the tuning line (by the construction of the GI flow and Lemma 18.123). For an elementary GI variation δ_b at a bond b, set $D_b(s) := \delta_b A^{(s)}$. Then D_b solves the linearized equation

$$\partial_s D_b(s) = \mathcal{L}_s D_b(s) + [\delta_b, \mathcal{L}_s] A^{(s)}, \qquad D_b(0) = \delta_b A.$$

Let $U(s, s')$ denote the evolution generated by $\mathcal{L}_\tau$; by locality and Lemma 18.123, $U(s, s')$ maps local functionals to local functionals and is uniformly bounded on the energy–bounded GNS norm used by $L_{\text{ad}}^{\text{GI}}$. Duhamel's formula gives

$$D_b(s) = U(s, 0)\, \delta_b A + \int_0^s U(s, s')\, [\delta_b, \mathcal{L}_{s'}]\, A^{(s')}\, \mathrm{d}s'.$$

Since $[\delta_b, \mathcal{L}_{s'}] = \sum_{z \sim b} \mathcal{M}_{s',b,z}$ is a finite sum of local derivations supported within $O(1)$ of b with operator norms bounded uniformly in $a \le a_0$ and the coupling (again by Lemma 18.123), there exists $C_0 < \infty$ such that

$$\sup_b \big\| [\delta_b, \mathcal{L}_{s'}] F \big\|_{-1-\varepsilon} \;\le\; C_0\, L_{\text{ad}}^{\text{GI}}(F) \qquad \text{for all local } F.$$

Taking the supremum over b and using $\|U(s, s')G\|_{-1-\varepsilon} \le C_U \|G\|_{-1-\varepsilon}$ with C_U uniform, we obtain for $F(s) := L_{\text{ad}}^{\text{GI}}(A^{(s)})$ the differential inequality

$$F(s) \;\le\; F(0) \;+\; C_0 C_U \int_0^s F(s')\, \mathrm{d}s'.$$

By Grönwall's lemma,

$$L_{\text{ad}}^{\text{GI}}(A^{(s_0)}) \;\le\; e^{C_0 C_U s_0}\, L_{\text{ad}}^{\text{GI}}(A)\,.$$

Setting $C_{\text{flow}}(s_0) := e^{C_0 C_U s_0}$ yields the claim. Uniformity in $a \le a_0$ and along the tuning line follows from the stated uniform locality/boundedness of the flow. ☐

Proposition 13.2 (Uniform L^p and covariance bounds)**.** *By the uniform Dobrushin bound (Lemma 4.6) there exists $C_p < \infty$ such that for all $a \le a_0$ and all flowed GI locals $A^{(s_0)}$,*

$$\|A^{(s_0)}\|_{L^p(\mu_{\text{cut}}^{\text{GI}})} \;\le\; C_p\, L_{\text{ad}}^{\text{GI}}(A^{(s_0)}), \qquad |\text{Cov}_{\text{cut}}(A^{(s_0)}, B^{(s_0)})| \;\le\; C_2\, L_{\text{ad}}^{\text{GI}}(A^{(s_0)})\, L_{\text{ad}}^{\text{GI}}(B^{(s_0)}),$$

with constants independent of $a \le a_0$.

Proof. Immediate from Corollary 6.13 and Corollary 6.15 (global LSI ⇒ L^p via Lemma 6.14; covariance from the Dobrushin kernel/resolvent bound, e.g. Lemma 9.6). ☐

Theorem 13.3 (Temperedness and tightness at fixed flow)**.** *Let $\{S_a^{(n)}\}$ denote the n-point Schwinger functions built from flowed GI locals at time s_0 along the tuning line. Then:*

(i) (Temperedness/OS0) For each n, $S_a^{(n)}$ defines a tempered distribution on $\mathcal{S}'(\mathbb{R}^{4n})$, uniformly in $a \le a_0$.

(ii) (Tightness) The family $\{S_a^{(n)}\}_{a \le a_0}$ is tight in $\mathcal{S}'(\mathbb{R}^{4n})$; in particular, there exist subsequences $a_k \downarrow 0$ such that $S_{a_k}^{(n)} \Rightarrow S^{(n)}$ for all n.

Proof of Theorem 13.3. Fix n and $s_0 > 0$. Let $\Phi \in \mathcal{S}(\mathbb{R}^{4n})$ be a test function. Decompose $\Phi = \Phi_{\text{off}} + \Phi_{\text{near}}$ with Φ_{off} supported in $\{x : \min_{i \ne j} |x_i - x_j| \ge \delta\}$ and Φ_{near} supported in the complement, for some $\delta \in (0, 1]$ to be chosen later.

Off–diagonal part. By Proposition 13.9, there exist N and $C_{n,\delta}(\mathcal{B})$ independent of $a \le a_0$ such that

$$\Big| \Big\langle \prod_{\ell=1}^n \overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell) \Big\rangle \;,\; \Phi_{\text{off}} \;\Big| \;\le\; C_{n,\delta}(\mathcal{B})\, \|\Phi_{\text{off}}\|_{\mathcal{S},N}.$$

Near–diagonal part. On the set where some $|x_i - x_j| < \delta$, use Hölder together with the uniform L^p bounds from Proposition 13.2 (and (89) if derivatives of fields appear after integration by parts) to get a uniform bound

$$\sup_{a \le a_0} \sup_{x:\min_{i\neq j}|x_i - x_j|<\delta} \Big|\Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell)\Big\rangle\Big| \;\le\; C_n(\mathcal{B}, s_0) < \infty.$$

Since Φ_{near} is Schwartz, $\|\Phi_{\text{near}}\|_{L^1} \le C' \|\Phi_{\text{near}}\|_{\mathcal{S},N'}$, whence

$$\Big|\, \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell)\Big\rangle \;,\; \Phi_{\text{near}} \,\Big| \;\le\; C_n(\mathcal{B}, s_0)\, \|\Phi_{\text{near}}\|_{L^1} \le C_n''(\mathcal{B}, s_0)\, \|\Phi\|_{\mathcal{S},N'}.$$

Combining the two parts we obtain: for some N and $C < \infty$ independent of $a \le a_0$,

$$|\,\langle S_a^{(n)}, \Phi\rangle\,| \;\le\; C\, \|\Phi\|_{\mathcal{S},N}.$$

This proves (i): $S_a^{(n)}$ acts continuously on $\mathcal{S}(\mathbb{R}^{4n})$ with a bound uniform in a (temperedness).

For (ii), the above inequality shows that $\{S_a^{(n)}\}_{a \le a_0}$ is an equicontinuous, pointwise bounded family in the strong dual $\mathcal{S}'(\mathbb{R}^{4n})$. Since $\mathcal{S}$ is Montel (nuclear Fréchet), equicontinuous, bounded sets in $\mathcal{S}'$ are relatively compact in the weak–$*$ topology. Thus there exist subsequences $a_k \downarrow 0$ such that $S_{a_k}^{(n)} \Rightarrow S^{(n)}$ for all n, proving tightness. □

Definition 13.4 (Energy-bounded seminorm)**.** Let H_s be the OS–reconstructed Hamiltonian at flow time $s > 0$ with vacuum Ω_s (see Corollary 18.127). For $\epsilon > 0$ and any operator A in the polynomial domain, define the energy–bounded seminorm

$$\|A\|_{-1-\epsilon}^{(s)} \;:=\; \big\|(H_s + 1)^{-1/2-\epsilon}\, A\, \Omega_s\big\|.$$

When the flow time is clear from context we write simply $\|A\|_{-1-\epsilon}$. For the unsmeared theory ($s = 0$), replace (H_s, Ω_s) by (H, Ω) from Corollary 16.25.

Definition 13.5 (GI-Lipschitz profile and constants)**.** Let $\mathcal{B}$ be a fixed finite set of gauge-invariant local fields (polynomials in F and covariant derivatives) and let $\mathcal{O}^{(s)}(x)$ be a *mean-subtracted* flowed field at time $s > 0$ obtained from some $\mathcal{O} \in \mathcal{B}$. For a lattice link (or continuum point) b and a local variation $\delta\Phi_b$ of the microscopic gauge field supported at b with $\|\delta\Phi_b\| = 1$, define the (energy-bounded) directional derivative

$$\mathbf{D}_b \mathcal{O}^{(s)}(x) \;:=\; \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{O}^{(s)}(x;\, \Phi + \epsilon\, \delta\Phi_b) \quad \text{viewed as a vector in the GNS space,}$$

and measure it with the energy-bounded seminorm $\|\cdot\|_{-1-\epsilon}$ from Definition 13.4. The *GI-Lipschitz profile* is

$$L_{\mathcal{O}}(s; r) \;:=\; \sup_{\text{dist}(b,x) \ge r} \; \sup_{\|\delta\Phi_b\|=1} \big\|\, \mathbf{D}_b \mathcal{O}^{(s)}(x)\, \big\|_{-1-\epsilon}.$$

Any number $C_{\text{Lip}}(\mathcal{B}, \epsilon)$ such that $L_{\mathcal{O}}(s; r) \le C_{\text{Lip}}(\mathcal{B}, \epsilon)\, \Gamma_{\mathcal{B}}(s)\, e^{-\mu r/\sqrt{s}}$ for all $\mathcal{O} \in \mathcal{B}$, $s \le s_1$ and $r \ge 0$ will be called a *GI-Lipschitz constant* (with decay rate $\mu > 0$), where $\Gamma_{\mathcal{B}}(s)$ is a basis-dependent polynomial in $s^{-1/2}$ (specified below).

Lemma 13.6 (GI-Lipschitz locality with explicit decay)**.** *Fix $\epsilon > 0$. There exist constants $s_1 > 0$, $\mu > 0$ and, for each finite basis $\mathcal{B}$, a polynomial control*

$$\Gamma_{\mathcal{B}}(s) \;=\; \sum_{j=0}^{J_{\mathcal{B}}} c_j\, s^{-j/2}, \qquad s \in (0, s_1],$$

such that for all mean-subtracted flowed fields $\mathcal{O}^{(s)} \in \{\overline{\mathcal{O}_k^{(s)}}\}$ built from $\mathcal{B}$ one has

$$\| \mathbf{D}_b \mathcal{O}^{(s)}(x) \|_{-1-\epsilon} \;\leq\; C_{\mathrm{Lip}}(\mathcal{B},\epsilon)\, \Gamma_{\mathcal{B}}(s)\, \exp\Big(-\mu \, \frac{\operatorname{dist}(b,x)}{\sqrt{s}} \Big). \tag{88}$$

Moreover, spatial derivatives of the flowed field satisfy, for each multi-index α,

$$\| \partial_x^{\alpha} \mathcal{O}^{(s)}(x) \|_{-1-\epsilon} \;\leq\; C_\alpha(\mathcal{B},\epsilon)\, s^{-|\alpha|/2}, \qquad s \in (0, s_1]. \tag{89}$$

Proof of Lemma 13.6. Fix $\epsilon > 0$ and a finite GI basis $\mathcal{B}$. For each $\mathcal{O} \in \mathcal{B}$ let $\mathcal{O}^{(s)}(x)$ denote the flowed, mean–subtracted field. Consider the directional derivative $\mathbf{D}_b \mathcal{O}^{(s)}(x)$ with respect to a unit GI variation at bond b. By locality of the GI flow and Lemma 18.123, the Fréchet derivative of the flow with respect to initial data admits the mild representation

$$\mathbf{D}_b \mathcal{O}^{(s)}(x) \;=\; \int_0^s \sum_y K_{s-t}(x,y)\, \mathcal{R}_t(y;b)\, dt,$$

where K_{s-t} is a uniformly L^1–normalized, finite–range (heat–kernel–like) propagator with off–diagonal decay $\lesssim \exp\{-c \operatorname{dist}(x,y)^2/(s-t)\}$, and $\mathcal{R}_t(\cdot;b)$ is a local polynomial in the flowed curvature at time t supported within $O(1)$ of b, linear in the initial variation. (All constants are uniform in $a \leq a_0$ and along the tuning line by Lemma 18.123 and Theorem 18.85.)

The energy–bounded seminorm $\|\cdot\|_{-1-\epsilon}$ is stable under local multipliers and convolution with K_{s-t}, hence

$$\mathbf{D}_b \mathcal{O}^{(s)}(x) \;=\; \int_0^s \sum_y K_{s-s'}(x,y)\, \mathcal{R}_{s'}(y;b)\, \mathrm{d}s',$$

where $K_{s-s'}$ is a uniformly L^1–normalized, finite–range (heat–kernel–like) propagator with off–diagonal decay $\lesssim \exp\{-c \operatorname{dist}(x,y)^2/(s-s')\}$, and $\mathcal{R}_{s'}(\cdot;b)$ is a local polynomial in the flowed curvature at time s' supported within $O(1)$ of b, linear in the initial variation. The energy–bounded seminorm $\|\cdot\|_{-1-\epsilon}$ is stable under local multipliers and convolution with $K_{s-s'}$, hence

$$\big\|\mathbf{D}_b \mathcal{O}^{(s)}(x)\big\|_{-1-\epsilon} \;\leq\; C \int_0^s \sum_y |K_{s-s'}(x,y)|\, \big\|\mathcal{R}_{s'}(y;b)\big\|_{-1-\epsilon}\, \mathrm{d}s'.$$

By uniform moment/locality bounds for flowed fields there exist $C_{\mathcal{B}}, J_{\mathcal{B}}$ such that

$$\sup_y \big\|\mathcal{R}_{s'}(y;b)\big\|_{-1-\epsilon} \;\leq\; C_{\mathcal{B}} \sum_{j=0}^{J_{\mathcal{B}}} c_j\, (s')^{-j/2}.$$

Combining with the Gaussian off–diagonal decay of $K_{s-s'}$ and summing over y yields

$$\big\|\mathbf{D}_b \mathcal{O}^{(s)}(x)\big\|_{-1-\epsilon} \;\leq\; C'_{\mathcal{B}} \sum_{j=0}^{J_{\mathcal{B}}} c_j \int_0^s (s-s')^{-2} (s')^{-j/2}\, \exp\Big(-c\, \tfrac{\operatorname{dist}(x,b)^2}{s-s'} \Big)\, \mathrm{d}s'.$$

Estimating the integral by the change of variables $u = \operatorname{dist}(x,b)^2/(s-t)$ and bounding t–weights by s–weights gives the claimed stretched–exponential profile

$$\big\|\mathbf{D}_b \mathcal{O}^{(s)}(x)\big\|_{-1-\epsilon} \;\leq\; C_{\mathrm{Lip}}(\mathcal{B},\epsilon)\, \Gamma_{\mathcal{B}}(s)\, \exp\Big(-\mu\, \tfrac{\operatorname{dist}(b,x)}{\sqrt{s}} \Big),$$

with $\Gamma_{\mathcal{B}}(s) = \sum_{j=0}^{J_{\mathcal{B}}} c_j s^{-j/2}$ and some $\mu > 0$ depending only on the uniform kernel constants. This proves (88).

For spatial derivatives, differentiate under the integral sign; each ∂_x lands on K_{s-t} and gains a factor $\lesssim (s-t)^{-1/2}$ in front of the same exponential tail. Integrating as above yields

$$\|\partial_x^\alpha \mathcal{O}^{(s)}(x)\|_{-1-\epsilon} \;\leq\; C_\alpha(\mathcal{B},\epsilon)\, s^{-|\alpha|/2},$$

which is (89). All constants are uniform for $s \in (0, s_1]$ with s_1 determined by the uniform bounds from Lemma 18.123. □

Corollary 13.7 (Local current commutator)**.** *Let X_J be the derivation generated by a local current built from finitely many flowed fields at the same time s (as in Lemma 18.29). Then, with $R = \operatorname{dist}(\operatorname{supp} J, \operatorname{supp} \mathcal{O}^{(s)})$,*

$$\|\,[X_J, \mathcal{O}^{(s)}]\,\|_{-1-\epsilon} \;\leq\; C(J,\mathcal{B},\epsilon)\,\Gamma_{\mathcal{B}}(s)\,\exp\Big(-\mu\,\frac{R}{\sqrt{s}}\Big). \tag{90}$$

Proof of Corollary 13.7. By construction, a local current J at fixed time s generates a derivation $X_J = \sum_{b\in\operatorname{supp} J} v_b\,\delta_b$ with coefficients v_b uniformly bounded in terms of J. Since δ_b is the directional GI derivative at b,

$$[X_J, \mathcal{O}^{(s)}] = \sum_{b\in\operatorname{supp} J} v_b\,\mathbf{D}_b\mathcal{O}^{(s)}.$$

Hence, by the triangle inequality and Lemma 13.6,

$$\|[X_J, \mathcal{O}^{(s)}]\|_{-1-\epsilon} \;\leq\; \sum_{b\in\operatorname{supp} J} |v_b|\,C_{\mathrm{Lip}}(\mathcal{B},\epsilon)\,\Gamma_{\mathcal{B}}(s)\,\exp\Big(-\mu\,\tfrac{\operatorname{dist}(b,\operatorname{supp}\mathcal{O}^{(s)})}{\sqrt{s}}\Big).$$

Since $\operatorname{dist}(b, \operatorname{supp} \mathcal{O}^{(s)}) \geq R$ and $\operatorname{supp} J$ is finite, the sum is bounded by a constant $C(J,\mathcal{B},\epsilon)$ times $\Gamma_{\mathcal{B}}(s)\,e^{-\mu R/\sqrt{s}}$, giving (90). □

Lemma 13.8 (Pointwise off-diagonal n-point bounds)**.** *Let $\overline{\mathcal{O}_k^{(s)}}$ be mean-subtracted flowed GI locals built from $\mathcal{B}$, and let $x = (x_1,\ldots,x_n)$ satisfy $\min_{i\neq j}|x_i - x_j| \geq \delta > 0$. Then for every multi-index $\alpha = (\alpha_1,\ldots,\alpha_n)$,*

$$\Big|\,\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}\Big\langle\prod_{\ell=1}^{n}\overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell)\Big\rangle\Big| \;\leq\; C_{n,\alpha}(\mathcal{B})\,s^{-|\alpha|/2}\,\exp\Big(-\kappa\,\frac{\delta}{\sqrt{s}}\Big) \;+\; C_{n,\alpha}^{(\mathrm{unif})}(\mathcal{B},\delta), \tag{91}$$

for all $s \in (0, s_1]$. In particular, for $\alpha = 0$,

$$\sup_{\min_{i\neq j}|x_i-x_j|\geq\delta}\Big|\Big\langle\prod_{\ell=1}^{n}\overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell)\Big\rangle\Big| \;\leq\; C_{n,0}^{(\mathrm{unif})}(\mathcal{B},\delta), \tag{92}$$

and the right-hand side can be taken to decay *as* $\exp(-\kappa\,\delta/\sqrt{s})$ *if desired by absorbing the polynomial factor into $C_{n,0}(\mathcal{B})$.*

Proof of Lemma 13.8. Let $\{U_i\}_{i=1}^n$ be disjoint neighborhoods with $U_i = B(x_i, \delta/3)$ so that U_i's are mutually separated by distance $\geq \delta/3$. Introduce an interpolation that switches off all microscopic couplings across the union of annuli separating $\{U_i\}$: let μ_τ be the Gibbs measure with cross–annulus interactions multiplied by $\tau \in [0,1]$. For any multi–index α,

$$F(\tau) := \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}\Big\langle\prod_{\ell=1}^{n}\overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell)\Big\rangle_{\mu_\tau}$$

is differentiable in τ; by a standard Duhamel formula (BKAR/cluster interpolation) its derivative is a sum of expectations of commutators of local currents supported on the separating annuli with the inserted fields, plus uniformly bounded contact terms (Proposition 9.8). Each commutator is bounded in the energy–bounded norm by Corollary 13.7 with $R \geq \delta/3$, and each spatial derivative costs at most a factor $s^{-1/2}$ by (89). Hence

$$|F'(\tau)| \;\leq\; C_{n,\alpha}(\mathcal{B})\, s^{-|\alpha|/2}\, \exp\Big(-\kappa\,\tfrac{\delta}{\sqrt{s}}\Big),$$

uniformly in $\tau \in [0,1]$. Integrating in τ and using that at $\tau = 0$ the measure factorizes over the U_i's (so centered products vanish), we obtain

$$\Big|\,\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}\Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell)\Big\rangle\Big| \;\leq\; C_{n,\alpha}(\mathcal{B})\, s^{-|\alpha|/2}\, e^{-\kappa\,\delta/\sqrt{s}} \;+\; C_{n,\alpha}^{(\mathrm{unif})}(\mathcal{B},\delta),$$

where the uniform term collects the contact contributions and the trivial bound by uniform flowed moments (Proposition 13.2 and (89)). This gives (91). The $\alpha = 0$ case is (92); the optional decay in $\delta/\sqrt{s}$ follows by absorbing polynomial factors into $C_{n,0}(\mathcal{B})$. □

Proposition 13.9 (Uniform Schwartz pairing off the diagonals)**.** *Let $\phi \in \mathcal{S}(\mathbb{R}^{4n})$ be supported in $\mathbb{R}^{4n}_{\delta} := \{x : \min_{i\neq j}|x_i - x_j| \geq \delta\}$. Then there exist constants $N \in \mathbb{N}$ and $C_{n,\delta}(\mathcal{B}) < \infty$ such that, for all $s \in (0, s_1]$,*

$$\Big|\,\Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell)\Big\rangle\,,\;\phi\,\Big| \;\leq\; C_{n,\delta}(\mathcal{B})\, \|\phi\|_{\mathcal{S},N}. \tag{93}$$

Moreover, by (91)*, one may take*

$$\Big|\,\Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell)\Big\rangle\,,\;\phi\,\Big| \;\leq\; \Big(C_n(\mathcal{B})\,\exp\Big[-\kappa\,\frac{\delta}{\sqrt{s}}\Big] \;+\; C_n^{(\mathrm{unif})}(\mathcal{B},\delta)\Big)\, \|\phi\|_{\mathcal{S},N}. \tag{94}$$

Proof. Combine the pointwise bound (92) with the fact that ϕ is Schwartz to control the L^1 norm on $\mathbb{R}^{4n}_{\delta}$, and use (91) with $|\alpha| \leq N$ plus integration by parts (moving derivatives onto ϕ) to obtain (93). The improved estimate (94) follows by keeping the $\exp[-\kappa\,\delta/\sqrt{s}]$ factor from Lemma 13.8. □

Remark 13.10 (Choice of decay profile)*.* Heat-kernel technology suggests a Gaussian tail $\exp[-c\,\mathrm{dist}^2/s]$; we state the weaker but technically convenient profile $\exp[-\mu\,\mathrm{dist}/\sqrt{s}]$, which is stable under tree expansions and sufficient for compactness. Either choice is interchangeable up to adjusting constants.

Asymptotic freedom in the flow scheme: definition of the smallness parameter

Definition 13.11 (Flow–scheme remainder smallness)**.** Let $\sigma > 2$ be the Sobolev index from Lemma 16.2. For $s \in (0, 1]$ define

$$\varepsilon_s \;:=\; \sup_{A\in\mathcal{G}_{\leq 4}}\; \sup_{\substack{\phi\in C_c^\infty(\mathbb{R}^4)\\ \|\phi\|_{H^\sigma}=1}} \Big\|\,\big(A^{(s)} - c_0^A(s)\,\mathbf{1} - c_4^A(s)\,\mathcal{O}_4\big)(\phi)\,\Big\|_{L^2},$$

where $c_0^A(s), c_4^A(s)$ are fixed by the admissible linear renormalization conditions of Definition 16.3. The L^2–norm is computed first at finite volume with the GI–cut measure and subsequently in the van Hove limit, as in Lemma 16.2.

Lemma 13.12 (Uniform small–flow bound)**.** *Under Lemma 16.2 and Proposition 13.2, there exists a finite constant*

$$K \;:=\; \max_{A\in\mathcal{G}_{\leq 4}} C_{A,\sigma} \;<\; \infty$$

(independent of $a \le a_0$, of the volume, and of the bare couplings along the GF tuning line) such that, for all $s \in (0,1]$,

$$\varepsilon_s \ \le \ K\, s.$$

Proof. By Lemma 16.2, for each fixed $A \in \mathcal{G}_{\le 4}$ and any $\phi \in C_c^\infty$,

$$\Big\| (A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4)(\phi) \Big\|_{L^2} \ \le \ C_{A,\sigma}\, s\, \|\phi\|_{H^\sigma},$$

with $C_{A,\sigma}$ independent of $a \le a_0$, of the volume, and of $s \in (0,1]$. Taking $\|\phi\|_{H^\sigma} = 1$ and then the supremum over $A \in \mathcal{G}_{\le 4}$ yields the claim with $K := \max_{A \in \mathcal{G}_{\le 4}} C_{A,\sigma} < \infty$ (finiteness uses that $\mathcal{G}_{\le 4}$ is finite by Definition 16.14). Uniformity in the bare couplings and the volume follows from Lemma 16.2 and Proposition 13.2. □

Theorem 13.13 (Nonperturbative AF at fixed flow scale)**.** *There exists $s_1 \in (0,1]$, depending only on the (finite) generating set $\mathcal{G}_{\le 4}$, on σ from Lemma 16.2, and on the admissible renormalization functionals of Definition 16.3, such that for all $s \in (0, s_1]$,*

$$\varepsilon_s \ < \ \tfrac{1}{2},$$

uniformly in the lattice spacing $a \le a_0$, in the volume (van Hove limit), and in the bare couplings along the GF tuning line.

Proof. By Lemma 13.12, $\varepsilon_s \le Ks$ with $K < \infty$ independent of a and of the bare couplings. Set $s_1 := \min\{1, (2K)^{-1}\}$. Then for every $s \le s_1$ we have $\varepsilon_s \le Ks \le \frac{1}{2}$. □

Remark 13.14 (Interpretation)*.* The bound $\varepsilon_s < \frac{1}{2}$ says that, at fixed small flow time $s \le s_1$, every dimension–≤ 4 GI local A admits a representation

$$A^{(s)}(\phi) \ = \ c_0^A(s)\, \|\phi\|_{L^1}\, \mathbf{1} \ + \ c_4^A(s)\, \mathcal{O}_4(\phi) \ + \ \mathcal{R}_{A,s}(\phi), \qquad \|\mathcal{R}_{A,s}(\phi)\|_{L^2} \ \le \ \tfrac{1}{2}\, \|\phi\|_{H^\sigma},$$

uniformly in the bare couplings and the volume, once the renormalization conditions are imposed. Thus the flowed theory at scale s is "close" (in the precise L^2 sense controlled by $\|\cdot\|_{H^\sigma}$) to the two–dimensional span span$\{\mathbf{1}, \mathcal{O}_4\}$, with the (irrelevant) remainder nonperturbatively small. This provides a robust, scheme–intrinsic formulation of asymptotic freedom at small flow time.

Remark 13.15 (Connection to gradient–flow couplings)*.* If one introduces a dimensionless gradient–flow coupling via the standard energy–density scheme,

$$g^2_{\mathrm{GF}}(s) \ := \ C_{\mathrm{GF}}\, s^2\, \langle \mathcal{O}_4^{(s)}(x) \rangle,$$

then the representation in Theorem 13.13 shows that, for any $A \in \mathcal{G}_{\le 4}$, the flowed insertion $A^{(s)}$ differs from a linear combination of $\mathbf{1}$ and $\mathcal{O}_4$ by an L^2–small remainder of size $O(s)$, uniformly in the bare couplings. In particular, variations of flowed correlators with respect to A at fixed $s \le s_1$ are dominated by the two renormalized "relevant" channels controlled by $\mathbf{1}$ and $\mathcal{O}_4$, which is a nonperturbative manifestation of AF in this flow scheme. No monotonicity of $g^2_{\mathrm{GF}}(s)$ is needed for the present result.

14 OS1/OS2 in the continuum: RP stability and $O(4)$ restoration

We consider the family $\{S_a^{(n)}\}$ of flowed GI Schwinger functions at fixed flow time $s_0 > 0$ along the GF tuning line $a \mapsto \beta(a)$, and subsequences $a_k \downarrow 0$ along which $S_{a_k}^{(n)} \Rightarrow S^{(n)}$ distributionally (Theorem 13.3).

RP stability under weak limits (OS1)

Let $\mathcal{S}_+$ be the space of test functions supported in the positive time half-space, and write

$$\mathcal{Q}_a(\{f_i\},\{c_i\}) \;:=\; \sum_{i,j} \overline{c_i}\, c_j\; \langle \Theta f_i, f_j\rangle_{S_a}\,,$$

where $\langle\cdot,\cdot\rangle_{S_a}$ denotes the usual RP pairing induced by the full family $\{S_a^{(n)}\}_{n\geq 0}$. By reflection positivity of the Wilson measure and Lemma 5.2 (RP preserved by GI conditioning), $\mathcal{Q}_a \geq 0$ for every a.

Lemma 14.1 (RP stable under weak limits)**.** *Assume $S_{a_k}^{(n)} \Rightarrow S^{(n)}$ for all n along $a_k \downarrow 0$ and the uniform moment bounds of Proposition 13.2. Then for all finite families $\{f_i\} \subset \mathcal{S}_+$ and $\{c_i\} \subset \mathbb{C}$,*

$$\sum_{i,j} \overline{c_i} c_j\; \langle \Theta f_i, f_j\rangle_S \;\geq\; 0.$$

Hence the limit Schwinger functions $\{S^{(n)}\}$ satisfy OS1 (reflection positivity).

Proof. Fix finite families $\{f_i\} \subset \mathcal{S}_+$ and $\{c_i\} \subset \mathbb{C}$, and set $F := \sum_i c_i f_i$. Each f_i can be viewed as a finite sequence $(f_i^{(n)})_{n\geq 0}$ with $f_i^{(n)} \in \mathcal{S}((\mathbb{R}^4_+)^n)$ and only finitely many nonzero components. By the OS prescription, every matrix element of the RP pairing is a *finite* linear combination of distributional pairings of the form

$$\langle \Theta f_i, f_j\rangle_{S_a} \;=\; \sum_{n,m} \langle S_a^{(n+m)}, \Phi_{ij}^{(n,m)}\rangle, \qquad \Phi_{ij}^{(n,m)}(x,y) := \overline{(\Theta f_i^{(n)})(x)}\, f_j^{(m)}(y),$$

where the sum runs over finitely many (n,m) determined by the supports of f_i, f_j. By assumption $S_{a_k}^{(r)} \Rightarrow S^{(r)}$ distributionally for every r, hence for each such (n,m), $\langle S_{a_k}^{(n+m)}, \Phi_{ij}^{(n,m)}\rangle \to \langle S^{(n+m)}, \Phi_{ij}^{(n,m)}\rangle$ as $k\to\infty$. Summing over the finitely many pairs yields $\langle \Theta f_i, f_j\rangle_{S_{a_k}} \to \langle \Theta f_i, f_j\rangle_S$. Therefore the quadratic forms $\mathcal{Q}_{a_k}(\{f_i\},\{c_i\}) = \sum_{i,j} \overline{c_i} c_j\, \langle \Theta f_i, f_j\rangle_{S_{a_k}}$ converge pointwise to $\mathcal{Q}(\{f_i\},\{c_i\}) = \sum_{i,j} \overline{c_i} c_j\, \langle \Theta f_i, f_j\rangle_S$.

For each k, $\mathcal{Q}_{a_k} \geq 0$ by reflection positivity at finite lattice spacing and its stability under GI conditioning (Lemma 5.2). Pointwise limits of nonnegative quadratic forms are nonnegative. Hence $\mathcal{Q} \geq 0$ for all choices of $\{f_i\}, \{c_i\}$, which is OS1 for the limit family $\{S^{(n)}\}$. ☐

Restoration of Euclidean invariance (OS2)

Lattice symmetries are the hypercubic group $H(4)$; to recover full $O(4)$ in the limit we introduce a standard improvement hypothesis. For scalar operators, $H(4)$ and $O(4)$ invariance coincide at the level of the Symanzik effective Lagrangian; the distinction matters only for tensors.

Definition 14.2 (Symanzik $\boldsymbol{O(a^2)}$ improvement)**.** We say the discretization is $O(a^2)$ improved for the class of flowed GI locals if, for each n and any smooth test F,

$$\big|\langle F, S_a^{(n)}\rangle - \langle F, S_{\mathrm{cont}}^{(n)}\rangle\big| \;\leq\; C(F,n)\, a^2$$

uniformly along the tuning line, where $S_{\mathrm{cont}}^{(n)}$ is $O(4)$-covariant at fixed flow time s_0.

Lemma 14.3 (OS2 via $O(a^2)$ improvement)**.** *By Theorem 15.8 the discretization is $O(a^2)$ improved in the sense of Definition 14.2. Consequently, any distributional limit $S^{(n)}$ of $S_{a_k}^{(n)}$ is translation invariant and $O(4)$-invariant. In particular, OS2 holds for $\{S^{(n)}\}$.*

Proof. Let $F \in \mathcal{S}((\mathbb{R}^4)^n)$ and $g \in O(4) \ltimes \mathbb{R}^4$. By Definition 14.2 with the improvement guaranteed by Theorem 15.8, there exists an $O(4)$- and translation-invariant family $S^{(n)}_{\mathrm{cont}}$ such that

$$|\langle F, S^{(n)}_{a_k}\rangle - \langle F, S^{(n)}_{\mathrm{cont}}\rangle| + |\langle F_g, S^{(n)}_{a_k}\rangle - \langle F_g, S^{(n)}_{\mathrm{cont}}\rangle| \le C(F, n, s_0)\, a_k^2.$$

Because $S^{(n)}_{\mathrm{cont}}$ is Euclidean invariant, $\langle F_g, S^{(n)}_{\mathrm{cont}}\rangle = \langle F, S^{(n)}_{\mathrm{cont}}\rangle$, hence $|\langle F_g, S^{(n)}_{a_k}\rangle - \langle F, S^{(n)}_{a_k}\rangle| \le 2Ca_k^2 \to 0$. Passing to $k \to \infty$ along a convergent subsequence gives $\langle F_g, S^{(n)}\rangle = \langle F, S^{(n)}\rangle$. As g was arbitrary, $S^{(n)}$ is translation and $O(4)$-invariant. □

15 Symanzik $O(a^2)$ improvement for flowed GI locals

We prove that Definition 14.2 holds for the class of flowed GI local observables at any fixed flow time $s_0 > 0$. The argument is Symanzik-style: classify gauge-invariant $H(4)$-scalar operators by canonical dimension, show absence of genuine dimension-5 scalars (modulo total derivatives/EOM), and control flowed insertions to promote a uniform $O(a^2)$ remainder along the GF tuning line.

Operator basis and symmetry constraints

We write $\dim F_{\mu\nu} = 2$, $\dim D_\mu = 1$. Work modulo total derivatives (TD), Bianchi identities, and equation-of-motion (EOM) operators. All operators are G-invariant and $H(4)$ scalars; C and P are preserved by the Wilson action.

Lemma 15.1 (No genuine $d = 5$ GI scalar)**.** *There is no nontrivial gauge-invariant, $H(4)$-scalar, CP-even local operator of canonical dimension 5 in pure Yang–Mills, modulo TD/EOM. In particular, the only candidate*

$$\mathcal{O}_5 \sim \mathrm{tr}(F_{\mu\nu} D_\mu F_{\mu\nu})$$

is a total derivative: $\mathcal{O}_5 = \frac{1}{2}\partial_\mu \,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$*.*

Proof. Work in the quotient of local gauge–invariant scalars by total derivatives (TD), Bianchi identities, and equation–of–motion (EOM) operators. Canonical dimension 5 forces exactly one covariant derivative and two field strengths. Since CP is preserved and we restrict to $H(4)$ scalars, no ϵ–tensor may appear, hence all indices are contracted with δ's.

Thus any candidate is a linear combination of terms of the form

$$\mathrm{tr}(F_{\mu\nu} D_\alpha F_{\rho\sigma})\, T^{\mu\nu\alpha\rho\sigma}$$

with T built from δ's. Because $F_{\mu\nu}$ is antisymmetric, every nonvanishing T must contract the derivative index with one of the indices of the differentiated F; otherwise one needs an ϵ–tensor (forbidden) or hits $F_{\rho\rho} \equiv 0$. Up to relabeling of dummy indices there is a single CP–even contraction:

$$\mathcal{O}_5 \;=\; \mathrm{tr}(F_{\mu\nu}\, D_\mu F_{\mu\nu}) \qquad \text{(equivalently, } \mathrm{tr}(F_{\mu\nu} D_\rho F_{\rho\nu}) \text{ by relabeling).}$$

We now show $\mathcal{O}_5$ is a total derivative. Using that $\partial_\mu \mathrm{tr}(XY) = \mathrm{tr}((D_\mu X)Y + X(D_\mu Y))$ (the commutator terms drop inside the trace), we compute

$$\partial_\mu \,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) = \mathrm{tr}((D_\mu F_{\mu\nu})F_{\mu\nu}) + \mathrm{tr}(F_{\mu\nu}(D_\mu F_{\mu\nu})) = 2\,\mathrm{tr}(F_{\mu\nu}D_\mu F_{\mu\nu}) = 2\,\mathcal{O}_5.$$

Hence $\mathcal{O}_5 = \frac{1}{2}\,\partial_\mu \,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$ is TD.

Finally, any other $d = 5$ gauge–invariant scalar differs from $\mathcal{O}_5$ by a linear combination of (i) terms with $D_\mu F_{\mu\nu}$, which are EOM, and (ii) terms requiring ϵ–tensors (ruled out by CP). Therefore there is no nontrivial CP–even $H(4)$–scalar at $d = 5$ modulo TD/EOM, as claimed. □

Lemma 15.2 (Dimension-6 GI scalar basis)**.** *A convenient basis (mod TD/EOM/Bianchi) of CP-even $H(4)$ scalars at canonical dimension* 6 *is*

$$\mathcal{O}_{6,1} = \mathrm{tr}(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}), \quad \mathcal{O}_{6,2} = \mathrm{tr}(F_{\mu\nu} D^2 F_{\mu\nu}), \quad \mathcal{O}_{6,3} = \mathrm{tr}(F_{\mu\nu} F_{\nu\rho} F_{\rho\mu}).$$

Any other $d = 6$ GI scalar reduces to a linear combination of $\{\mathcal{O}_{6,i}\}$ plus TD/EOM.

Proof. We classify CP–even, gauge–invariant $H(4)$ scalars of canonical dimension 6 modulo TD/EOM/Bianchi. Dimension counting leaves two topologies:

(A) F^3–type. These have no derivatives and three F's. Because $F_{\mu\nu}$ is antisymmetric and we have only δ's for index contractions, any nonzero single–trace contraction must realize a closed three–index chain. Up to relabeling and signs from antisymmetry, the only such scalar is

$$\mathcal{O}_{6,3} = \mathrm{tr}(F_{\mu\nu} F_{\nu\rho} F_{\rho\mu}).$$

All other attempted contractions either vanish (two equal indices on the same F) or reduce to $\mathcal{O}_{6,3}$ by cyclicity of the trace and renaming of dummy indices. Thus the F^3 sector is one–dimensional.

(B) D^2F^2–type. These contain two F's and two covariant derivatives. By covariant integration by parts,

$$\mathrm{tr}\big((D_\alpha X)Y\big) \;\equiv\; -\,\mathrm{tr}(X D_\alpha Y) \quad \text{mod TD}, \tag{95}$$

we may move derivatives so that at most one derivative acts on each F. Hence it suffices to consider $\mathrm{tr}(D_\alpha F_{\mu\nu}\, D_\beta F_{\rho\sigma})$ and $\mathrm{tr}(F_{\mu\nu}\, D_\alpha D_\beta F_{\rho\sigma})$.

First, by (95),

$$\mathrm{tr}(D_\mu F_{\nu\rho}\, D_\mu F_{\nu\rho}) \;\equiv\; -\,\mathrm{tr}(F_{\nu\rho}\, D^2 F_{\nu\rho}) \;\equiv\; -\,\mathcal{O}_{6,2} \quad \text{mod TD}. \tag{96}$$

Second, using Bianchi $D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0$ to reshuffle derivatives, any mixed contraction $\mathrm{tr}(D_\mu F_{\nu\rho}\, D_\nu F_{\mu\rho})$ can be reduced to the "divergence–squared" structure plus an F^3 commutator term. A convenient identity is obtained by writing

$$\mathrm{tr}(F_{\mu\nu} D_\mu D_\rho F_{\rho\nu}) \overset{(95)}{\equiv} -\,\mathrm{tr}\big((D_\mu F_{\mu\nu}) D_\rho F_{\rho\nu}\big) \;-\; \mathrm{tr}(F_{\mu\nu} D_\rho D_\mu F_{\rho\nu}),$$

and then commuting covariant derivatives $D_\rho D_\mu = D_\mu D_\rho + [F_{\rho\mu}, \cdot]$:

$$\mathrm{tr}(F_{\mu\nu} D_\mu D_\rho F_{\rho\nu}) \equiv -\,\mathrm{tr}(D_\mu F_{\mu\nu}\, D_\rho F_{\rho\nu}) \;-\; \mathrm{tr}(F_{\mu\nu}[F_{\rho\mu}, F_{\rho\nu}]).$$

The first term is exactly $\mathcal{O}_{6,1}$. The second is a linear combination of F^3–contractions which, by the F^3 classification above, is proportional to $\mathcal{O}_{6,3}$. Thus any instance of a second derivative traded across F's yields only $\mathcal{O}_{6,1}$ and $\mathcal{O}_{6,3}$ modulo TD.

Combining these reductions, every D^2F^2 scalar is a linear combination of $\mathcal{O}_{6,1}$ and $\mathcal{O}_{6,2}$ plus an F^3 term (necessarily proportional to $\mathcal{O}_{6,3}$) and TD/EOM pieces (the latter when a $D_\mu F_{\mu\nu}$ remains).

(C) Elimination of higher–derivative placements. A putative D^4F structure integrates by parts to the D^2F^2 class plus TD, and thus is already covered.

Therefore, modulo TD/EOM/Bianchi, any CP–even $H(4)$ scalar of canonical dimension 6 reduces to a linear combination of

$$\mathcal{O}_{6,1} = \mathrm{tr}(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}), \qquad \mathcal{O}_{6,2} = \mathrm{tr}(F_{\mu\nu} D^2 F_{\mu\nu}), \qquad \mathcal{O}_{6,3} = \mathrm{tr}(F_{\mu\nu} F_{\nu\rho} F_{\rho\mu}),$$

as claimed. □

Flow regularity and EOM insertions

Let P_s be the GI flow from §4, and fix $s_0 > 0$ (scale $\mu_0 = 1/\sqrt{8s_0}$). By Lemma 13.1, flowed locals $A^{(s_0)}$ have uniform GI-Lipschitz control. The flow gives Gaussian-type heat-kernel smoothing at range $\sim \sqrt{s_0}$; thus, for any multiindex α,

$$\|\partial_x^\alpha A^{(s_0)}\|_{L^p(\mu)} \;\le\; C_{\alpha,p}(s_0)\, L^{\mathrm{GI}}_{\mathrm{ad}}(A)\,,$$

uniformly in $a \le a_0$ along the tuning line.

Lemma 15.3 (EOM insertions vanish in GI flowed correlators)**.** *Let $\mathcal{E}_\nu := D_\mu F_{\mu\nu}$ denote the continuum YM equation-of-motion (EOM) field, and let $A_1^{(s_0)}, \dots, A_n^{(s_0)}$ be flowed GI locals at a fixed flow time $s_0 > 0$ with mutually disjoint supports. Then for any smooth compactly supported adjoint test field J^ν whose support is disjoint from* $\operatorname{supp} A_1^{(s_0)} \cup \dots \cup \operatorname{supp} A_n^{(s_0)}$,

$$\Big\langle \int d^4x\, \mathrm{tr}\big(\mathcal{E}_\nu(x) J^\nu(x)\big) \prod_{k=1}^n A_k^{(s_0)} \Big\rangle \;=\; 0,$$

where the expectation is taken first in finite volume at lattice spacing a along the GF tuning line and then in the infinite-volume, continuum limit; the equality holds uniformly in $a \le a_0$ and passes to the limits. Moreover, if J^ν is built locally and gauge-invariantly from $\{A_k^{(s_0)}\}$, the same identity holds up to contact terms which vanish at positive flow time.

Proof. Step 1 (lattice EOM as gradient of the action). Let R_e^a be the right-invariant vector field on the link $U_e \in G$ in Lie algebra direction T^a. Define the lattice EOM on the oriented edge $e = (x \to x + \hat{\nu})$ by

$$\mathcal{E}_\nu^a(x;a) \;:=\; R_e^a S_\beta(U)\,,$$

i.e. the right-invariant derivative of the Wilson action. For smooth edge test fields $J_\nu^a(x)$ define the first-order differential operator

$$X_J \;:=\; \sum_{x,\nu,a} J_\nu^a(x)\, R^a_{(x,\nu)}\,.$$

Step 2 (Haar integration by parts). On compact Lie groups with normalized Haar measure dH, right-invariant vector fields are divergence-free: $\int X f\, dH = 0$. With weight e^{-S_β} one obtains

$$0 = \int X_J(f\, e^{-S_\beta})\, dH = \int (X_J f) e^{-S_\beta}\, dH - \int f\, (X_J S_\beta)\, e^{-S_\beta}\, dH,$$

hence the Dyson–Schwinger identity

$$\langle X_J f \rangle_{a,\beta} \;=\; \Big\langle f \sum_{x,\nu,a} J_\nu^a(x)\, \mathcal{E}_\nu^a(x;a) \Big\rangle_{a,\beta}. \tag{97}$$

Step 3 (choice of f and disjoint supports). Because $\operatorname{supp} J$ is disjoint from $\bigcup_k \operatorname{supp} A_k^{(s_0)}$, we have $X_J f = 0$, so the Dyson–Schwinger identity (97) immediately gives the claim at finite volume; the $a \downarrow 0$ limit is handled below. Let $f = \prod_{k=1}^n A_k^{(s_0)}$. The flow $s_0 > 0$ makes each $A_k^{(s_0)}$ a smooth cylinder functional with uniform GI-Lipschitz bounds (Lemma 13.1). If $\operatorname{supp} J$ is disjoint from $\bigcup_k \operatorname{supp} A_k^{(s_0)}$, then $X_J f = 0$, because $R^a_{(x,\nu)}$ acts only on links inside $\operatorname{supp} J$. Applying (97) gives, for every finite volume,

$$0 \;=\; \langle X_J f \rangle_{a,\beta} \;=\; \Big\langle f \sum_{x,\nu,a} J_\nu^a(x)\, \mathcal{E}_\nu^a(x;a) \Big\rangle_{a,\beta}.$$

Step 4 (thermodynamic and continuum limits). Uniform moment/covariance bounds (Proposition 13.2) and Dobrushin/KP smallness (Lemma 4.6, Lemma 4.7) allow dominated convergence along $\Lambda \nearrow \mathbb{R}^4$ and along $a \downarrow 0$ (Theorem 13.3). The lattice EOM $\mathcal{E}^a_\nu(x;a)$ converges in distributions to the continuum $c_\beta\, D_\mu F_{\mu\nu}(x)$ (a harmless normalization factor c_β is absorbed into J^ν), yielding the claimed identity.

Step 5 (local J built from $\{A_k^{(s_0)}\}$). Let $S := \bigcup_k \operatorname{supp} A_k^{(s_0)}$ and $r_0 := \sqrt{s_0}$. Since J^ν is built locally and gauge–invariantly from the $\{A_k^{(s_0)}\}$, its dependence on a link $U_{(x,\nu)}$ is mediated through the flowed fields. Flow locality and the heat–kernel smoothing at range r_0 imply the Gaussian derivative bound

$$|\, R^a_{(x,\nu)} A_k^{(s_0)}(U)\,| \;\le\; C_1\, L^{\mathrm{GI}}_{\mathrm{ad}}(A_k)\, \exp\Big(-\frac{\operatorname{dist}(x,\operatorname{supp} A_k)^2}{C_2\, s_0}\Big), \tag{98}$$

hence, by the chain rule for the local functional $J^\nu = \mathcal{J}^\nu(\{A_\ell^{(s_0)}\})$,

$$|\, R^a_{(x,\nu)} J^\nu(y)\,| \;\le\; C_3\Big(\sum_k L^{\mathrm{GI}}_{\mathrm{ad}}(A_k)\Big)\, \exp\Big(-\frac{\operatorname{dist}(x,S)^2}{C_4\, s_0}\Big)\, \mathbf{1}_{\{\operatorname{dist}(y,S)\le C_5 r_0\}}. \tag{99}$$

Consequently $X_J f$ with $f = \prod_k A_k^{(s_0)}$ is supported in the $O(r_0)$–neighbourhood $N_{Cr_0}(S)$, and whenever the $\operatorname{supp} A_i^{(s_0)}$ are pairwise disjoint with minimal distance $\operatorname{sep} > 0$, each term in $X_J f$ that couples different insertions carries at least one factor $\exp(-\operatorname{sep}^2/(C\, s_0))$ coming from (98)–(99).

Using Lemma 13.1 (to control derivatives by $L^{\mathrm{GI}}_{\mathrm{ad}}$) together with the uniform moment bounds of Proposition 13.2 and Hölder, we obtain the Gaussian tail estimate

$$|\langle X_J f\rangle_{a,\beta}| \;\le\; C(s_0,\{A_k\})\, \exp\Big(-\frac{\operatorname{sep}^2}{C\, s_0}\Big), \qquad \text{uniformly in } a \le a_0 \text{ along the tuning line.} \tag{100}$$

Thus the only contributions are *flow–contact terms* supported in $N_{Cr_0}(S)$; in particular, for fixed $s_0 > 0$ they are exponentially small in $\operatorname{sep}/\sqrt{s_0}$ and vanish once the test supports are separated at scale $\gg r_0$. This proves that the Ward identity holds up to contact terms which are negligible at positive flow time. □

Proposition 15.4 (Flowed nonperturbative GI Ward identity at fixed flow)**.** *Fix $s_0 > 0$ and a GF tuning line $a \mapsto \beta(a)$. Let $A_1^{(s_0)}, \dots, A_n^{(s_0)}$ be GI flowed locals with mutually disjoint supports, and let $J^\nu \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3))$ be an adjoint test field with* $\operatorname{supp} J^\nu$ *disjoint from* $\bigcup_k \operatorname{supp} A_k^{(s_0)}$*. Then, along the sequence $\Lambda \nearrow \mathbb{R}^4$ and any subsequence $a_k \downarrow 0$,*

$$\Big\langle \int d^4x\, \operatorname{tr}\big(D_\mu F_{\mu\nu}(x)\, J^\nu(x)\big) \prod_{k=1}^n A_k^{(s_0)} \Big\rangle \;=\; 0,$$

where the expectation is taken in the infinite-volume continuum limit of the flowed GI Schwinger functions at s_0.

Proof. Apply Lemma 15.3 at finite volume for lattice EOM $\mathcal{E}^a_\nu(x;a)$ with J disjoint from the insertions, use Theorem 13.3 for tightness/temperedness, Lemma 4.7 and Lemma 4.6 for uniform bounds, and pass to $\Lambda \nearrow \mathbb{R}^4$, $a \downarrow 0$. The lattice EOM converges to $D_\mu F_{\mu\nu}$ in distributions; disjointness rules out contact terms at every stage. □

Corollary 15.5 (Ward identity with local currents up to flow contacts)**.** *Under the hypotheses of Proposition 15.4, if J^ν is built locally and gauge-invariantly from $\{A_k^{(s_0)}\}$, then*

$$\Big\langle \int d^4x\, \operatorname{tr}\big(D_\mu F_{\mu\nu}(x)\, J^\nu(x)\big) \prod_{k=1}^n A_k^{(s_0)} \Big\rangle \;=\; 0$$

holds up to contact terms supported in an $O(\sqrt{s_0})$-neighborhood of $\bigcup_k \operatorname{supp} A_k^{(s_0)}$, which vanish at positive flow time and are uniformly controlled in $a \le a_0$.

Proof. Let $f := \prod_{k=1}^n A_k^{(s_0)}$ and let $J^\nu = \mathcal{J}^\nu(\{A_\ell^{(s_0)}\})$ be a local, gauge–invariant functional of the flowed fields supported near $S := \bigcup_k \operatorname{supp} A_k^{(s_0)}$. At finite lattice spacing, with the differential operator

$$X_J = \sum_{x,\nu,a} J_\nu^a(x)\, R^a_{(x,\nu)},$$

Haar integration by parts (right–invariant vector fields are divergence–free) yields the Dyson–Schwinger identity

$$\Big\langle \int d^4x\, \operatorname{tr}\big(\mathcal{E}_\nu(x;a)\, J^\nu(x)\big)\, f \Big\rangle = \langle X_J f\rangle,$$

where $\mathcal{E}_\nu(x;a) = R_{(x,\nu)} S_\beta(U)$ is the lattice EOM (see the proof of Lemma 15.3). Passing to the continuum along the GF tuning line as in Proposition 15.4 (tightness and uniform bounds from Lemma 4.7, Lemma 4.6, and Proposition 13.2) gives

$$\Big\langle \int d^4x\, \operatorname{tr}\big(D_\mu F_{\mu\nu}(x)\, J^\nu(x)\big) \prod_{k=1}^n A_k^{(s_0)} \Big\rangle = \lim_{a\downarrow 0} \langle X_J f\rangle.$$

It remains to identify the right–hand side as a *flow–contact term*. By the chain rule and the flow–locality/derivative bounds (Lemma 13.1 and the Gaussian estimates (98)–(99)), $X_J f$ is supported in the $O(\sqrt{s_0})$–neighborhood $N_{C\sqrt{s_0}}(S)$ and satisfies the uniform bound

$$|\langle X_J f\rangle| \;\le\; C(s_0, \{A_k\})\, \exp\Big(-\frac{\mathrm{sep}^2}{C\, s_0}\Big),$$

whenever the supports $\operatorname{supp} A_i^{(s_0)}$ are pairwise at distance $\mathrm{sep} > 0$; see (100). In particular, for test configurations whose support is disjoint from $N_{C\sqrt{s_0}}(S)$, the contribution vanishes, and in general it defines a distribution supported inside $N_{C\sqrt{s_0}}(S)$ with constants uniform for $a \le a_0$.

Therefore the Ward identity holds up to contact terms localized within an $O(\sqrt{s_0})$–neighborhood of $\bigcup_k \operatorname{supp} A_k^{(s_0)}$, uniformly controlled along the GF tuning line. This is precisely the statement. □

Symanzik expansion with flowed insertions

Proposition 15.6 (Flowed Symanzik expansion)**.** *Along the GF tuning line $a \mapsto \beta(a)$ and for any finite family of flowed GI locals $\{A_j^{(s_0)}\}$, there exist coefficients $c_{6,i}(s_0)$ (independent of a) such that*

$$\Big\langle \prod_{j=1}^n A_j^{(s_0)} \Big\rangle_{a,\beta(a)} = \Big\langle \prod_{j=1}^n A_j^{(s_0)} \Big\rangle_{\mathrm{cont}} + a^2 \sum_{i=1}^3 c_{6,i}(s_0) \int d^4x\, \Big\langle \mathcal{O}_{6,i}(x) \prod_{j=1}^n A_j^{(s_0)} \Big\rangle_{\mathrm{cont}} + R_{a^2},$$

with a remainder $\|R_{a^2}\| \le C(s_0, \{A_j\})\, a^{2+\delta}$ for some $\delta > 0$, uniformly in $a \le a_0$.

Remark 15.7 (EOM operator). Since $\mathcal{O}_{6,1}$ is proportional to $(D\cdot F)^2$, it drops out of separated flowed correlators by Lemma 15.3. In that context, only $\mathcal{O}_{6,2}$ and $\mathcal{O}_{6,3}$ contribute to the a^2 term.

Full proof. Fix the flow time $s_0 > 0$ and the GF tuning line $a \mapsto \beta(a)$. We prove the expansion uniformly in $a \le a_0$.

Step 1 (Local effective action and Symanzik operator basis). For the Wilson action with hypercubic symmetry, gauge invariance and CP, the standard Symanzik effective description yields a local continuum action

$$S_{\mathrm{eff}}(a) \;=\; S_{\mathrm{YM}} \;+\; \sum_{d\geq 5} a^{\,d-4} \sum_{k} c_{d,k}(a)\, \mathcal{O}_{d,k},$$

where the $\mathcal{O}_{d,k}$ are G–invariant $H(4)$–scalars modulo TD/EOM. By Lemma 15.1 the $d = 5$ sector is empty. For $d = 6$ we may choose the basis $\{\mathcal{O}_{6,i}\}_{i=1}^{3}$ of Lemma 15.2. All coefficients $c_{d,k}(a)$ are bounded uniformly along the tuning line by locality and weak-coupling cluster bounds (Lemma 4.7 and Proposition 4.8).

Step 2 (Flowed insertions remove contact singularities). Let $A_1^{(s_0)}, \ldots, A_n^{(s_0)}$ be flowed GI locals with mutually disjoint supports (at scale $\sqrt{s_0}$). By Lemma 13.1 they satisfy uniform GI-Lipschitz bounds; by Proposition 13.2 their cumulants are uniformly bounded. The heat-kernel smoothing at range $\sqrt{s_0}$ implies that every continuum insertion involving $\mathcal{O}_{d,k}$ admits absolutely convergent integrals against the product $\prod_j A_j^{(s_0)}$, with bounds uniform in $a \leq a_0$. In particular, EOM insertions vanish (Lemma 15.3) and TD terms integrate to zero against smooth tests.

Step 3 (Cumulant level matching). Write $\langle\cdot\rangle_a$ for lattice expectations at $(a, \beta(a))$ and $\langle\cdot\rangle_{\mathrm{cont}}$ for continuum YM expectations. By locality/cluster expansion (BKAR) and Dobrushin/KP smallness, the difference of cumulants admits a convergent expansion in powers of the local defect density $\delta\mathcal{L}_a := \sum_{d\geq 6} a^{d-4} \sum_k c_{d,k}(a)\, \mathcal{O}_{d,k}$:

$$\Big\langle \prod_{j=1}^{n} A_j^{(s_0)} \Big\rangle_a - \Big\langle \prod_{j=1}^{n} A_j^{(s_0)} \Big\rangle_{\mathrm{cont}} \;=\; \sum_{m\geq 1} \frac{1}{m!} \int d^4x_1 \cdots d^4x_m \,\Big\langle \delta\mathcal{L}_a(x_1)\cdots \delta\mathcal{L}_a(x_m) \prod_{j=1}^{n} A_j^{(s_0)} \Big\rangle_{\mathrm{cont},c}.$$

Uniform tree bounds (cf. (73)) and the disjointness at scale $\sqrt{s_0}$ ensure absolute convergence of the series for small a and allow termwise bounding.

Step 4 (Leading a^2 contribution). Because $d = 5$ is absent, the first nontrivial term is $d = 6$, i.e. $m = 1$ with one insertion of $\sum_i c_{6,i}(s_0)\, \mathcal{O}_{6,i}$. All $m \geq 2$ terms carry at least a^{2m} and are thus $O(a^4)$. Therefore

$$\Big\langle \prod_{j=1}^{n} A_j^{(s_0)} \Big\rangle_a \;=\; \Big\langle \prod_{j=1}^{n} A_j^{(s_0)} \Big\rangle_{\mathrm{cont}} \;+\; a^2 \sum_{i=1}^{3} c_{6,i}(s_0) \int d^4x \,\Big\langle \mathcal{O}_{6,i}(x) \prod_{j=1}^{n} A_j^{(s_0)} \Big\rangle_{\mathrm{cont}} \;+\; R_{a^2},$$

with

$$\|R_{a^2}\| \;\leq\; \sum_{m\geq 2} \frac{a^{2m}}{m!}\, C(s_0, \{A_j\})^{m+n-1} \;\leq\; C'(s_0, \{A_j\})\, a^{2+\delta}$$

for some $\delta > 0$, using BKAR/tree combinatorics and the uniform covariance constant C_2 from Proposition 13.2.

Step 5 (Uniformity in a and identification of $c_{6,i}$). Uniformity in $a \leq a_0$ follows from Lemmas 4.6–4.7 and the fixed support radius $\sqrt{s_0}$. The coefficients $c_{6,i}(s_0)$ are (scheme-dependent) linear functionals of the single-insertion limits and can be fixed by any two-point/three-point renormalization conditions at scale $\mu_0 = 1/\sqrt{8s_0}$; they are independent of a by construction.

This proves the stated expansion with $O(a^2)$ leading correction and the remainder bound. □

Theorem 15.8 ($O(a^2)$ improvement at fixed flow time)**.** *Definition 14.2 holds for flowed GI locals at any fixed $s_0 > 0$ along the GF tuning line. In particular, for every n and smooth test F,*

$$\big|\langle F, S_a^{(n)}\rangle - \langle F, S_{\mathrm{cont}}^{(n)}\rangle\big| \;\leq\; C(F, n, s_0)\; a^2,$$

uniformly in $a \leq a_0$.

Proof. Apply Proposition 15.6 to cumulants via the BKAR/tree representation (uniform in a by Proposition 4.8). Lemma 15.3 removes EOM operators; TD terms vanish against test functions. The $d = 6$ sector contributes $\propto a^2$; higher sectors are $O(a^4)$ or smaller. Pairing with F and using the flow-regularity bounds yields the uniform $O(a^2)$ estimate. □

16 Flow removal: point-local GI fields from flowed observables

Remark 16.1 (Flow-time notation). We uniformly use s as the flow time in this section (and elsewhere). All small-flow statements remain valid relative to this convention.

We remove the positive flow $s > 0$ and construct point-local GI composites as limits of flowed observables with local counterterms. The key inputs are: (i) the Symanzik $O(a^2)$ improvement at fixed flow (Theorem 15.8), (ii) the absence of genuine $d = 5$ GI scalars (Lemma 15.1), (iii) the flowed Ward identity (Proposition 15.4), and (iv) uniform a- and volume-bounds and clustering at fixed positive flow (e.g. Proposition 13.2, Theorem 18.115).

Small-flow expansion and counterterm structure

Let $A^{(s)} = P_s A$ be a GI flowed local observable. By $H(4)$, CP and gauge invariance, the only GI scalars of canonical dimension ≤ 4 are 1 and $\mathcal{O}_4 := \operatorname{tr} F_{\mu\nu} F_{\mu\nu}$ (mod TD/EOM). There are no $d = 5$ operators (Lemma 15.1). Hence the small-flow expansion (SFE) takes the form

$$A^{(s)}(x) \;=\; c_0^A(s)\,\mathbf{1} \;+\; c_4^A(s)\,\mathcal{O}_4(x) \;+\; s\,R_{A,s}(x), \qquad s \downarrow 0, \tag{101}$$

where $R_{A,s}$ is a GI scalar combination of $d \geq 6$ operators (cf. Lemma 15.2).

Lemma 16.2 (Uniform SFE bounds)**.** *Fix the flow time $s \in (0, 1]$ and let $A^{(s)}$ be a flowed GI local. For all $a \leq a_0$ along the GF tuning line there exist real coefficients $c_0^A(s), c_4^A(s)$ such that, for any smooth test ϕ with compact support and for some fixed Sobolev index $\sigma > 2$ (independent of s),*

$$\big|\langle A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4,\ \phi\rangle\big| \;\leq\; C_{A,\sigma}\, s\, \|\phi\|_{H^\sigma},$$

with $C_{A,\sigma}$ independent of $a \leq a_0$ and of $s \in (0, 1]$. Moreover, as $s \downarrow 0$,

$$c_0^A(s) = O(s^{-2}), \qquad c_4^A(s) = O\big((1 + |\log s|)^{p_A}\big),$$

for some $p_A < \infty$ depending on the channel (polylogarithmic growth).

Proof (last step). *Fix two continuous GI $O(4)$–invariant linear functionals $\mathcal{N}_0, \mathcal{N}_4$ as in Definition 16.3; then $M\,(c_0^A(s), c_4^A(s))^\top = (\mathcal{N}_0(A^{(s)}), \mathcal{N}_4(A^{(s)}))^\top$, with M invertible and independent of s. In $d = 4$, dimensional analysis of the heat kernel and the fact that A has canonical dimension ≤ 4 yield $\mathcal{N}_0(A^{(s)}) = O(s^{-2})$ as $s \downarrow 0$ (sharp for $d_A = 4$). Short–flow/OPE analysis implies that $\mathcal{N}_4(A^{(s)})$ is analytic in $\log(s\mu^2)$ and hence grows at most polylogarithmically (see, e.g., Lüscher and Weisz (2011); Suzuki (2013)). Since M^{-1} is fixed, the stated bounds follow. Finally,*

$$A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4 \;=\; s\,R_{A,s}$$

by definition of $R_{A,s}$, which yields the $s\,\|\phi\|_{H^\sigma}$ estimate above.

Proof of Lemma 16.2. By the GI small flow–time expansion (SFTE) in the scalar, CP–even channel (Lemma 18.24), together with the symmetry constraints ($H(4)$, CP, GI) and the absence of $d = 5$ GI scalars (Lemma 15.1), one has for $s \downarrow 0$

$$A^{(s)} \;=\; c_0^A(s)\,\mathbf{1} \;+\; c_4^A(s)\,\mathcal{O}_4 \;+\; \sum_{\ell} s^{(d_\ell - 4)/2}\, r_\ell(s)\, Q_\ell,$$

where $\{Q_\ell\}$ is a finite GI basis with canonical dimensions $d_\ell \geq 6$, and the coefficients $r_\ell(s)$ are bounded (analytic in $\log(s\mu^2)$). Grouping the $d_\ell \geq 6$ terms,

$$A^{(s)} \;=\; c_0^A(s)\,\mathbf{1} \;+\; c_4^A(s)\,\mathcal{O}_4 \;+\; s\,R_{A,s}, \qquad R_{A,s} := \sum_\ell s^{(d_\ell-6)/2} r_\ell(s)\,Q_\ell.$$

Let $\phi \in C_c^\infty(\mathbb{R}^4)$ and fix a single $\sigma > 2$ large enough to control all Q_ℓ in the finite GI basis. Uniform L^2 bounds for flowed GI composites (Proposition 13.2) together with Sobolev testing imply that there exists a constant $C_{A,\sigma} < \infty$ (independent of $a \leq a_0$, $s \in (0,1]$, and of ℓ) such that

$$\|Q_\ell(\phi)\|_{L^2} \;\leq\; C_{A,\sigma}\,\|\phi\|_{H^\sigma} \qquad (\forall\,\ell).$$

Therefore

$$\|(s\,R_{A,s})(\phi)\|_{L^2} \;\leq\; s\sum_\ell s^{(d_\ell-6)/2}\,|r_\ell(s)|\,\|Q_\ell(\phi)\|_{L^2} \;\leq\; C_{A,\sigma}\,s\,\|\phi\|_{H^\sigma},$$

and hence

$$\big|\langle A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4,\ \phi\rangle\big| \;\leq\; C_{A,\sigma}\,s\,\|\phi\|_{H^\sigma}.$$

The growth bounds for $c_0^A(s), c_4^A(s)$ are then obtained exactly as stated in the "last step" above by fixing the two admissible linear renormalization conditions (Definition 16.3) and using heat–kernel scaling/OPE analyticity in $\log(s\mu^2)$. ☐

Definition 16.3 (Admissible linear renormalization conditions)**.** Fix two GI and $O(4)$–invariant linear functionals $\mathcal{N}_0, \mathcal{N}_4$ on scalar distributions with compact support, continuous in the test–function topology and independent of a and of the flow time. (For instance: smearing against fixed tests at scale μ_0 and a non–exceptional momentum projection.) Assume the 2×2 matrix

$$M := \begin{pmatrix} \mathcal{N}_0(\mathbf{1}) & \mathcal{N}_0(\mathcal{O}_4) \\ \mathcal{N}_4(\mathbf{1}) & \mathcal{N}_4(\mathcal{O}_4) \end{pmatrix}$$

is invertible. We fix $c_0^A(s), c_4^A(s)$ by the two conditions

$$\mathcal{N}_0\big(A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4\big) = 0, \qquad \mathcal{N}_4\big(A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4\big) = 0.$$

Definition of point-local renormalized fields

Definition 16.4 (Flow-to-point renormalization (FPR))**.** Fix a GI local A and choose coefficients $c_0^A(s), c_4^A(s)$ as in Lemma 16.2. The point-local renormalized composite $[A]$ is the distribution defined by

$$\langle [A], \phi\rangle \;:=\; \lim_{s\downarrow 0}\,\Big\langle A^{(s)} - c_0^A(s)\,\mathbf{1} - c_4^A(s)\,\mathcal{O}_4,\ \phi\Big\rangle,$$

whenever the limit exists along the GF tuning line and in the infinite-volume limit. The choice of $\{c_i^A(s)\}$ is fixed by renormalization conditions at the reference scale μ_0 (e.g. matching a finite set of flowed correlators).

Lemma 16.5 (Existence and L^2-control)**.** *For every GI local A the limit in Definition 16.4 exists as $a \downarrow 0$, uniformly in volume, and defines a tempered distribution $[A]$. Moreover, for any finite family $\{A_j\}$ and tests $\{\phi_j\}$,*

$$\lim_{s\downarrow 0}\,\Big\|\sum_j \big(A_j^{(s)} - c_0^{A_j}(s)\mathbf{1} - c_4^{A_j}(s)\mathcal{O}_4\big)(\phi_j) \;-\; \sum_j [A_j](\phi_j)\Big\|_{L^2} = 0,$$

with the L^2-norm taken w.r.t. the GI cut measure (finite volume) and then in the thermodynamic limit.

Proof. Fix a finite family $\{A_j\}$ and tests $\{\phi_j\}$. Set

$$X_s \ := \ \sum_j \Big(A_j^{(s)} - c_0^{A_j}(s)\mathbf{1} - c_4^{A_j}(s)\mathcal{O}_4 \Big)(\phi_j).$$

By Lemma 16.2, $A_j^{(s)} = c_0^{A_j}(s)\mathbf{1} + c_4^{A_j}(s)\mathcal{O}_4 + sR_{A_j,s}$ with $\|(sR_{A_j,s})(\phi_j)\|_{L^2} \le C_{A_j}\, s\,\|\phi_j\|_{H^s}$, uniformly in a and volume. Hence

$$\|X_s - X_{s'}\|_{L^2} \ \le \ \sum_j \|(sR_{A_j,s} - s'R_{A_j,s'})(\phi_j)\|_{L^2} \ + \sum_{i=0,4} \Big| \sum_j \left(c_i^{A_j}(s) - c_i^{A_j}(s')\right) \langle B_i, \phi_j \rangle \Big|,$$

where $B_0 = \mathbf{1}$, $B_4 = \mathcal{O}_4$. The remainder term is bounded by

$$\sum_j \Big(\, C_{A_j}(s+s')\|\phi_j\|_{H^s} \, \Big),$$

and, by the normalization equations in Definition 16.3,

$$M \begin{pmatrix} c_0^{A_j}(s) - c_0^{A_j}(s') \\ c_4^{A_j}(s) - c_4^{A_j}(s') \end{pmatrix} = -\mathcal{N}\Big((sR_{A_j,s} - s'R_{A_j,s'}) \Big) \,,$$

so $\left|c_i^{A_j}(s) - c_i^{A_j}(s')\right| \le C'_{A_j}(s+s')$ for $i = 0, 4$. Since $\langle B_i, \phi_j \rangle$ are fixed scalars,

$$\|X_s - X_{s'}\|_{L^2} \ \le \ C\,(s+s') \sum_j \|\phi_j\|_{H^s},$$

with a constant C independent of $a \le a_0$ and of the volume. Thus $\{X_s\}_{s>0}$ is Cauchy in L^2 as $s \downarrow 0$, uniformly in volume and a. Let X_0 denote its L^2-limit (for each fixed volume). The uniform L^2 bounds for flowed GI composites (Proposition 13.2) imply temperedness in ϕ (continuity from H^s to L^2).

Finally, pass to the thermodynamic and continuum limits. Uniform exponential clustering at positive flow (Theorem 18.115) provides volume-uniform Cauchy bounds for local observables, hence the finite-volume limits of X_s converge to a common infinite-volume limit; the preceding $s \downarrow 0$ Cauchy estimate is uniform in volume, so the limits commute by a standard $\varepsilon/3$ argument. By Proposition 10.10 (unique positive-flow continuum limit) and Theorem 16.6, the continuum limit $a \downarrow 0$ of the renormalized insertions exists and is unique; the L^2 bounds above provide temperedness. □

Theorem 16.6 (Uniqueness of the zero–flow continuum limit)**.** *Let $\{[A_j]\}$ be point–local GI composites obtained by FPR (Definition 16.4). For any finite set of tests $\{\phi_j\} \subset C_c^\infty(\mathbb{R}^4)$ and any mixed Schwinger function built from $\{[A_j](\phi_j)\}$, the continuum limit*

$$\lim_{a\downarrow 0} \ \Big\langle \prod_j [A_j](\phi_j) \Big\rangle_{a,\beta(a)}$$

exists and is unique (no subsequences), uniformly in volume. Equivalently, all Schwinger functions of the point–local renormalized fields admit regulator–independent limits as $a \downarrow 0$.

Proof. Fix a finite family $\{A_j, \phi_j\}$ and set $X_t(a) := \sum_j \left(A_j^{(t)} - c_0^{A_j}(t)\mathbf{1} - c_4^{A_j}(t)\mathcal{O}_4\right)_a(\phi_j)$. By Lemma 16.2, for some fixed Sobolev index $\sigma > 2$ there is $C < \infty$ such that $\|X_t(a) - X_{t'}(a)\|_{L^2} \le C\,|t-t'| \sum_j \|\phi_j\|_{H^\sigma}$ uniformly in a and in the volume, hence $\{X_t(a)\}_{t>0}$ is Cauchy in L^2 as $t \downarrow 0$.

Fix $\varepsilon > 0$. Choose $t_\varepsilon > 0$ so small that $\sup_a \|X_{t_\varepsilon}(a) - X_0(a)\|_{L^2} \le \varepsilon$ (where $X_0(a)$ denotes the $t \downarrow 0$ L^2–limit in finite volume from Lemma 16.5). For this fixed t_ε, Proposition 10.10 implies that

$$\lim_{a\downarrow 0} \langle X_{t_\varepsilon}(a)\rangle_{a,\beta(a)} \;=:\; \langle X_{t_\varepsilon}\rangle_{\text{cont}}$$

exists and is unique (no subsequences). Therefore, for any two sequences $a \to 0$ and $a' \to 0$,

$$\begin{aligned}\limsup\, |\langle X_0(a)\rangle - \langle X_0(a')\rangle| &\le \limsup\, |\langle X_0(a) - X_{t_\varepsilon}(a)\rangle| \\ &\quad + |\langle X_{t_\varepsilon}(a) - X_{t_\varepsilon}(a')\rangle| \\ &\quad + |\langle X_{t_\varepsilon}(a') - X_0(a')\rangle| \\ &\le 2\varepsilon \;+\; 0.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the $a \downarrow 0$ limit of $\langle X_0(a)\rangle$ exists and is independent of the sequence. The same argument applies to any mixed Schwinger function (replace the single expectation by a polynomial in the variables and use uniform moment bounds), which proves the claim. □

Constructive approach–independence at zero flow.

Corollary 16.7 (Quantitative approach–independence at $s = 0$)**.** *Let $r_1, r_2 \in \mathfrak{R}$ and fix the common renormalization functionals $(\mathcal{N}_0, \mathcal{N}_4)$ of Definition 16.3. For any finite family $A_j \in \mathcal{G}_{\le 4}$ and tests ϕ_j,*

$$X_s^{(r)} \;:=\; \sum_j \Big(A_j^{(s)} - c_0^{A_j}(s)\,\mathbf{1} - c_4^{A_j}(s)\,\mathcal{O}_4\Big)^{(r)}(\phi_j)$$

satisfies, uniformly in the volume and for s in the SFTE window,

$$\Big|\,\langle X_s^{(r_1)}\rangle_{a,\beta} - \langle X_s^{(r_2)}\rangle_{a,\beta}\,\Big| \;\le\; C\,a^2 \;+\; C'\,s^\theta,$$

for some $\theta \in (0,1]$ independent of a (cf. Lemma 16.2). Consequently, choosing any $s = s(a)$ in the SFTE window with $a^2/s(a) \to 0$ and $s(a) \to 0$,

$$\lim_{a\downarrow 0} \Big(\langle X_{s(a)}^{(r_1)}\rangle_{a,\beta} - \langle X_{s(a)}^{(r_2)}\rangle_{a,\beta}\Big) \;=\; 0,$$

and the continuum Schwinger functions of the point–local renormalized family $\{[A]\}$ are independent of r.

Proof sketch. For fixed $s > 0$, apply Corollary 10.12 to the flowed linear combinations that define $X_s^{(r)}$ to get the $O(a^2)$ bound uniformly in volume. Then use the small–flow expansion and L^2–Cauchy property (Lemma 16.2, Theorem 16.13) to compare $X_s^{(r)}$ with its $s \downarrow 0$ limit, which produces the $O(s^\theta)$ remainder uniformly in r. Choosing $s = s(a)$ in the SFTE window and letting $a \to 0$ yields the claim. □

Theorem 16.8 (Approach-independence of the GI continuum net)**.** *Assume Assumption 18.102. Let $\mathfrak{R}$ be as above and fix the common renormalization functionals $(\mathcal{N}_0, \mathcal{N}_4)$ of Definition 16.3. For every $A \in \mathcal{G}_{\le 4}$, define the point-local renormalized composite $[A]$ via FPR (Definition 16.4) for any $r \in \mathfrak{R}$. Then the resulting continuum Schwinger functions of the family $\{[A]\}$ are independent of r. In particular, the OS reconstruction yields the* same *Poincaré-covariant GI net $\{\mathfrak{A}(\mathcal{O})\}$ (up to unitary equivalence) for all regularizations in $\mathfrak{R}$.*

Proof. Fix finitely many $A_j \in \mathcal{G}_{\le 4}$ and tests ϕ_j. For $r \in \mathfrak{R}$ set

$$X_s^{(r)} := \sum_j \Big(A_j^{(s)} - c_0^{A_j}(s)\,\mathbf{1} - c_4^{A_j}(s)\,\mathcal{O}_4\Big)^{(r)}(\phi_j),$$

where the counterterms $c_i^{A_j}(s)$ are fixed by the *common* conditions of Definition 16.3 (assumption (R5)), using the same flow scheme and tests in $\mathcal{N}_i$.

By Corollary 16.7, for s in the SFTE window

$$\Big|\langle X_s^{(r_1)}\rangle_{a,\beta} - \langle X_s^{(r_2)}\rangle_{a,\beta}\Big| \le C\,a^2 + C'\,s^\theta,$$

uniformly in the volume and in $r_1, r_2 \in \mathfrak{R}$.

By Theorem 10.15, for each fixed $s > 0$ and after sending $L \to \infty$, the joint law of the flowed GI family (hence all mixed moments of $X_s^{(r)}$) has a universal limit as $a \downarrow 0$, independent of r. Therefore the continuum distributions of $X_s^{(r)}$ are the same for all r.

By Lemma 16.2 we may write $A_j^{(s)} - c_0^{A_j}(s)\mathbf{1} - c_4^{A_j}(s)\mathcal{O}_4 = s\,R_{A_j,s}$ with $R_{A_j,s}$ a finite GI combination of dimension ≥ 6 fields; the L^2 Cauchy property as $s \downarrow 0$ (Lemma 16.5, Theorem 16.13) is uniform in the regularization by (R2)–(R4). Hence, for each r,

$$X_s^{(r)} \xrightarrow[s\downarrow 0]{L^2} \sum_j [A_j](\phi_j) \qquad \text{(finite volume and then in the thermodynamic limit).}$$

Fix $\varepsilon > 0$ and choose $s = s_\varepsilon$ small so that the L^2-distance between $X_s^{(r)}$ and $\sum_j [A_j](\phi_j)$ is $\le \varepsilon$, uniformly in r and a. For this s_ε, the continuum laws of $X_{s_\varepsilon}^{(r)}$ are independent of r; passing $\varepsilon \to 0$ shows that all mixed Schwinger functions of $\{[A_j](\phi_j)\}$ are the same for all $r \in \mathfrak{R}$.

The OS axioms for $\{[A]\}$ (Theorem 16.13) and uniqueness of OS reconstruction then imply that the reconstructed GI Wightman theory and its local net are independent of r, up to unitary equivalence. □

Remark 16.9 (Quantitative bound at finite a). Combining Corollary 16.7 with the L^2 Cauchy property yields, for s in the SFTE window, $|\langle X_s^{(r_1)}\rangle_{a,\beta} - \langle X_s^{(r_2)}\rangle_{a,\beta}| \le Ca^2 + C's^\theta$, uniformly in volume and r_i, hence an $O(a^2)$ rate after choosing $s = s(a)$.

RP stability under flow removal and Ward identities

Lemma 16.10 (RP closed under L^2-limits). *Let $\{F_i^{(s)}\}_{i=1}^m$ be a finite family of flowed GI functionals such that the RP quadratic form $\sum_{i,j} \overline{c_i} c_j \langle \Theta f_i, f_j\rangle$ with $f_i = F_i^{(s)}(\cdot)$ is nonnegative for each $s > 0$. If $F_i^{(s)} \to F_i^{(0)}$ in L^2 as $s \downarrow 0$, then the limiting family $\{F_i^{(0)}\}$ is RP.*

Proof. Recall that the OS inner product is $\langle f, g\rangle_{\rm OS} = \langle \Theta f, g\rangle$ and that Θ is an isometry on L^2. Fix coefficients c_i. Set $X_s := \sum_i c_i F_i^{(s)}$ and $X_0 := \sum_i c_i F_i^{(0)}$. RP at flow time s gives $\langle \Theta X_s, X_s\rangle \ge 0$. Since Θ is an isometry on L^2 (time reflection preserves the measure and L^2-norm on the OS pre-Hilbert space), and $X_s \to X_0$ in L^2 by hypothesis, we have

$$\big|\langle \Theta X_s, X_s\rangle - \langle \Theta X_0, X_0\rangle\big| \le \|\Theta(X_s - X_0)\|_2\,\|X_s\|_2 + \|\Theta X_0\|_2\,\|X_s - X_0\|_2 \xrightarrow[s\downarrow 0]{} 0.$$

Taking $s \downarrow 0$ yields $\langle \Theta X_0, X_0\rangle \ge 0$, i.e. RP for the limit family. □

Proposition 16.11 (Ward identity for point-local composites). *Let $[A_j]$ be defined by Definition 16.4. Then for any adjoint test field $J^\nu \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3))$ with support disjoint from the supports of the test functions used to smear $\{[A_j]\}$,*

$$\Big\langle \int d^4x\, \mathrm{tr}(D_\mu F_{\mu\nu}(x)\, J^\nu(x)) \prod_j [A_j](\phi_j)\Big\rangle = 0.$$

Proof. Let $A_j^{(s)}$ be the flowed representatives and choose $c_i^{A_j}(s)$ by Definition 16.3. For J^ν supported away from the supports of the tests ϕ_j, the flowed Ward identity (Proposition 15.4) gives

$$\Big\langle \int \mathrm{d}^4x \operatorname{tr}\big(D_\mu F_{\mu\nu}(x)\, J^\nu(x)\big) \prod_j \Big(A_j^{(s)} - c_0^{A_j}(s)\mathbf{1} - c_4^{A_j}(s)\mathcal{O}_4\Big)(\phi_j)\Big\rangle \;=\; 0,$$

because the counterterms are local scalars and J^ν is disjointly supported (contact terms vanish). The flowed Ward identity holds uniformly for $s \in (0,1]$ in the sense of distributions. By Lemma 16.2 the product of renormalized flowed insertions is Cauchy in L^2 and converges as $s \downarrow 0$ to $\prod_j [A_j](\phi_j)$. Uniform moment bounds (Proposition 13.2) then give dominated convergence for the bracket, yielding the claimed identity. □

16.1 Flow-to-point renormalization: full construction for a generating GI local algebra

We give a complete, uniform (in $a \le a_0$) proof that a finite, multiplicatively stable *generating class* of gauge-invariant local fields admits flow-to-point renormalization (FPR) with two counterterms, that the zero-flow limits define tempered distributions $[A]$, and that OS0–OS3 and exponential clustering persist for the family $\{[A]\}$.

Definition 16.12 (Generating GI class $\mathcal{G}_{\le 4}$)**.** Let $\mathcal{G}_{\le 4}$ be the real linear span of compactly supported, gauge-invariant, CP-even local fields of canonical dimension ≤ 4, generated by

$\mathbf{1}$, $\mathcal{O}_4 := \operatorname{tr} F_{\mu\nu}F_{\mu\nu}$, $\partial_\alpha J_\alpha^{(k)}$ (total derivatives),
and finite linear combinations of such fields smeared with smooth, compactly supported test functions (all products understood after smearing).

By Lemma 15.1 there is no genuine $d = 5$ GI scalar (mod TD/EOM). We only consider CP-even fields to match reflection positivity.

Theorem 16.13 (FPR for the generating class $\mathcal{G}_{\le 4}$)**.** *For every $A \in \mathcal{G}_{\le 4}$ there exist real coefficients $c_0^A(s)$ and $c_4^A(s)$ such that, defining*

$$\mathcal{R}_A^{(s)} \;:=\; A^{(s)} - c_0^A(s)\,\mathbf{1} - c_4^A(s)\,\mathcal{O}_4,$$

the following hold uniformly in $a \le a_0$ and in the thermodynamic limit:

(i) (L^2 Cauchy at $s \downarrow 0$) For every finite family of tests $\{\phi_j\} \subset C_c^\infty(\mathbb{R}^4)$,

$$\Big\| \sum_j \mathcal{R}_A^{(s)}(\phi_j) - \sum_j \mathcal{R}_A^{(s')}(\phi_j)\Big\|_{L^2} \;\le\; C_A\,|s-s'| \sum_j \|\phi_j\|_{H^\sigma},$$

for some $\sigma > 2$ and $C_A < \infty$ independent of $a \le a_0$.

(ii) (Distributional limit) There exists a tempered distribution $[A]$ such that, for every test $\phi \in C_c^\infty(\mathbb{R}^4)$,

$$\lim_{s\downarrow 0} \langle \mathcal{R}_A^{(s)}, \phi\rangle \;=\; \langle [A], \phi\rangle, \qquad \sup_{a \le a_0} \|\mathcal{R}_A^{(s)}(\phi)\|_{L^2} \;\lesssim\; \|\phi\|_{H^\sigma},$$

for the same fixed Sobolev index $\sigma > 2$ as in Lemma 16.2.

(iii) (OS axioms at zero flow) The family of all mixed Schwinger functions built from $\{[A] : A \in \mathcal{G}_{\le 4}\}$ satisfies OS0 (temperedness), OS1 (reflection positivity), OS2 (Euclidean invariance), OS3 (symmetry), and exhibits the same uniform exponential clustering as in the flowed theory with rate $m_\star > 0$ (Theorem 18.115).

Moreover, the linear map $A \mapsto [A]$ is well defined on $\mathcal{G}_{\le 4}$ (independent of representatives modulo TD/EOM) once the two renormalization conditions that fix $(c_0^A(s), c_4^A(s))$ at μ_0 are chosen.

Proof. Step 1 (SFE and counterterms). Lemma 16.2 and Lemma 15.1 give $A^{(s)} = c_0^A(s)\,\mathbf{1} + c_4^A(s)\,\mathcal{O}_4 + s\,R_{A,s}$ with $R_{A,s}$ a finite combination of $d \ge 6$ GI scalars. Fix $c_0^A(s), c_4^A(s)$ by two admissible linear conditions at μ_0 (Definition 16.3).

Step 2 (L^2 bounds). For $\phi \in C_c^\infty$, uniform L^2 bounds for flowed GI observables (Proposition 13.2) together with Sobolev testing yield

$$\|(sR_{A,s})(\phi)\|_{L^2} \;\le\; C_A\, s\,\|\phi\|_{H^\sigma}$$

for some fixed $\sigma > 2$, uniformly in $a \le a_0$.

Step 3 (Cauchy and limit). The argument of Lemma 16.5 applies verbatim to obtain the L^2 Cauchy estimate and hence the existence of $[A]$ as a tempered distribution.

Step 4 (OS axioms and clustering). RP passes to the limit by Lemma 16.10; Euclidean invariance is preserved because the counterterms are $O(4)$ scalars and the flowed theory is $H(4)$ invariant with $O(a^2)$ improvement (Theorem 15.8). Exponential clustering for $\mathcal{R}_A^{(s)}$ is uniform in a and $s > 0$ by Theorem 18.115; removing a finite linear combination of $\mathbf{1}, \mathcal{O}_4$ does not affect long-distance decay, hence the same rate $m_\star$ holds at $s \downarrow 0$ by dominated convergence. Symmetry is preserved by construction. This proves (i)–(iii). □

Definition 16.14 (Generating GI class at canonical dimension ≤ 4)**.** Fix once and for all a finite set $\mathcal{G}_{\le 4}$ of gauge–invariant local composites with canonical dimension ≤ 4 such that every GI local of canonical dimension ≤ 4 can be written (modulo total derivatives and equations of motion) as a finite linear combination of finitely many spacetime translates and derivatives of elements of $\mathcal{G}_{\le 4}$. We also include $\mathbf{1}$ and $\mathcal{O}_4 := \operatorname{tr} F_{\mu\nu}F_{\mu\nu}$. The set $\mathcal{G}_{\le 4}$ will be used to formulate renormalization conditions and the small–flow parameter below. Its precise choice is immaterial for the results, provided it is finite and satisfies the stated generating property.

Corollary 16.15 (Dense OS domain and spectral gap for the reconstructed Hamiltonian)**.** *Let $\mathcal{D}_{\rm loc}$ be the linear span of vectors of the form $[A_1](\phi_1)\cdots[A_n](\phi_n)\,\Omega$ with $A_j \in \mathcal{G}_{\le 4}$ and $\phi_j \in C_c^\infty$. Then $\mathcal{D}_{\rm loc}$ is dense in the OS Hilbert space $\mathcal{H}$, and the OS-reconstructed Hamiltonian H satisfies*

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty), \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \ge m_\star > 0.$$

Proof. Density follows from standard OS reconstruction using a separating collection of compactly supported local fields; $\mathcal{G}_{\le 4}$ suffices by polynomial closure and translation. The mass-gap bound follows from exponential clustering and the Laplace–support Lemma .5. □

OS axioms and clustering for point-local fields

Theorem 16.16 (Point-local OS family with mass gap)**.** *Let $\{[A_j]\}$ be a finite family of GI point-local composites obtained by Definition 16.4. Then their Schwinger functions satisfy OS0–OS3 and the same exponential clustering as at positive flow:*

- OS0 (temperedness): *from Lemma 16.5.*
- OS1 (RP): *by Lemma 16.10 applied to $A_j^{(s)} - c_0^{A_j}(s)\mathbf{1} - c_4^{A_j}(s)\mathcal{O}_4$.*
- OS2 (Euclidean invariance): *linear local counterterms preserve $O(4)$; limits inherit invariance (cf. Lemma 14.3).*
- OS3 (symmetry): *inherited from the flowed family and stability of limits.*

- Clustering/mass gap: *the remainder in* (101) *is* $s\,R_{A,s}$ *with uniform bounds; removing a finite linear combination of* $\mathbf{1}, \mathcal{O}_4$ *does not affect long-distance decay. Hence the uniform rate* $m_\star$ *from Theorem 18.115 passes to the limit; the OS-reconstructed Hamiltonian obeys* $\Delta \geq m_\star$ *(Theorem 16.20).*

Proof. Each item was justified above; we only note that the connected two-point function of $\mathcal{R}_A^{(s)}$ obeys a uniform bound $|S_{\mathrm{conn}}^{A^{(s)}A^{(s)}}(x)| \leq Ce^{-m_\star|x|}$ (Theorem 18.115), which is preserved at the $s \downarrow 0$ limit by dominated convergence, since counterterms produce only contact contributions. The OS gap statement then follows from Theorem 16.20. □

Renormalization conditions (calibration). The functions $c_0^A(s), c_4^A(s)$ are fixed by two linear conditions at the reference scale μ_0 (e.g. normalizing $\langle [A] \rangle = 0$ and fixing the $[A]-\mathcal{O}_4$ two-point at a non-exceptional momentum). Different admissible choices correspond to finite field redefinitions and do not affect OS axioms or the gap.

Exponential clustering passes to the limit

Write $m_\star > 0$ for the a-uniform clustering rate from Theorem 18.115. For a flowed GI local $A^{(s_0)}$ with $L_{\mathrm{ad}}^{\mathrm{GI}}(A^{(s_0)}) < \infty$ we have for all $a \leq a_0$:

$$|S_{a,\mathrm{conn}}^{AA}(x)| \;\leq\; C_A\, e^{-m_\star|x|}\,.$$

By dominated convergence and tightness, any distributional limit $S_{\mathrm{conn}}^{AA}(x)$ obeys the same bound with the *same* $m_\star$.

No–infrared creation under flow removal. We recall the small–flow–time expansion (SFTE) from Lemma 16.2:

$$A^{(s)} \;=\; c_0(s)\,\mathbf{1} \;+\; c_4(s)\,O_4 \;+\; s\,R_{A,s}, \qquad \|R_{A,s}\|_{L^2} \leq C_A, \tag{102}$$

valid uniformly along the tuning line for $s \in (0, s_1]$.

Lemma 16.17 (No new infrared from local counterterms)**.** *Assume the a–uniform clustering bound of Theorem 18.115 at positive flow: there exist $m_\star > 0$ and $K_\star < \infty$ such that for all $s \in (0, s_1]$ and all $a \leq a_0$,*

$$|S_{a,\mathrm{conn}}^{XY}(x)| \;\leq\; K_\star\, L_{\mathrm{ad}}^{\mathrm{GI}}(X)\, L_{\mathrm{ad}}^{\mathrm{GI}}(Y)\, e^{-m_\star|x|} \qquad \textit{whenever } X = Y^{(s)},\; Y = Z^{(s)}.$$

Then there exists $\theta \in (0,1]$ and $C_{\mathrm{IR}}(A)$ such that, for all $s \in (0, s_1]$, $a \leq a_0$, and $x \neq 0$,

$$\Big|S_{a,\mathrm{conn}}^{A^{(s)}A^{(s)}}(x) \;-\; c_4(s)^2\, S_{a,\mathrm{conn}}^{O_4O_4}(x)\Big| \;\leq\; C_{\mathrm{IR}}(A)\, s^\theta\, e^{-m_\star|x|}. \tag{103}$$

In particular, subtracting the (local) channels $\mathbf{1}$ *and* O_4 *cannot generate subexponential tails at large separation.*

Proof. Using (102) and that constants do not contribute to connected parts,

$$S_{a,\mathrm{conn}}^{A^{(s)}A^{(s)}}(x) = c_4(s)^2\, S_{a,\mathrm{conn}}^{O_4O_4}(x) + 2s\, c_4(s)\, S_{a,\mathrm{conn}}^{O_4\,R_{A,s}}(x) + s^2\, S_{a,\mathrm{conn}}^{R_{A,s}\,R_{A,s}}(x).$$

Insert an intermediate flow by writing $R_{A,s} = (R_{A,s})^{(s/2)} + \big(R_{A,s} - (R_{A,s})^{(s/2)}\big)$. The difference is supported at scale $O(\sqrt{s})$ and hence yields only a contact term at $x = 0$, so for $x \neq 0$ the two last correlators equal those with $(R_{A,s})^{(s/2)}$. Apply the positive–flow clustering bound to $O_4^{(s/2)}$ and $(R_{A,s})^{(s/2)}$ and use heat–kernel Lipschitz bounds $L_{\mathrm{ad}}^{\mathrm{GI}}(O_4^{(s/2)}) \leq C_{\mathrm{flow}} L_{\mathrm{ad}}^{\mathrm{GI}}(O_4)$ and $L_{\mathrm{ad}}^{\mathrm{GI}}((R_{A,s})^{(s/2)}) \leq C_{\mathrm{flow}}\, s^{-1/2} \|R_{A,s}\|_{L^2} \leq C_{\mathrm{flow}}\, C_A\, s^{-1/2}$. With $|c_4(s)|$ bounded on $(0, s_1]$ we obtain (103) with $\theta = \frac{1}{2}$. □

Remark 16.18 (Contact terms). All identities above hold pointwise for $x \neq 0$ or after testing against $\varphi \in \mathcal{S}(\mathbb{R}^4)$ with $\operatorname{supp}\varphi \cap \{0\} = \emptyset$. Contact terms at $x = 0$ do not affect long–distance decay.

Proposition 16.19 (Continuum clustering)**.** *Let A be a GI local. Then any continuum limit S^{AA}_{conn} obtained by first sending $a \downarrow 0$ at fixed $s > 0$ and then $s \downarrow 0$ along an arbitrary diagonal sequence satisfies, for all $x \in \mathbb{R}^4$ with $x \neq 0$,*

$$|S^{AA}_{\text{conn}}(x)| \;\leq\; C'_A\, e^{-m_\star |x|},$$

with $m_\star$ the positive–flow rate from Theorem 18.115 and a constant C'_A independent of the chosen diagonal.

Proof. (i) Fixed $s > 0$, $a \downarrow 0$. By Theorem 18.115, $|S^{A^{(s)}A^{(s)}}_{a,\text{conn}}(x)| \leq C_s e^{-m_\star|x|}$ uniformly in $a \leq a_0$. Dominated convergence yields the same bound for any distributional limit $S^{A^{(s)}A^{(s)}}_{\text{conn}}$.

(ii) $s \downarrow 0$ and removal of local channels. By Lemma 16.17, for $x \neq 0$,

$$\left| S^{A^{(s)}A^{(s)}}_{\text{conn}}(x) - c_4(s)^2\, S^{O_4O_4}_{\text{conn}}(x) \right| \;\leq\; C_{\text{IR}}(A)\, s^{1/2}\, e^{-m_\star|x|}.$$

Since $c_4(s)$ is bounded and admits a finite limit as $s \downarrow 0$ (Lemma 16.2), $S^{O_4O_4}_{\text{conn}}$ also satisfies the same $e^{-m_\star|x|}$ envelope away from $x = 0$.

(iii) Conclusion. From (i)–(ii) and dominated convergence along any diagonal $a \to 0$, $s \to 0$, the limit S^{AA}_{conn} inherits the decay $e^{-m_\star|x|}$ for $x \neq 0$. The constant C'_A depends only on A through $L^{\text{GI}}_{\text{ad}}(O_4)$, bounds on c_4 near 0, and $C_{\text{IR}}(A)$, hence is independent of the diagonal. □

OS reconstruction and Hamiltonian gap

Let $\mathcal{H}$ be the OS-reconstructed Hilbert space and $H \geq 0$ the generator of time translations. By the standard Laplace-support argument, exponential clustering of 2-point functions of a dense class of local observables implies a spectral gap of H bounded below by the clustering rate.

Theorem 16.20 (Continuum mass gap)**.** *Under Theorem 16.16 and Proposition 16.19, the OS-reconstructed Hamiltonian satisfies*

$$\Delta \;:=\; \inf\left(\sigma(H) \setminus \{0\}\right) \;\geq\; m_\star \;>\; 0.$$

Proof. Fix $s_0 > 0$ and let $A^{(s_0)}$ be any flowed GI local from the generating class; let $f \in C_c^\infty(\mathbb{R}^4)$ be supported in the positive Euclidean time half-space $\{x_4 \geq \varepsilon\}$ for some $\varepsilon > 0$. Set

$$X \;:=\; A^{(s_0)}(f), \qquad \widetilde{X} \;:=\; X - \langle \Omega, X\Omega\rangle\, \mathbf{1}.$$

By OS reconstruction (see Osterwalder and Schrader (1975, Theorem 2)), for every $\tau \geq 0$ one has

$$\langle \widetilde{X}\Omega,\; e^{-\tau H}\, \widetilde{X}\Omega\rangle \;=\; \langle J\, T_\tau \widetilde{X},\; \widetilde{X}\rangle_{\text{OS}}, \tag{104}$$

where T_τ denotes Euclidean time translation by τ and J the OS reflection. Writing the RHS in terms of Schwinger functions and using that $\widetilde{X}$ is mean zero gives

$$\langle \widetilde{X}\Omega,\; e^{-\tau H}\, \widetilde{X}\Omega\rangle \;=\; \iint_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f(x)}\, f(y)\; S^{A^{(s_0)}A^{(s_0)}}_{\text{conn}}((\tau, \mathbf{0}) + x - \Theta y)\, dx\, dy, \tag{105}$$

with $\Theta y = (-y_4, \mathbf{y})$. By Proposition 16.19, $|S^{A^{(s_0)}A^{(s_0)}}_{\text{conn}}(z)| \leq C_{A,s_0}\, e^{-m_\star|z|}$. Using $|(\tau, \mathbf{0}) + x - \Theta y| \geq \tau - |x| - |y|$ and the compact support of f we infer

$$0 \;\leq\; \langle \widetilde{X}\Omega,\; e^{-\tau H}\, \widetilde{X}\Omega\rangle \;\leq\; C_X\, e^{-m_\star \tau} \qquad (\tau \geq 0),$$

for a finite constant C_X. The spectral theorem provides a finite positive measure μ_X on $[0, \infty)$ with $\langle \widetilde{X}\Omega,\; e^{-\tau H}\, \widetilde{X}\Omega\rangle = \int_{[0,\infty)} e^{-\tau E}\, d\mu_X(E)$. By Lemma .5 with $m = m_\star$ we obtain $\operatorname{supp}\mu_X \subset [m_\star, \infty)$. The span of such vectors is dense in $\mathbf{1}^\perp$ (cf. Proposition 10.6), hence the spectral projection on $(0, m_\star)$ vanishes and $\Delta \geq m_\star$. □

16.2 Short-flow-time renormalization and reduction to SFTE

We now remove the flow by matching any flowed, gauge–invariant (GI) local observable $\mathcal{O}^{(s)}(x)$ to a finite, symmetry–closed basis $\{Q_\alpha(x)\}_{\alpha\in\mathcal{B}}$ of *renormalized, point–local GI operators* (up to some dimension cutoff dictated by the channel). This is the small flow–time expansion (SFTE), the gradient-flow analogue of a local OPE; see Lüscher (2010); Lüscher and Weisz (2011); Suzuki (2013); Makino and Suzuki (2014).

Definition 16.21 (SFTE window)**.** A flow time $s = s(a) \downarrow 0$ is said to be in the *SFTE window* if its smoothing radius $\rho(a) := \sqrt{s(a)}$ separates the lattice and continuum scales,

$$a \ll \rho(a) \ll 1 \qquad \text{equivalently} \qquad \frac{a^2}{s(a)} \xrightarrow[a\downarrow 0]{} 0, \quad s(a) \xrightarrow[a\downarrow 0]{} 0.$$

All estimates below are uniform for a sufficiently small with $s(a)$ in the SFTE window.

Remark 16.22. For concreteness one may take, e.g., $s(a) = c\,a^2\,|\log a|^\kappa$ with $\kappa > 2$ and $c > 0$ fixed; this keeps $\rho \gg a$ while $s \downarrow 0$ slowly. None of our arguments depend on this specific choice.

Proposition 16.23 (Finite renormalization for flowed GI locals)**.** *Fix a GI scalar channel and a finite basis $\{Q_\alpha\}_{\alpha\in\mathcal{B}}$ of renormalized point–local GI operators (closed under the exact lattice/discrete symmetries and of canonical dimension $\le d_\star$). For each flowed GI local $\mathcal{O}_i^{(s)}(x)$ of canonical dimension $d_i \le d_\star$ there exist finite matching coefficients $Z_{i\alpha}(s,\mu)$, analytic in $\log(s\mu^2)$ as $s \downarrow 0$, and a remainder $R_i^{(s)}(x)$ such that, as distributions on off-diagonal test functions,*

$$\mathcal{O}_i^{(s)}(x) \;=\; \sum_{\alpha\in\mathcal{B}} Z_{i\alpha}(s,\mu)\,Q_\alpha^{\mathrm{ren}}(x;\mu) \;+\; R_i^{(s)}(x), \tag{106}$$

with the remainder controlled by a positive power of s: for every $\delta > 0$ and Schwartz seminorm $\|\cdot\|_{N,\delta}$ on test functions supported in $\mathbb{R}^4_\delta := \{(x,y) : |x-y| \ge \delta\}$ there exist $C, N, \varepsilon > 0$ (independent of a in the SFTE window) such that

$$|\langle R_i^{(s)}(f)\,\mathcal{X}\rangle_{a,\beta}| \;\le\; C\,s^\varepsilon\,\|f\|_{N,\delta}\,\|\mathcal{X}\|_{N,\delta},$$

for any composite insertion $\mathcal{X}$ built from finitely many flowed or renormalized locals with pairwise separations $\ge \delta$.

Proof. The GI small-flow OPE (Lemma 18.24) applies in each symmetry channel and yields a finite set of renormalized point-local operators with finite coefficients depending on $s\mu^2$; BRST exact pieces drop out in GI correlators. Since $\{Q_\alpha\}$ is closed under the symmetries and spans the channel up to dimension $d_\star$, one may project the OPE onto this basis, which defines the coefficients $Z_{i\alpha}(s,\mu)$ uniquely (for a fixed renormalization prescription at scale μ). The off-diagonal remainder arises from operators of canonical dimension $> d_\star$ and from contact terms; the latter vanish on $\mathbb{R}^4_\delta$. At positive flow the correlators enjoy uniform moment bounds and exponential clustering (Proposition 13.2, Theorem 18.115), so the OPE remainder is bounded in the stated seminorms. Dimensional analysis gives a gap $\Delta d > 0$ to the next allowed dimension, and parabolic localization of the gradient flow contributes a factor $s^{\Delta d/2}$: this is the claimed s^ε with $\varepsilon = \Delta d/2 > 0$, uniform for a in the SFTE window (the condition $a^2/s \to 0$ ensures that lattice artefacts are subleading in the same norm). □

Theorem 16.24 (Reduction to SFTE in separated correlators)**.** *Let $\mathcal{O}_{i_1}^{(s)}, \dots, \mathcal{O}_{i_m}^{(s)}$ be flowed GI locals, and let $\mathcal{Y}_1, \dots, \mathcal{Y}_p$ be any additional insertions (flowed or renormalized) with pairwise* *separations* $\geq \delta > 0$. *In the SFTE window and for* $s \downarrow 0$,

$$\Big\langle \prod_{j=1}^{m} \mathcal{O}_{i_j}^{(s)}(x_j) \prod_{k=1}^{p} \mathcal{Y}_k(y_k) \Big\rangle_{a,\beta}$$
$$= \sum_{\alpha_1,\dots,\alpha_m} \prod_{j=1}^{m} Z_{i_j\alpha_j}(s,\mu) \Big\langle \prod_{j=1}^{m} Q_{\alpha_j}^{\rm ren}(x_j;\mu) \prod_{k=1}^{p} \mathcal{Y}_k(y_k) \Big\rangle_{a,\beta} + O(s^{\varepsilon}),$$

with $O(s^{\varepsilon})$ *uniform in* a *(for* a *small) and in the separations* $\geq \delta$. *Equivalently, the generating functionals with flowed sources converge to those with renormalized point-local sources after the finite linear map* $\mathcal{O}^{(s)} \mapsto \sum_{\alpha} Z(s,\mu)\, Q_{\alpha}^{\rm ren}$.

Proof of Theorem 16.24. Expand each $\mathcal{O}_{i_j}^{(s)}$ using (106) and multiply out. The product equals the finite linear combination of correlators with $Q_{\alpha_j}^{\rm ren}$ insertions plus a finite sum of terms that contain at least one remainder $R_{i_j}^{(s)}$. For each such term, Theorem 18.115 and Proposition 13.2 yield uniform bounds on mixed correlators of separated local fields, hence

$$|\langle R_{i_j}^{(s)}(f_j)\, \mathcal{Z}\rangle| \;\leq\; C\, s^{\varepsilon}\, \|f_j\|_{N,\delta}\, \|\mathcal{Z}\|_{N,\delta},$$

with $\mathcal{Z}$ the product of the remaining insertions. Summing the finitely many such contributions gives the $O(s^{\varepsilon})$ remainder, uniformly in a in the SFTE window and in the separation parameter δ. The coefficient functions $Z_{i\alpha}(s,\mu)$ are finite and depend only on $s\mu^2$ by Proposition 16.23, which completes the proof. □

Corollary 16.25 (Unsmeared OS/Wightman theory)**.** *The limiting Schwinger functions* $S_{i_1,\dots,i_n}^{\rm ren}(\cdot;R)$ *from Theorem 16.24 reconstruct a Wightman theory via OS (unique up to field redefinitions within the finite span). The vacuum is unique (clustering passes to the limit), and the fields* $\mathcal{O}_j^{\rm ren}(\cdot;R)$ *are the corresponding unsmeared gauge–invariant local operators.*

Proof. At each $s > 0$ the flowed GI family satisfies OS0–OS3 and exhibits exponential clustering (Theorem 18.115). By Theorem 16.24 the $s \downarrow 0$ limits of separated correlators exist and coincide with correlators of renormalized point-local fields. OS0–OS3 are stable under such limits (cf. Lemma 16.10 for RP and the $H(4)$ invariance for OS2), so the OS reconstruction theorem applies and yields a Wightman theory; vacuum uniqueness follows from clustering. □

Corollary 16.26 (Flow removal for the variational interpolator)**.** *Let* $A_\star^{(s_0)}$ *be the principal interpolator obtained at positive flow* $s_0 > 0$ *from the GEVP/variational construction (Theorem 18.111). There exists a finite renormalized point-local operator* $A_\star^{(0),\rm ren}$ *(a linear combination of* $\{Q_\alpha^{\rm ren}\}$*) such that, in separated correlators and for* $s \downarrow 0$ *inside the SFTE window,*

$$\langle A_\star^{(s)}(x)\, A_\star^{(s)}(y)\rangle = \langle A_\star^{(0),\rm ren}(x;\mu)\, A_\star^{(0),\rm ren}(y;\mu)\rangle + O(s^{\varepsilon}).$$

In particular the strictly positive one-particle residue at mass $m_\star$ *persists in the unsmeared limit.*

Proof. Fix a finite symmetry–closed renormalized GI basis $\{Q_\alpha^{\rm ren}\}_{\alpha\in\mathcal{B}}$ for the scalar channel and, for $s > 0$ in the SFTE window (Def. 16.21), set $\Phi_\alpha^{(s)} := G_s * Q_\alpha^{\rm ren}$. By the variational/GEVP construction (Proposition 18.110), we may take the principal interpolator at flow s in the span of $\{\Phi_\alpha^{(s)}\}$:

$$A_\star^{(s)}(x) \;=\; \sum_{\alpha\in\mathcal{B}} v_\alpha^{(s)}\, \Phi_\alpha^{(s)}(x), \qquad v^{(s)} \text{ solves } C^{(s)}(\tau)\, v = \lambda^{(s)}\, C^{(s)}(\tau_0)\, v,$$

with $0 < \tau_0 < \tau$ fixed and $C^{(s)}(t)_{\alpha\beta} := \langle \Omega, \Phi_\alpha^{(s)}(t)\Phi_\beta^{(s)}(0)\Omega\rangle$.

Step 1 (SFTE reduction of Gram matrices). By Proposition 16.23 (and Lemma 18.24), for separated insertions

$$\Phi_\alpha^{(s)} \;=\; \sum_\beta Z_{\alpha\beta}(s)\, Q_\beta^{\text{ren}} \;+\; \partial\cdot\Upsilon_\alpha^{(s)} \;+\; R_\alpha^{(s)},$$

where $Z(s)$ is analytic in $\log(s\mu^2)$ as $s \downarrow 0$, and the remainders obey $\|R_\alpha^{(s)}\| = O(s^\varepsilon)$ in matrix elements, uniformly in a within the SFTE window. The improvement term $\partial\cdot\Upsilon_\alpha^{(s)}$ contributes only contact terms, so it drops out of connected two–point functions at noncoincident points. Therefore, for the correlation matrices

$$C^{(s)}(t)_{\alpha\beta} := \langle\Omega, \Phi_\alpha^{(s)}(t)\,\Phi_\beta^{(s)}(0)\Omega\rangle \quad \text{and} \quad G(t)_{\alpha\beta} := \langle\Omega, Q_\alpha^{\text{ren}}(t)\,Q_\beta^{\text{ren}}(0)\Omega\rangle,$$

we have the factorization

$$C^{(s)}(t) \;=\; Z(s)\,G(t)\,Z(s)^T \;+\; E^{(s)}(t), \qquad \|E^{(s)}(t)\| \le C\,s^\varepsilon, \tag{107}$$

with the operator norm taken on the finite index space and the bound uniform in a and for $t \in \{\tau_0, \tau\}$ used in the GEVP.

Step 2 (transport of the GEVP and existence of the $s \downarrow 0$ limit). For s sufficiently small, $Z(s)$ is invertible on the GI quotient (Theorem 18.35). Define $w^{(s)} := Z(s)^T v^{(s)}$. Using (107) and multiplying the GEVP $C^{(s)}(\tau)\,v^{(s)} = \lambda^{(s)}\,C^{(s)}(\tau_0)\,v^{(s)}$ on the left by $Z(s)^{-T}$ gives

$$\big(G(\tau) + \widetilde{E}^{(s)}(\tau)\big)\,w^{(s)} \;=\; \lambda^{(s)}\,\big(G(\tau_0) + \widetilde{E}^{(s)}(\tau_0)\big)\,w^{(s)}, \qquad \widetilde{E}^{(s)}(t) := Z(s)^{-T}E^{(s)}(t)Z(s)^{-1}.$$

Since $Z(s)$ and $Z(s)^{-1}$ are bounded for small s (analyticity and invertibility on the GI quotient), the same estimate holds: $\|\widetilde{E}^{(s)}(t)\| \le C\,s^\varepsilon$ for $t \in \{\tau_0, \tau\}$, uniformly in a. By Proposition 18.110 (stability of the principal generalized eigenpair) together with the uniform spectral gap in the scalar GI channel (Corollary 16.15 and Theorem 16.20), there exist limits

$$\lambda^{(s)} \underset{s\downarrow 0}{\longrightarrow} \lambda_\star = e^{-m_\star(\tau-\tau_0)}, \qquad w^{(s)} \underset{s\downarrow 0}{\longrightarrow} w^{(0)} \neq 0,$$

after fixing the normalization $w^{(s)T}G(\tau_0)w^{(s)} = 1$. We then define the renormalized point–local interpolator

$$A_\star^{(0),\text{ren}}(x;\mu) \;:=\; \sum_{\alpha\in\mathcal{B}} w_\alpha^{(0)}\,Q_\alpha^{\text{ren}}(x;\mu).$$

Step 3 (two–point reduction with $O(s^\varepsilon)$ remainder). For x, y with $|x-y| \ge \delta > 0$,

$$\langle A_\star^{(s)}(x)A_\star^{(s)}(y)\rangle \;=\; v^{(s)T}C^{(s)}(x^0{-}y^0)\,v^{(s)} = w^{(s)T}\,G(x^0{-}y^0)\,w^{(s)} \;+\; O(s^\varepsilon),$$

by (107). Passing to the limit $s \downarrow 0$ and using $w^{(s)} \to w^{(0)}$ gives

$$\langle A_\star^{(s)}(x)A_\star^{(s)}(y)\rangle = \;\langle A_\star^{(0),\text{ren}}(x;\mu)\,A_\star^{(0),\text{ren}}(y;\mu)\rangle \;+\; O(s^\varepsilon),$$

uniformly in the SFTE window and in the separation $\ge \delta$; this is the stated reduction.

Step 4 (persistence of the one–particle residue). In the renormalized point–local scalar channel, Theorem 18.133 yields an operator (e.g. $\text{tr}(F^2)_R$) with strictly positive 0^{++} LSZ residue at $m_\star$. By Proposition 18.110, the variational maximizer for the pair $(G(\tau), G(\tau_0))$—which is precisely $A_\star^{(0),\text{ren}}$ constructed above—has residue not smaller than that benchmark and hence strictly positive. Therefore the one–particle residue at $m_\star$ persists in the $s \downarrow 0$ (unsmeared) limit. □

OS axioms at $s = 0$: checklist and pointers

We summarize where each Osterwalder–Schrader axiom is verified *at zero flow* (after flow–to–point renormalization). We follow the common convention:

OS1: Reflection positivity (RP), OS2: Euclidean invariance (O(4) & translations),
OS3: Symmetry (Bose), OS4: Cluster property, OS5: Regularity/temperedness.

Axiom	**Content at $s = 0$**	**Where proved / input**
OS1 (RP)	RP holds for the renormalized point–local family $\{[A]\}$; RP is closed under L^2 limits of flowed GI locals with counterterms.	*Lemma 16.10* (RP closed under L^2 limits) applied to $\mathcal{R}_A^{(s)} := A^{(s)} - c_0^A(s)\mathbf{1} - c_4^A(s)\mathcal{O}_4$; RP at $s > 0$ is standard (Sec. 17 and Lemma 5.2). For weak limits see also Lemma 14.1.
OS2 (Euclidean invariance)	$O(4)$ and translation invariance are restored at $a \downarrow 0$ and are preserved under flow removal because the counterterms are $O(4)$ scalars.	*Theorem 15.8* ($O(a^2)$ improvement & $H(4) \to O(4)$ at positive flow) + Step 4 of *Theorem 16.13(iii)* (limits inherit $O(4)$); see also the proof of *Theorem 16.16* (OS2 item).
OS3 (Symmetry)	Full Bose symmetry (permutation invariance) of Schwinger functions at $s = 0$; this encodes *locality* after OS reconstruction (spacelike commutativity).	*Theorem 16.13(iii)* (OS3 item) and *Theorem 16.16.* Point–locality of the fields $[A]$ from *Definition 16.4* and *Theorem 16.13(ii)* ensures that symmetry implies Haag–Kastler locality after OS.
OS4 (Cluster property)	Exponential clustering persists at $s = 0$ with the same rate $m_\star > 0$ as at positive flow.	*Proposition 16.19* (clustering passes to the limit) and *Theorem 16.16* (clustering item). Consequence: *Theorem 16.20* (Hamiltonian gap $\Delta \geq m_\star$).
OS5 (Regularity / temperedness)	Schwinger functions are tempered distributions; dependence on tests is continuous; dense OS domain exists at $s = 0$.	*Lemma 16.5* and *Theorem 16.13(ii)* (tempered limits), plus *Corollary 16.15* (dense OS domain).

Closure under limits (one line). OS1–OS5 at $s = 0$ are obtained by combining: (i) uniform positive–flow bounds (RP, $O(4)$, clustering) and (ii) stability under $s \downarrow 0$ of *renormalized* flowed insertions $\mathcal{R}_A^{(s)}$, via the L^2 Cauchy/limit statements in *Lemma 16.5* and the RP closure *Lemma 16.10.* The only inputs beyond RP are $O(4)$ restoration (*Theorem 15.8*) and point–locality of $[A]$ (FPR), which together ensure OS2–OS3 (hence locality in the Wightman theory).

Reader's map to Haag–Kastler. OS1–OS5 at $s = 0 \Rightarrow$ OS reconstruction (Sec. 17) $\Rightarrow$ Wightman fields with mass gap (*Theorem 16.20*) $\Rightarrow$ Haag–Kastler net (isotony/covariance/locality) for the GI sector; independence of regularization is in *Theorem 16.8.*

17 From OS to Wightman: Reconstruction and Haag–Kastler Net

We now pass from the Euclidean OS family of point-local gauge-invariant fields constructed in §16 to a Lorentzian Wightman theory. Throughout, we work with the generating class $\mathcal{G}_{\leq 4}$ and its flow-to-point renormalized representatives $[A]$ from Theorem 16.13; these satisfy OS0–OS3 and exponential clustering with rate $m_\star > 0$ (Theorem 16.13(iii), Corollary 16.15), and enjoy full $O(4)$ invariance (Theorem 15.8).

Theorem 17.1 (OS⇒Wightman for the GI sector). *Let $\{S^{(n)}\}$ be the Euclidean Schwinger functions of the family $\{[A] : A \in \mathcal{G}_{\leq 4}\}$ obtained in Theorem 16.13. Assume OS0–OS3 and $O(4)$ invariance (Theorem 15.8), and exponential clustering with rate $m_\star > 0$ (Corollary 16.15). Then there exist:*

- *a Hilbert space $\mathcal{H}$ with cyclic vacuum Ω;*
- *a strongly continuous unitary representation U of the proper orthochronous Poincaré group on $\mathcal{H}$;*
- *for each $A \in \mathcal{G}_{\leq 4}$, a scalar Wightman field $x \mapsto \widehat{A}(x)$ (an operator-valued tempered distribution on a common invariant dense domain $\mathcal{D} \subset \mathcal{H}$);*

such that the Wightman axioms hold on the net generated by $\{\widehat{A}\}$:

(W0) *Temperedness: all vacuum expectation values of products of smeared $\widehat{A}$ are tempered distributions.*

(W1) *Poincaré covariance: $U(\Lambda, a)\,\widehat{A}(x)\,U(\Lambda, a)^{-1} = \widehat{A}(\Lambda x + a)$ for all (Λ, a).*

(W2) *Spectral condition: the joint spectrum of the translation generators lies in the closed forward light cone; in particular, the Hamiltonian H is positive.*

(W3) *Locality (microcausality): $[\widehat{A}(x), \widehat{B}(y)] = 0$ for all $A, B \in \mathcal{G}_{\leq 4}$ whenever $(x - y)^2 < 0$.*

(W4) *Existence and uniqueness of the vacuum: Ω is U-invariant and unique up to phase.*

For every bounded open region $\mathcal{O} \Subset \mathbb{R}^{1,3}$, the vacuum Ω is cyclic and separating for the local von Neumann algebra

$$\mathcal{A}(\mathcal{O}) \;:=\; \{\widehat{A}(f) \;:\; A \in \mathcal{G}_{\leq 4},\ \operatorname{supp} f \subset \mathcal{O}\}'',$$

the classical Reeh–Schlieder property, see Reeh and Schlieder (1961). Moreover, the time-translation generator coincides with the OS Hamiltonian from §11, and the mass gap transfers:

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty) \qquad \Rightarrow \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \geq m_\star > 0.$$

Finally, the Minkowski n-point Wightman distributions $\{W^{(n)}\}$ are the boundary values of functions analytic in the forward tube and are related to $\{S^{(n)}\}$ by the standard Wick rotation.

Full proof. OS data ⇒ reconstruction. By Theorem 16.13 and Theorem 15.8, the Euclidean Schwinger functions $\{S^{(n)}\}$ of the family $\{[A] : A \in \mathcal{G}_{\leq 4}\}$ satisfy OS0 (temperedness), OS1 (reflection positivity), OS2 ($O(4)$ invariance), OS3 (symmetry), and OS4 (cluster) thanks to exponential clustering at rate $m_\star > 0$ (Corollary 16.15). The Osterwalder–Schrader reconstruction therefore yields: (i) a Hilbert space $\mathcal{H}$ with cyclic vacuum Ω; (ii) a strongly continuous unitary representation of the Euclidean group with generator of Euclidean time translations $H \geq 0$; (iii) Wightman distributions $\{W^{(n)}\}$ obtained by analytic continuation to the forward tubes.

Poincaré covariance and fields. $O(4)$ invariance analytically continues to a unitary representation U of the proper orthochronous Poincaré group, with $U(a) = e^{iP\cdot a}$ and $P^0 = H \geq 0$, verifying *(W1)–(W2).* For each $A \in \mathcal{G}_{\leq 4}$ we obtain an operator-valued tempered distribution $x \mapsto \widehat{A}(x)$ on the invariant dense domain $\mathcal{D}$ generated by finite polynomials of smeared fields acting on Ω. Temperedness *(W0)* is inherited from OS0.

Locality. Local commutativity *(W3)* follows from OS1+OS3 via the edge-of-the-wedge analyticity of the vacuum distributions and the standard OS locality argument. Since the $[A]$ are CP-even GI scalars, the fields are bosonic.

Vacuum. Ω is U-invariant by construction and unique up to phase by clustering (OS4), giving *(W4).*

Identification of H and the gap. The time-translation generator coincides with the OS Hamiltonian constructed from the RP completion; $U(it) = e^{-tH}$ on $\mathcal{H}_+$. Exponential Euclidean clustering at rate $m_\star$ implies, via the Laplace–support lemma (the standard OS spectral–support argument), that

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty), \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \geq m_\star > 0.$$

Finally, $\{W^{(n)}\}$ are boundary values of functions analytic in the forward tubes and agree with the Wick rotations of $\{S^{(n)}\}$, concluding the proof. □

Common polynomial domain. Let

$$\mathcal{D}_{\text{poly}} \;:=\; \operatorname{span}\Big\{\, \widehat{A}_1(f_1)\cdots\widehat{A}_n(f_n)\,\Omega \;:\; A_j \in \mathcal{G}_{\leq 4},\; f_j \in \mathcal{S}(\mathbb{R}^{1,3}),\; n \in \mathbb{N} \,\Big\}.$$

By the OS reconstruction and the Reeh–Schlieder property for Wightman fields, $\mathcal{D}_{\text{poly}}$ is dense, invariant under $U(\Lambda, a)$, and invariant under left multiplication by each $\widehat{A}(f)$.

Lemma 17.2 (Subgaussian moment bounds and Nelson analyticity)**.** *For each $A \in \mathcal{G}_{\leq 4}$ and $\phi \in C_c^\infty(\mathbb{M})$ there exist constants $\lambda_0 > 0$ and $\Sigma = \Sigma(A, \phi) < \infty$ such that*

$$\langle \Omega,\, e^{\lambda\,\widehat{A}(\phi)}\,\Omega\rangle \;\leq\; \exp\!\big(\tfrac{1}{2}\Sigma^2\lambda^2\big) \qquad \textit{for all } |\lambda| \leq \lambda_0. \tag{108}$$

Consequently, for every $\psi \in \mathcal{D}_{\text{poly}}$ there exists $r = r(A, \phi, \psi) > 0$ with

$$\sum_{n=0}^{\infty} \frac{r^n}{n!}\,\|\,\widehat{A}(\phi)^n\,\psi\| \;<\; \infty,$$

so ψ is an entire analytic vector for $\widehat{A}(\phi)$ in the sense of Nelson.

Proof. Step 1: Flowed subgaussian control (uniform in a). Fix $s \in (0, s_0]$. By the global logarithmic Sobolev inequality (Proposition 6.12) and the Herbst argument, any flowed GI local $F^{(s)}(\phi)$ with finite GI–Lipschitz seminorm satisfies a subgaussian bound

$$\Big\langle \exp\big(\lambda(F^{(s)}(\phi) - \langle F^{(s)}(\phi)\rangle)\big)\Big\rangle \;\leq\; \exp\!\Big(\tfrac{1}{2}\Sigma_{F,s}^2\,\lambda^2\Big), \qquad |\lambda| \leq \lambda_*,$$

with $\lambda_* > 0$ and $\Sigma_{F,s} \lesssim L_{\text{ad}}^{\text{GI}}(F^{(s)}(\phi))$, uniformly in the volume and along the GF tuning line $a \leq a_0$ (cf. Lemma 13.1, Proposition 13.2). Apply this to $F = A$ and to $F = \mathcal{O}_4 := \operatorname{tr} F_{\mu\nu}F_{\mu\nu}$ to get

$$\Big\langle \exp\big(\lambda(A^{(s)}(\phi) - \langle A^{(s)}(\phi)\rangle)\big)\Big\rangle \leq e^{\frac{1}{2}\Sigma_{A,s}^2\lambda^2}, \quad \Big\langle \exp\big(\lambda(\mathcal{O}_4(\phi) - \langle \mathcal{O}_4(\phi)\rangle)\big)\Big\rangle \leq e^{\frac{1}{2}\Sigma_4^2\lambda^2}.$$

Step 2: Counterterms and ψ_2–triangle (flowed version). Define the centered combination with a *flowed* quartic counterterm

$$X_s \;:=\; A^{(s)}(\phi) \;-\; c_0^A(s)\,\|\phi\|_{L^1} \;-\; c_4^A(s)\,\mathcal{O}_4^{(s)}(\phi).$$

By Step 1, both $A^{(s)}(\phi)$ and $\mathcal{O}_4^{(s)}(\phi)$ enjoy subgaussian MGFs with parameters controlled by their GI–Lipschitz seminorms, uniformly in the volume and along the tuning line. Hence, by the ψ_2 triangle inequality, for $|\lambda| \le \lambda_*$,

$$\Big\langle \exp\big(\lambda\,(X_s - \langle X_s\rangle)\big)\Big\rangle \;\le\; \exp\Big(\tfrac12\big(\Sigma_{A,s} + |c_4^A(s)|\,\Sigma_{4,s}\big)^2\lambda^2\Big),$$

where $\Sigma_{A,s}, \Sigma_{4,s} < \infty$ are uniform in volume and $a \le a_0$. As in Lemma 16.2, $|c_4^A(s)| \lesssim (1+|\log s|)^{p_A}$, so although the MGF radius may shrink as $s \downarrow 0$, for each fixed n we have the uniform moment bound

$$\sup_{0<s\le s_0} \langle |X_s - \langle X_s\rangle|^n\rangle \;\le\; C_n(A,\phi,s_0) \;<\; \infty. \tag{109}$$

Step 3: Passage to the OS limit. Using the small flow–time expansion $\mathcal{O}_4^{(s)} = \mathcal{O}_4 + s\,R_{4,s}$ with $R_{4,s}$ a finite combination of GI scalars of dimension ≥ 6 (hence uniformly L^2–bounded when tested against ϕ), we may replace

$$X_s \;=\; A^{(s)}(\phi) - c_0^A(s)\|\phi\|_{L^1} - c_4^A(s)\,\mathcal{O}_4^{(s)}(\phi)$$

by

$$\widetilde{X}_s \;:=\; A^{(s)}(\phi) - c_0^A(s)\|\phi\|_{L^1} - c_4^A(s)\,\mathcal{O}_4(\phi)$$

at the cost of an L^2–error bounded by $C\,s\,\|\phi\|_{H^\sigma}$ (the same $\sigma > 2$ as in Lemma 16.2). By Lemma 16.5, $\widetilde{X}_s \to \widehat{A}(\phi)$ in L^2 as $s \downarrow 0$. Moreover, the subgaussian control for X_s from Step 2 implies, for each fixed n, uniform moment bounds and hence uniform integrability for $\{\widetilde{X}_s^n\}_{s\le s_0}$ (use $|\widetilde{X}_s|^n \le 2^{n-1}(|X_s|^n + |X_s - \widetilde{X}_s|^n)$ and the L^2–estimate for the difference). Therefore

$$\langle \widehat{A}(\phi)^n\rangle = \lim_{s\downarrow 0}\langle \widetilde{X}_s^n\rangle \qquad \text{for each } n \in \mathbb{N}.$$

Choose $\lambda_0 > 0$ so that $\sum_{n\ge0}|\lambda|^n \sup_{s\le s_0}\langle|\widetilde{X}_s|^n\rangle/n!$ is finite for $|\lambda| \le \lambda_0$; then dominated convergence passes the limit under the power series for the exponential:

$$\langle e^{\lambda\,\widehat{A}(\phi)}\rangle = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\,\langle \widehat{A}(\phi)^n\rangle = \lim_{s\downarrow0}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\,\langle \widetilde{X}_s^n\rangle \;\le\; \exp(\tfrac12\Sigma^2\lambda^2),$$

for $|\lambda| \le \lambda_0$ and some $\Sigma < \infty$ (depending on A, ϕ, s_0), yielding (108).

Step 4: Nelson analyticity on $\mathcal{D}_{\rm poly}$. From (108) (with λ real) and Cauchy's estimates for power series, the even moments obey $\langle \Omega, \widehat{A}(\phi)^{2n}\,\Omega\rangle \;\le\; (2n)!\,C^n$ for some $C = C(A,\phi)$. Hence $\|\widehat{A}(\phi)^n\Omega\| \;\le\; C_1^n\,n!$. If $\psi \in \mathcal{D}_{\rm poly}$ is a finite polynomial in smeared GI fields applied to Ω, repeated Cauchy–Schwarz together with the uniform mixed-moment bounds (Proposition 13.2, transported through OS) gives $\|\widehat{A}(\phi)^n\psi\| \;\le\; C(\psi)\,C_2^n\,n!$. Therefore, for $r < C_2^{-1}$, $\sum_{n\ge0}\frac{r^n}{n!}\,\|\widehat{A}(\phi)^n\psi\| < \infty$, so every $\psi \in \mathcal{D}_{\rm poly}$ is an entire analytic vector for $\widehat{A}(\phi)$. This completes the proof. □

Proposition 17.3 (Essential self-adjointness on a common core)**.** *For every $A \in \mathcal{G}_{\le4}$ and real $\phi \in C_c^\infty(\mathbb{M})$ the operator $\widehat{A}(\phi)$ is symmetric on $\mathcal{D}_{\rm poly}$ and essentially self-adjoint there. Denote its closure by $\overline{\widehat{A}(\phi)}$.*

Proof. Symmetry on $\mathcal{D}_{\text{poly}}$ holds because $\widehat{A}$ is Hermitian and ϕ is real. By Lemma 17.2, $\mathcal{D}_{\text{poly}}$ consists of entire analytic vectors for $\widehat{A}(\phi)$. Nelson's analytic vector theorem implies essential self-adjointness on $\mathcal{D}_{\text{poly}}$. □

Lemma 17.4 (Strong commutativity at spacelike separation)**.** *Let $A, B \in \mathcal{G}_{\leq 4}$ and let $\phi, \psi \in C_c^\infty(\mathbb{M})$ be real test functions with* $\operatorname{supp}\phi \subset \mathcal{O}$ *and* $\operatorname{supp}\psi \subset \mathcal{O}'$*, where $\mathcal{O}$ and $\mathcal{O}'$ are spacelike separated regions. Then the self-adjoint closures $\overline{\widehat{A}(\phi)}$ and $\overline{\widehat{B}(\psi)}$* strongly commute*, i.e. their spectral measures commute; equivalently,*

$$e^{is\,\overline{\widehat{A}(\phi)}}\,e^{it\,\overline{\widehat{B}(\psi)}} \;=\; e^{it\,\overline{\widehat{B}(\psi)}}\,e^{is\,\overline{\widehat{A}(\phi)}} \qquad (\forall s, t \in \mathbb{R}).$$

Proof. By locality (W3) the smeared fields $\widehat{A}(\phi)$ and $\widehat{B}(\psi)$ commute as operators on the common invariant polynomial domain $\mathcal{D}_{\text{poly}}$ (defined above Theorem 17.1). By Lemma 17.2, $\mathcal{D}_{\text{poly}}$ consists of entire analytic vectors for each $\widehat{C}(\eta)$ with $C \in \mathcal{G}_{\leq 4}$ and real test function η; in particular, $\mathcal{D}_{\text{poly}}$ is a common invariant set of entire analytic vectors for $\widehat{A}(\phi)$ and $\widehat{B}(\psi)$. By Proposition 17.3, both are essentially self-adjoint on $\mathcal{D}_{\text{poly}}$, with closures $\overline{\widehat{A}(\phi)}$ and $\overline{\widehat{B}(\psi)}$.

Let $X := \widehat{A}(\phi)$ and $Y := \widehat{B}(\psi)$. On $\mathcal{D}_{\text{poly}}$ we have $[X, Y] = 0$. For $\xi \in \mathcal{D}_{\text{poly}}$, analyticity allows us to expand

$$e^{isX}\,e^{itY}\,\xi \;=\; \sum_{m,n\geq 0} \frac{(is)^m (it)^n}{m!\,n!}\,X^m Y^n \xi \;=\; \sum_{m,n\geq 0} \frac{(is)^m (it)^n}{m!\,n!}\,Y^n X^m \xi \;=\; e^{itY}\,e^{isX}\,\xi.$$

Since $\mathcal{D}_{\text{poly}}$ is a core for both closures and the exponentials are unitary (hence bounded), the equality extends by continuity to all of $\mathcal{H}$ with X, Y replaced by their closures. This is an instance of Nelson's commutativity theorem: if two essentially self-adjoint operators commute on a common dense set of entire analytic vectors for both, then their closures strongly commute. □

Definition 17.5 (Local von Neumann algebras)**.** We adopt Definition 17.22 as the canonical definition of $\mathfrak{A}(\mathcal{O})$ for double cones $\mathcal{O} \subset \mathbb{R}^{1,3}$. For a general bounded open region $\mathcal{O} \subset \mathbb{M}$, define

$$\mathfrak{A}(\mathcal{O}) \;:=\; \Big(\bigcup_{\substack{\mathcal{O}' \subset \mathcal{O} \\ \mathcal{O}' \text{ double cone}}} \mathfrak{A}(\mathcal{O}') \Big)''.$$

This agrees with Definition 17.22 when $\mathcal{O}$ is itself a double cone and generates the same quasilocal C^*-algebra.

Theorem 17.6 (Haag–Kastler net for the GI sector)**.** *The assignment $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ defines a Haag–Kastler net on $(\mathcal{H}, \Omega)$ with the following properties:*

1. *(Isotony) If $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.*
2. *(Locality) If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, then $\big[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)\big] = \{0\}$.*
3. *(Poincaré covariance) With U from Theorem 17.1, $U(\Lambda, a)\,\mathfrak{A}(\mathcal{O})\,U(\Lambda, a)^{-1} = \mathfrak{A}(\Lambda\mathcal{O} + a)$.*
4. *(Vacuum cyclicity and separating properties) Ω is cyclic for each $\mathfrak{A}(\mathcal{O})$ and separating for $\mathfrak{A}(\mathcal{O})'$.*
5. *(Spectrum condition) The time-translation generator H is positive, with $\sigma(H) \subset \{0\} \cup [m_\star, \infty)$ from Theorem 17.1.*

Proof. (1) Isotony is immediate from Definition 17.5.

(2) Locality: for $\mathcal{O}_1 \perp \mathcal{O}_2$, Lemma 17.4 gives strong commutativity of the self-adjoint generators, hence the unitary groups commute and so do the generated von Neumann algebras.

(3) Covariance: The Wightman covariance (Theorem 17.1) gives $U(\Lambda, a)\, \widehat{A}(\phi)\, U(\Lambda, a)^{-1} = \widehat{A}(\phi_{(\Lambda,a)})$ with $\operatorname{supp} \phi_{(\Lambda,a)} = \Lambda \operatorname{supp} \phi + a$. Essential self-adjointness and functional calculus yield $U(\Lambda, a)\, e^{i\, \overline{\widehat{A}(\phi)}}\, U(\Lambda, a)^{-1} = e^{i\, \overline{\widehat{A}(\phi_{(\Lambda,a)})}}$, so the double commutant transforms accordingly.

(4) Reeh–Schlieder: For Wightman fields with locality and spectral condition, the vacuum is cyclic for each bounded region (standard Reeh–Schlieder). Since $\mathfrak{A}(\mathcal{O})$ is generated by exponentials of local fields, cyclicity transfers; separating for the commutant follows by locality.

(5) Spectrum condition and gap: from Theorem 17.1. □

Proposition 17.7 (Inner regularity and weak additivity)**.** *Let $\mathfrak{A}(\mathcal{O})$ be the net from Theorem 17.6.*

(i) (Inner regularity) If $\mathcal{O}_n \nearrow \mathcal{O}$ is an increasing sequence of bounded open regions with $\overline{\mathcal{O}_n} \subset \mathcal{O}$, then

$$\mathfrak{A}(\mathcal{O}) \;=\; \Big(\bigcup_{n \in \mathbb{N}} \mathfrak{A}(\mathcal{O}_n) \Big)''.$$

(ii) (Weak additivity) For any nonempty bounded open $\mathcal{O}$,

$$\Big(\bigcup_{a \in \mathbb{R}^4} \mathfrak{A}(\mathcal{O} + a) \Big)'' \;=\; \mathcal{B}(\mathcal{H}).$$

Equivalently, $\overline{\operatorname{span}}\{\, \mathfrak{A}(\mathcal{O} + a)\Omega : a \in \mathbb{R}^4 \,\} = \mathcal{H}$.

Proof. (i) Let $A \in \mathcal{G}_{\le 4}$ and $\phi \in C_c^\infty(\mathbb{M}, \mathbb{R})$ with $\operatorname{supp} \phi \subset \mathcal{O}$. Choose $\phi_n \in C_c^\infty(\mathbb{M}, \mathbb{R})$ with $\operatorname{supp} \phi_n \subset \mathcal{O}_n$ and $\phi_n \to \phi$ in the test-function topology. By Lemma 17.2 and Proposition 17.3, the entire-analytic core $\mathcal{D}_{\mathrm{poly}}$ is common for all smearings and $\phi \mapsto \widehat{A}(\phi)$ is continuous in the strong resolvent sense on that core. By temperedness, $\phi \mapsto \widehat{A}(\phi)\xi$ is continuous for each $\xi \in \mathcal{D}_{\mathrm{poly}}$; since $\phi_n \to \phi$ in the test topology and $\mathcal{D}_{\mathrm{poly}}$ is a common core, $e^{i\, \overline{\widehat{A}(\phi_n)}} \to e^{i\, \overline{\widehat{A}(\phi)}}$ strongly by continuity of the exponential series on entire analytic vectors. Strong closure of $\mathfrak{A}(\mathcal{O})$ then gives the claim. Since $\mathfrak{A}(\mathcal{O})$ is generated by such exponentials and is strongly closed, (i) follows.

(ii) Suppose $\Psi \in \mathcal{H}$ is orthogonal to $\mathfrak{A}(\mathcal{O} + a)\Omega$ for every $a \in \mathbb{R}^4$. By Kaplansky density, it suffices to consider vectors of the form $e^{i\, \overline{\widehat{A}(\phi_a)}}\Omega$ with $\operatorname{supp} \phi_a \subset \mathcal{O} + a$. The function $F(a) := \langle \Psi, e^{i\, \overline{\widehat{A}(\phi_a)}}\Omega \rangle$ is continuous in a by strong continuity of translations and the strong resolvent continuity in (i), and $F(a) = 0$ for all a. Differentiating at $a = 0$ along coordinate directions (Nelson analyticity on $\mathcal{D}_{\mathrm{poly}}$ allows termwise differentiation under the vacuum expectation), we obtain $\langle \Psi, \widehat{C}(\eta)\Omega \rangle = 0$ for all $C \in \mathcal{G}_{\le 4}$ and all real test functions η; by polynomiality and density of $\mathcal{D}_{\mathrm{poly}}$, this forces $\Psi = 0$. Hence the translates of $\mathfrak{A}(\mathcal{O})$ act cyclically on Ω and the double commutant is all of $\mathcal{B}(\mathcal{H})$. □

Proposition 17.8 (Exponential clustering from the mass gap)**.** *Assume Theorem 17.1 yields a spectral gap $m_\star > 0$ above the vacuum. Then for all $A, B \in \mathfrak{A}_{\mathrm{loc}} := \bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$ there exist constants $C_{A,B} < \infty$ and $\mu \in (0, m_\star)$ such that, for all spacelike $x \in \mathbb{R}^4$,*

$$\big|\, \langle \Omega, A\, U(x)\, B\, \Omega \rangle \;-\; \langle \Omega, A\Omega \rangle \, \langle \Omega, B\Omega \rangle \,\big| \;\le\; C_{A,B}\, e^{-\mu\, |x|}.$$

Full proof. Let $A, B \in \mathfrak{A}_{\mathrm{loc}}$ and set $A_0 := A - \langle \Omega, A\Omega \rangle \mathbf{1}$. Then

$$F(x) \;:=\; \langle \Omega,\, A_0\, U(x)\, B\, \Omega \rangle$$

is the boundary value of a function analytic in the forward tube $\{x+iy : \ y \in V_+\}$ and tempered on the real axis (Wightman axioms). By the spectral condition, the Fourier transform $\widetilde{F}(p)$ is a finite complex Borel measure supported in the closed forward cone with $\operatorname{supp} \widetilde{F} \subset \{p : \ p^2 \geq m_\star^2, \ p^0 \geq 0\}$ because $E(\{0\})A_0\Omega = 0$ and $\sigma(H) \setminus \{0\} \subset [m_\star, \infty)$ (Theorem 17.1).

Fix spacelike x and choose a Lorentz frame in which $x = (0, \mathbf{r})$ with $R := |\mathbf{r}| = \sqrt{-x^2}$. Then

$$F(x) \ = \ \int e^{-ip\cdot x} \, d\widetilde{F}(p) \ = \ \int e^{-i\mathbf{p}\cdot\mathbf{r}} \, d\widetilde{F}(p).$$

Since $\operatorname{supp} \widetilde{F}$ lies above the mass threshold $m_\star$, the Paley–Wiener/Jost–Lehmann–Dyson bound yields exponential decay in spacelike directions:

$$|F(x)| \ \leq \ C_{A,B} \, e^{-\mu R} \qquad \text{for any } \mu < m_\star,$$

with $C_{A,B} < \infty$ depending on suitable energy norms of A, B (finite by Lemma 17.2). Restoring the subtracted means gives the stated clustering estimate. □

Corollary 17.9 (Uniqueness and purity of the vacuum)**.** *If $\Psi \in \mathcal{H}$ is invariant under all translations $U(a)$, then $\Psi = \langle \Psi, \Omega \rangle \, \Omega$. In particular, the vacuum is unique and the vacuum state $A \mapsto \langle \Omega, A\Omega \rangle$ is a pure state on the quasilocal algebra $\mathfrak{A} := \overline{\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})}^{\|\cdot\|}$.*

Proof. Let $A \in \mathfrak{A}_{\text{loc}}$. Using translation invariance of Ψ and Proposition 17.8 with $B := A^*$,

$$\langle \Psi, A\Omega \rangle = \lim_{|x|\to\infty,\, x^2<0} \langle \Psi, U(x) A \, \Omega \rangle = \lim_{|x|\to\infty,\, x^2<0} \langle \Omega, A \, U(-x)\Psi \rangle = \langle \Omega, A\Omega \rangle \, \langle \Psi, \Omega \rangle.$$

By density of $\{A\Omega : A \in \mathfrak{A}_{\text{loc}}\}$ this implies $\Psi = \langle \Psi, \Omega \rangle \, \Omega$. Purity follows since any translation-invariant vector implementing a decomposition of the vacuum state would contradict uniqueness. □

Remark 17.10 (What this buys us next)*.* Proposition 17.7 and Corollary 17.9 are standard inputs for Haag–Ruelle scattering. Together with the gap and Nelson analyticity, they allow us to construct multi-particle asymptotic states once an isolated mass shell is identified. We avoid a standing hypothesis: the isolated one-particle mass shell will be obtained below from the mass gap and the *nonzero residue theorem* (Theorem 18.111), see Theorem 17.20.

Haag–Ruelle scattering in the GI sector

We work under the *conclusions* of Theorem 17.20 (proved below): there is an isolated mass hyperboloid $\mathscr{H}_{m_\star} = \{p : \ p^2 = m_\star^2, \ p^0 > 0\}$ with nontrivial one-particle subspace $\mathcal{H}_1 := E(\mathscr{H}_{m_\star})\mathcal{H} \neq \{0\}$. For $x = (t, \mathbf{x}) \in \mathbb{R}^4$ write

$$\alpha_x(A) \ := \ U(x) \, A \, U(x)^{-1} \qquad (A \in \mathfrak{A}_{\text{loc}}),$$

and denote by $E(\cdot)$ the joint spectral measure of translations. Let $\omega_{m_\star}(\mathbf{p}) := \sqrt{m_\star^2 + |\mathbf{p}|^2}$.

Definition 17.11 (HR wave packets and velocity support)**.** For $f \in \mathcal{S}(\mathbb{R}^3)$ with Fourier transform $\widetilde{f}$, define the positive-frequency Klein–Gordon packet

$$f_t^{(m_\star)}(\mathbf{x}) \ := \ (2\pi)^{-3/2} \int_{\mathbb{R}^3} d^3\mathbf{p} \ \widetilde{f}(\mathbf{p}) \, e^{\, i\left(\omega_{m_\star}(\mathbf{p})\, t - \mathbf{p}\cdot\mathbf{x}\right)}.$$

Its *velocity support* is

$$\operatorname{Vel}(f) \ := \ \Big\{ \ \mathbf{v}(\mathbf{p}) := \frac{\mathbf{p}}{\omega_{m_\star}(\mathbf{p})} \ : \ \mathbf{p} \in \operatorname{supp} \widetilde{f} \ \Big\} \ \subset \ \{ \, \mathbf{v} \in \mathbb{R}^3 : |\mathbf{v}| < 1 \, \}.$$

Definition 17.12 (Energy–momentum filter)**.** Let $\Delta \Subset \mathbb{R}^4$ be a compact neighborhood of the mass hyperboloid $\mathscr{H}_{m_*}$ with $\Delta \cap \sigma(U) = \mathscr{H}_{m_*}$. Pick $h \in \mathcal{S}(\mathbb{R}^4)$ with $\widehat{h} \in C_c^\infty(\mathbb{R}^4)$ satisfying

$$\operatorname{supp} \widehat{h} \ \subset \ \Delta, \qquad \widehat{h} \equiv 1 \text{ on a neighborhood of } \mathscr{H}_{m_*}.$$

For $B \in \mathfrak{A}(\mathcal{O})$ define the (almost local) filtered operator

$$B_h \ := \ \int_{\mathbb{R}^4} d^4x \ h(x)\, \alpha_x(B) \qquad \text{(strong Bochner integral).}$$

Lemma 17.13 (One-particle limit)**.** *Assume Theorem 17.20. Let $B \in \mathfrak{A}(\mathcal{O})$ be such that $E(\mathscr{H}_{m_\star})B\Omega \neq 0$. Then for every $f \in \mathcal{S}(\mathbb{R}^3)$,*

$$\lim_{t\to\pm\infty} B_{h,t}(f)\,\Omega \ =: \ \psi_f^{\pm} \ \in \ \mathcal{H}_1, \qquad B_{h,t}(f) \ := \ \int_{\mathbb{R}^3} d^3\mathbf{x} \ f_t^{(m_\star)}(\mathbf{x}) \ \alpha_{(t,\mathbf{x})}(B_h).$$

Moreover, $\psi_f^{\pm}$ equals the one-particle wave packet determined by $E(\mathscr{H}_{m_\star})B\Omega$:

$$\psi_f^{\pm} \ = \ \int_{\mathscr{H}_{m_\star}} \widetilde{f}(\mathbf{p}) \ E(dp)\, B\Omega,$$

and $\|B_{h,t}(f)\Omega - \psi_f^{\pm}\| = O(|t|^{-N})$ as $t \to \pm\infty$ for every $N \in \mathbb{N}$ (rates depend on B, h, f).

Full proof. Identical to the proof given previously (replace m by m_* and $\mathscr{H}_m$ by $\mathscr{H}_{m_*}$), using that $E(\Delta^c)B_h\Omega = 0$ and $E(\mathscr{H}_{m_*})B_h\Omega = E(\mathscr{H}_{m_*})B\Omega \neq 0$ by construction of h. ☐

Proposition 17.14 (Asymptotic commutator decay)**.** *Let $B_k \in \mathfrak{A}(\mathcal{O}_k)$ and $f_k \in \mathcal{S}(\mathbb{R}^3)$ ($k = 1, 2$). If $\mathrm{Vel}(f_1) \cap \mathrm{Vel}(f_2) = \varnothing$, then for all $N \in \mathbb{N}$ there exists $C_N < \infty$ such that*

$$\| \ [B_{1,h_1,t}(f_1), \ B_{2,h_2,t}(f_2)] \ \| \ \le \ C_N\,(1 + |t|)^{-N} \qquad (t \to \pm\infty).$$

Full proof. As before (with m_* in place of m), using locality and the disjointness of the large-$|t|$ velocity supports. ☐

Theorem 17.15 (Existence of multi-particle in/out states)**.** *Under the conclusions of Theorem 17.20, let $B_1, \ldots, B_n \in \mathfrak{A}_{\mathrm{loc}}$ with $E(\mathscr{H}_{m_\star})B_j\Omega \neq 0$ and choose $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^3)$ with pairwise disjoint velocity supports. Then the limits*

$$\Psi^{\mathrm{out}} \ := \ \lim_{t\to+\infty} B_{1,h_1,t}(f_1) \cdots B_{n,h_n,t}(f_n)\,\Omega, \qquad \Psi^{\mathrm{in}} \ := \ \lim_{t\to-\infty} B_{1,h_1,t}(f_1) \cdots B_{n,h_n,t}(f_n)\,\Omega$$

exist and depend only on the one-particle vectors $\psi_{f_j} := \lim_{t\to\pm\infty} B_{j,h_j,t}(f_j)\Omega \in \mathcal{H}_1$. Moreover,

$$\Psi^{\mathrm{out/in}} \ = \ \psi_{f_1} \ \overset{s}{\otimes} \ \cdots \ \overset{s}{\otimes} \ \psi_{f_n},$$

the symmetric tensor (bosonic) product in the Fock space over $\mathcal{H}_1$, and the limit is independent of the choices of B_j, h_j as long as they yield the same ψ_{f_j}.

Full proof. Cook's method with Proposition 17.14; identical to the argument previously given, with m_* in place of m. ☐

Corollary 17.16 (Møller operators and S-matrix)**.** *Let $\Gamma_s(\mathcal{H}_1)$ be the symmetric Fock space over $\mathcal{H}_1$. There exist isometries*

$$\Omega^{\mathrm{out/in}} : \ \Gamma_s(\mathcal{H}_1) \longrightarrow \mathcal{H}$$

such that for simple tensors $\psi_1 \overset{s}{\otimes} \cdots \overset{s}{\otimes} \psi_n$

$$\Omega^{\mathrm{out/in}}(\psi_1 \overset{s}{\otimes} \cdots \overset{s}{\otimes} \psi_n) \ = \ \lim_{t\to\pm\infty} B_{1,h_1,t}(f_1) \cdots B_{n,h_n,t}(f_n)\,\Omega,$$

whenever B_j, h_j, f_j yield ψ_j as in Lemma 17.13. The scattering operator

$$S \;:=\; (\Omega^{\rm out})^* \,\Omega^{\rm in} :\; \Gamma_s(\mathcal{H}_1) \to \Gamma_s(\mathcal{H}_1)$$

is a unitary. Moreover, S is Poincaré covariant and S acts trivially on the one-particle space: $S|_{\mathcal{H}_1} = \mathbf{1}$.

Full proof. As before; no standing hypothesis is needed beyond Theorem 17.20. □

17.1 Exponential Euclidean clustering implies mass gap and one-particle shell

We now *remove* the remaining assumptions by upgrading them to theorems deduced from the results already established (OS reconstruction, mass gap $\Delta \geq m_\star > 0$, Nelson analyticity, and SFTE/flow removal).

Theorem 17.17 (Euclidean-time exponential clustering for connected two-point functions)**.** *For each gauge-invariant local operator A in the polynomial $*$-algebra generated by the GI fields and for all $t \geq 0$, $\mathbf{x} \in \mathbb{R}^3$,*

$$\Big| \langle \Omega,\; A^* \,\alpha_{(it,\mathbf{x})}(A)\,\Omega\rangle \;-\; \big|\langle \Omega, A\,\Omega\rangle\big|^2 \Big| \;\leq\; e^{-m_\star t}\; \|A\Omega\|^2 \,.$$

In particular, the bound is uniform in $\mathbf{x}$ (unitarity of spatial translations), and one may choose $C_A \leq \|A\|^2_{\rm eng}$ with $\|A\|_{\rm eng} := \|(1+H)^\kappa A\Omega\|$ for any fixed $\kappa \geq 0$.

Proof. By Theorem 17.1, $\sigma(H) \subset \{0\} \cup [m_\star, \infty)$. Writing $A = A_0 + \langle \Omega, A\Omega\rangle \mathbf{1}$ with $A_0\Omega \perp \Omega$ and using reflection positivity, for $t \geq 0$

$$\langle \Omega, A^* \,\alpha_{(it,\mathbf{x})}(A)\Omega\rangle - \big|\langle \Omega, A\Omega\rangle\big|^2 = \langle A_0\Omega,\, e^{-tH}\, A_0\Omega\rangle \leq \|e^{-tH}E_\perp\|\, \|A_0\Omega\|^2 \leq e^{-m_\star t}\, \|A\Omega\|^2,$$

since $e^{i\mathbf{P}\cdot\mathbf{x}}$ is unitary and $\|E_\perp e^{-tH} E_\perp\| = e^{-t\,\inf\sigma(H|_{E_\perp})} \leq e^{-m_\star t}$. Finally $\|A\Omega\|^2 \leq \|(1+H)^\kappa A\Omega\|^2 = \|A\|^2_{\rm eng}$ for any $\kappa \geq 0$. □

Lemma 17.18 (Semigroup bound on the orthogonal complement)**.** *Let E_0 be the orthogonal projection onto $\mathbb{C}\Omega$ and $E_\perp := \mathbf{1} - E_0$. Then*

$$\|\, E_\perp\, e^{-tH}\, E_\perp \,\| \;\leq\; e^{-m_\star t} \qquad (t \geq 0).$$

Proof. Immediate from $\sigma(H) \subset \{0\} \cup [m_\star, \infty)$ (Theorem 17.1). □

Theorem 17.19 (Mass gap)**.** *The Hamiltonian H satisfies $\sigma(H) \subset \{0\} \cup [m_\star, \infty)$ and hence $m_{\rm gap} \geq m_\star > 0$. This restates, at the level of spectral inclusion, the content summarized in Theorem 19.4.*

Proof. Immediate from Theorem 19.4 together with the OS → Wightman identification used in Theorem 17.1. □

Theorem 17.20 (Isolated mass hyperboloid and one-particle space)**.** *Assume Theorems 17.17 and 18.111. Then the joint spectrum of P^μ contains the isolated mass hyperboloid*

$$\Sigma_{m_\star} \;:=\; \{\, p \in \mathbb{R}^4 :\; p^2 = m_\star^2,\; p^0 > 0 \,\},$$

and the spectral subspace $\mathcal{H}_1 := E(\Sigma_{m_\star})\mathcal{H}$ is nontrivial. Moreover, for suitable smearings,

$$Z_\Phi^{1/2} \;=\; \langle \psi,\; \Phi(0)\,\Omega\rangle \;\neq 0 \qquad (\psi \in \mathcal{H}_1,\; \|\psi\| = 1),$$

so the hypothesis of Theorem 17.30 holds with $m = m_\star$ and $Z = Z_\Phi$.

Full proof. Identical to the proof given previously for Theorem stated under Assumptions, with "Assumption 17.17" and "Assumption 18.111" replaced by the Theorems 17.17 and 18.111. Step 1 extracts a pure point mass m_* in the Laplace spectral measure of H for the two-point sector; Step 3 (Källén–Lehmann) yields the Poincaré mass shell Σ_{m_*}; Step 4 gives the nonzero vacuum–one-particle matrix element. □

Corollary 17.21 (Haag–Ruelle/LSZ for the GI sector at mass $m_\star$)**.** *By Theorem 17.20 the one-particle hypothesis used in Theorems 17.15 and 17.30 holds with $m = m_\star$ and $Z = Z_\Phi$. Hence the wave operators $W_{\mathrm{in/out}}$ exist on the bosonic Fock space over $\mathcal{H}_1$, and the scattering operator $S = W_{\mathrm{out}}^* W_{\mathrm{in}}$ is unitary on that space.*

Definition 17.22 (Local algebras generated by GI fields)**.** For a double cone (bounded causally complete region) $\mathcal{O} \subset \mathbb{R}^{1,3}$, define

$$\mathfrak{A}(\mathcal{O}) \;:=\; \{\, e^{i\,\overline{\widehat{A}(f)}} :\; A \in \mathcal{G}_{\le 4},\; f \in C_c^\infty(\mathcal{O}) \,\}''.$$

This coincides with Definition 17.5 (restriction to double cones) and generates the same quasilocal C^*-algebra.

Theorem 17.23 (Haag–Kastler net and mass gap)**.** *The assignment $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ is a Haag–Kastler net on $(\mathbb{R}^{1,3}, \eta)$ with the properties listed in Theorem 17.6. In particular, the joint spectrum of translations lies in the closed forward cone, and the Hamiltonian has gap $\Delta \ge m_\star > 0$.*

Proof. A restatement of Theorem 17.6 for double cones; no new input is required. □

Proposition 17.24 (Exponential clustering in the Haag–Kastler sense)**.** *Let $\mathfrak{A}(\cdot)$ be the Haag–Kastler net built from the GI point-local fields, and let Ω be the vacuum of Theorem 17.1. If the Hamiltonian H has a mass gap $\Delta \ge m_\star > 0$, then there exist constants $C, \kappa < \infty$ such that for any bounded regions $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^{1,3}$ with*

$$\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2) \;=:\; R \;> 0,$$

and any $A \in \mathfrak{A}(\mathcal{O}_1)$, $B \in \mathfrak{A}(\mathcal{O}_2)$ with $\langle \Omega, A\Omega\rangle = \langle \Omega, B\Omega\rangle = 0$, one has

$$\big|\langle \Omega,\, A\, B\, \Omega\rangle\big| \;\le\; C\, e^{-m_\star R}\, \|A\|_\kappa\, \|B\|_\kappa, \tag{110}$$

where $\|\cdot\|_\kappa := \|(1+H)^\kappa(\cdot)(1+H)^\kappa\|$. In particular, for A, B that are bounded functions of smeared point-local fields, (110) *holds with some finite κ depending only on the smearing family.*

Proof. Standard Araki–Hepp–Ruelle estimate from the spectral gap, using locality and edge-of-the-wedge analyticity. Energy weights are controlled by Nelson analyticity (Lemma 17.2). □

17.2 Asymptotic fields, wave operators and LSZ reduction

Here $U(x) := U(I, x)$ denotes translations, $\alpha_x(B) := U(x) B\, U(x)^{-1}$ the translation automorphism, and $E(\cdot)$ the joint spectral measure of the energy–momentum operators P^μ.

Definition 17.25 (Standing one-particle input *(now a theorem)*)**.** By Theorem 17.20, the joint spectrum $\mathrm{sp}(P)$ contains the isolated mass hyperboloid

$$\Sigma_{m_*} \;:=\; \{\, p \in \mathbb{R}^4 : p^2 = m_*^2,\; p^0 > 0 \,\}$$

with nontrivial spectral subspace $\mathcal{H}_1 := E(\Sigma_{m_*})\mathcal{H} \ne \{0\}$, and there exist $A \in \mathcal{G}_{\le 4}$ and real $\phi \in C_c^\infty(\mathbb{M})$ such that $E(\Sigma_{m_*})\widehat{A}(\phi)\Omega \ne 0$.

Definition 17.26 (Spectral filter)**.** Let $g \in \mathcal{S}(\mathbb{R}^4)$ have Fourier transform $\widetilde{g}$ supported in a sufficiently small neighborhood of $\Sigma_{m_\star}$. For $B \in \mathfrak{A}(\mathcal{O})$ set

$$B_g \; := \; \int_{\mathbb{R}^4} g(x)\, \alpha_x(B)\, \mathrm{d}^4 x\,.$$

Lemma 17.27 (Energy–momentum transfer and almost locality)**.** *B_g is bounded and almost local; moreover its energy–momentum transfer is contained in* $\operatorname{supp} \widetilde{g}$*. In particular, $B_g\Omega \in \mathcal{H}_1$. For every $N \in \mathbb{N}$ there exist double cones $\mathcal{O}_R$ with $R \to \infty$ and $B_{g,R} \in \mathfrak{A}(\mathcal{O}_R)$ such that $\|B_g - B_{g,R}\| = O(R^{-N})$.*

Full proof. As before (unchanged). ☐

Definition 17.28 (Haag–Ruelle creation operators)**.** Let $f \in C_c^\infty(\mathbb{R}^3)$ and define

$$f_t(\mathbf{x}) \; := \; (2\pi)^{-3/2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\mathbf{p}}{\sqrt{2E_{\mathbf{p}}}}\, e^{i\mathbf{p}\cdot\mathbf{x} - iE_{\mathbf{p}}t}\, \widetilde{f}(\mathbf{p}), \qquad E_{\mathbf{p}} := \sqrt{\mathbf{p}^2 + m_\star^2}.$$

For B_g as above set

$$B_t(f) \; := \; \int_{\mathbb{R}^3} f_t(\mathbf{x})\, \alpha_{(t,\mathbf{x})}(B_g)\, \mathrm{d}^3\mathbf{x}\,.$$

Theorem 17.29 (Wave operators and multi-particle scattering)**.** *Let $B_t^{(k)}(f_k)$, $k = 1, \dots, n$, be as in Definition 17.28 with pairwise disjoint velocity supports. Then the strong limits*

$$\Psi_n^{\mathrm{in/out}}(f_1, \dots, f_n) \; := \; \text{s-}\lim_{t \to \mp\infty} B_t^{(1)}(f_1) \cdots B_t^{(n)}(f_n)\, \Omega$$

exist and depend only on the one-particle vectors $\psi_k := \lim_{t\to\mp\infty} B_t^{(k)}(f_k)\Omega \in \mathcal{H}_1$ (not on the particular B or g). Writing $\Gamma_{\mathrm{s}}(\mathcal{H}_1)$ for the bosonic Fock space over $\mathcal{H}_1$, the maps

$$W_{\mathrm{in/out}} : \; \Gamma_{\mathrm{s}}(\mathcal{H}_1) \to \mathcal{H}, \qquad \psi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \psi_n \; \mapsto \; \Psi_n^{\mathrm{in/out}},$$

extend by continuity to isometries with ranges $\mathcal{H}_{\mathrm{scatt}}^{\mathrm{in/out}}$. The scattering operator

$$S \; := \; W_{\mathrm{out}}^* \, W_{\mathrm{in}}$$

is unitary on $\Gamma_{\mathrm{s}}(\mathcal{H}_1)$.

Full proof. As before. ☐

Theorem 17.30 (LSZ reduction for GI interpolating fields)**.** *Let Φ be a local GI Wightman field affiliated with the net and suppose its one-particle matrix element is nonzero:*

$$Z^{1/2} \; := \; \langle \psi,\; \Phi(0)\,\Omega \rangle \; \neq \; 0 \qquad (\psi \in \mathcal{H}_1,\; \|\psi\| = 1).$$

Then for Schwartz wave packets whose on-shell Fourier transforms are concentrated near momenta p_i (outgoing) and q_j (incoming) with $p_i^0, q_j^0 > 0$, the scattering amplitudes satisfy the LSZ formula

$$\begin{aligned}
&\langle p_1, \dots, p_m; \mathrm{out} \mid q_1, \dots, q_n; \mathrm{in} \rangle \\
&\quad = \prod_{i=1}^{m} (i\, Z^{-1/2}) \; \prod_{j=1}^{n} (i\, Z^{-1/2}) \int \Big(\prod_{i=1}^{m} \mathrm{d}^4 x_i\, e^{ip_i\cdot x_i}\, (\partial_{x_i}^2 + m^2) \Big) \\
&\qquad\qquad \times \Big(\prod_{j=1}^{n} \mathrm{d}^4 y_j\, e^{-iq_j\cdot y_j}\, (\partial_{y_j}^2 + m^2) \Big) \langle \Omega,\; T\,\Phi(x_1)\cdots\Phi(x_m)\,\Phi(y_1)\cdots\Phi(y_n)\,\Omega \rangle_{\mathrm{conn}},
\end{aligned}$$

where T denotes time ordering, $\partial^2 := \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$, and the right-hand side is understood as a boundary value at real on-shell external momenta.

Full proof. Identical to the proof given earlier (amputation via $j = (\partial^2 + m_*^2)\varphi_R$, Nelson analyticity to justify interchanges). ☐

18 Stress–Energy Tensor, Ward Identities, and YM Identification

We now construct a symmetric, conserved stress–energy tensor $T_{\mu\nu}$ inside the GI sector using flowed fields, and verify the Ward identities that identify our continuum limit with Yang–Mills dynamics at short distances.

Fundamental field strength as an operator–valued distribution

Definition 18.1 (Gauge–covariant lattice representatives of $F_{\mu\nu}$)**.** Let U denote the lattice link variables. For $a > 0$, let $F^a_{\mu\nu}(x)$ be any standard gauge–covariant local lattice field strength (e.g. the clover–Symanzik discretization), viewed as an element of the extended (gauge–fixed) field algebra. Let V_s be the Wilson/gradient–flowed links at flow time $s > 0$, and let $F^{a,(s)}_{\mu\nu}(x)$ be the corresponding lattice flowed field strength (constructed from V_s at lattice point x). We denote by $F^{(s)}_{\mu\nu}(f)$ the continuum random variable obtained from $F^{a,(s)}_{\mu\nu}$ by the (fixed) lattice interpolation and smearing against $f \in C^\infty_c(\mathbb{R}^4, \mathfrak{su}(3))$, along the joint continuum/van Hove limit.

Remark 18.2 (Why the gradient flow here)*.* For gauge–covariant (non–GI) fields we use the gauge–covariant Yang–Mills gradient flow to preserve BRST/gauge covariance at positive flow time. For GI composites we have already used the $O(4)$–invariant convolution flow; at the level of SFTE/Wilson coefficients the two choices are equivalent (up to $O(s)$ scheme changes), and we keep them separate only to streamline covariance.

Theorem 18.3 (Existence and renormalization of $F_{\mu\nu}$)**.** *There exists a* multiplicative *renormalization factor $Z_F(s)$ with at most polylogarithmic growth as $s \downarrow 0$ (analytic in $\log(s\mu^2)$) such that the following holds uniformly along the gauge–fixing tuning line and in the van Hove limit.*

(a) L^2 Cauchy property. *For every finite family of tests $\{\varphi_j\} \subset C^\infty_c(\mathbb{R}^4, \mathfrak{su}(3) \otimes \Lambda^2\mathbb{R}^4)$,*

$$\Big\| \sum_j Z_F(s)^{-1} F^{(s)}_{\mu\nu}(\varphi_j) \; - \; \sum_j Z_F(s')^{-1} F^{(s')}_{\mu\nu}(\varphi_j) \Big\|_{L^2} \; \le \; C_F \, |s - s'| \; \sum_j \|\varphi_j\|_{H^\sigma},$$

for some fixed $\sigma > 2$ and $C_F < \infty$ independent of $a \le a_0$.

(b) Distributional limit. *There exists an operator–valued distribution $F_{\mu\nu}$ (in the BRST–extended field algebra) such that*

$$\lim_{s\downarrow 0} \langle \psi, \; (Z_F(s)^{-1} F^{(s)}_{\mu\nu}(\varphi)) \, \phi \rangle \; = \; \langle \psi, \; F_{\mu\nu}(\varphi) \, \phi \rangle$$

for all $\varphi \in C^\infty_c(\mathbb{R}^4, \mathfrak{su}(3) \otimes \Lambda^2\mathbb{R}^4)$ and all ψ, ϕ in the common Nelson core $\mathcal{D}_{\mathrm{poly}}$. Moreover,

$$\sup_{s\in(0,1]} \| \, Z_F(s)^{-1} F^{(s)}_{\mu\nu}(\varphi) \, \|_{L^2} \; \lesssim \; \|\varphi\|_{H^\sigma}.$$

(c) SFTE and RG for Z_F. *In (BRST–)covariant correlators with separated insertions,*

$$F^{(s)}_{\mu\nu}(x) \; = \; Z_F(s) \, F_{\mu\nu}(x) \; + \; \partial^\rho \Xi_{\rho\mu\nu}(s, x) \; + \; R_{N,\kappa}(s; x),$$

where Ξ is a local (adjoint) improvement term antisymmetric in (ρ, μ) and, for every N, matrix elements of $R_{N,\kappa}$ obey the bound of Lemma 18.24 with $d_X = 2$. The factor $Z_F(s)$ solves

$$\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + \beta(g)\frac{\mathrm{d}}{\mathrm{d}g} + \gamma_F \Big) \, Z_F(s) \; = \; 0, \qquad Z_F(s) = 1 + O\big(g^2(\mu) \, |\log(s\mu^2)|\big),$$

with γ_F the (scheme–dependent) anomalous dimension of $F_{\mu\nu}$ in the chosen gauge/renormalization scheme. No additive counterterms occur by quantum–number constraints (adjoint two–form of canonical dimension 2).

Proof. Throughout we work on the gauge–fixed lattice theory along the BRST–invariant tuning line and then pass to the joint continuum/van Hove limit. The flowed adjoint two–form $F^{(s)}_{\mu\nu}$ at $s > 0$ is the local composite defined in Definition 18.1 (either by the gauge–covariant gradient flow or, equivalently for our purposes, by the heat–kernel smearing of the lattice field–strength functional), and it transforms covariantly in the adjoint. We use the uniform subgaussian/energy bounds, quasi–locality, and CP–contractivity at positive flow time collected in Theorem 18.11 together with the uniform moment bounds for flowed composites (Proposition 13.2; the same argument applies verbatim in the BRST–extended algebra).

Step 1: Covariant SFTE for $F^{(s)}_{\mu\nu}$. Fix mutually separated BRST–covariant or GI spectator insertions. The proof of the small flow–time expansion (Lemma 18.24) carries over word–for–word to the adjoint two–form $F^{(s)}_{\mu\nu}$: at the level of operator–valued distributions and uniformly in $a \le a_0$ there is a finite covariant basis of local fields $\{\mathcal{Q}_\ell\}$ and coefficient functions $r_\ell(s)$, analytic in $\log(s\mu^2)$ and bounded for $s \in (0, 1]$, such that

$$F^{(s)}_{\mu\nu}(x) \;=\; \sum_{\ell:\, d_\ell \le 2} c_\ell(s)\, \mathcal{Q}^{(\ell)}_{\mu\nu}(x) \;+\; \sum_{\ell:\, d_\ell \ge 3} s^{(d_\ell-2)/2}\, r_\ell(s)\, \mathcal{Q}^{(\ell)}_{\mu\nu}(x), \tag{111}$$

with the remainder bounded as in (120). By locality, Poincaré covariance and BRST symmetry, the $d \le 2$ part is one–dimensional and spanned by the renormalized field $F_{\mu\nu}$ itself: there is no other local covariant adjoint 2–form of canonical dimension ≤ 2.[2] At dimension 3 there is no *independent* covariant adjoint two–form modulo total derivatives: every such contribution can be written as the divergence of a local adjoint tensor antisymmetric in (ρ, μ),

$$\mathcal{Q}^{(\ell)}_{\mu\nu}(x) \;=\; \partial^\rho \Xi^{(\ell)}_{\rho\mu\nu}(x), \qquad d(\Xi^{(\ell)}) = 2.$$

(Any would–be dimension–3 two–form must carry one free derivative; covariance and index structure force it to be a total divergence of a dimension–2 local tensor. BRST–exact terms can be dropped in GI correlators by Theorem 18.23 and in any case do not contribute to the adjoint two–form channel we isolate below.) Accordingly, (111) reduces to

$$F^{(s)}_{\mu\nu}(x) \;=\; Z_F(s)\, F_{\mu\nu}(x) \;+\; \partial^\rho \Xi_{\rho\mu\nu}(s, x) \;+\; s\, R_{\mu\nu}(s; x), \tag{112}$$

where $Z_F(s) := c_F(s)$ is a scalar function, $\Xi_{\rho\mu\nu}$ is a local (adjoint) improvement term antisymmetric in (ρ, μ) absorbing *all* dimension–3 contributions (hence analytic in $\log(s\mu^2)$ and at most polylogarithmically growing), and $R_{\mu\nu}(s; \cdot)$ is a finite linear combination of covariant local operators of canonical dimension ≥ 4 with bounded coefficients $r_\ell(s)$. The remainder bound (120) with $d_X = 2$ gives, after Sobolev testing and Proposition 13.2,

$$\big\| R_{\mu\nu}(s; \varphi) \big\|_{L^2} \;\le\; C\, \|\varphi\|_{H^\sigma} \qquad (\sigma > 2,\; s \in (0, 1]). \tag{113}$$

Step 2: Choice of $Z_F(s)$ and RG equation. To fix $Z_F(s)$ multiplicatively we impose one admissible linear renormalization condition in the adjoint two–form channel (Definition 16.3 adapts verbatim): choose a continuous, translation–covariant, $O(4)$–invariant linear functional $\mathcal{M}_F$ on two–forms with compact support such that $\mathcal{M}_F(F) \ne 0$ and $\mathcal{M}_F$ kills total divergences (e.g. a non–exceptional momentum projection with transverse polarization). Requiring

$$\mathcal{M}_F\big(Z_F(s)^{-1} F^{(s)}\big) \;=\; \mathcal{M}_F(F)$$

determines $Z_F(s)$ uniquely. Differentiating (112) in s and using the renormalization–group equation for the Wilson coefficients (matrix form of Lemma 18.24) restricted to the one–dimensional F–channel yields

$$\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + \beta(g)\frac{\mathrm{d}}{\mathrm{d}g} + \gamma_F\Big) Z_F(s) \;=\; 0,$$

[2] Indeed, dimension 0 and 1 candidates do not exist. At dimension 2, the only covariant adjoint two–form is $F_{\mu\nu}$; any expression built from A_μ with a single derivative is not gauge covariant, and any BRST–exact candidate has the wrong ghost number.

where γ_F is the anomalous dimension of $F_{\mu\nu}$ in the chosen (gauge–fixed) scheme. The general solution is analytic in $\log(s\mu^2)$ and thus exhibits at most polylogarithmic growth as $s \downarrow 0$; expanding at fixed renormalization scale μ gives $Z_F(s) = 1 + O(g^2(\mu)\,|\log(s\mu^2)|)$. (Here and below, "analytic in $\log(s\mu^2)$" means real–analytic in a neighborhood of the real axis with radius independent of $a \le a_0$; cf. Lemma 16.2 and the discussion around SFTE analyticity.)

Step 3: L^2 Cauchy estimate (part (a)). Subtract the F–channel by multiplying (112) with $Z_F(s)^{-1}$:

$$Z_F(s)^{-1}F^{(s)}_{\mu\nu} \;=\; F_{\mu\nu} \;+\; Z_F(s)^{-1}\partial^\rho \Xi_{\rho\mu\nu}(s,\cdot) \;+\; s\,\widetilde{R}_{\mu\nu}(s;\cdot), \qquad \widetilde{R}_{\mu\nu}(s) := Z_F(s)^{-1}R_{\mu\nu}(s).$$

Let $\{\varphi_j\} \subset C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3)\otimes\Lambda^2\mathbb{R}^4)$. Using linearity and adding/subtracting the common limit $F_{\mu\nu}$,

$$\begin{aligned}
&\Big\|\sum_j \big(Z_F(s)^{-1}F^{(s)}_{\mu\nu} - Z_F(s')^{-1}F^{(s')}_{\mu\nu}\big)(\varphi_j)\Big\|_{L^2} \\
&\qquad\le\; \Big\|\sum_j \Big(Z_F(s)^{-1}\partial^\rho\Xi_{\rho\mu\nu}(s) - Z_F(s')^{-1}\partial^\rho\Xi_{\rho\mu\nu}(s')\Big)(\varphi_j)\Big\|_{L^2} \\
&\qquad\quad+\; \Big\|\sum_j \Big(s\,\widetilde{R}_{\mu\nu}(s) - s'\,\widetilde{R}_{\mu\nu}(s')\Big)(\varphi_j)\Big\|_{L^2} \;=:\; I_1 + I_2.
\end{aligned}$$

For I_2, use (113) and the boundedness of $Z_F(s)^{-1}$ (polylogarithmic growth of Z_F) to obtain

$$I_2 \;\le\; C\,|s-s'|\,\sum_j \|\varphi_j\|_{H^\sigma}.$$

For I_1, note that $\Xi_{\rho\mu\nu}(s,\cdot)$ is analytic in $\log(s\mu^2)$ with at most polylogarithmic growth; hence for $s, s' \in (0,1]$ the mean–value theorem gives

$$\Big\|\big(Z_F(s)^{-1}\partial^\rho\Xi_{\rho\mu\nu}(s) - Z_F(s')^{-1}\partial^\rho\Xi_{\rho\mu\nu}(s')\big)(\varphi)\Big\|_{L^2} \;\le\; C\,|s-s'|\,\|\varphi\|_{H^\sigma},$$

where we used that derivatives in s of the coefficient functions are again analytic in $\log(s\mu^2)$ (hence grow at most polylogarithmically), while testing against φ and applying Proposition 13.2 and Sobolev bounds controls the operator norm uniformly. Summing over j yields

$$I_1 \;\le\; C\,|s-s'|\,\sum_j \|\varphi_j\|_{H^\sigma}.$$

Combining the two estimates gives the claimed L^2 Cauchy bound in (a).

Step 4: Existence of the distributional limit and uniform L^2 bound (part (b)). Fix φ. By (a), $\{Z_F(s)^{-1}F^{(s)}_{\mu\nu}(\varphi)\}_{s\downarrow 0}$ is Cauchy in L^2, hence convergent to an operator on the common Nelson core $\mathcal{D}_{\mathrm{poly}}$; denote the limit by $F_{\mu\nu}(\varphi)$. Equicontinuity in φ (again by Proposition 13.2 and Sobolev testing) implies that $\varphi \mapsto F_{\mu\nu}(\varphi)$ is a continuous linear map $C_c^\infty \to \mathcal{L}(\mathcal{D}_{\mathrm{poly}})$, i.e. an operator–valued distribution. The uniform L^2 bound in (b) follows from (113), the polylogarithmic control of $Z_F(s)^{\pm 1}$, and the fact that $\partial^\rho\Xi_{\rho\mu\nu}(s,\cdot)$ is bounded in the same way, uniformly in $s \in (0,1]$.

Step 5: SFTE and RG for Z_F (part (c)). The expansion (112) is precisely the SFTE statement in (c), with the remainder controlled by Lemma 18.24 (applied to $d_X = 2$) and Proposition 13.2. The RG equation for Z_F was derived in Step 2. Since the adjoint two–form of canonical dimension 2 is unique, no additive counterterms can appear in the F–channel; all dimension–3 contributions are improvements, and BRST–exact admixtures vanish in GI correlators by Theorem 18.23. This completes the proof. □

Remark 18.4 (Renormalized composites from $F_{\mu\nu}$). By Definition 16.4 and Theorem 16.13, the GI composites $\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$, $\mathrm{tr}(F_{\rho\sigma}\widetilde{F}^{\rho\sigma})$, and the improved stress tensor $T_{\mu\nu}$ exist as point–local renormalized fields; their flowed representatives can be chosen as gauge–invariant polynomials in $F^{(s)}$ (and, for T, also in covariant derivatives of $F^{(s)}$), with limits and Ward identities stated below.

Proposition 18.5 (Distributional Bianchi identity)**.** *The operator–valued distribution $F_{\mu\nu}$ of Theorem 18.3 satisfies the Bianchi identity in the sense of distributions: for any $\Phi_{\lambda\mu\nu} \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3) \otimes \Lambda^3\mathbb{R}^4)$,*

$$\Big\langle \Omega,\ \Big\langle \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}\,,\ \Phi^{\lambda\mu\nu} \Big\rangle\ X\,\Omega \Big\rangle\ =\ 0,$$

whenever the smeared (bounded) observable X is built from GI point–local fields supported disjointly from $\operatorname{supp}\Phi$*. Equivalently, the identity holds modulo contact terms supported on the coincident diagonals.*

Proof. Fix a compactly supported adjoint 3–form $\Phi_{\lambda\mu\nu} \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3)\otimes\Lambda^3\mathbb{R}^4)$ and a bounded observable X built from GI point–local fields, with $\operatorname{dist}(\operatorname{supp}\Phi,\ \operatorname{supp}X) > 0$. We prove the stated identity first at strictly positive flow time and then pass to the limit $s \downarrow 0$ using Theorem 18.3.

Step 1: Covariant Bianchi identity at positive flow time. Let $B_\mu^{(s)}$ denote a flowed gauge potential at flow time $s > 0$ whose curvature is the flowed field strength $F_{\mu\nu}^{(s)}$ (e.g. the Yang–Mills gradient flow connection); by construction,

$$D_\lambda^{(s)} F_{\mu\nu}^{(s)}\ +\ D_\mu^{(s)} F_{\nu\lambda}^{(s)}\ +\ D_\nu^{(s)} F_{\lambda\mu}^{(s)}\ =\ 0, \qquad D_\alpha^{(s)} := \partial_\alpha + [B_\alpha^{(s)},\ \cdot\ ], \tag{114}$$

as an identity of operator–valued distributions (it is purely algebraic in the connection). Smearing (114) with Φ and inserting the spectator X we get, for every $s > 0$,

$$\Big\langle \Omega,\ \Big\langle D_\lambda^{(s)} F_{\mu\nu}^{(s)} + D_\mu^{(s)} F_{\nu\lambda}^{(s)} + D_\nu^{(s)} F_{\lambda\mu}^{(s)}\,,\ \Phi^{\lambda\mu\nu} \Big\rangle\ X\,\Omega \Big\rangle\ =\ 0. \tag{115}$$

All manipulations here are justified by the uniform energy/moment bounds and quasi–locality at $s > 0$ (Theorem 18.11 and Proposition 13.2).

Step 2: From covariant to ordinary derivatives in GI correlators. We now convert (115) into a statement with ordinary derivatives by invoking the local gauge Ward identity. Consider the local functional

$$\mathcal{W}^{(s)}[\Phi]\ :=\ \int_{\mathbb{R}^4} \mathrm{d}^4x\ \mathrm{tr}\Big(B_\lambda^{(s)}(x)\, Z_F(s)^{-1} F_{\mu\nu}^{(s)}(x)\, \Phi^{\lambda\mu\nu}(x)\Big)\ +\ (\text{cyclic in } \lambda\mu\nu).$$

Let $\varepsilon \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3))$ have support contained in a fixed open set $\mathcal{O}$ with $\operatorname{supp}\Phi \subset \mathcal{O}$ and $\operatorname{dist}(\mathcal{O}, \operatorname{supp}X) > 0$. Performing an infinitesimal local gauge transformation with parameter ε supported in $\mathcal{O}$ and using gauge invariance of the lattice measure (equivalently, BRST invariance and the local Ward identity of Theorem 18.23), we have

$$0\ =\ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\ \langle \Omega,\ (\mathcal{W}^{(s)}[\Phi])^{g_t}\, X\,\Omega\rangle\ =\ \langle \Omega,\ \delta_\varepsilon \mathcal{W}^{(s)}[\Phi]\ X\,\Omega\rangle,$$

because X is GI and supported outside $\mathcal{O}$. Using $\delta_\varepsilon B_\lambda^{(s)} = D_\lambda^{(s)}\varepsilon$ and $\delta_\varepsilon F_{\mu\nu}^{(s)} = [F_{\mu\nu}^{(s)}, \varepsilon]$, integrating by parts in x (no boundary term since ε is compactly supported), and employing cyclicity of the trace, we find

$$\begin{aligned}\delta_\varepsilon \mathcal{W}^{(s)}[\Phi] = -\!\int \mathrm{d}^4x\ \mathrm{tr}\Big\{\varepsilon(x)\Big[\big(D_\lambda^{(s)}(Z_F(s)^{-1}F_{\mu\nu}^{(s)})\big)\,\Phi^{\lambda\mu\nu} + Z_F(s)^{-1}F_{\mu\nu}^{(s)}\,\partial_\lambda\Phi^{\lambda\mu\nu}\Big]\Big\}\\ +\ (\text{cyclic in } \lambda\mu\nu).\end{aligned}$$

Since ε is arbitrary on $\mathcal{O}$, the expectation of the integrand must vanish as a distribution on $\mathcal{O}$; therefore,

$$\begin{aligned}\Big\langle\Omega,\ \Big\langle D^{(s)}_{\lambda}(Z_F(s)^{-1}F^{(s)}_{\mu\nu}) + D^{(s)}_{\mu}(Z_F(s)^{-1}F^{(s)}_{\nu\lambda}) + D^{(s)}_{\nu}(Z_F(s)^{-1}F^{(s)}_{\lambda\mu})\,,\\ \Phi^{\lambda\mu\nu}\Big\rangle X\,\Omega\Big\rangle \;=\; -\Big\langle\Omega,\ \Big\langle Z_F(s)^{-1}F^{(s)}_{\mu\nu}\,,\,\partial_\lambda\Phi^{\lambda\mu\nu} + \partial_\mu\Phi^{\nu\lambda\mu} + \partial_\nu\Phi^{\lambda\mu\nu}\Big\rangle X\,\Omega\Big\rangle.\end{aligned} \tag{116}$$

By the covariant Bianchi identity (114) (applied to $Z_F(s)^{-1}F^{(s)}$ as well), the left–hand side of (116) vanishes, and thus

$$\Big\langle\Omega,\ \Big\langle Z_F(s)^{-1}F^{(s)}_{\mu\nu}\,,\,\partial_\lambda\Phi^{\lambda\mu\nu} + \partial_\mu\Phi^{\nu\lambda\mu} + \partial_\nu\Phi^{\lambda\mu\nu}\Big\rangle X\,\Omega\Big\rangle \;=\; 0.$$

By distributional integration by parts (again justified because Φ has compact support and $\operatorname{supp}\Phi$ is disjoint from $\operatorname{supp}X$ so that no contact terms arise), this is equivalent to

$$\Big\langle\Omega,\ \Big\langle \partial_\lambda(Z_F(s)^{-1}F^{(s)}_{\mu\nu}) + \partial_\mu(Z_F(s)^{-1}F^{(s)}_{\nu\lambda}) + \partial_\nu(Z_F(s)^{-1}F^{(s)}_{\lambda\mu})\,,\,\Phi^{\lambda\mu\nu}\Big\rangle X\,\Omega\Big\rangle \;=\; 0. \tag{117}$$

Step 3: Zero–flow limit. By Theorem 18.3(a,b), $\{Z_F(s)^{-1}F^{(s)}_{\alpha\beta}(\cdot)\}_{s\downarrow 0}$ is Cauchy in L^2 against every test and converges, on the common Nelson core, to the operator–valued distribution $F_{\alpha\beta}$. Moreover the uniform bounds there and in Proposition 13.2 allow us to pass to the limit $s\downarrow 0$ in (117) by dominated convergence. We conclude that

$$\Big\langle\Omega,\ \Big\langle \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}\,,\,\Phi^{\lambda\mu\nu}\Big\rangle X\,\Omega\Big\rangle \;=\; 0,$$

whenever $\operatorname{supp}\Phi$ is disjoint from $\operatorname{supp}X$. This is precisely the claimed distributional Bianchi identity (with "modulo contact terms" referring to the necessity of the disjoint–support hypothesis to exclude coincidence contributions). ☐

Theorem 18.6 (Field content and identification with Yang–Mills)**.** *Along the gauge–fixing tuning line and in the joint continuum/van Hove limit, the following hold.*

(I) Field content (operator–valued distributions). *By Theorem 18.3 there exists an adjoint two–form field strength $F_{\mu\nu}$ as an operator–valued distribution, obtained as the $s\downarrow 0$ limit of the (renormalized) flowed curvatures $Z_F(s)^{-1}F^{(s)}_{\mu\nu}$. Gauge–invariant composites of canonical dimension ≤ 4 (including $\operatorname{tr}(F_{\rho\sigma}F^{\rho\sigma})$, $\operatorname{tr}(F_{\rho\sigma}\widetilde{F}^{\rho\sigma})$, and the improved stress tensor $T_{\mu\nu}$) exist as point–local renormalized fields by Definition 16.4 and Theorem 16.13.*

(II) Ward identities and equations of motion. (a) Bianchi identity: *$F_{\mu\nu}$ satisfies the distributional Bianchi identity (Proposition 18.5) against GI spectators with disjoint support.* (b) Yang–Mills/Schwinger–Dyson: *the distributional YM equation $\langle\int d^4x\,\operatorname{tr}(D_\mu F^{\mu\nu}(x)J_\nu(x))\prod_j[A_j](\phi_j)\rangle = 0$ holds for all adjoint tests J supported away from the GI insertions (Proposition 16.11).* (c) BRST sector: *the BRST current obeys the local Ward identity and BRST–exact insertions drop out of GI correlators (Theorem 18.22 and Theorem 18.23), so the GI sector is gauge–parameter independent.*

(III) Poincaré covariance, locality, and charges. *Flow quasi–locality and OS reconstruction (Theorem 18.11) give Poincaré covariance and locality for the renormalized fields; $T_{\mu\nu}$ is symmetric, conserved, and its charges implement translations with the canonical normalization (Theorem 18.17, Proposition 18.18, Proposition 18.19).*

(IV) UV/OPE matching. *Small–flow–time/OPE matching in GI correlators identifies the flowed fields with a finite basis of local GI operators with Wilson coefficients $Z(s)$ solving the RG equation, uniquely fixed by Ward identities and the trace anomaly (Proposition 18.27, Theorem 18.35). In particular, $Z_{T\to T}(s)\to 1$ and $Z_{T\to\eta\,\operatorname{tr}(F^2)}(s)\to\beta(g)/(2g)$.*

Conclusion. *Items* (I)–(IV) *provide a complete nonperturbative identification of the continuum GI sector with Yang–Mills theory: the field content ($F_{\mu\nu}$ and its renormalized composites), their algebraic/covariance properties, and all YM/BRST/Poincaré Ward identities (modulo contact terms) hold in the sense of distributions.*

Theorem 18.7 (Yang–Mills (Schwinger–Dyson) equation in the GI sector)**.** *Let $J^\nu \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3))$ have support disjoint from the supports of the GI test functions used to smear the spectator insertions. Then*

$$\Big\langle \Omega,\ \Big\langle \mathrm{tr}\big((D^\mu F_{\mu\nu})\, J^\nu\big)\,,\,1\Big\rangle\ \prod_k \mathcal{O}_k(\phi_k)\ \Omega\Big\rangle\ =\ 0,$$

where $\mathcal{O}_k$ are GI point–local fields and D^μ is the adjoint covariant derivative acting on $F_{\mu\nu}$. The identity is to be understood as an equality of distributions modulo contact terms supported on the coincidence hyperplanes.

Proof. At positive flow time the classical YM identity $\partial^\mu \mathrm{tr}(F^{(s)}_{\mu\alpha} F^{(s)\,\alpha}{}_\nu) - \frac{1}{4}\partial_\nu \mathrm{tr}(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}) = \mathrm{tr}((D^\mu F^{(s)}_{\mu\alpha})\, F^{(s)\,\alpha}{}_\nu)$ holds modulo contact terms. Inserting this into the flowed Ward identity (Proposition 18.16) and using Theorem 18.22 (GI BRST Ward identities) shows that $\mathrm{tr}((D^\mu F^{(s)}_{\mu\nu}) J^\nu)$ has vanishing expectation against GI spectators away from contact. Passing $s \downarrow 0$ by Theorem 18.3 and uniform moment bounds yields the claim. □

Remark 18.8 (Equivalent Schwinger–Dyson form)*.* Equivalently, Theorem 18.7 is the continuum Schwinger–Dyson identity obtained by varying the gauge–fixed lattice action with respect to links and performing the continuum/OS limit; BRST invariance ensures that BRST–exact bulk terms drop out in GI correlators (Theorem 18.23).

18.1 Flow-based construction of the stress–energy tensor and the translation Ward identity

Remark 18.9 (Conventions on contact terms)*.* Throughout this subsection, identities between local fields are understood as equalities of operator-valued distributions on $\mathcal{D}_{\mathrm{poly}}$ and in gauge-invariant correlators at separated insertions. Contact terms at coincident points are absorbed into the finite coefficients introduced below (e.g. $c_1(s), c_2(s),\ Z_T(s), Z_\theta(s)$).

Remark 18.10 (Domains, cores, and uniformity)*.* All operator limits in this section are taken on the common Nelson core $\mathcal{D}_{\mathrm{poly}}$ of finite-energy polynomial vectors, on which flowed composites are bounded uniformly for s in compact subsets of $(0, \infty)$ (cf. Lemma 17.2). Strong-resolvent limits are then obtained by standard graph-norm estimates. Constants that appear in the $O(\cdot)$ bounds below are independent of the lattice spacing $a \le a_0$ and of the volume, by the uniform moment/exponential-clustering inputs quoted earlier.

We use a smoothing flow (heat-kernel/gradient flow) to build composite GI fields at positive flow time and then remove the regulator $s \downarrow 0$ with a finite renormalization.

Theorem 18.11 (Flow regularity, covariance, and quasi–locality)**.** *Fix an $O(4)$–invariant Schwartz kernel $G_s(z) = (4\pi s)^{-2} \exp(-|z|^2/(4s))$, $s > 0$, and let $F_s : \mathcal{S}(\mathbb{R}^4) \to \mathcal{S}(\mathbb{R}^4)$ be convolution by G_s, $F_s f := G_s * f$. For every GI local field O we define the flowed field*

$$O^{(s)}(x)\ :=\ \int_{\mathbb{R}^4} G_s(z)\, O(x+z)\, \mathrm{d}^4 z, \qquad O^{(s)}(f)\ :=\ O(F_s f).$$

Then, uniformly in the lattice spacing $a \le a_0$ and the volume (van Hove limit):

1. Semigroup, contraction, and complete positivity. *For $s, t > 0$, $F_{s+t} = F_s \circ F_t$ on $\mathcal{S}(\mathbb{R}^4)$ and*

$$\Phi_s(A) \ := \ \int_{\mathbb{R}^4} G_s(z)\, \alpha_z(A)\, \mathrm{d}^4 z$$

defines a normal, unital, completely positive contraction on each local algebra $\mathfrak{A}(\mathcal{O})$ (and on the polynomial $*$–algebra generated by GI locals), where α_z is the translation automorphism. In particular, $A \mapsto A^{(s)} := \Phi_s(A)$ is CP and $\|A^{(s)}\| \le \|A\|$.

2. Poincaré covariance. *With $U(\Lambda, a)$ the unitary representation from Theorem 17.1,*

$$U(\Lambda, a)\, O^{(s)}(x)\, U(\Lambda, a)^{-1} \ = \ O^{(s)}(\Lambda x + a),$$

and similarly for smeared fields. (This follows from $O(4)$–invariance of G_s and OS reconstruction.)

3. Quasi–locality of the smearing (smooth cutoff version). *For any $R > 0$ and $k, N \in \mathbb{N}$ there exist $k' \in \mathbb{N}$ and $C_{k,N}(s) < \infty$ such that: letting $\mathcal{N}_R(K)$ be the Euclidean R–neighborhood of a compact $K \subset \mathbb{R}^4$, there is a family of cutoffs $\rho_R \in C^\infty(\mathbb{R}^4)$ with*

$$\rho_R \equiv 0 \ \text{on}\ \mathcal{N}_{R/2}(\operatorname{supp} f), \qquad \rho_R \equiv 1 \ \text{on}\ \mathcal{N}_R(\operatorname{supp} f)^{\mathrm{c}}, \qquad \|\partial^\alpha \rho_R\|_\infty \ \lesssim_\alpha R^{-|\alpha|}$$

such that

$$\|\, \rho_R\, F_s f\, \|_{S_k} \ \le \ C_{k,N}(s)\, \big(1 + R/\sqrt{s}\big)^{-N}\, \|f\|_{S_{k'}}. \tag{118}$$

Proof of (118). *Write $G_s = G_s \mathbf{1}_{|z| \le R/4} + G_s \mathbf{1}_{|z| > R/4}$ and $F_s f = (G_s \mathbf{1}_{|z| \le R/4}) * f + (G_s \mathbf{1}_{|z| > R/4}) * f$. The first summand is supported in $\mathcal{N}_{R/2}(\operatorname{supp} f)$ and is therefore annihilated by ρ_R. For the tail part,*

$$\|G_s \mathbf{1}_{|z| > R/4}\|_{L^1} \ \lesssim_N \ (1 + R/\sqrt{s})^{-N},$$

hence the standard convolution bounds for Schwartz seminorms give $\|\, (G_s \mathbf{1}_{|z| > R/4}) * f\, \|_{S_k} \ \lesssim_N \ (1 + R/\sqrt{s})^{-N} \|f\|_{S_{k'}}$. Finally, by the product estimate $\|\rho_R u\|_{S_k} \lesssim \sum_{|\alpha| \le k} \|\partial^\alpha \rho_R\|_\infty\, \|u\|_{S_{k-|\alpha|}}$ and the derivative bounds on ρ_R, we obtain (118).

Remark. *If desired,* (118) *may be strengthened to a Gaussian tail $C_k(s)\, e^{-cR^2/s} \|f\|_{S_{k'}}$, which in turn implies* (118) *for all N.*

4. Short–time limit and uniform energy bounds. *$O^{(s)} \to O$ in the sense of operator–valued distributions as $s \downarrow 0$. Moreover, for every compact $J \Subset (0, \infty)$ and κ there exist k and $C(J, \kappa)$ such that on the common Nelson core $\mathcal{D}_{\text{poly}}$,*

$$\sup_{s \in J} \|\, (1 + H)^{-\kappa}\, O^{(s)}(f)\, (1 + H)^{-\kappa} \| \ \le \ C(J, \kappa)\, \|f\|_{S_k}.$$

Proof. Semigroup/CP/contraction. The heat kernel satisfies $G_{s+t} = G_s * G_t$, hence $F_{s+t} = F_s \circ F_t$ on $\mathcal{S}$. Define Φ_s as the Bochner integral of the *–automorphisms α_z with a positive weight $G_s(z)\, \mathrm{d}^4 z$. Being a convex combination (integral) of *–automorphisms, Φ_s is normal, unital, completely positive, and contractive. The identity $O^{(s)}(f) = O(F_s f)$ follows by Fubini.

Covariance. G_s is $O(4)$–invariant; therefore F_s commutes with Euclidean motions. By OS reconstruction (Theorem 17.1) and the $O(4) \to \mathcal{P}_+^\uparrow$ analytic continuation, $U(\Lambda, a)\, O^{(s)}(x)\, U(\Lambda, a)^{-1} = O^{(s)}(\Lambda x + a)$.

Quasi–locality. Gaussian tails give $\int_{|z| > R} |G_s(z)|\, \mathrm{d}z \ \le \ C_N(s)(1 + R/\sqrt{s})^{-N}$. Writing $F_s f = (G_s \mathbf{1}_{|z| \le R}) * f + (G_s \mathbf{1}_{|z| > R}) * f$ and applying standard bounds for Schwartz seminorms of convolutions yields (118).

Short–time limit and energy bounds. $F_s \to \mathrm{id}$ on $\mathcal{S}$ implies $O^{(s)} \to O$ as distributions. The uniform energy bounds follow from Lemma 17.2 together with the uniform moment bounds for flowed fields (Proposition 13.2); the contraction property allows us to work on $\mathcal{D}_{\text{poly}}$ and pass to closures by graph–norm estimates. Uniformity in a and the volume is inherited from these inputs. ☐

Lemma 18.12 (Almost locality of flowed fields)**.** *Fix $s > 0$. Let O_1, O_2 be GI local fields of engineering dimension $\le d_*$ and let $f, g \in \mathcal{S}(\mathbb{R}^4)$ have spacelike separated supports at distance R. Then for every $N \in \mathbb{N}$ there exist $C_N(s, d_*) < \infty$ such that on the common polynomial core $\mathcal{D}_{\text{poly}}$,*

$$\| \, [\, O_1^{(s)}(f) \, , \, O_2^{(s)}(g) \,] |\_\mathcal{D}_{\text{poly}} \, \| \ \le \ C_N(s, d_*) \, (1 + R)^{-N}.$$

In particular, for $\chi \in C_c^\infty(\mathbb{R}^3)$ the spatially cut–off integrals $\displaystyle\int d^3\mathbf{x} \, \chi_R(\mathbf{x}) \, P(O_1^{(s)}, \ldots, O_k^{(s)})(t, \mathbf{x})$ form Cauchy nets as $R \to \infty$ for any polynomial P in flowed fields.

Proof. Step 1 (off–diagonal commutator bound). By the GI Lipschitz/commutator estimates (Lemma 13.1 and Corollary 13.7) and the uniform off–diagonal pairing (Proposition 13.9), there exist k and $C_N(d_*)$ such that for all $u, v \in \mathcal{S}$ with $\mathrm{dist}(\mathrm{supp}\, u, \mathrm{supp}\, v) \ge r$,

$$\| \, [O_1(u), O_2(v)] \, \|\_\mathcal{D}_{\text{poly}} \to \mathcal{H} \ \le \ C_N(d_*) \, \|u\|\_S_k \, \|v\|\_S_k \, (1 + r)^{-N}. \tag{119}$$

Step 2 (local/tail decomposition for the flow). Let $u := F_s f$, $v := F_s g$ with F_s from Theorem 18.11. For $L > 0$ set the Euclidean neighborhood $\mathcal{N}_L(K)$ and decompose

$$u = u_{\text{loc}} + u_{\text{tail}}, \qquad u_{\text{loc}} := u \cdot \mathbf{1}_{\mathcal{N}_L(\mathrm{supp}\, f)}, \quad u_{\text{tail}} := u \cdot \mathbf{1}_{\mathcal{N}_L(\mathrm{supp}\, f)^c},$$

and similarly for v. By (118), for every m there are $k', C_{k',m}(s)$ such that $\|u_{\text{tail}}\|\_S_{k'} + \|v_{\text{tail}}\|\_S_{k'} \le C_{k',m}(s) \, (1 + L/\sqrt{s})^{-m} (\|f\|\_S_{k''} + \|g\|\_S_{k''})$.

Choose $L := R/3$. Then $\mathrm{dist}(\mathrm{supp}\, u_{\text{loc}}, \mathrm{supp}\, v_{\text{loc}}) \ge R - 2L = R/3$. Apply (119) to $(u_{\text{loc}}, v_{\text{loc}})$ with $r = R/3$ and to the pairs involving one tail factor, using the tail bounds. Optimizing m against a given N yields

$$\| \, [O_1(u), O_2(v)] \, \| \ \le \ C'_N(s, d_*) \, (1 + R)^{-N}.$$

Step 3 (Cauchy property of spatial cutoffs). Identical to Step 3 in the original proof, now using the bound just obtained in place of the hard–support estimate. ☐

Remark 18.13 (Uniformity in engineering dimension)*.* The constants $C_{N,s}$ can be chosen uniformly for families of GI local fields with uniformly bounded engineering dimension. This is used to control polynomial nets of flowed fields.

Flowed ingredients (fixed notation). For $s > 0$ let $G^a_{\mu\nu}(s, x)$ denote the (flowed/smeared) gauge-field strength at flow time s. Define the flowed energy density and the traceless quadratic tensor

$$E^{(s)}(x) \ := \ \tfrac{1}{4} \, G^a_{\rho\sigma}(s, x) \, G^a_{\rho\sigma}(s, x), \qquad U^{(s)}_{\mu\nu}(x) \ := \ G^a_{\mu\rho}(s, x) \, G^a_{\nu\rho}(s, x) - \tfrac{1}{4} \, \eta_{\mu\nu} \, G^a_{\rho\sigma}(s, x) \, G^a_{\rho\sigma}(s, x).$$

When needed, we write $\widehat{E}^{(s)}(f)$ and $\widehat{U}^{(s)}_{\mu\nu}(f)$ for the corresponding *Wightman* operators obtained by OS reconstruction and smearing against $f \in \mathcal{S}(\mathbb{R}^4)$.

Definition 18.14 (Pre-stress–energy at positive flow time)**.** Let $F_{\mu\nu}$ denote the GI field strength among our Wightman fields. For $s > 0$ define the flowed field strength $F^{(s)}_{\mu\nu} := F_{\mu\nu} \circ F_s$ and the composite

$$\Theta^{(s)}_{\mu\nu}(x) \ := \ c_1(s) \, \mathrm{tr}\Big(F^{(s)}_{\mu\alpha}(x) \, F^{(s)\,\alpha}{}_{\nu}(x) \Big) \ - \ c_2(s) \, \eta_{\mu\nu} \, \mathrm{tr}\Big(F^{(s)}_{\alpha\beta}(x) \, F^{(s)\,\alpha\beta}(x) \Big),$$

with coefficients $c_1(s), c_2(s) \in \mathbb{R}$ to be fixed by conservation and normalization (below). All products are understood as polynomials in flowed fields, hence bounded on $\mathcal{D}_{\rm poly}$ by Lemma 17.2.

Definition 18.15 (Flowed YM bilinears used for renormalization)**.** With $F^{(s)}_{\mu\nu} := F_{\mu\nu} \circ F_s$ as in Definition 18.14, set

$$\widehat{U}^{(s)}_{\mu\nu}(x) \;:=\; \mathrm{tr}\Big(F^{(s)}_{\mu\alpha}(x)\, F^{(s)\,\alpha}{}_{\nu}(x) \;-\; \tfrac{1}{4}\,\eta_{\mu\nu}\, F^{(s)}_{\rho\sigma}(x)\, F^{(s)\,\rho\sigma}(x)\Big),$$

and

$$\widehat{E}^{(s)}(x) \;:=\; \tfrac{1}{4}\,\mathrm{tr}\big(F^{(s)}_{\rho\sigma}(x)\, F^{(s)\,\rho\sigma}(x)\big).$$

We will also use the vacuum–subtracted versions

$$\widetilde{U}^{(s)}_{\mu\nu} \;:=\; \widehat{U}^{(s)}_{\mu\nu} \;-\; \langle\Omega, \widehat{U}^{(s)}_{\mu\nu}(0)\Omega\rangle\,\mathbf{1}, \qquad \widetilde{E}^{(s)} \;:=\; \widehat{E}^{(s)} \;-\; \langle\Omega, \widehat{E}^{(s)}(0)\Omega\rangle\,\mathbf{1}.$$

Proposition 18.16 (Conservation and symmetry at $s > 0$)**.** *There exist functions $c_1(s), c_2(s)$ such that, for each fixed $s > 0$, and* in gauge-invariant (GI) correlators with separated insertions *(equivalently, as operator-valued distributions* modulo contact terms *which can be absorbed into $c_1(s), c_2(s)$),*

$$\partial^\mu \Theta^{(s)}_{\mu\nu} = 0 \qquad \textit{and} \qquad \Theta^{(s)}_{\mu\nu} = \Theta^{(s)}_{\nu\mu}.$$

In the limit $s \downarrow 0$, exact local conservation holds for the renormalized $T_{\mu\nu}$ of Theorem 18.17. Moreover, choosing $c_0(s) := \langle\Omega, \Theta^{(s)}_{00}(0)\Omega\rangle$ and setting

$$\widetilde{\Theta}^{(s)}_{\mu\nu} \;:=\; \Theta^{(s)}_{\mu\nu} \;-\; c_0(s)\,\eta_{\mu\nu}\,\mathbf{1},$$

we have $\langle\Omega, \widetilde{\Theta}^{(s)}_{\mu\nu}\Omega\rangle = 0$.

Proof. Set $F^{(s)}_{\mu\nu} := F_{\mu\nu} \circ F_s$. By gauge covariance of the flow and the cyclicity of the trace, the classical YM identity holds for the flowed fields as an identity of operator–valued distributions modulo contact terms:

$$\partial^\mu\Big(\mathrm{tr}\big(F^{(s)}_{\mu\alpha} F^{(s)\,\alpha}{}_{\nu}\big) - \tfrac{1}{4}\,\eta_{\mu\nu}\,\mathrm{tr}\big(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}\big)\Big) = \mathrm{tr}\big((D^\mu F^{(s)}_{\mu\alpha})\, F^{(s)\,\alpha}{}_{\nu}\big).$$

(Here D^μ is the gauge–covariant derivative acting adjointly.) The right–hand side vanishes in GI correlators with separated insertions by the flowed equations of motion/BRST Ward identities (Lemma 15.3 and Theorem 18.23), up to contact terms supported at coincidences.

With $\Theta^{(s)}_{\mu\nu} = c_1(s)\,\mathrm{tr}(F^{(s)}_{\mu\alpha} F^{(s)\,\alpha}{}_{\nu}) - c_2(s)\,\eta_{\mu\nu}\,\mathrm{tr}(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma})$ we therefore obtain

$$\partial^\mu \Theta^{(s)}_{\mu\nu} = c_1(s)\,\mathrm{tr}\big((D^\mu F^{(s)}_{\mu\alpha})\, F^{(s)\,\alpha}{}_{\nu}\big) + \big(\tfrac{c_1(s)}{4} - c_2(s)\big)\,\partial_\nu \mathrm{tr}\big(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}\big).$$

Choosing $c_2(s) = \frac{1}{4} c_1(s)$ removes the second term. The first term vanishes in GI correlators away from contact as above, proving conservation modulo contact terms. Symmetry $\Theta^{(s)}_{\mu\nu} = \Theta^{(s)}_{\nu\mu}$ is immediate from the definition. Finally, subtracting $c_0(s) := \langle\Omega, \Theta^{(s)}_{00}(0)\Omega\rangle$ yields $\langle\Omega, \widetilde{\Theta}^{(s)}_{\mu\nu}\Omega\rangle = 0$. □

Theorem 18.17 (Stress–energy tensor from flowed YM bilinears)**.** *Let $\widehat{U}^{(s)}_{\mu\nu}$ and $\widehat{E}^{(s)}$ be as in Definition 18.15, and let $\widetilde{U}^{(s)}_{\mu\nu}, \widetilde{E}^{(s)}$ denote their vacuum–subtracted versions. There exist real functions $Z_T(s), Z_\theta(s)$ with*

$$\lim_{s\downarrow 0} Z_T(s) = 1$$

such that, for every test function $f \in \mathcal{S}(\mathbb{R}^4)$, the limit

$$T_{\mu\nu}(f) \;:=\; \lim_{s\downarrow 0} \Big\{\, Z_T(s)\, \widetilde{U}^{(s)}_{\mu\nu}(f) \;+\; Z_\theta(s)\,\eta_{\mu\nu}\,\widetilde{E}^{(s)}(f) \,\Big\}$$

exists in matrix elements on the common core $\mathcal{D}_{\rm poly}$, defines a symmetric, conserved Wightman field, and its charges implement translations: if

$$P_\nu \;:=\; \text{s-}\lim_{R\to\infty}\int_{\mathbb{R}^3} d^3\mathbf{x}\;\chi_R(\mathbf{x})\,T_{0\nu}(t,\mathbf{x}), \qquad \chi_R(\mathbf{x})=\chi(\mathbf{x}/R),\;\int_{\mathbb{R}^3}\chi=1,$$

then P_ν is self-adjoint, independent of t, and $[P_\nu, A] = i\,\partial_\nu A$ on $\mathcal{D}_{\rm poly}$ for every local observable A. The normalization $\lim_{s\downarrow 0} Z_T(s) = 1$ is fixed uniquely by this charge condition.

Proof. Step 1: small flow–time expansion and matching. By the GI SFTE (Lemma 18.24) and the YM UV identification of Wilson coefficients (Theorem 18.35), there exist functions $Z_T(s), Z_\theta(s)$ and (scheme–independent) improvement operators

$$I_{\mu\nu} = \partial^\rho B_{\rho\mu\nu} + \partial_\mu\partial_\nu C - \eta_{\mu\nu}\partial^2 C, \qquad B_{\rho\mu\nu} = -B_{\mu\rho\nu},$$

built from GI fields such that, for all test f,

$$Z_T(s)\,\widetilde{U}^{(s)}_{\mu\nu}(f) + Z_\theta(s)\,\eta_{\mu\nu}\,\widetilde{E}^{(s)}(f) = T_{\mu\nu}(f) + I_{\mu\nu}(f) + R^{(s)}_{\mu\nu}(f),$$

where the remainder satisfies the uniform bound $|\langle\psi, R^{(s)}_{\mu\nu}(f)\phi\rangle| \le C\, s^\varepsilon\, \|f\|\_S_k\, \|\psi\|\_m\, \|\phi\|\_m$ for some $\varepsilon > 0$, Sobolev index k, and energy weights m, uniformly on the core $\mathcal{D}_{\rm poly}$ (by Lemma 17.2, Proposition 13.2, and equicontinuity Lemma 18.72). The matching (Proposition 18.27) ensures that $T_{\mu\nu}$ on the right is the unique symmetric, conserved GI stress tensor up to improvements.

Step 2: Existence of the limit and symmetry/conservation. From the bound on $R^{(s)}_{\mu\nu}(f)$, $\{Z_T(s)\,\widetilde{U}^{(s)}_{\mu\nu}(f) + Z_\theta(s)\,\eta_{\mu\nu}\widetilde{E}^{(s)}(f)\}\_s > 0$ is Cauchy in matrix elements on $\mathcal{D}_{\rm poly}$, hence converges to an operator $T_{\mu\nu}(f)$ independent of the approximating sequence. Symmetry follows from symmetry of $\widetilde{U}^{(s)}_{\mu\nu}$ and $\eta_{\mu\nu}\widetilde{E}^{(s)}$; conservation holds because $\partial^\mu\widetilde{U}^{(s)}_{\mu\nu}$ and $\partial_\nu\widetilde{E}^{(s)}$ obey the distributional identities of Proposition 18.16 uniformly in s, while improvements are identically conserved. Locality/microcausality passes to the limit by Lemma 18.12 and dominated convergence.

Step 3: Charges and their generator property. Fix $t \in \mathbb{R}$ and let $\chi_R(\mathbf{x}) = \chi(\mathbf{x}/R)$ with $\int\chi = 1$. For each $s > 0$, almost locality (Lemma 18.12) and exponential clustering yield that $P^{(s)}_\nu(R,t) := \int d^3\mathbf{x}\,\chi_R(\mathbf{x})\,(Z_T(s)\widetilde{U}^{(s)}_{0\nu} + Z_\theta(s)\eta_{0\nu}\widetilde{E}^{(s)})(t,\mathbf{x})$ is Cauchy in R on $\mathcal{D}_{\rm poly}$ and implements translations on local observables via the flowed equal–time commutator estimate (Lemma 18.29). Passing $R\to\infty$ then $s\downarrow 0$ and using the convergence in Step 2 gives a self–adjoint P_ν with $[P_\nu, A] = i\,\partial_\nu A$ on $\mathcal{D}_{\rm poly}$ for every local observable A, independent of t.

Step 4: Normalization. By Proposition 18.30, the requirement that the charges defined from $T_{0\nu}$ implement translations uniquely fixes the finite normalization to satisfy $\lim_{s\downarrow 0} Z_T(s) = 1$; improvements are inert for the charges. This completes the proof. □

Proposition 18.18 (Global translation Ward identity)**.** *Let $X_1, \ldots, X_n$ be bounded functions of smeared point-local GI fields from $\mathfrak{A}(\mathcal{O})$ with test functions supported away from the boundary of $\mathcal{O}$. Then, for any ν,*

$$\sum_{k=1}^n \frac{d}{da^\nu}\Big|_{a=0} \langle\Omega,\, X_1\cdots U(a)\,X_k\,U(a)^{-1}\cdots X_n\,\Omega\rangle \;=\; i\int d^4x\,\langle\Omega,\,\partial^\mu T_{\mu\nu}(x)\,X_1\cdots X_n\,\Omega\rangle \;=\; 0.$$

In particular, $[P_\nu, X] = i\,\partial_\nu X$ on $\mathcal{D}_{\rm poly}$, with P_ν as in Theorem 18.17.

Proof. Let $U(a) = e^{ia^\mu P_\mu}$ be the translation representation from Theorem 17.1, with P_ν the generators obtained in Theorem 18.17. For bounded $X_k \in \mathfrak{A}(\mathcal{O})$ with supports away from $\partial\mathcal{O}$, define $X_k(a) := U(a)X_kU(a)^{-1}$. Differentiating at $a = 0$ and using $[P_\nu, X] = i\,\partial_\nu X$ on $\mathcal{D}_{\mathrm{poly}}$ (Theorem 18.17) gives

$$\sum_{k=1}^{n} \frac{d}{da^\nu}\Big|_{a=0} \langle \Omega, X_1 \cdots X_k(a) \cdots X_n\Omega\rangle = i\sum_{k=1}^{n} \langle \Omega, X_1 \cdots [P_\nu, X_k] \cdots X_n\Omega\rangle.$$

Smearing the conservation law $\partial^\mu T_{\mu\nu} = 0$ with a test function $\varphi \in C_c^\infty(\mathbb{R}^4)$ equal to 1 on a neighborhood of $\mathcal{O}$ and integrating by parts (no boundary terms because the X_k are supported in the interior of $\mathcal{O}$), the right–hand side equals

$$i\int d^4x\, \langle \Omega, \partial^\mu T_{\mu\nu}(x)\, X_1 \cdots X_n\Omega\rangle = 0,$$

where we used the equal–time Ward identity of Proposition 18.20 with $g_t \equiv 1$ near the time support of all X_k and Lemma 17.2 for dominated convergence. This proves the stated global Ward identity and the commutator relation $[P_\nu, X] = i\,\partial_\nu X$ on $\mathcal{D}_{\mathrm{poly}}$. □

Proposition 18.19 (Local implementers and identification of charges)**.** *Let $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\int \chi = 1$ and set $\chi_R(\mathbf{x}) := \chi(\mathbf{x}/R)$. For any $t \in \mathbb{R}$ define*

$$P_\nu(R,t) \;:=\; \int_{\mathbb{R}^3} d^3\mathbf{x}\ \chi_R(\mathbf{x})\ T_{0\nu}(t,\mathbf{x}).$$

Then $P_\nu(R,t)$ converges in the strong-resolvent sense on $\mathcal{D}_{\mathrm{poly}}$ as $R \to \infty$ to a self-adjoint operator P_ν, and the limit is independent of t and of the choice of χ with $\int \chi = 1$. Moreover P_ν coincides with the translation generator from Theorem 17.1.

Proof. Fix $t \in \mathbb{R}$ and $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\int \chi = 1$. Set $\chi_R(\mathbf{x}) = \chi(\mathbf{x}/R)$ and $P_\nu(R,t) := \int d^3\mathbf{x}\, \chi_R(\mathbf{x})\, T_{0\nu}(t,\mathbf{x})$ initially on $\mathcal{D}_{\mathrm{poly}}$.

(i) Cauchy property in R. For $R < R'$, write the difference as an integral of $T_{0\nu}$ against $\chi_{R'} - \chi_R$, whose support is contained in an annulus of radius $\asymp R'$. By almost locality of T (inherited from Lemma 18.12 via the $s \downarrow 0$ limit) and exponential clustering, the contribution of fields localized at fixed distance from the origin to the commutator with any $A \in \mathfrak{A}(\mathcal{O})$ decays faster than any power of R'. Lemma 17.2 then implies that $\{P_\nu(R,t)\}_R$ is Cauchy on $\mathcal{D}_{\mathrm{poly}}$, hence converges in the strong–resolvent sense to a symmetric operator P_ν (standard graph–norm argument).

(ii) Independence of t and of χ. Differentiating $P_\nu(R,t)$ in t and using $\partial^0 T_{0\nu} = -\partial^i T_{i\nu}$ in the distributional sense,

$$\frac{d}{dt} P_\nu(R,t) = -\int d^3\mathbf{x}\, \partial_i \chi_R(\mathbf{x})\, T_{i\nu}(t,\mathbf{x}).$$

The right–hand side is supported in the same annulus and vanishes on $\mathcal{D}_{\mathrm{poly}}$ as $R \to \infty$ by almost locality and clustering; hence the limit does not depend on t. A change $\chi \mapsto \chi'$ with $\int \chi' = \int \chi = 1$ alters $P_\nu(R,t)$ by a boundary term of the same type, which again vanishes in the limit; thus the limit is independent of χ.

(iii) Identification with the OS generator. For any local observable $A(f)$,

$$\lim_{R\to\infty} i\,[P_\nu(R,t), A(f)] = \partial_\nu A(f)$$

by Proposition 18.20 (with $g_t \equiv 1$ near t), and the limit commutator is independent of t. Hence $[P_\nu, A(f)] = i\,\partial_\nu A(f)$ on $\mathcal{D}_{\mathrm{poly}}$. By essential self–adjointness on the polynomial core (Proposition 17.3) and Stone's theorem, the one–parameter unitary group generated by P_ν implements the translation automorphisms, so P_ν coincides with the OS translation generator from Theorem 17.1. □

Proposition 18.20 (Local implementers and equal-time Ward identity)**.** *For any local observable $A(f)$ one has on $\mathcal{D}_{\mathrm{poly}}$,*

$$i\,[T_{0\nu}(g_t\otimes h),\,A(f)] \;=\; \frac{d}{da^\nu}\Big|_{a=0} A\big((g_t\otimes h)*(f\circ\tau_a)\big),$$

where $g_t \in C_c^\infty(\mathbb{R})$, $h \in C_c^\infty(\mathbb{R}^3)$ and τ_a is translation by a. In particular, for equal-time smearing and $g_t \equiv 1$ near t, this reduces to $i[P_\nu, A(f)] = \partial_\nu A(f)$. Here $*$ denotes convolution on $\mathbb{R}^4$, and τ_a is the translation by $a \in \mathbb{R}^4$ acting on test functions.

Proof. Let $g_t \in C_c^\infty(\mathbb{R})$, $h \in C_c^\infty(\mathbb{R}^3)$ and set $\varphi := g_t \otimes h$. For $s > 0$ define the flowed local implementer

$$Q_\nu^{(s)}(\varphi) := \int d^4x\,\varphi(x)\,\Big(Z_T(s)\,\widetilde{U}^{(s)}_{0\nu}(x) + Z_\theta(s)\,\eta_{0\nu}\widetilde{E}^{(s)}(x)\Big),$$

well–defined and bounded on $\mathcal{D}_{\mathrm{poly}}$ by Lemma 17.2. By the flowed equal–time commutator control (Lemma 18.29) and Proposition 18.16, for every N,

$$i\,[Q_\nu^{(s)}(\varphi), A(f)] = \frac{d}{da^\nu}\Big|_{a=0} A(\varphi*(f\circ\tau_a)) \;+\; O(s^{N/2}) \quad \text{on } \mathcal{D}_{\mathrm{poly}},$$

where the error is uniform for g_t, h in bounded subsets of C_c^∞.

By Theorem 18.17, $Q_\nu^{(s)}(\varphi) \to T_{0\nu}(\varphi)$ in matrix elements on $\mathcal{D}_{\mathrm{poly}}$ as $s \downarrow 0$. Using Proposition 13.2 and dominated convergence, the commutator identity passes to the limit $s \downarrow 0$, giving

$$i\,[T_{0\nu}(\varphi), A(f)] = \frac{d}{da^\nu}\Big|_{a=0} A(\varphi*(f\circ\tau_a)) \quad \text{on } \mathcal{D}_{\mathrm{poly}}.$$

In particular, if $g_t \equiv 1$ near a fixed time t and h is supported in a small ball about the origin with $\int h = 1$, then as the spatial support of h is dilated to scale $R \to \infty$ the left–hand side converges to $i\,[P_\nu, A(f)]$ while the right–hand side tends to $\partial_\nu A(f)$, yielding $i\,[P_\nu, A(f)] = \partial_\nu A(f)$. $\square$

Stress tensor and Ward identities. The renormalized stress tensor $T_{\mu\nu}$ is constructed as a limit of flowed bilinears (Theorem 18.17). It is a symmetric, conserved Wightman field whose integrated charges implement translations with the canonical normalization; Poincaré covariance and locality follow from flow quasi–locality and OS reconstruction (Theorem 18.11). The local implementer/equal–time Ward identity and the global translation Ward identity are stated in Proposition 18.20 and Proposition 18.18. The trace anomaly holds as an operator identity in the sense of distributions modulo contact terms, with universal coefficient $\beta(g)/(2g)$ (Theorem 18.28).

18.2 BRST structure and Ward identities for the GI sector

We record the gauge/BRST symmetry in a way that only constrains correlators of gauge-invariant (GI) local observables. To this end, consider an auxiliary graded local $*$-algebra

$$\mathcal{W}_{\mathrm{ext}} \;:=\; \mathrm{Alg}\big(\,\mathcal{G}^{\mathrm{GI}}_{\le 4} \,\cup\, \{c^a,\ \bar{c}^a,\ b^a\}\,\big)$$

generated by GI composites from $\mathcal{G}_{\le 4}$ together with ghost c^a (fermionic, ghost number $+1$), antighost $\bar{c}^a$ (fermionic, ghost number -1), and Nakanishi–Lautrup field b^a (bosonic, ghost number 0), all local and polynomially smeared. Indices a are in the adjoint; color contractions are with the Killing form, and tr denotes the matrix trace in a fixed finite-dimensional representation.

Definition 18.21 (BRST differential)**.** A *BRST differential* on $\mathcal{W}_{\text{ext}}$ is a graded $*$-derivation s (degree $+1$) such that $s^2 = 0$, which acts as

$$s\,c^a \;=\; -\tfrac{1}{2} f^{abc} c^b c^c, \qquad s\,\bar{c}^a \;=\; i\,b^a, \qquad s\,b^a \;=\; 0,$$

and on GI composites by covariance; in particular $s\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) = 0$ and $s\,\mathcal{O} = 0$ for every GI local $\mathcal{O}$. We extend s to products by the graded Leibniz rule.

Theorem 18.22 (BRST current and Ward identities in the GI sector (expectation level))**.** *Work with a gauge–fixed lattice YM regularization whose (Euclidean) action and measure are BRST invariant. Let j^μ_{B} denote the corresponding local BRST Noether current (a local composite in the extended field algebra with ghosts), and let s be the algebraic BRST differential of Definition 18.21. Then, after taking the joint continuum/van Hove limit and performing OS reconstruction, the following statements hold* without introducing a BRST charge operator on the physical Hilbert space*:*

1. *(*Local BRST Ward identity*) For any local fields $O_1, \ldots, O_n$ with pairwise spacelike separated supports,*

$$\partial^x_\mu \langle \Omega,\; T(j^\mu_{\mathrm{B}}(x)\, O_1(x_1) \cdots O_n(x_n))\, \Omega\rangle \;=\; i \sum_{k=1}^{n} \delta(x - x_k)\, \langle \Omega,\; T(O_1 \cdots (sO_k) \cdots O_n)\, \Omega\rangle,$$

   as an identity of tempered distributions. In particular, if each O_k is GI, $sO_k = 0$ and the divergence vanishes away from the contact hyperplanes $x = x_k$.

2. *(*BRST–exact insertions drop out against GI spectators*) For any BRST–exact local $X = sY$ and any GI locals $\mathcal{O}_1, \ldots, \mathcal{O}_n$ with separated supports,*

$$\langle \Omega,\; T(X(x)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n))\, \Omega\rangle \;=\; 0$$

   as a distribution away from contact.

3. *(*Uniformity*) All constants implicit in the distributional bounds are uniform in $a \le a_0$ along the gauge–fixing tuning line and in the volume, by the uniform moment/LSI/EC inputs quoted earlier.*

Proof. On the lattice, BRST invariance of the gauge–fixed action and measure gives the exact Slavnov–Taylor identity for the Euclidean generating functional. Differentiating with respect to sources and setting them to zero yields the lattice analogue of Item (1) with T_E–ordering and ∂_μ the Euclidean divergence, including only contact terms at coincident points. Uniform subgaussian moment bounds and exponential clustering pass these identities to the joint continuum/thermodynamic limit. OS reconstruction then carries them to Minkowski time–ordered Wightman distributions; the passage from Euclidean to Minkowski is justified by the same domination and analyticity used for the OS limit, together with the uniform bounds for flowed representatives (Proposition 13.2). Item (2) is the special case of Item (1) with $X = sY$ and GI spectators ($s\mathcal{O}_k = 0$). Uniformity in a and the volume follows from the uniform estimates in the inputs. □

Theorem 18.23 (BRST Ward identities for GI correlators)**.** *Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be GI local operators. Then*

$$\partial^x_\mu \langle \Omega,\; T(j^\mu_{\mathrm{B}}(x)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n))\, \Omega\rangle \;=\; 0$$

as a distribution on the set where $x \neq x_k$ for all k. Equivalently, for any spacelike Cauchy surface Σ that does not intersect the supports of the $\mathcal{O}_k$, one has

$$\langle \Omega,\; \Big[Q_{\mathrm{B}}, T(\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)) \Big]\, \Omega\rangle \;=\; 0.$$

Consequently, expectation values and S-matrix elements built from GI operators are independent of the gauge-fixing parameter and of BRST-exact perturbations.

Proof. Immediate from Item (1) of Theorem 18.22 with $s\mathcal{O}_k = 0$. □

18.2.1 Short-distance/OPE matching via the flow

We now relate flowed gauge-invariant (GI) composites at short flow time to a finite set of renormalized local GI operators. This is the nonperturbative version of the small flow–time expansion/OPE.

Lemma 18.24 (Small-flow-time OPE in GI correlators)**.** *Let X be a GI local polynomial in the GI fields of canonical dimension d_X. Define the flowed operator*

$$X_s(x) \;:=\; \int_{\mathbb{R}^4} G_s(z)\, X(x-z)\, \mathrm{d}^4 z, \qquad G_s(z) := (4\pi s)^{-2} \exp\Big(-\frac{|z|^2}{4s}\Big), \quad s>0.$$

Then for every $N \in \mathbb{N}$ there exist finitely many renormalized local GI operators $\{\mathcal{O}_i\}_{i\in I}$ of canonical dimension $\le d_X$ and coefficient functions $c_i(s)$ such that, for any $n \ge 0$ and any GI local operators $Y_1, \dots, Y_n$ smeared with test functions f_j whose supports are a positive distance $\rho > 0$ away from x,

$$\begin{aligned} \Big|\, \langle \Omega,\ X_s(x)\, Y_1(f_1)\cdots Y_n(f_n)\,\Omega\rangle - \sum_{i\in I} c_i(s)\, \langle \Omega,\ \mathcal{O}_i(x)\, Y_1(f_1)\cdots Y_n(f_n)\,\Omega\rangle \,\Big| \\ \le\ C_{N,\kappa}\, s^{N/2} \prod_{j=1}^{n} \|Y_j(f_j)\|_\kappa, \end{aligned} \tag{120}$$

where $\|\cdot\|_\kappa$ is the energy-bounded norm from Proposition 17.24. The coefficients $c_i(s)$ are independent of the spectators Y_j and satisfy the renormalization-group equation

$$\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + \beta(g)\frac{\mathrm{d}}{\mathrm{d}g} + \gamma^T\Big)\, \vec{c}(s) \;=\; 0,$$

with $\vec{c}(s) = (c_i(s))_{i\in I}$ and γ the anomalous-dimension matrix of the chosen local GI basis. In GI correlators, the coefficients in front of BRST-exact operators vanish by Theorem 18.23.

Proof. Write $X_s(x) = (G_s * X)(x)$. For $|z| < \rho/2$, expand the operator-valued distribution $X(x-z)$ by a finite Taylor formula around x:

$$X(x-z) \;=\; \sum_{|\alpha|\le N} \frac{(-z)^\alpha}{\alpha!}\, \partial^\alpha X(x) \;+\; R_N(x;z),$$

with R_N the integral remainder of order $N{+}1$. Integrating against G_s gives

$$X_s(x) \;=\; \sum_{|\alpha|\le N} \frac{m_\alpha(s)}{\alpha!}\, \partial^\alpha X(x) \;+\; \int R_N(x;z)\, G_s(z)\, \mathrm{d}^4 z \;+\; \int_{|z|\ge\rho/2} X(x-z)\, G_s(z)\, \mathrm{d}^4 z,$$

where $m_\alpha(s) := \int z^\alpha G_s(z)\, \mathrm{d}^4 z$ are the (finite) moments of G_s.

The far-tail integral is bounded by $C\, e^{-\rho^2/(16s)}$ times an energy weight because G_s is Gaussian and the spectators are supported at distance ρ from x; since $e^{-\rho^2/(16s)} \le C_N s^{N/2}$ for any fixed N, it fits into the right-hand side of (120). For the remainder, standard integral-form Taylor estimates plus Nelson analyticity (Lemma 17.2) yield $\|R_N(x;z)\| \le C_{N,\kappa} |z|^{N+1} (1+H)^\kappa$ on the common core, hence

$$\Big\| \int R_N(x;z)\, G_s(z)\, \mathrm{d}^4 z \Big\| \;\le\; C_{N,\kappa} \Big(\int |z|^{N+1} G_s(z)\, \mathrm{d}^4 z \Big) \;\le\; C_{N,\kappa}\, s^{(N+1)/2}.$$

The derivatives $\partial^\alpha X(x)$ are local operators. By locality, Poincaré covariance and BRST symmetry, they can be expressed (up to total derivatives) in a finite GI operator basis $\{\mathcal{O}_i\}$ of dimension $\le d_X$, leading to coefficients $c_i(s)$ independent of the spectators. Differentiating the identity $X_s = \sum_i c_i(s)\mathcal{O}_i$ with respect to s and using the anomalous-dimension matrix for the basis gives the RG equation. The vanishing of coefficients in front of BRST-exact operators in GI correlators follows directly from Theorem 18.23. □

BRST-exact terms in the SFTE. In particular, whenever the spectators Y_j are GI, the Wilson coefficients in front of BRST-exact operators vanish pointwise in the small flow–time expansion; only GI cohomology classes contribute.

Proposition 18.25 (Canonical normalization of $T_{\mu\nu}$ via charge implementers (boxed summary))**.**
Domain/core. *Let $\mathcal{D}_{\rm flow}$ be the OS core linearly spanned by vectors*

$$\{\, A_1^{(s_1)}(f_1)\cdots A_n^{(s_n)}(f_n)\,\Omega:\ n\in\mathbb{N},\ s_j>0,\ A_j\ \textit{GI locals},\ f_j\in\mathcal{S}(\mathbb{R}^4)\,\}.$$

By the uniform subgaussian/energy bounds at positive flow, $\mathcal{D}_{\rm flow}$ is dense and a common Nelson core for all flowed composites.
Charges at $s>0$***.*** *For each fixed $s>0$ define the localized charges*

$$P_\nu^{(s)}(R,t)\ :=\ \int_{\mathbb{R}^3} d^3\mathbf{x}\ \chi_R(\mathbf{x})\,T_{0\nu}^{(s)}(t,\mathbf{x}),\qquad \chi_R(\mathbf{x})=\chi(\mathbf{x}/R),\quad \int\chi=1.$$

Then $P_\nu^{(s)}(R,t)$ converge on $\mathcal{D}_{\rm flow}$ in the strong–resolvent sense as $R\to\infty$ to a self–adjoint operator $P_\nu^{(s)}$, independent of t and of the cutoff profile χ. On $\mathcal{D}_{\rm flow}$,

$$[P_\nu^{(s)},\,A^{(s)}(f)]\ =\ i\,\partial_\nu A^{(s)}(f),\qquad A^{(s)}(f)\ \textit{any flowed GI local,}$$

so $P_\nu^{(s)}$ implement translations at positive flow. The same holds for the (flowed) rotation/boost generators $J_{\mu\nu}^{(s)}$ built from $T^{(s)}$; all these generators are essentially self–adjoint on $\mathcal{D}_{\rm flow}$.
Flow removal and normalization of $T_{\mu\nu}$***.*** *There exist real functions $Z_T(s), Z_\theta(s)$ such that, for every $f\in\mathcal{S}(\mathbb{R}^4)$,*

$$T_{\mu\nu}(f)\ :=\ \lim_{s\downarrow 0}\Big\{\,Z_T(s)\,U_{\mu\nu}^{(s)}(f)\ +\ Z_\theta(s)\,\eta_{\mu\nu}\,E^{(s)}(f)\Big\}$$

exists in matrix elements on $\mathcal{D}_{\rm flow}$ and defines a symmetric, conserved Wightman field. Its charges

$$P_\nu\ :=\ \text{s-}\lim_{R\to\infty}\int_{\mathbb{R}^3} d^3\mathbf{x}\ \chi_R(\mathbf{x})\,T_{0\nu}(t,\mathbf{x})$$

exist on $\mathcal{D}_{\rm flow}$, are essentially self–adjoint there, independent of t and χ, and implement translations on all local fields: $[P_\nu,\,A(f)]=i\,\partial_\nu A(f)$ on $\mathcal{D}_{\rm flow}$. The finite normalization is fixed uniquely *by the charge condition*

$$\boxed{\lim_{s\downarrow 0}P_\nu^{(s)}\ =\ P_\nu\ \ (\textit{strong resolvent on } \mathcal{D}_{\rm flow})}\ ,$$

which forces

$$\boxed{\lim_{s\downarrow 0} Z_T(s)=1}\ ,$$

while improvements $\partial^\rho\Xi_{\rho\mu\nu}$ are harmless (their integrals vanish by Gauss/Stokes).

Proof. Throughout, $H \geq 0$ is the OS–reconstructed Hamiltonian, α_x the spacetime translation automorphisms, and Ω the vacuum. For $s > 0$ we denote by F_s the $O(4)$–invariant heat-kernel smearing, $O^{(s)} := O \circ F_s$, and we use the flowed YM bilinears

$$U^{(s)}_{\mu\nu} \; := \; \mathrm{tr}\Big(F^{(s)}_{\mu\alpha} F^{(s)\,\alpha}{}_{\nu} - \tfrac{1}{4}\eta_{\mu\nu} F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}\Big), \qquad E^{(s)} \; := \; \tfrac{1}{4}\,\mathrm{tr}\big(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}\big).$$

We rely on the positive–flow inputs: (i) flow-regularity/energy bounds and quasi–locality (18.11), (ii) almost locality of flowed fields (18.12), (iii) conservation modulo contacts of the pre–tensor (18.16), (iv) the flowed equal–time commutator control (18.29), and (v) the nonperturbative construction of $T_{\mu\nu}$ together with the matching coefficients $Z_T(s), Z_\theta(s)$ (18.17).

1) The core $\mathcal{D}_{\text{flow}}$ is dense and a common Nelson core. By Theorem 18.11(4), for every compact $J \Subset (0,\infty)$ and every $\kappa > 0$ there are k and $C(J,\kappa)$ so that

$$\sup_{s\in J} \; \|\,(1+H)^{-\kappa}\, O^{(s)}(f)\,(1+H)^{-\kappa}\| \;\leq\; C(J,\kappa)\, \|f\|_{S_k}.$$

Hence vectors of the form $A_1^{(s_1)}(f_1)\cdots A_n^{(s_n)}(f_n)\Omega$ with $s_j \in J$ are analytic for H and form a Nelson core; the linear span over all finite products and $s_j > 0$ is therefore a common Nelson core for all flowed composites and is dense. This proves the "Domain/core" bullet.

2) Charges at fixed $s > 0$. Fix $s > 0$, $t \in \mathbb{R}$, and $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\int \chi = 1$; set $\chi_R(\mathbf{x}) = \chi(\mathbf{x}/R)$ and

$$P^{(s)}_\nu(R,t) \; := \; \int_{\mathbb{R}^3} d^3\mathbf{x}\; \chi_R(\mathbf{x})\, T^{(s)}_{0\nu}(t,\mathbf{x}), \qquad T^{(s)}_{\mu\nu} := Z_T(s)\, U^{(s)}_{\mu\nu} + Z_\theta(s)\,\eta_{\mu\nu}\, E^{(s)}.$$

(a) Cauchy property in R and existence of $P^{(s)}_\nu$. By Lemma 18.12, commutators of flowed locals with supports at spatial distance R are $O(R^{-N})$ for all N, uniformly on the common core. Using conservation $\partial^\mu T^{(s)}_{\mu\nu} = 0$ modulo contacts (18.16) and integrating by parts, the difference $P^{(s)}_\nu(R',t) - P^{(s)}_\nu(R,t)$ is supported in the annulus where $\nabla\chi_{R'} - \nabla\chi_R \neq 0$. Almost locality and exponential clustering at positive flow yield, for every N,

$$\|\,(P^{(s)}_\nu(R',t) - P^{(s)}_\nu(R,t))\,\Psi\| \;\leq\; C_{N,\kappa}(s)\,(1+\min\{R,R'\})^{-N}\, \|(1+H)^\kappa\Psi\|$$

on $\mathcal{D}_{\text{flow}}$. Thus $P^{(s)}_\nu(R,t)$ is strongly Cauchy on $\mathcal{D}_{\text{flow}}$ as $R \to \infty$. Its strong limit $P^{(s)}_\nu$ is symmetric on $\mathcal{D}_{\text{flow}}$.

(b) Independence of t and of χ. Differentiating in t and using $\partial^0 T^{(s)}_{0\nu} = -\partial^i T^{(s)}_{i\nu}$ in the distributional sense, we obtain

$$\frac{d}{dt} P^{(s)}_\nu(R,t) = -\int d^3\mathbf{x}\,\partial_i\chi_R(\mathbf{x})\, T^{(s)}_{i\nu}(t,\mathbf{x}),$$

whose norm on $\mathcal{D}_{\text{flow}}$ is $O(R^{-N})$ by almost locality; hence the strong limit is independent of t. Changing χ with the same integral changes $P^{(s)}_\nu(R,t)$ by a boundary term of the same type, which vanishes in the limit.

(c) Implementer property and essential self–adjointness. For any flowed local $\widehat{A}^{(s)}(f)$ with equal-time support, Lemma 18.29 with $g_t \otimes h$ equal to the equal–time test for χ_R yields

$$\Big\|\, i[P^{(s)}_\nu(R,t),\, \widehat{A}^{(s)}(f)] - \partial_\nu \widehat{A}^{(s)}(f)\Big\| \;\leq\; C_{N,\kappa}(s)\, R^{-N}\, \|\widehat{A}^{(s)}(f)\|_\kappa.$$

Letting $R \to \infty$ gives on $\mathcal{D}_{\text{flow}}$ $[P^{(s)}_\nu,\, \widehat{A}^{(s)}(f)] = i\,\partial_\nu \widehat{A}^{(s)}(f)$. By Nelson's commutator theorem (with H as control operator and the uniform energy bounds from Theorem 18.11), $P^{(s)}_\nu$ is essentially self–adjoint on $\mathcal{D}_{\text{flow}}$. The same argument applied to the densities $x_\mu T^{(s)}_{0\nu} - x_\nu T^{(s)}_{0\mu}$

gives the flowed rotation/boost generators and their implementer identity. This proves the "Charges at $s > 0$" bullet.

3) Flow removal and construction of $T_{\mu\nu}$. By Theorem 18.17 there exist real functions $Z_T(s), Z_\theta(s)$ such that

$$T_{\mu\nu}(f) \;=\; \lim_{s\downarrow 0} \Big\{ Z_T(s)\, U^{(s)}_{\mu\nu}(f) + Z_\theta(s)\, \eta_{\mu\nu} E^{(s)}(f) \Big\}$$

exists in matrix elements on $\mathcal{D}_{\rm flow}$, and $T_{\mu\nu}$ is symmetric, conserved, local, and Poincaré covariant. Define the localized charges

$$P_\nu(R,t) \;:=\; \int_{\mathbb{R}^3} d^3\mathbf{x}\; \chi_R(\mathbf{x})\, T_{0\nu}(t,\mathbf{x}), \qquad P_\nu \;:=\; \text{s-}\lim_{R\to\infty} P_\nu(R,t).$$

Exactly as at positive flow (now using Proposition 18.19), $P_\nu(R,t)$ converge in the strong–resolvent sense on $\mathcal{D}_{\rm flow}$ to a self–adjoint P_ν, independent of t and χ, and $[P_\nu, A(f)] = i\partial_\nu A(f)$ on $\mathcal{D}_{\rm flow}$ for every local $A(f)$. This proves existence and the implementer property in the third bullet.

4) Strong–resolvent limit $P_\nu^{(s)} \to P_\nu$ and fixing $\lim_{s\downarrow 0} Z_T(s) = 1$. Let $g_t \otimes h$ be an equal–time test with $g_t \equiv 1$ near t and h compactly supported, and set $Q_\nu^{(s)}(g_t \otimes h) := \int (g_t \otimes h)\, T^{(s)}_{0\nu}$. By Lemma 18.29,

$$\Big\| i[Q_\nu^{(s)}(g_t {\otimes} h),\, \widehat{A}(f)] \;-\; \partial_\nu \widehat{A}(f) \Big\| \;\le\; C_{N,\kappa}\, s^{N/2}\, \|\widehat{A}(f)\|_\kappa \qquad \text{on } \mathcal{D}_{\rm flow}. \tag{121}$$

Letting the spatial support of h tend to all space (as in the proof of Part 2) shows that

$$\lim_{R\to\infty} Q_\nu^{(s)}(g_t \otimes \chi_R) \;=\; P_\nu^{(s)} \quad \text{and} \quad \lim_{R\to\infty} T_{0\nu}(g_t \otimes \chi_R) \;=\; P_\nu$$

in the strong resolvent sense on $\mathcal{D}_{\rm flow}$. Using (121) and the matrix–element convergence in Theorem 18.17 we obtain, for every N,

$$\|\, (P_\nu^{(s)} - P_\nu)\, \Psi \| \;\le\; C_{N,\kappa}\, s^{N/2}\, \|(1+H)^\kappa \Psi\| \qquad (\Psi \in \mathcal{D}_{\rm flow}),$$

which implies $P_\nu^{(s)} \to P_\nu$ in the strong–resolvent sense on $\mathcal{D}_{\rm flow}$.

Now suppose, for contradiction, that $\lim_{s\downarrow 0} Z_T(s) = 1 + \delta$ with $\delta \neq 0$. Write, at fixed s,

$$T^{(s)}_{0\nu} \;=\; Z_T(s)\, T_{0\nu} \;+\; Z_\theta(s)\, \eta_{0\nu}\, E \;+\; \partial^\rho \Xi_{\rho 0\nu}(s,\cdot) \;+\; R^{(s)}_{0\nu},$$

with $R^{(s)}_{0\nu} = O(s^\varepsilon)$ in matrix elements and the improvement $\partial^\rho \Xi$ integrating to a boundary term. Smearing at equal time against χ_R and sending $R \to \infty$,

$$P_\nu^{(s)} \;=\; Z_T(s)\, P_\nu \;+\; Z_\theta(s)\, \delta_{0\nu} \int_{\mathbb{R}^3} E(t,\mathbf{x})\, d^3\mathbf{x} \;+\; o(1) \quad (s \downarrow 0),$$

where $o(1) \to 0$ strongly on $\mathcal{D}_{\rm flow}$ (remainder/improvement statements). The E–term does not contribute to the commutator with spatially localized equal–time fields (it is a scalar density), hence comparing the implementer identities on $\mathcal{D}_{\rm flow}$ yields

$$[P_\nu^{(s)}, \cdot] \underset{s\downarrow 0}{\longrightarrow} [P_\nu, \cdot] \qquad \Longrightarrow \qquad Z_T(s) \underset{s\downarrow 0}{\longrightarrow} 1.$$

If $Z_T(s) \to 1 + \delta \neq 1$, we would have $[P_\nu^{(s)}, \cdot] \to (1+\delta)[P_\nu, \cdot]$, contradicting the limit of the commutators. Therefore

$$\boxed{\lim_{s\downarrow 0} Z_T(s) = 1}.$$

5) Improvements are harmless. Any local improvement $\partial^\rho \Xi_{\rho\mu\nu}$ is a divergence of a local tensor antisymmetric in $\rho\mu$. Its equal–time spatial integral reduces to a boundary term on spheres of radius R, which vanishes as $R \to \infty$ by almost locality and clustering. Hence improvements neither affect the existence of the charges nor their commutators with local fields; in particular they play no role in the normalization fixed by the implementer condition.

Collecting the conclusions of Parts 1–5 proves all claims of Proposition 18.25.

□

OPE matching and trace coefficient. For the flowed stress tensor we use, in GI correlators with separated insertions,

$$T^{(s)}_{\mu\nu} \;=\; Z_T(s)\, T_{\mu\nu} \;+\; Z_\theta(s)\, \eta_{\mu\nu}\, \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}) \;+\; \partial^\rho \Xi_{\rho\mu\nu}(s,\cdot) \;+\; R^{(s)}_{\mu\nu}, \qquad R^{(s)}_{\mu\nu} = O(s^\varepsilon).$$

Lemma 18.26 (Trace matching)**.**

1. CS in step–scaling/GF. *In the GF scheme, the Callan–Symanzik equation* $\partial_{\ln s} \Sigma(u,s) = \beta_{\mathrm{GF}}(\Sigma)$ *with* $\beta_{\mathrm{GF}}(v) = -2b_0 v^2 + \cdots$ *implies that* $\mu\partial_\mu$*–variations of correlators are generated by insertions of the trace* $T^\mu{}_\mu$*.*
2. SFTE for $E^{(s)}$ and the Ward identity. *By the small–flow expansion,* $E^{(s)} = c_E(s)\, O_4 +$*higher, with* $O_4 := \mathrm{tr}(F^2)$*, and* $c_E(s)$ *analytic in* $\log s$*; the dilation/translation Ward identities give* $T^\mu{}_\mu = \frac{\beta(g)}{2g}\, O_4 + \partial\cdot(\textit{improvement})$ *once* $T_{\mu\nu}$ *is charge–normalized. Comparing with the flowed OPE for* $T^{(s)}_{\mu\nu}$ *forces* $\boxed{\lim_{s\downarrow 0} Z_\theta(s) = \beta(g)/(2g)}$ *(scheme–independent on the GI quotient).*

Proposition 18.27 (OPE matching for the stress tensor)**.** *Let* $T^{(s)}_{\mu\nu}$ *be the flowed, symmetric, conserved stress tensor constructed in this section. Then as* $s \downarrow 0$ *one has, in GI correlators with separated insertions,*

$$T^{(s)}_{\mu\nu}(x) \;=\; Z_T(s)\, T_{\mu\nu}(x) \;+\; Z_\theta(s)\, \eta_{\mu\nu}\, \mathrm{tr}\big(F_{\rho\sigma}F^{\rho\sigma}\big)(x) \;+\; \partial^\rho \Xi_{\rho\mu\nu}(s,x) \;+\; R_{N,\kappa}(s;x), \tag{122}$$

where $\Xi_{\rho\mu\nu}$ *is a local improvement term (antisymmetric in* $\rho\mu$*) and, for every* N*, matrix elements of* $R_{N,\kappa}$ *satisfy the bound* (120) *with* $X = T_{\mu\nu}$*. Moreover*

$$\lim_{s\downarrow 0} Z_T(s) \;=\; 1, \qquad \lim_{s\downarrow 0} Z_\theta(s) \;=\; \frac{\beta(g)}{2g}\,. \tag{123}$$

The overall normalization of $\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$ *follows the convention* $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$*; other conventions shift* Z_θ *by an obvious factor.*

Anomaly coefficient is scheme independent. Improvements $\partial^\rho \Xi_{\rho\mu\nu}$ are traceless up to total derivatives in GI correlators; once the charge normalization of $T_{\mu\nu}$ is fixed by Proposition 18.30, the coefficient of $\mathrm{tr}(F^2)$ in $T^\mu{}_\mu$ is scheme independent. Thus the limits in (123) are universal.

Proof. Apply Lemma 18.24 with $X = T_{\mu\nu}$. By symmetry, Poincaré covariance, gauge invariance and dimension ≤ 4, the only GI local tensors with the quantum numbers of $T_{\mu\nu}$ are $T_{\mu\nu}$ itself, $\eta_{\mu\nu}\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$, and total derivatives $\partial^\rho \Xi_{\rho\mu\nu}$. This proves (122).

Conservation of $T^{(s)}_{\mu\nu}$ and of $T_{\mu\nu}$ implies that the only possible nontrivial scalar admixture is $\eta_{\mu\nu}\mathrm{tr}(F^2)$; its coefficient can affect only the trace. Taking the trace of (122) and using that improvements are traceless up to total derivatives, we obtain in GI correlators

$$T^{(s)\,\mu}{}_\mu(x) \;=\; 4\, Z_\theta(s)\, \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})(x) \;+\; (\text{total derivatives}) \;+\; R_{N,\kappa}(s;x).$$

On the other hand, the BRST Ward identities together with scale breaking yield the Yang–Mills trace anomaly in GI correlators:

$$T^{\mu}{}_{\mu}(x) \;=\; \frac{\beta(g)}{2g}\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})(x).$$

Matching the coefficients of $\mathrm{tr}(F^2)$ in the $s\downarrow 0$ limit gives $\lim_{s\downarrow 0} Z_\theta(s) = \beta(g)/(2g)$. The limit $\lim_{s\downarrow 0} Z_T(s) = 1$ is fixed by the requirement that the Poincaré charges $P_\nu = \int \mathrm{d}^3x\, T^{(s)}_{0\nu}(t,\mathbf{x})$ (defined on the common Nelson core and then by closure) implement translations on the local fields with the standard commutation relations; any residual finite renormalization would violate this normalization. □

Theorem 18.28 (Trace anomaly as an operator identity modulo contact)**.** *Let $T_{\mu\nu}$ be the renormalized stress tensor constructed in Theorem 18.17, normalized so that its charges implement translations (Proposition 18.19). Then there exists a local operator–valued distribution Σ_ρ (a divergence of an improvement current) such that, as operator–valued distributions,*

$$T^{\mu}{}_{\mu} \;=\; \frac{\beta(g)}{2g}\,\mathrm{tr}\big(F_{\rho\sigma}F^{\rho\sigma}\big) \;+\; \partial^\rho \Sigma_\rho, \tag{124}$$

with the following precise insertion statement: for any test $\varphi \in C_c^\infty(\mathbb{R}^4)$ and any finite family of gauge–invariant (GI) point–local fields $[A_j](\phi_j)$ whose supports are a positive distance away from $\operatorname{supp}\varphi$*,*

$$\Big\langle \Omega,\ \Big(T^{\mu}{}_{\mu}(\varphi) \;-\; \tfrac{\beta(g)}{2g}\,\mathrm{tr}(F^2)(\varphi) \;-\; \partial^\rho\Sigma_\rho(\varphi)\Big)\ \prod_j [A_j](\phi_j)\,\Omega\Big\rangle \;=\; 0. \tag{125}$$

Equivalently, (124) *holds* modulo contact terms *supported on the coincident diagonals with the GI insertions. The coefficient $\beta(g)/(2g)$ is universal (scheme independent) once the charge normalization of $T_{\mu\nu}$ is fixed.*

Normalization reminder. *The identification of the coefficient follows from the flowed OPE/matching for $T^{(s)}_{\mu\nu}$, see Proposition 18.27, where* $\lim_{s\downarrow 0} Z_T(s) = 1$ *and* $\lim_{s\downarrow 0} Z_\theta(s) = \beta(g)/(2g)$*.*

Proof. By Theorem 18.17 there exist functions $Z_T(s), Z_\theta(s)$, with $Z_T(s)\to 1$ as $s\downarrow 0$, such that for any $f \in \mathcal{S}(\mathbb{R}^4)$

$$T_{\mu\nu}(f) \;=\; \lim_{s\downarrow 0}\ \Big\{ Z_T(s)\,\widetilde{U}^{(s)}_{\mu\nu}(f) \;+\; Z_\theta(s)\,\eta_{\mu\nu}\,\widetilde{E}^{(s)}(f) \Big\}$$

in matrix elements on the common core $\mathcal{D}_{\mathrm{poly}}$. Here $\widetilde{U}^{(s)}_{\mu\nu}$ is (by construction) traceless, so

$$T^{\mu}{}_{\mu}(f) \;=\; \lim_{s\downarrow 0}\ 4\,Z_\theta(s)\,\widetilde{E}^{(s)}(f) \qquad \text{in matrix elements on } \mathcal{D}_{\mathrm{poly}}. \tag{126}$$

Next invoke the GI small–flow–time/OPE matching for the stress tensor (Proposition 18.27): in GI correlators with separated insertions,

$$T^{(s)}_{\mu\nu}(x) \;=\; Z_T(s)\,T_{\mu\nu}(x) \;+\; Z_\theta(s)\,\eta_{\mu\nu}\,\mathrm{tr}(F^2)(x) \;+\; \partial^\rho \Xi_{\rho\mu\nu}(s,x) \;+\; R_{N,\kappa}(s;x),$$

where $R_{N,\kappa}$ is $O(s^{N/2})$ in matrix elements uniformly on bounded energy vectors, and $\Xi_{\rho\mu\nu}$ is a local improvement (antisymmetric in $\rho\mu$). Taking the trace and using tracelessness of $U^{(s)}_{\mu\nu}$ yields, as distributions in GI correlators,

$$T^{(s)\,\mu}{}_{\mu}(x) \;=\; 4\,Z_\theta(s)\,\mathrm{tr}(F^2)(x) \;+\; \partial^\rho \Lambda_\rho(s,x) \;+\; R^{\mathrm{tr}}_{N,\kappa}(s;x), \tag{127}$$

with $\Lambda_\rho(s,x) := \eta^{\mu\nu}\Xi_{\rho\mu\nu}(s,x)$ a local vector operator and $R^{\rm tr}_{N,\kappa}$ satisfying the same $O(s^{N/2})$ bound.

Let $\varphi \in C_c^\infty(\mathbb{R}^4)$ and let the GI insertions $[A_j](\phi_j)$ have supports disjoint from $\operatorname{supp}\varphi$. Smearing (127) with φ and integrating by parts,

$$T^{(s)\,\mu}{}_\mu(\varphi) \;-\; 4\,Z_\theta(s)\,\mathrm{tr}(F^2)(\varphi) \;=\; -\,\Lambda_\rho(s,\partial^\rho\varphi) \;+\; R^{\rm tr}_{N,\kappa}(s;\varphi).$$

By uniform moment/energy bounds at positive flow time and quasi–locality (Theorem 18.11) together with the disjoint–support hypothesis, all correlators in which $\Lambda_\rho(s,\partial^\rho\varphi)$ and $R^{\rm tr}_{N,\kappa}(s;\varphi)$ appear are uniformly bounded and dominated. Hence, taking expectation against $\prod_j[A_j](\phi_j)$ and using dominated convergence plus $R^{\rm tr}_{N,\kappa} = O(s^{N/2})$, we obtain

$$\lim_{s\downarrow 0}\,\Big\langle \Omega,\ \Big(T^{(s)\,\mu}{}_\mu(\varphi) - 4\,Z_\theta(s)\,\mathrm{tr}(F^2)(\varphi)\Big)\,\prod_j[A_j](\phi_j)\,\Omega\Big\rangle \;=\; -\,\lim_{s\downarrow 0}\,\langle \Omega,\ \Lambda_\rho(s,\partial^\rho\varphi)\,\prod_j[A_j](\phi_j)\,\Omega\rangle.$$

On the other hand, by (126) we also have

$$\lim_{s\downarrow 0}\,\langle\Omega,\ T^{(s)\,\mu}{}_\mu(\varphi)\,\prod_j[A_j](\phi_j)\,\Omega\rangle \;=\; \langle\Omega,\ T^{\mu}{}_\mu(\varphi)\,\prod_j[A_j](\phi_j)\,\Omega\rangle.$$

Combining the last two displays and using the anomaly matching $\lim_{s\downarrow 0} Z_\theta(s) = \beta(g)/(2g)$ from Proposition 18.27 gives

$$\Big\langle\Omega,\ \Big(T^{\mu}{}_\mu(\varphi) \;-\; \tfrac{\beta(g)}{2g}\,\mathrm{tr}(F^2)(\varphi)\Big)\,\prod_j[A_j](\phi_j)\,\Omega\Big\rangle \;=\; -\,\lim_{s\downarrow 0}\,\langle\Omega,\ \Lambda_\rho(s,\partial^\rho\varphi)\,\prod_j[A_j]\Omega\rangle.$$

Define the operator–valued distribution Σ_ρ by its action on tests $\psi \in C_c^\infty(\mathbb{R}^4)$ via the distributional limit

$$\Sigma_\rho(\psi) \;:=\; \text{w-}\lim_{s\downarrow 0}\ \Lambda_\rho(s,\psi),$$

which exists in matrix elements against GI spectators with disjoint support by the same domination (the family is Cauchy due to the SFTE with coefficients analytic in $\log(s\mu^2)$ and uniform energy bounds; cf. Lemma 18.24 and Theorem 16.13). With this definition and the arbitrariness of φ, we have established (125). Equivalently, (124) holds as an identity of distributions modulo contact terms (integration by parts moves the divergence onto φ and no boundary terms arise because of compact support and disjointness). The universality of the coefficient $\beta(g)/(2g)$ follows from Proposition 18.27 and the charge normalization of $T_{\mu\nu}$ (Theorem 18.17), which fixes $Z_T(s)\to 1$ and removes any residual finite renormalization. □

18.2.2 Canonical normalization of the stress–energy tensor via charges

We now fix the finite normalization of the stress tensor by requiring that its charges implement the given unitary representation U (Theorem 17.1) on the local fields.

Lemma 18.29 (Localized charges from the flowed tensor)**.** *Let $T^{(s)}_{\mu\nu}$ be the flowed conserved symmetric tensor constructed above. For $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\chi \equiv 1$ on a neighborhood of* $\operatorname{supp} f$*, define*

$$P^{(s)}_\nu[\chi] \;:=\; \int_{\mathbb{R}^3} T^{(s)}_{0\nu}(t,\mathbf{x})\,\chi(\mathbf{x})\,\mathrm{d}^3\mathbf{x}.$$

Then for any smeared local GI field $\widehat{A}(f)$ with $\operatorname{supp} f \subset \{t\}\times\mathbb{R}^3$ *and for every $N \in \mathbb{N}$ there exist κ and $C_{N,\kappa} < \infty$ such that, on the common core $\mathcal{D}_{\rm poly}$,*

$$\Big\|\,i\,[P^{(s)}_\nu[\chi],\,\widehat{A}(f)] - \partial_\nu\widehat{A}(f)\,\Big\| \;\le\; C_{N,\kappa}\,s^{N/2}\,\|\widehat{A}(f)\|_\kappa, \qquad \|\widehat{A}(f)\|_\kappa := \big\|(1{+}H)^\kappa\,\widehat{A}(f)\,(1{+}H)^\kappa\big\|. \tag{128}$$

In particular, $P^{(s)}_\nu[\chi] \to P_\nu$ in the strong resolvent sense on $\mathcal{D}_{\rm poly}$ as $s\downarrow 0$, where P_ν is the generator of translations from U.

Proof. Use conservation $\partial^\mu T^{(s)}_{\mu\nu} = 0$ and integrate by parts in the equal-time commutator with a space cutoff $\chi \equiv 1$ on $\operatorname{supp} f$, which eliminates surface terms (locality). Insert the OPE (122) for $T^{(s)}_{0\nu}$ near $\operatorname{supp} f$. The improvement term integrates to a boundary contribution which vanishes by the choice of χ. The remainder $R_{N,\kappa}$ is controlled by (120). The only surviving local piece is $Z_T(s)\, T_{0\nu}$, whose equal-time commutator with $\widehat{A}(f)$ is the standard one, $i[T_{0\nu}(t,\mathbf{x}), \widehat{A}(f)] = \partial_\nu \widehat{A}(f)$ on $\mathcal{D}_{\text{poly}}$. This yields (128) with an extra factor $|Z_T(s) - 1|$ in front of the leading term. Since $\lim_{s\downarrow 0} Z_T(s) = 1$ by Proposition 18.27, the right-hand side is $O(s^{N/2})$, and strong resolvent convergence follows from standard graph-norm estimates on $\mathcal{D}_{\text{poly}}$ and essential self-adjointness (Proposition 17.3). The constants $C_{N,\kappa}$ can be chosen independent of $s \in (0, s_0]$ by the uniform moment bounds and almost-locality at positive flow (Lemmas 18.12, 18.72). □

Proposition 18.30 (Uniqueness of finite normalization)**.** *Among all local, symmetric, conserved tensors that differ from $T^{(s)}_{\mu\nu}$ by finite local counterterms (linear combinations of $\eta_{\mu\nu}\operatorname{tr}(F_{\rho\sigma}F^{\rho\sigma})$ and improvements $\partial^\rho \Xi_{\rho\mu\nu}$), the choice fixed by*

$$\lim_{s\downarrow 0} \int_{\mathbb{R}^3} T^{(s)}_{0\nu}(t,\mathbf{x})\, \chi(\mathbf{x})\, \mathrm{d}^3\mathbf{x} \;=\; P_\nu \qquad (\forall\, \chi \equiv 1 \textit{ near the region of interest})$$

is unique. Equivalently, the limit condition forces $\lim_{s\downarrow 0} Z_T(s) = 1$ *in* (122)*, while the improvement freedom remains but does not affect the charges.*

Charge constraint. In particular, the localized-charge condition forces $\lim_{s\downarrow 0} Z_T(s) = 1$ in (122); improvements $\partial^\rho \Xi_{\rho\mu\nu}$ drop out of the charges by Gauss' law.

Proof. Suppose we changed $T^{(s)}_{\mu\nu}$ by $\delta Z_T(s)\, T_{\mu\nu} + \delta Z_\theta(s)\, \eta_{\mu\nu}\operatorname{tr}(F^2) + \partial^\rho \Delta\Xi_{\rho\mu\nu}(s)$. The integrated improvement term vanishes by Gauss/Stokes and the support choice for χ. If $\lim_{s\downarrow 0} \delta Z_T(s) = \delta \neq 0$, then the limiting charge would be $(1+\delta)P_\nu$, contradicting the fact that the translation generator is fixed by U. Hence $\lim_{s\downarrow 0} \delta Z_T(s) = 0$. The scalar admixture $\eta_{\mu\nu}\operatorname{tr}(F^2)$ cannot contribute to the spatial momenta ($\nu = i$) and would add a multiple of $\int \operatorname{tr}(F^2)$ to P_0; this would change the equal-time commutators with some local fields, again contradicting Lemma 18.29. Thus the normalization is unique modulo improvements, which leave the charges invariant. □

Rotation/boost charges and the Poincaré algebra

Define the (Euclidean) angular–momentum densities

$$J_{\lambda\mu\nu}(x) \;:=\; x_\mu\, T_{\lambda\nu}(x) \;-\; x_\nu\, T_{\lambda\mu}(x), \qquad J_{\mu\nu} := J_{0\mu\nu}.$$

Let $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\chi \equiv 1$ near the origin and set $\chi_R(\mathbf{x}) := \chi(\mathbf{x}/R)$.

Lemma 18.31 (Localized rotation/boost charges from the flowed tensor)**.** *Let $T^{(s)}_{\mu\nu}$ be the flowed, canonically normalized tensor (i.e. with $Z_T(s) \to 1$ as $s \downarrow 0$ by Proposition 18.30). Define*

$$J^{(s)}_{\mu\nu}[\chi_R] \;:=\; \int_{\mathbb{R}^3} d^3\mathbf{x}\; \chi_R(\mathbf{x}) \left(x_\mu\, T^{(s)}_{0\nu}(t,\mathbf{x}) \;-\; x_\nu\, T^{(s)}_{0\mu}(t,\mathbf{x}) \right).$$

For any smeared local GI field $\widehat{A}(f)$ with $\operatorname{supp} f \subset \{t\} \times \mathbb{R}^3$ *and any $N \in \mathbb{N}$ there exist κ and $C_{N,\kappa} < \infty$ such that, on the common core $\mathcal{D}_{\text{poly}}$,*

$$\left\| \, i\,[J^{(s)}_{\mu\nu}[\chi_R],\, \widehat{A}(f)] \;-\; (x_\mu \partial_\nu - x_\nu \partial_\mu)\widehat{A}(f) \, \right\| \;\le\; C_{N,\kappa} \left(R^{-1} + s^{N/2} \right) \|\widehat{A}(f)\|_\kappa. \tag{129}$$

In particular, $J^{(s)}_{\mu\nu}[\chi_R] \to J^{(s)}_{\mu\nu}$ strongly as $R \to \infty$ on $\mathcal{D}_{\text{poly}}$, and $J^{(s)}_{\mu\nu} \to M_{\mu\nu}$ in the strong resolvent sense as $s \downarrow 0$, where $M_{\mu\nu}$ implements rotations/boosts on local fields.

Proof. Integrate the conservation law $\partial^\lambda T^{(s)}_{\lambda\nu} = 0$ against $x_\mu \chi_R(\mathbf{x})$ and integrate by parts at equal time t. Boundary terms at $|\mathbf{x}| \sim R$ are controlled by almost locality and exponential clustering for flowed fields (Lemma 18.12, Theorem 18.115), giving the R^{-1} decay. Insert the flowed OPE

$$T^{(s)}_{\alpha\beta} \;=\; Z_T(s)\, T_{\alpha\beta} \;+\; Z_\theta(s)\, \eta_{\alpha\beta}\, \mathrm{tr}(F^2) \;+\; \partial^\rho \Xi_{\rho\alpha\beta}(s,\cdot) \;+\; R_{N,\kappa}(s;\cdot)$$

(Proposition 18.27) into the commutator with $\widehat{A}(f)$. The improvement term integrates to a boundary contribution that vanishes by the choice of χ_R. The remainder is bounded by $\|R_{N,\kappa}\| = O(s^{N/2})$. The only surviving leading piece is $Z_T(s)\, T_{0\nu}$, yielding the standard equal-time commutator with coefficient $Z_T(s)$; since $Z_T(s) \to 1$, (129) follows. Strong limits then follow by standard graph-norm estimates on $\mathcal{D}_{\mathrm{poly}}$ (cf. Lemma 18.29). □

Theorem 18.32 (Rotation/boost generators and the Poincaré algebra)**.** *Let P_μ be the translation generators from Theorem 18.17. The limits*

$$M_{\mu\nu} \;:=\; \text{s-}\lim_{R\to\infty}\, \lim_{s\downarrow 0}\, J^{(s)}_{\mu\nu}[\chi_R]$$

exist on $\mathcal{D}_{\mathrm{poly}}$, are essentially self-adjoint on this core, are independent of the time slice t, and satisfy

$$[M_{\mu\nu},\, A] \;=\; i\,(x_\mu\partial_\nu - x_\nu\partial_\mu)A \qquad \text{on } \;\mathcal{D}_{\mathrm{poly}}$$

for every local observable A. Moreover, on $\mathcal{D}_{\mathrm{poly}}$,

$$[P_\rho, M_{\mu\nu}] \;=\; i\,(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu), \qquad [M_{\mu\nu}, M_{\rho\sigma}] \;=\; i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}),$$

which becomes the usual Poincaré Lie algebra after OS reconstruction.

Proof. The equal-time commutator relation follows from Lemma 18.31 by first sending $R \to \infty$ then $s \downarrow 0$. Time-independence of the charges follows by differentiating w.r.t. t and using $\partial^\lambda T_{\lambda\nu} = 0$ in the sense of distributions, with boundary terms vanishing as above. The commutators with P_μ are computed by replacing one T with ∂ of the integrated density and integrating by parts; improvement terms drop out, and remainders vanish as $s \downarrow 0$. The $[M, M]$ algebra follows by iterating the same argument (or by the standard current algebra for the Noether densities with almost-local smearing). Essential self-adjointness on $\mathcal{D}_{\mathrm{poly}}$ holds by Nelson's commutator theorem (the bounds of Lemma 18.31 give the required graph norm estimates). □

Proposition 18.33 (Global rotation Ward identity)**.** *Let $X_1, \ldots, X_n$ be bounded functions of smeared point-local GI fields from $\mathfrak{A}(\mathcal{O})$ with supports strictly inside $\mathcal{O}$. Then for any antisymmetric $\omega^{\mu\nu}$,*

$$\begin{aligned}\sum_{k=1}^{n} \frac{d}{d\theta}\Big|_{\theta=0} \Big\langle \Omega,\, X_1 \cdots e^{\frac{i\theta}{2}\omega^{\mu\nu}M_{\mu\nu}}\, X_k\, e^{-\frac{i\theta}{2}\omega^{\mu\nu}M_{\mu\nu}} \cdots X_n\, \Omega \Big\rangle \\ = \; \frac{i}{2}\,\omega^{\mu\nu} \int d^4x\, \langle \Omega,\, \partial^\lambda J_{\lambda\mu\nu}(x)\, X_1 \cdots X_n\, \Omega \rangle \\ = \; 0.\end{aligned}$$

In particular, $[M_{\mu\nu}, X] = i(x_\mu\partial_\nu - x_\nu\partial_\mu)X$ on $\mathcal{D}_{\mathrm{poly}}$.

Proof. Identical to Proposition 18.18, using Lemma 18.31 in place of Lemma 18.29 and the fact that improvements are divergence-free modulo total derivatives in GI correlators. □

Corollary 18.34 (Trace anomaly)**.** *With the canonical normalization of $T_{\mu\nu}$ fixed by the charges,*

$$T^{\mu}{}_{\mu}(x) = \frac{\beta(g)}{2g}\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})(x) + \partial^{\mu}J_{\mu}(x),$$

where the divergence term is irrelevant in GI correlators at separated points.

Proof. Insert the small flow–time expansion of Proposition 18.27:

$$T^{(s)}_{\mu\nu} \;=\; Z_T(s)\,T_{\mu\nu} \;+\; Z_\theta(s)\,\eta_{\mu\nu}\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}) \;+\; \partial^{\rho}\Xi_{\rho\mu\nu}(s,\cdot) \;+\; R_{N,\kappa}(s;\cdot),$$

valid in GI correlators with separated insertions and with $\|R_{N,\kappa}\| = O(s^{N/2})$. Taking the trace and using that improvements are traceless up to total derivatives in GI correlators,

$$T^{(s)\,\mu}{}_{\mu} \;=\; 4\,Z_\theta(s)\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}) \;+\; \partial^{\mu}J^{(s)}_{\mu} \;+\; R^{\mathrm{tr}}_{N,\kappa}(s;\cdot).$$

By the charge normalization (Proposition 18.30), $\lim_{s\downarrow 0} Z_T(s) = 1$, while Proposition 18.27 yields $\lim_{s\downarrow 0} Z_\theta(s) = \beta(g)/(2g)$. Since $R^{\mathrm{tr}}_{N,\kappa}(s;\cdot) \to 0$ in matrix elements between vectors from the common Nelson core, $T^{(s)\,\mu}{}_{\mu} \to T^{\mu}{}_{\mu}$ in the distributional sense on GI correlators as $s \downarrow 0$. Absorbing the limit of the improvement trace into $\partial^{\mu}J_{\mu}$, we conclude

$$T^{\mu}{}_{\mu} \;=\; \frac{\beta(g)}{2g}\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}) \;+\; \partial^{\mu}J_{\mu}$$

in GI correlators at separated points. *This is precisely a corollary of Theorem 18.28, with the coefficient fixed by the OPE normalization* (122) *and* (123). ☐

18.2.3 YM short-distance identification of the GI sector

We now formulate the precise UV matching statement we will use subsequently.

Theorem 18.35 (YM short-distance identification in GI correlators)**.** *Let $\{\mathcal{O}_i\}_{i\in I}$ be a finite basis of renormalized GI local operators of canonical dimension ≤ 4 closed under Poincaré and discrete symmetries, containing $T_{\mu\nu}$ and $\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$. For each i define the flowed operator $\mathcal{O}_i^{(s)} := G_s * \mathcal{O}_i$ as in Lemma 18.24. Then, for any GI correlator with mutually separated insertions, one has the small flow–time expansion*

$$\mathcal{O}_i^{(s)}(x) \;=\; \sum_{j\in I} Z_{ij}(s)\,\mathcal{O}_j(x) \;+\; \partial^{\rho}\Upsilon^{(i)}_{\rho}(s,x) \;+\; R^{(i)}_{N,\kappa}(s;x), \tag{130}$$

where (i) the remainders obey the bound (120) *uniformly in the spectators; (ii) the coefficient matrix $Z(s) = (Z_{ij}(s))$ satisfies the RG equation*

$$\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + \beta(g)\frac{\mathrm{d}}{\mathrm{d}g} + \gamma^{T}\Big)\,Z(s) \;=\; 0,$$

with γ the anomalous-dimension matrix of the basis; (iii) $Z(s)$ is uniquely determined by the Ward identities of Section 18 together with the canonical normalization of $T_{\mu\nu}$ (Proposition 18.30) and the trace-anomaly matching (Proposition 18.27); in particular,

$$Z_{T\to T}(s) \;\xrightarrow[s\downarrow 0]{}\; 1, \qquad Z_{T\to\eta\,\mathrm{tr}(F^2)}(s) \;\xrightarrow[s\downarrow 0]{}\; \frac{\beta(g)}{2g}, \tag{131}$$

and coefficients multiplying BRST-exact operators vanish in GI correlators by Theorem 18.23. Moreover, when the YM β-function and anomalous dimensions are inserted (pure YM: asymptotically free), $Z(s)$ coincides to all orders in the formal weak-coupling expansion with the Wilson coefficient matrix of continuum YM at renormalization scale $\mu = (8s)^{-1/2}$.

Proof. Equation (130) with remainder control follows from Lemma 18.24 applied to each $\mathcal{O}_i$. The RG equation is the matrix form of the scalar equation in Lemma 18.24, using that the chosen basis closes under renormalization. The Ward identities (Poincaré, BRST, and the trace anomaly) impose linear constraints on $Z(s)$ which fix the components in (131). Proposition 18.30 eliminates any residual finite normalization ambiguity for $T_{\mu\nu}$, and Theorem 18.23 removes BRST-exact admixtures in GI correlators, yielding uniqueness of $Z(s)$ on the GI quotient. Finally, expanding the RG equation perturbatively at $\mu = (8s)^{-1/2}$ and solving with the same boundary/normalization conditions gives the YM Wilson coefficients order by order in $g(\mu)$; uniqueness of solutions to the first-order system ensures equality of the formal series. ☐

Remark 18.36. The improvement terms $\partial^\rho \Upsilon^{(i)}_\rho$ in (130) never affect integrated charges or on-shell scattering and can be fixed by conventional choices (e.g. Belinfante). The identification in Theorem 18.35 is precisely what we need to transport YM short-distance information (trace anomaly, operator mixings, UV dimensions) into the nonperturbative GI sector built earlier.

18.2.4 Associativity of the GI OPE from the SFTE

Theorem 18.37 (Associativity of the gauge–invariant OPE)**.** *Let $\{Q^{\mathrm{ren}}_\alpha\}_{\alpha\in\mathcal{B}}$ be a finite symmetry–closed basis of renormalized GI local operators of canonical dimension ≤ 4 as in Theorem 18.35. Define the (point–local) OPE inside GI correlators with separated insertions by*

$$Q^{\mathrm{ren}}_i(x)\, Q^{\mathrm{ren}}_j(y) \;\sim\; \sum_{n\in\mathcal{B}} C^n_{ij}(x-y;\mu)\, Q^{\mathrm{ren}}_n(y) \qquad (x \to y),$$

where "$\sim$" means equality when paired with any GI test configuration whose other insertions are a positive distance away from $\{x, y\}$. Then the Wilson coefficients satisfy, for hierarchical configurations $0 < |x-y| \ll |y-z|$,

$$\sum_{m\in\mathcal{B}} C^m_{ij}(x-y;\mu)\, C^n_{mk}(y-z;\mu) \;=\; \sum_{m\in\mathcal{B}} C^m_{jk}(y-z;\mu)\, C^n_{im}(x-y;\mu), \tag{132}$$

as an identity of distributions on the off–diagonal region $\{(x, y, z) : x \ne y \ne z\}$ in GI correlators. Coefficients multiplying BRST–exact operators vanish in GI correlators (Theorem 18.23). Moreover, $\{C^n_{ij}\}$ obey the Callan–Symanzik equation with anomalous–dimension matrix of the chosen basis (Theorem 18.35).

Proof. **1) Normalization functionals and notation.** Fix GI, $O(4)$–invariant linear functionals $\{\mathcal{N}_\alpha\}_{\alpha\in\mathcal{B}}$ supported in a small ball around the origin as in Definition 16.3, with

$$M \;:=\; \big(\mathcal{N}_\alpha(Q^{\mathrm{ren}}_\beta)\big)_{\alpha,\beta\in\mathcal{B}} \quad \text{invertible.}$$

Translate by y via $\mathcal{N}^{(y)}_\alpha(X) := \mathcal{N}_\alpha(\tau_{-y} X \tau_y)$. All pairings below are well–defined by temperedness/tightness (Theorem 13.3, Corollary 16.25) together with the off–diagonal bounds (Lemma 13.8, Proposition 13.9).

2) Nonperturbative definition of C^n_{ij}. For $x \ne y$, define the coefficient vector $\mathbf{C}_{ij}(x-y;\mu) = (C^n_{ij}(x-y;\mu))_{n\in\mathcal{B}}$ by the *projector equation*

$$\big(\mathcal{N}^{(y)}_\alpha\big(Q^{\mathrm{ren}}_i(x)\, Q^{\mathrm{ren}}_j(y)\big)\big)_{\alpha\in\mathcal{B}} \;=\; M\; \mathbf{C}_{ij}(x-y;\mu). \tag{133}$$

Since M is invertible, $\mathbf{C}_{ij}$ exists and is unique as a vector–valued distribution on $\{x \ne y\}$. Equation (133) is equivalent to the stated OPE in the sense of pairings with all $\mathcal{N}^{(y)}_\alpha$, therefore inside any GI correlator with other insertions kept away from $\{x, y\}$. By Theorem 18.23, BRST–exact operators are invisible in GI correlators, so coefficients are defined on the GI cohomology.

3) Associativity at the renormalized, point–local level. Consider $Q_i^{\rm ren}(x)Q_j^{\rm ren}(y)Q_k^{\rm ren}(z)$ with $x \neq y \neq z$ and apply $\mathcal{N}_\nu^{(z)}$ for arbitrary $\nu \in \mathcal{B}$. Using (133) twice and algebra associativity,

$$\begin{aligned}\mathcal{N}_\nu^{(z)}(Q_i(x)Q_j(y)Q_k(z)) &= \sum_m C_{ij}^m(x-y;\mu)\ \mathcal{N}_\nu^{(z)}(Q_m(y)Q_k(z))\\ &= \sum_m C_{ij}^m(x-y;\mu)\ \big(M\,\mathbf{C}_{mk}(y-z;\mu)\big)_\nu,\end{aligned}$$

and similarly

$$\mathcal{N}_\nu^{(z)}(Q_i(x)Q_j(y)Q_k(z)) = \sum_m C_{jk}^m(y-z;\mu)\ \big(M\,\mathbf{C}_{im}(x-y;\mu)\big)_\nu.$$

Subtracting and using that this holds for all ν gives, in vector form,

$$\sum_m C_{ij}^m(x-y;\mu)\,M\,\mathbf{C}_{mk}(y-z;\mu) \;=\; \sum_m C_{jk}^m(y-z;\mu)\,M\,\mathbf{C}_{im}(x-y;\mu).$$

Left–multiplying by M^{-1} yields (132). All distributions are tested off the diagonals, justified by the cited temperedness and off–diagonal control.

4) Compatibility with the SFTE and RG. Let $\mathcal{O}_i^{(s)} := G_s * Q_i^{\rm ren}$ be the flowed representatives (Lemma 18.24). For $s > 0$ in the SFTE window (Definition 16.21), Theorem 16.24 (together with Theorem 18.35) gives, in separated correlators,

$$\mathcal{O}_i^{(s)}(x) = \sum_{\alpha\in\mathcal{B}} Z_{i\alpha}(s,\mu)\,Q_\alpha^{\rm ren}(x;\mu) + R_i^{(s)}(x), \qquad \|R_i^{(s)}\| = O(s^\varepsilon),$$

with $Z(s,\mu)$ analytic in $\log(s\mu^2)$ and invertible on the GI quotient. Define flowed coefficients by the same projector prescription,

$$\big(\mathcal{N}_\alpha^{(y)}(\mathcal{O}_i^{(s)}(x)\,\mathcal{O}_j^{(s)}(y))\big)_\alpha \;=\; M\ \widetilde{\mathbf{C}}_{ij}(x-y;s,\mu).$$

Expanding $\mathcal{O}^{(s)}$ twice and using the $O(s^\varepsilon)$ off–diagonal bounds (Proposition 13.9),

$$\widetilde{\mathbf{C}}_{ij}(x-y;s,\mu) \;=\; Z(s,\mu)\,\mathbf{C}_{ij}(x-y;\mu) \;+\; O(s^\varepsilon),$$

uniformly on compact off–diagonal sets. The same algebraic argument as in Step 2 yields *exact* associativity for $\widetilde{C}$ at fixed $s > 0$; letting $s \downarrow 0$ in the SFTE window and using invertibility of $Z(s,\mu)$ on the GI quotient delivers (132). The Callan–Symanzik equation for $\mathbf{C}_{ij}$ follows from the RG for $Z(s,\mu)$ in Theorem 18.35. □

Remark 18.38. The proof uses only: (a) existence of a separating GI/$O(4)$–invariant family $\{\mathcal{N}_\alpha\}$ with invertible M (Definition 16.3); (b) associativity of the product on a common polynomial domain; (c) off–diagonal continuity/temperedness (Theorem 13.3, Lemma 13.8, Proposition 13.9); (d) SFTE reduction and YM UV identification (Lemma 18.24, Theorem 16.24, Theorem 18.35). Improvement terms contribute only contacts and do not affect (132) for separated insertions.

Lemma 18.39 (Flow preserves BRST conservation and almost locality in the extended algebra)**.** *Let $j_{\rm B}^\mu$ be the lattice BRST Noether current (ghosts included) and define its flowed version by convolution: $j_{\rm B}^{\mu,(s)} := G_s * j_{\rm B}^\mu$. Then $\partial_\mu j_{\rm B}^{\mu,(s)} = 0$ in the sense of operator–valued distributions, and the almost–locality bound of Lemma 18.12 holds verbatim with the graded commutator $[\,\cdot\,,\,\cdot\,]_{\rm gr}$ and with O_i replaced by arbitrary local composites in the extended (ghost) algebra with uniformly bounded engineering dimension.*

Proof. Conservation: $\partial_\mu j_{\mathrm{B}}^{\mu,(s)} = \partial_\mu(G_s * j_{\mathrm{B}}^\mu) = G_s * (\partial_\mu j_{\mathrm{B}}^\mu) = 0$. Almost locality: the proof of Lemma 18.12 only uses (i) the tail bound (118), (ii) off–diagonal graded commutator bounds for locals, and (iii) the flow/local–tail decomposition. These extend to the ghost sector with the graded commutator and the same dimension bookkeeping. □

Definition 18.40 (Localized flowed BRST charge)**.** Fix $s > 0$, $t \in \mathbb{R}$, and $\chi_R \in C_c^\infty(\mathbb{R}^3)$ with $\chi_R \equiv 1$ on $B_R(0)$ and $\|\partial^\alpha \chi_R\|_\infty \lesssim_\alpha R^{-|\alpha|}$. Set

$$Q_{\mathrm{B}}^{(s)}[\chi_R; t] \;:=\; \int_{\mathbb{R}^3} j_{\mathrm{B}}^{0,(s)}(t, \mathbf{x})\, \chi_R(\mathbf{x})\, \mathrm{d}^3\mathbf{x},$$

initially defined on the common polynomial core $\mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s)$ generated by flowed extended locals.

Proposition 18.41 (Implementer property, independence of cutoff, and closability)**.** *Let X be any local composite in the extended algebra and let $f \in \mathcal{S}(\mathbb{R}^4)$ have $\operatorname{supp} f \subset \{t\} \times \mathbb{R}^3$. Then, for every $N \in \mathbb{N}$, there exist κ and $C_{N,\kappa}(s) < \infty$ such that on $\mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s)$,*

$$\Big\| i\,[Q_{\mathrm{B}}^{(s)}[\chi_R; t],\, X^{(s)}(f)]_{\mathrm{gr}} \;-\; (sX)^{(s)}(f) \Big\| \;\le\; C_{N,\kappa}(s)\,(1+R)^{-N}\, \|(1+H)^\kappa X^{(s)}(f)(1+H)^\kappa\|. \tag{134}$$

Consequently, $\{Q_{\mathrm{B}}^{(s)}[\chi_R; t]\}_{R\to\infty}$ is a Cauchy net in the strong operator topology on $\mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s)$, with limit $Q_{\mathrm{B}}^{(s)}$ independent of χ_R and t. The operator $Q_{\mathrm{B}}^{(s)}$ is closable, $\mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s)$ is a core for its closure, and

$$i\,[Q_{\mathrm{B}}^{(s)},\, X^{(s)}(f)]_{\mathrm{gr}} \;=\; (sX)^{(s)}(f) \qquad \text{on } \mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s). \tag{135}$$

Moreover, $Q_{\mathrm{B}}^{(s)}\Omega = 0$ and, on $\mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s)$, $(Q_{\mathrm{B}}^{(s)})^2 = 0$.

Proof. Integrate the conservation law $\partial_\mu j_{\mathrm{B}}^{\mu,(s)} = 0$ against a spacetime test of the form $g_t \otimes \chi_R$ with $g_t \equiv 1$ near t and use graded locality to convert spatial derivatives to boundary terms supported where $\nabla\chi_R \neq 0$. Applying Lemma 18.39 yields the $(1+R)^{-N}$ decay of those boundary contributions. The local BRST Ward identity (Theorem 18.22(1) with the graded bracket) identifies the remaining contact term with $(sX)^{(s)}(f)$, giving (134). The Cauchy property and cutoff independence follow by taking $R \to \infty$. Closability is standard from (135) and the uniform energy bounds (Theorem 18.11(4)). The vacuum invariance $Q_{\mathrm{B}}^{(s)}\Omega = 0$ follows by testing the Ward identity with GI spectators and letting $R \to \infty$. Finally, on $\mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s)$, (135) and $s^2 = 0$ imply $-[Q_{\mathrm{B}}^{(s)}, [Q_{\mathrm{B}}^{(s)}, X^{(s)}(f)]_{\mathrm{gr}}]_{\mathrm{gr}} = (s^2X)^{(s)}(f) = 0$, and with $Q_{\mathrm{B}}^{(s)}\Omega = 0$ this yields $(Q_{\mathrm{B}}^{(s)})^2 = 0$ on the polynomial core. □

Corollary 18.42 (Operator–level Ward/ST identities at fixed flow)**.** *At $s > 0$, on $\mathcal{D}_{\mathrm{poly}}^{\mathrm{ext}}(s)$, the graded commutator with $Q_{\mathrm{B}}^{(s)}$ implements the BRST differential as in* (135)*. In particular, insertions of BRST–exact flowed locals vanish against GI spectators away from contact, and the STI for the flowed 1PI functional holds with the usual antifield sources. Upon passing to $s \downarrow 0$ via the FPR of Theorem 16.13, these reduce to the expectation–level Ward/ST identities of Theorem 18.23 and Proposition 18.60.*

18.3 Scalar (0^{++}) channel: canonical interpolator, θ–$\mathrm{tr}(F^2)$ matching, and spectral sum rule

Set $\theta := T^\mu{}_\mu$. By Corollary 18.34 we have, in gauge-invariant (GI) correlators,

$$\theta(x) \;=\; \frac{\beta(g)}{2g}\, \mathrm{tr}\big(F_{\rho\sigma}F^{\rho\sigma}\big)(x). \tag{136}$$

18.3.1 Canonical 0++ interpolating field and LSZ residue

Let $\mathcal{H}_1$ be the one-particle space for mass $m_\star$ from Theorem 17.20 and let $\mathcal{H}_1^{(0^{++})}$ denote its scalar, positive-parity, charge-conjugation even subspace (possibly trivial).

Lemma 18.43 (Covariant one-particle form factor of $T_{\mu\nu}$)**.** *If $\mathcal{H}_1^{(0^{++})} \neq \{0\}$, then for any normalized $\psi \in \mathcal{H}_1^{(0^{++})}$ with momentum p,*

$$\langle \Omega,\, T_{\mu\nu}(0)\, \psi(p) \rangle \;=\; f_\theta\, p_\mu p_\nu, \qquad \langle \Omega,\, \theta(0)\, \psi(p) \rangle \;=\; f_\theta\, m_\star^2,$$

for a constant $f_\theta \in \mathbb{R}$ (the scalar gravitational form factor). For non-scalar spins, the vacuum–one-particle matrix element of $T_{\mu\nu}$ vanishes by covariance and parity.

Proof. Wigner covariance and conservation ($\partial^\mu T_{\mu\nu} = 0$) imply that a vacuum–one-particle matrix element must be a symmetric tensor built from p_μ; Lorentz and parity invariance force the structure $A\, p_\mu p_\nu$. Taking the trace gives the second relation. For non-scalar spins, there is no invariant vector, hence the matrix element must vanish (Schur's lemma). □

Proposition 18.44 (Canonical 0^{++} interpolator and LSZ normalization)**.** *Assume $\mathcal{H}_1^{(0^{++})} \neq \{0\}$. Fix a small flow time $s > 0$ and define*

$$\mathcal{S}^{(s)}(x) \;:=\; \mathrm{tr}\big(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}\big)(x), \qquad \Phi^{(s)}_{0^{++}}(x) \;:=\; c_s\, \mathcal{S}^{(s)}(x),$$

with $c_s \in \mathbb{R}$ chosen so that the Källén–Lehmann residue of the two-point function of $\Phi^{(s)}_{0^{++}}$ at $p^2 = m_\star^2$ equals $+1$. Then the Haag–Ruelle creation operators built from $\Phi^{(s)}_{0^{++}}$ produce asymptotic one-particle states in $\mathcal{H}_1^{(0^{++})}$ with canonical *LSZ normalization (unit residue); the resulting in/out scalar asymptotic fields are independent of s and c_s (once normalized), and differ by at most a phase from those constructed with θ.*

Proof. Small flow–time expansion and Theorem 18.35 imply that $\mathcal{S}^{(s)} = Z_{SS}(s)\,\mathrm{tr}(F^2) + \partial \cdot (\cdots) +$ remainder with remainder controlled as in (120). The HR limits (Theorem 17.29) are insensitive to total derivatives and $O(s^{N/2})$ remainders. Adjust c_s to normalize the residue to 1. Canonical HR/LSZ theory then yields asymptotic fields with the standard single-particle normalization; uniqueness up to phase follows from the equivalence of interpolating fields with the same one-particle residue. □

Corollary 18.45 (θ–$\mathrm{tr}(F^2)$ matching on the one-particle shell)**.** *On $\mathcal{H}_1^{(0^{++})}$ one has*

$$P_1^{(0^{++})}\, \theta(f)\, \Omega \;=\; \frac{\beta(g)}{2g}\, P_1^{(0^{++})}\, \mathrm{tr}(F^2)(f)\, \Omega,$$

for any test function f, where $P_1^{(0^{++})}$ is the spectral projection onto the scalar one-particle shell. In particular, θ and $\mathrm{tr}(F^2)$ define equivalent scalar interpolators up to the anomaly factor $\beta(g)/(2g)$.

Proof. Take the vacuum–one-particle matrix elements of (136). Improvement terms vanish after projection to $\mathcal{H}_1^{(0^{++})}$; flowed representatives converge by Lemma 18.24. The statement follows by density of one-particle wave packets. □

18.3.2 Spectral representation and anomaly sum rule in the scalar channel

Define the connected Wightman two-point function of θ,

$$W_\theta(x) \;:=\; \langle \Omega,\, \theta(x)\, \theta(0)\, \Omega\rangle^{\rm conn},$$

and its (tempered) Fourier transform $\widehat{W}_\theta(p)$. By reflection positivity and OS reconstruction (Theorem 17.1) there exists a positive measure ρ_θ on $[0,\infty)$ such that

$$\widehat{W}_\theta(p) \;=\; \int_0^\infty \rho_\theta(\sigma)\, \delta(p^2-\sigma)\, \theta(p^0)\, \mathrm{d}\sigma, \qquad \rho_\theta(\sigma) \;\geq\; 0. \tag{137}$$

If a mass gap $m_\star > 0$ exists (Theorem 17.19), then $\operatorname{supp}\rho_\theta \subset [m_\theta^2,\infty)$ with $m_\theta \geq m_\star$, and $m_\theta = m_\star$ iff ρ_θ has an atom at $m_\star^2$.

Proposition 18.46 (Anomaly sum rule at zero momentum)**.** *Assume the subtracted Euclidean correlator of θ is integrable at long distances (which holds under the mass gap and exponential clustering). Then*

$$\int_0^\infty \frac{\rho_\theta(\sigma)}{\sigma}\, \mathrm{d}\sigma \;=\; -\,4\, \langle \Omega,\, \theta(0)\, \Omega\rangle\,, \tag{138}$$

where the right-hand side equals -16 times the vacuum energy density in our convention. Moreover, using (136) *one can rewrite the left-hand side as $\big(\frac{\beta(g)}{2g}\big)^2$ times the corresponding moment of the* $\mathrm{tr}(F^2)$ *spectral density in GI correlators.*
With Minkowski signature $(+,-,-,-)$ and a Lorentz-invariant vacuum with pressure $p=-\varepsilon_{\rm vac}$, one has $\langle\theta\rangle = 4\,\varepsilon_{\rm vac}$; hence (138) reads $\int_0^\infty \rho_\theta(\sigma)\,\sigma^{-1}\,\mathrm{d}\sigma = -16\,\varepsilon_{\rm vac}$.

Proof. Let $G_\theta(x) := \langle \Omega, \theta(x)\theta(0)\Omega\rangle^{\rm conn}$ in Euclidean signature and let $\widehat{G}_\theta(p)$ be its Fourier transform. By reflection positivity and OS reconstruction (Theorem 17.1), there exists a positive spectral measure ρ_θ such that, up to local contact polynomials supported at $x=0$,

$$\widehat{G}_\theta(p_E) \;=\; \int_0^\infty \frac{\rho_\theta(\sigma)}{p_E^2+\sigma}\, \mathrm{d}\sigma,$$

whence at zero momentum

$$\widehat{G}_\theta(0) \;=\; \int_0^\infty \frac{\rho_\theta(\sigma)}{\sigma}\, \mathrm{d}\sigma, \tag{139}$$

with the understanding that the constant (contact) term has been subtracted; this subtraction is uniquely fixed by our normalization of $T_{\mu\nu}$ and the GI Ward identities (Proposition 18.30, Theorem 18.23, Corollary 18.62). Exponential clustering (Proposition 17.24) and the mass gap (Theorem 17.19) ensure integrability of $G_\theta(x)$ at large $|x|$.
Weyl Ward identity. Consider a uniform Euclidean Weyl rescaling $g_{\mu\nu} \mapsto g^\lambda_{\mu\nu} := e^{2\lambda} g_{\mu\nu}$ with $\lambda\in\mathbb{R}$. By the variational definition of $T_{\mu\nu}$ (Theorem 18.17) and the GI Ward identities, for any local GI observable X one has

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} \langle X\rangle_{g^\lambda} \;=\; -\int_{\mathbb{R}^4} \langle \theta(x)\, X(0)\rangle^{\rm conn}\, \mathrm{d}x, \tag{140}$$

where the right-hand side is the connected distribution with the same subtraction of local contacts as in (139). Apply (140) with $X=\theta(0)$. On the other hand, θ is the trace of the improved, conserved stress tensor with charge normalization fixed in Proposition 18.30; hence under a *global* Weyl rescaling it has Weyl weight $+4$ and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} \langle \theta(0)\rangle_{g^\lambda} \;=\; 4\, \langle\theta(0)\rangle, \tag{141}$$

while total-derivative (improvement) terms do not contribute in GI correlators (Corollary 18.62). Combining (140)–(141) yields the coordinate-space sum rule

$$\int_{\mathbb{R}^4} G_\theta(x)\,\mathrm{d}x \;=\; -\,4\,\langle\theta(0)\rangle. \tag{142}$$

From position to spectral variables. By definition of the Fourier transform at $p_E = 0$, the left-hand side of (142) equals $\widehat{G}_\theta(0)$ with the same contact subtraction. Using (139) we obtain

$$\int_0^\infty \frac{\rho_\theta(\sigma)}{\sigma}\,\mathrm{d}\sigma \;=\; -\,4\,\langle\Omega,\theta(0)\Omega\rangle,$$

which is (138). Finally, (136) (Corollary 18.34) gives the stated rewriting of the left-hand side as $\big(\frac{\beta(g)}{2g}\big)^2$ times the corresponding moment of the $\mathrm{tr}(F^2)$ spectral density in GI correlators. □

Remark 18.47. Equation (138) and $\rho_\theta \geq 0$ imply that the left-hand side is strictly positive whenever $\langle\Omega,\theta\,\Omega\rangle < 0$ (negative vacuum energy density), hence *some* scalar spectral weight must occur. If $\mathcal{H}_1^{(0^{++})} \neq \{0\}$, the $(\sigma = m_\star^2)$ contribution is precisely the one-particle residue $|\,\langle\Omega,\theta(0)\,\psi\rangle\,|^2$ integrated over the mass shell; by Corollary 18.45 this is nonzero iff $\mathrm{tr}(F^2)$ has nonzero one-particle overlap in the scalar channel. Thus the anomaly enforces scalar strength in the IR and ties its normalization to $\beta(g)$.

18.4 Scalar-channel effective-mass and Laplace bounds; two-sided bracket for m_θ

Let $\theta = T^\mu{}_\mu$ and define the *flowed* connected Euclidean-time correlator at zero spatial separation

$$S_\theta^{(s)}(\tau) \;:=\; \langle\Omega,\; \theta^{(s)}(\tau,0)\,\theta^{(s)}(0,0)\,\Omega\rangle^{\mathrm{conn}} \qquad (\tau \geq 0), \tag{143}$$

where $\theta^{(s)}$ is the flowed representative fixed in Proposition 18.30. By the small flow–time expansion (Lemma 18.24) and exponential clustering (Proposition 17.24), $S_\theta^{(s)}(\tau)$ is finite for all $\tau \geq 0$, strictly positive for $\tau > 0$, and has the same large-τ decay rate as the unflowed correlator.

Definition 18.48 (Effective mass)**.** For $\tau > 0$ set

$$m_{\mathrm{eff}}^{(s)}(\tau) \;:=\; -\frac{\mathrm{d}}{\mathrm{d}\tau}\log S_\theta^{(s)}(\tau), \qquad m_{\mathrm{eff}}^{(s)}(\tau;\Delta) \;:=\; \frac{1}{\Delta}\log\frac{S_\theta^{(s)}(\tau)}{S_\theta^{(s)}(\tau+\Delta)} \quad (\Delta > 0).$$

Lemma 18.49 (Complete monotonicity and log-convexity)**.** *There exists a positive measure $\nu_\theta^{(s)}$ on $[m_\theta,\infty)$ such that*

$$S_\theta^{(s)}(\tau) \;=\; \int_{m_\theta}^\infty e^{-E\,\tau}\,\mathrm{d}\nu_\theta^{(s)}(E)\,, \tag{144}$$

with $\mathrm{supp}\,\nu_\theta^{(s)} \subset [m_\theta,\infty)$ *and* $m_\theta \geq m_\star$ *(the spectral gap from Theorem 17.19). Hence* $(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} S_\theta^{(s)}(\tau) \geq 0$ *for all* $n \in \mathbb{N}$ *and* $\tau > 0$*, and* $S_\theta^{(s)}$ *is log-convex. Moreover*

$$\lim_{\tau\to\infty} m_{\mathrm{eff}}^{(s)}(\tau) \;=\; m_\theta, \qquad m_{\mathrm{eff}}^{(s)}(\tau;\Delta) \;\searrow\; m_\theta \quad \textit{as} \quad \tau\to\infty \quad (\Delta \textit{ fixed}).$$

Proof. By the spectral theorem,

$$S_\theta^{(s)}(\tau) \;=\; \langle\Omega,\; \theta^{(s)}\,e^{-H\tau}\,\theta^{(s)}\,\Omega\rangle_{\mathrm{conn}} \;=\; \int_{[0,\infty)} e^{-E\tau}\,\mathrm{d}\langle\Omega,\; \theta^{(s)}\,E(\mathrm{d}E)\,\theta^{(s)}\,\Omega\rangle,$$

which yields (144) with a positive measure supported in $[m_\theta,\infty)$ (the connected projection removes the vacuum piece). Complete monotonicity and log-convexity are standard for Laplace transforms of positive measures, and the limit of the logarithmic derivative equals the infimum of the support. □

In addition, for fixed $\Delta > 0$, the discrete effective mass $m^{(s)}_{\mathrm{eff}}(\tau;\Delta)$ is a decreasing function of τ.

Proposition 18.50 (Two-sided bracket and practical upper bounds for m_θ)**.** *For all $\tau > 0$ and $\Delta > 0$,*

$$m_\star \;\le\; m_\theta \;\le\; m^{(s)}_{\mathrm{eff}}(\tau) \;\le\; m^{(s)}_{\mathrm{eff}}(\tau;\Delta), \tag{145}$$

and the following additional (computable) bounds hold:

$$m_\theta \;\le\; \inf_{\tau>0} m^{(s)}_{\mathrm{eff}}(\tau), \tag{146}$$

$$m_\theta \;\le\; \inf_{\tau>0} \frac{S^{(s)}_\theta(\tau)}{\displaystyle\int_\tau^\infty S^{(s)}_\theta(t)\,\mathrm{d}t}. \tag{147}$$

and, writing $K_\theta := \int_0^\infty S^{(s)}_\theta(t)\,\mathrm{d}t$ (well-defined at positive flow $s > 0$),

$$K_\theta \;=\; \int_{m_\theta}^\infty \frac{1}{E}\,\mathrm{d}\nu^{(s)}_\theta(E). \tag{148}$$

For the fully space–time integrated *connected Euclidean correlator one has*

$$\int_{\mathbb{R}^4} G_\theta(x)\,\mathrm{d}^4x \;=\; \widehat{G}_\theta(0) \;=\; \int_0^\infty \frac{\rho_\theta(\sigma)}{\sigma}\,\mathrm{d}\sigma \;=\; -4\,\langle\Omega,\theta(0)\Omega\rangle,$$

as stated in Proposition 18.46. (The last identity involves also the spatial integration; it is not *identical to K_θ, which integrates over Euclidean time only at fixed spatial point.)*

Proof. The lower bound $\mu \le m_\theta$ follows from Theorem 17.19. For the first upper bound, using (144) and $\operatorname{supp}\nu^{(s)}_\theta \subset [m_\theta,\infty)$,

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\log S^{(s)}_\theta(\tau) = \frac{\int E\,e^{-E\tau}\,\mathrm{d}\nu}{\int e^{-E\tau}\,\mathrm{d}\nu} \;\ge\; m_\theta.$$

The discrete bound is the same argument with the ratio $\frac{S(\tau)}{S(\tau+\Delta)} = \frac{\int e^{-E\tau}\,\mathrm{d}\nu}{\int e^{-E(\tau+\Delta)}\,\mathrm{d}\nu}$ and monotonicity of $E \mapsto e^{E\Delta}$. For (146) take the infimum in τ. For (147), for $t \ge \tau$ we have $S(t) = \int e^{-E(t-\tau)}e^{-E\tau}\,\mathrm{d}\nu \le e^{-m_\theta(t-\tau)}S(\tau)$, hence $\int_\tau^\infty S(t)\,\mathrm{d}t \le S(\tau)/m_\theta$, i.e. $m_\theta \le S(\tau)/\int_\tau^\infty S(t)\,\mathrm{d}t$. Finally, Fubini with $\int_0^\infty e^{-Et}\,\mathrm{d}t = 1/E$ gives $K_\theta = \int(1/E)\,\mathrm{d}\nu$, and the anomaly sum rule relates it to $-2\langle\Omega,\theta\Omega\rangle$ as stated. ☐

Remark 18.51 (Flow-stability of bounds). By Lemma 18.24, for each fixed $\tau_0 > 0$ there exists $N \in \mathbb{N}$ and $C_{\tau_0} < \infty$ such that

$$\sup_{\tau\ge\tau_0} \left|\,S^{(s)}_\theta(\tau) - S^{(0)}_\theta(\tau)\,\right| \;\le\; C_{\tau_0}\,s^{N/2}.$$

Consequently, $m^{(s)}_{\mathrm{eff}}(\tau)$, the tail ratio in (147), and the integral K_θ are all $O(s^{N/2})$-close (uniformly for $\tau \ge \tau_0$) to their unflowed counterparts. Thus the bounds are insensitive to the auxiliary flow regulator.

Corollary 18.52 (Operational bracket for the lightest scalar)**.** *Combining Theorem 17.19 with Proposition 18.50,*

$$\boxed{\;m_\star \;\le\; m_\theta \;\le\; \inf_{\tau>0,\ \Delta>0} m^{(s)}_{\mathrm{eff}}(\tau;\Delta)\;},$$

with equality throughout if and only if the scalar spectral measure consists of a single mass shell. The anomaly identity (148) *(as corrected below) provides a cross–check on $S^{(s)}_\theta$.*

Proof. By Theorem 17.19, the scalar threshold obeys $\mu \le m_\theta$. Proposition 18.50 yields, for all $\tau > 0$ and $\Delta > 0$,

$$m_\theta \;\le\; m_{\text{eff}}^{(s)}(\tau) \;\le\; m_{\text{eff}}^{(s)}(\tau;\Delta).$$

Taking the infimum over τ and Δ gives the displayed bracket

$$\mu \;\le\; m_\theta \;\le\; \inf_{\tau>0,\ \Delta>0} m_{\text{eff}}^{(s)}(\tau;\Delta).$$

If the scalar spectral measure is a single mass shell, $\rho_\theta(\sigma) = Z\,\delta(\sigma - m_\theta^2)$, then $S_\theta^{(s)}(\tau)$ is a pure exponential and all inequalities are equalities. Conversely, if equality holds throughout, the monotonicity and log-convexity from Lemma 18.49 force $m_{\text{eff}}^{(s)}(\tau)$ to be constant in τ, which is only possible for a pure exponential, i.e. for a single shell. The identity (148) provides the stated consistency check. □

18.5 Spin-2 (tensor) channel: traceless-symmetric projection, positivity, and bounds

Write the spatial components of the flowed stress–energy tensor as $T_{ij}^{(s)}$ $(i,j = 1,2,3)$ and the flowed trace as $\theta^{(s)} := T^{(s)\mu}{}_{\mu}$, with the normalization fixed in Proposition 18.30. Define the *traceless-symmetric* representative

$$\mathsf{T}_{ij}^{(s)} \;:=\; T_{ij}^{(s)} \;-\; \frac{1}{3}\,\delta_{ij}\,\theta^{(s)}\,, \qquad \mathsf{T}_{ij}^{(s)} = \mathsf{T}_{ji}^{(s)}\,,\quad \delta_{ij}\mathsf{T}_{ij}^{(s)} = 0. \tag{149}$$

Let $P_{ij,kl}^{(2)}$ denote the standard projector onto symmetric traceless rank-2 tensors in $\mathbb{R}^3$,

$$P_{ij,kl}^{(2)} \;:=\; \frac{1}{2}\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big) \;-\; \frac{1}{3}\,\delta_{ij}\delta_{kl}. \tag{150}$$

Equivalently, choose any orthonormal basis $\{e_{ij}^{(a)}\}_{a=1}^{5}$ of the $J = 2$ subspace (symmetric traceless 3×3 matrices) and note

$$P_{ij,kl}^{(2)} \;=\; \sum_{a=1}^{5} e_{ij}^{(a)}\,e_{kl}^{(a)}\,. \tag{151}$$

Spin-2 Euclidean correlator. Define the flowed connected Euclidean-time correlator at zero spatial separation by

$$S_2^{(s)}(\tau) \;:=\; P_{ij,kl}^{(2)}\,\langle\Omega,\ \mathsf{T}_{ij}^{(s)}(\tau,0)\ \mathsf{T}_{kl}^{(s)}(0,0)\ \Omega\rangle^{\text{conn}} \qquad (\tau \ge 0). \tag{152}$$

By (151) and reflection positivity, $S_2^{(s)}(\tau) = \sum_{a=1}^{5}\langle\Omega,\ \mathcal{O}^{(a)}(\tau)\,\mathcal{O}^{(a)}(0)\,\Omega\rangle$ with $\mathcal{O}^{(a)} := e_{ij}^{(a)}\,\mathsf{T}_{ij}^{(s)}$, hence $S_2^{(s)}(\tau) > 0$ for $\tau > 0$. The small flow–time expansion (Lemma 18.24) and exponential clustering (Proposition 17.24) guarantee finiteness for all $\tau \ge 0$ and that the large-τ decay rate is flow-independent.

Lemma 18.53 (Spectral/Laplace representation and complete monotonicity)**.** *There exists a positive measure $\nu_2^{(s)}$ on $[m_2,\infty)$ (with $m_2 \ge m_\star$) such that*

$$S_2^{(s)}(\tau) \;=\; \int_{m_2}^{\infty} e^{-E\,\tau}\,\mathrm{d}\nu_2^{(s)}(E)\,, \tag{153}$$

hence $(-1)^n\,\partial_\tau^n S_2^{(s)}(\tau) \ge 0$ for all $n \in \mathbb{N}$ and $\tau > 0$ (complete monotonicity), and $S_2^{(s)}$ is log-convex. Moreover,

$$\lim_{\tau\to\infty}\Big(-\frac{\mathrm{d}}{\mathrm{d}\tau}\log S_2^{(s)}(\tau)\Big) \;=\; \inf\operatorname{supp}\nu_2^{(s)} \;=:\; m_2.$$

Proof. Using (151) and OS reconstruction, for each a we have the standard spectral decomposition

$$\langle\Omega,\ \mathcal{O}^{(a)}(\tau)\,\mathcal{O}^{(a)}(0)\,\Omega\rangle \;=\; \sum_n |\langle n, \mathcal{O}^{(a)}\Omega\rangle|^2\, e^{-E_n\tau}$$

with $E_n \geq \mu$ by Theorem 17.19. Summing over a produces (153) with a positive measure supported in $[\mu, \infty)$. The remaining statements are standard properties of Laplace transforms of positive measures. □

Definition 18.54 (Spin-2 effective mass)**.** For $\tau > 0$ and $\Delta > 0$ set

$$m^{(s)}_{\mathrm{eff},2}(\tau) \;:=\; -\frac{\mathrm{d}}{\mathrm{d}\tau}\log S^{(s)}_2(\tau), \qquad m^{(s)}_{\mathrm{eff},2}(\tau;\Delta) \;:=\; \frac{1}{\Delta}\,\log\frac{S^{(s)}_2(\tau)}{S^{(s)}_2(\tau+\Delta)}.$$

Proposition 18.55 (Two-sided bracket and practical bounds for m_2)**.** *For all $\tau > 0$ and $\Delta > 0$,*

$$m_\star \;\leq\; m_2 \;\leq\; m^{(s)}_{\mathrm{eff},2}(\tau) \;\leq\; m^{(s)}_{\mathrm{eff},2}(\tau;\Delta), \tag{154}$$

and

$$m_2 \;\leq\; \inf_{\tau>0} m^{(s)}_{\mathrm{eff},2}(\tau), \tag{155}$$

$$m_2 \;\leq\; \inf_{\tau>0} \frac{S^{(s)}_2(\tau)}{\displaystyle\int_\tau^\infty S^{(s)}_2(t)\,\mathrm{d}t}. \tag{156}$$

Moreover, for any fixed $\tau_0 > 0$ there exist $N \in \mathbb{N}$ and $C_{\tau_0} < \infty$ such that

$$\sup_{\tau\geq\tau_0} \left| m^{(s)}_{\mathrm{eff},2}(\tau) - m^{(0)}_{\mathrm{eff},2}(\tau)\right| \;\leq\; C_{\tau_0}\, s^{N/2},$$

and similarly for the discrete and tail–ratio versions; hence the bounds are flow-stable.

Proof. The lower bound $\mu \leq m_2$ follows from the spectral gap. The inequalities in (154) and (155) are immediate from (153) (Jensen/monotonicity for Laplace averages). For (156) use $S^{(s)}_2(t) \leq e^{-m_2(t-\tau)} S^{(s)}_2(\tau)$ for $t \geq \tau$ and integrate in t. Flow-stability follows from the small flow–time expansion and energy bounds (Lemma 18.24 and Proposition 17.24), which control the difference $S^{(s)}_2 - S^{(0)}_2$ uniformly on $[\tau_0, \infty)$ and hence the induced differences of logarithmic derivatives. □

Remark 18.56 (Independence from improvements and trace mixing). Any improvement of $T_{\mu\nu}$ by derivatives of a local operator adds to T_{ij} a combination of total derivatives and multiples of $\delta_{ij}\,\theta$. The projector $P^{(2)}$ eliminates the trace, and total derivatives contribute only contact terms to $S^{(s)}_2(\tau)$, which are smoothed by the flow and irrelevant for large τ. Thus m_2 and the bounds above are insensitive to the improvement freedom in $T_{\mu\nu}$.

Theorem 18.57 (Nonzero spin-2 one-particle residue (variationally and flow-stably))**.** *Fix $s_0 > 0$ and let $\mathcal{O}^{(a)} := e^{(a)}_{ij}\,\mathsf{T}^{(s_0)}_{ij}$ as above. For a smooth spatial smearing $\eta \in C^\infty_c(\mathbb{R}^3)$ (with unit integral and support $\ll \sqrt{s_0}$), consider the 5×5 correlator matrix*

$$C_{ab}(\tau) \;:=\; \langle\Omega,\ \mathcal{O}^{(a)}(\tau,\mathbf{0})[\eta]\ \mathcal{O}^{(b)}(0,\mathbf{0})[\eta]\ \Omega\rangle \qquad (\tau \geq 0),$$

and the generalized eigenvalue problem $C(\tau)\,v = \lambda(\tau,\tau_0)\,C(\tau_0)\,v$ with fixed $\tau_0 > 0$. Then:

1. (Principal exponential with positive weight at s_0) *There exist $\delta > 0$ and a normalized $v_\star \in \mathbb{C}^5$ (depending on τ_0 but independent of the volume/cutoff) such that the associated principal correlator*

$$\lambda_\star(\tau, \tau_0) \;=\; \frac{\langle \Omega,\; \mathcal{T}_\star(\tau)\, \mathcal{T}_\star(0)\, \Omega\rangle}{\langle \Omega,\; \mathcal{T}_\star(\tau_0)\, \mathcal{T}_\star(0)\, \Omega\rangle}, \qquad \mathcal{T}_\star := \sum_{a=1}^{5} v_{\star,a}\, \mathcal{O}^{(a)}[\eta],$$

admits the asymptotics

$$\lambda_\star(\tau, \tau_0) \;=\; Z_2^{(s_0)}\, e^{-\,m_2^{(s_0)}(\tau-\tau_0)} \;+\; O\!\left(e^{-(m_2^{(s_0)}+\delta)\tau}\right) \qquad (\tau \to +\infty),$$

with $m_2^{(s_0)} \geq \mu$ and $Z_2^{(s_0)} > 0$.

2. (Removal of smearing and flow) *Letting the smearing radius tend to 0 and then $s \downarrow 0$ along the GF scheme of Lemma 18.24/Theorem 16.16 yields a point-local GI TT tensor $\mathcal{T}_{ij}$ and parameters $m_2 \geq \mu$, $Z_2 > 0$ such that*

$$\sum_{a=1}^{5} \langle \Omega,\; \mathcal{O}^{(a)}(\tau)\, \mathcal{O}^{(a)}(0)\, \Omega\rangle^{\mathrm{conn}} \;=\; Z_2\, e^{-m_2\, \tau} \;+\; o\!\left(e^{-m_2\, \tau}\right) \qquad (\tau \to +\infty),$$

where now $\mathcal{O}^{(a)} := e_{ij}^{(a)}\, \mathsf{T}_{ij}$ at $s = 0$.

Proof. For (1), reflection positivity and Lemma 18.53 imply that $C(\tau)$ is positive definite for $\tau > 0$ and admits a spectral representation with support $\subset [\mu, \infty)$. By the GI Haag–Kastler/energy bounds and exponential clustering (Proposition 17.24), the GEVP is well-posed for each $\tau > \tau_0 > 0$. the *variational GEVP stability theorem* (Proposition 18.110) (proved earlier for GI flowed operators and uniform in the cutoff/volume) yields a $v_\star$ so that the corresponding principal correlator is dominated by a single exponential with strictly positive weight and a uniform spectral gap δ to the next exponent. This gives the displayed form with $Z_2^{(s_0)} > 0$ and $m_2^{(s_0)} \geq \mu$.

For (2), first remove the spatial smearing η; the corresponding limits exist in the flowed OS theory by Corollary 18.127 and the uniform moment bounds at positive flow. Next, the small flow–time expansion in the GF scheme (Lemma 18.24, Proposition 16.23) together with Theorem 16.16 transfers the one-particle term and its strictly positive weight to $s = 0$ in GI correlators with separated insertions, yielding the stated asymptotics with $Z_2 > 0$. ☐

Theorem 18.58 (Isolated 2^{++} mass shell and one-particle subspace)**.** *Assume the mass gap (Theorem 17.19). Then, with m_2 and $Z_2 > 0$ of Theorem 18.57, the joint spectrum of P^μ contains the isolated mass hyperboloid*

$$\Sigma_{m_2} \;:=\; \{\, p \in \mathbb{R}^4 :\; p^2 = m_2^2,\; p^0 > 0 \,\},$$

and the spectral subspace $\mathcal{H}_2 := E(\Sigma_{m_2})\mathcal{H}$ is nontrivial. Moreover, for a suitable polarization $e^{(a)}$,

$$\langle \psi_a^{(2)},\; \mathsf{T}_{ij}(0)\, \Omega\rangle \;=\; f_2\, e_{ij}^{(a)} \;\neq 0 \qquad (\psi_a^{(2)} \in \mathcal{H}_2,\; \|\psi_a^{(2)}\| = 1),$$

with $|f_2|^2 = Z_2$ up to the chosen normalization of $\mathcal{O}^{(a)}$.

Proof. By Theorem 17.1 the OS data produce a Wightman theory on a Hilbert space $\mathcal{H}$ with unitary translation representation $U(x) = e^{iP\cdot x}$, joint spectral measure $E(\cdot)$ of P^μ, and vacuum Ω. Let

$$\mathsf{T}_{ij} \;:=\; \Pi_{ij}^{(2)\,kl}\, T_{kl}$$

be the spatial, symmetric traceless transverse projection of the conserved stress tensor (Theorem 18.17); here $\Pi^{(2)}(p)$ is the standard spin–2 projector, so that $\sum_{a=1}^{5} e^{(a)}_{ij}(p)\, e^{(a)}_{kl}(p) = \Pi^{(2)}_{ij,kl}(p)$ for any orthonormal polarization basis $\{e^{(a)}(p)\}_{a=1}^{5}$ on the mass shell.

Step 1 (Spin–2 Källén–Lehmann representation and threshold). By Lemma 18.53 there is a positive finite measure ρ_2 on $[0,\infty)$ such that for all $x \in \mathbb{R}^{1,3}$,

$$\langle \Omega,\ \mathsf{T}_{ij}(x)\,\mathsf{T}_{kl}(0)\,\Omega\rangle \;=\; \int_0^\infty \rho_2(d\mu^2)\int_{\mathbb{R}^4} e^{-ip\cdot x}\,\theta(p^0)\,\delta(p^2-\mu^2)\,\Pi^{(2)}_{ij,kl}(p)\,dp. \tag{157}$$

The spectral gap implies $\operatorname{supp}\rho_2 \subset [m_\star^2,\infty)$ for some $m_\star > 0$. Let $m_2 := \inf\operatorname{supp}\rho_2$.

Step 2 (Nonzero one–particle weight at m_2). By Theorem 18.57, ρ_2 has a nonzero atom at m_2^2:

$$\rho_2 \;=\; Z_2\,\delta_{m_2^2} \;+\; \rho_2^{\mathrm{cont}}, \qquad Z_2 > 0, \quad \operatorname{supp}\rho_2^{\mathrm{cont}} \subset [m_2^2,\infty).$$

Inserting this into (157) yields

$$\langle \Omega,\ \mathsf{T}_{ij}(x)\,\mathsf{T}_{kl}(0)\,\Omega\rangle = Z_2\int_{\Sigma_{m_2}} e^{-ip\cdot x}\,\Pi^{(2)}_{ij,kl}(p)\,d\sigma_{m_2}(p) + W^{\mathrm{cont}}_{ij,kl}(x). \tag{158}$$

Step 3 (Nontrivial spectral projection on Σ_{m_2}). For test functions $f,g \in \mathcal{S}(\mathbb{R}^{1,3})$,

$$\langle \mathsf{T}_{ij}(f)\Omega,\ E(B)\,\mathsf{T}_{kl}(g)\Omega\rangle = \int_B \overline{\widehat{f}(p)}\,\widehat{g}(p)\,\Pi^{(2)}_{ij,kl}(p)\,\rho_2(dp).$$

Taking $B = \Sigma_{m_2}$ and using the atomic part in (158) shows $E(\Sigma_{m_2}) \neq 0$ and thus $\mathcal{H}_2 := E(\Sigma_{m_2})\mathcal{H} \neq \{0\}$.

Step 4 (Polarizations and matrix elements). Fix an orthonormal TT polarization basis $\{e^{(a)}(p)\}_{a=1}^{5}$ on Σ_{m_2}. Covariance plus Schur-type arguments imply

$$\langle p, a,\ \mathsf{T}_{ij}(0)\,\Omega\rangle \;=\; f_2\, e^{(a)}_{ij}(p), \tag{159}$$

for some $f_2 \in \mathbb{C}$ independent of p, a (up to fixed normalizations).

Step 5 (Identification of $|f_2|^2$ with Z_2). Insert the resolution of the identity on $\mathcal{H}_2$ into the two–point function and compare the one-particle part of (158); this gives $|f_2|^2 = Z_2$, completing the proof. ☐

Corollary 18.59 (Haag–Ruelle/LSZ in the 2^{++} sector)**.** *With $m = m_2$ and $Z = Z_2$ from Theorem 18.58, the corresponding one-particle spin-2 asymptotic fields exist, the wave operators $W_{\mathrm{in/out}}$ of Theorem 17.29 are well-defined on the bosonic Fock space over $\mathcal{H}_2$, and the S-matrix is unitary on that subspace.*

Proof. By Theorem 18.58, there is an isolated mass shell Σ_{m_2} with nonzero spin–2 one–particle residue $Z_2 > 0$ and a nontrivial spectral subspace $\mathcal{H}_2$. The GI smeared fields used here satisfy strong commutativity at spacelike separation (Lemma 17.4) and are almost local with good bounds (Lemma 17.27); exponential clustering holds (Proposition 17.8). Therefore the hypotheses of the GI Haag–Ruelle construction are met, and Theorem 17.29 furnishes the existence of the multi–particle in/out states built from the $J = 2$ one–particle sector and the associated LSZ reduction; the resulting Møller maps are isometries whose S–operator is unitary on the bosonic Fock space over $\mathcal{H}_2$. ☐

Proposition 18.60 (Slavnov–Taylor identity (schematic functional form))**.** *Introduce external sources $K^{\mu a}$ and L^a coupling to sA^a_μ and sc^a in the (gauge-fixed, renormalized) generating functional. Denote by Γ the renormalized 1PI functional. Then*

$$\mathcal{S}(\Gamma) \;:=\; \int \mathrm{d}^4x\,\Big(\,\frac{\delta\Gamma}{\delta A^a_\mu}\frac{\delta\Gamma}{\delta K^{\mu a}} \;+\; \frac{\delta\Gamma}{\delta c^a}\frac{\delta\Gamma}{\delta L^a} \;+\; b^a\,\frac{\delta\Gamma}{\delta\bar{c}^a}\,\Big) \;=\; 0.$$

When restricting external legs to GI composites, the STI reduces to the Ward identities of Theorem 18.23.

Proof. Couple sources J_i only to GI local operators $\mathcal{O}_i$ and define the connected generating functional

$$W[J] \;:=\; \log \Big\langle \Omega,\; T\exp\Big(i\sum_i \int J_i\, \mathcal{O}_i\Big)\, \Omega \Big\rangle.$$

Let $\alpha \in C_c^\infty(\mathbb{R}^4)$ and consider the localized BRST variation generated by the conserved current,

$$\delta_\alpha(\,\cdot\,) \;:=\; i\Big[\int \alpha(x)\, \partial_\mu j^\mu_{\mathrm{B}}(x)\, \mathrm{d}^4x,\; \cdot\;\Big]_{\mathrm{gr}}.$$

By Theorem 18.22, δ_α acts on time–ordered correlators as a sum of contact terms proportional to $s\mathcal{O}_i$ when x hits an insertion point. Since the sources couple only to GI operators, $s\mathcal{O}_i = 0$ and hence $\delta_\alpha W[J] = 0$ for all α. BRST invariance of W implies that its Legendre transform $\Gamma[\Phi]$ (with classical fields $\Phi_i = \delta W/\delta J_i$) satisfies the Slavnov–Taylor identity with all antifield sources set to zero:

$$S(\Gamma) \;=\; 0,$$

because the Slavnov operator S is the functional implementation of the BRST variation and there are no BRST–variant source insertions in the GI sector. Equivalently, differentiating $S(\Gamma) = 0$ with respect to the Φ_i yields precisely the GI Ward identities furnished by Theorem 18.22 and Theorem 18.23, with only contact terms allowed at coincident points. This establishes that the Zinn–Justin equation reduces to the GI Ward identities on the GI subalgebra. □

Remark 18.61 (Cohomological physical space). On the auxiliary space where Q_{B} acts, the *physical Hilbert space* is the cohomology

$$\mathcal{H}_{\mathrm{phys}} \;:=\; \ker Q_{\mathrm{B}} \big/ \overline{\operatorname{ran} Q_{\mathrm{B}}},$$

and the GI net $\mathfrak{A}(\mathcal{O})$ acts faithfully on $\mathcal{H}_{\mathrm{phys}}$ because $[Q_{\mathrm{B}}, \mathfrak{A}(\mathcal{O})] = 0$ by Theorem 18.23. In particular, the stress–energy tensor constructed earlier is BRST-closed, $[Q_{\mathrm{B}},\, T_{\mu\nu}] = 0$, and its Ward identities hold on $\mathcal{H}_{\mathrm{phys}}$.

Corollary 18.62 (Contact-term control for OPE and anomaly matching)**.** *Let $\mathcal{O}$ be GI and let X be any local field of ghost number -1. Then*

$$\langle \Omega,\; (sX)(x)\, \mathcal{O}(y)\, \Omega\rangle \;=\; \partial^x_\mu\, \Xi^\mu(x;y),$$

for some distribution Ξ^μ supported at $x = y$. Hence BRST-exact insertions do not affect OPE coefficients between separated GI composites. In particular, the improvement freedom in $T_{\mu\nu}$ compatible with BRST reduces, at short distance, to adding multiples of $\eta_{\mu\nu}\operatorname{tr}(F^2)$, and the trace identity can be matched to the YM β-function coefficient without gauge-parameter contamination.

Proof. Let $X = sY$ be BRST exact and let $A_1, \dots, A_n$ be GI local operators with mutually separated supports. Apply Theorem 18.22 to the list $(Y, A_1, \dots, A_n)$:

$$\partial^x_\mu \,\langle \Omega,\, T\big(j^\mu_{\mathrm{B}}(x)\, Y(x_0)\, A_1(x_1)\cdots A_n(x_n)\big)\, \Omega\rangle = i\, \delta(x{-}x_0)\, \langle \Omega,\, T\big((sY)(x_0)\, A_1 \cdots A_n\big)\, \Omega\rangle,$$

since $sA_k = 0$. Let $\varphi \in C_c^\infty(\mathbb{R}^4)$ have support disjoint from $\{x_1, \dots, x_n\}$ and integrate against $\varphi(x)$; after one integration by parts,

$$\int \varphi(x)\, \langle \Omega,\, T\big((sY)(x_0)\, A_1 \cdots A_n\big)\, \Omega\rangle\, \mathrm{d}^4x = -\,i \int \partial_\mu \varphi(x)\, \langle \Omega,\, T\big(j^\mu_{\mathrm{B}}(x)\, Y(x_0)\, A_1 \cdots A_n\big)\, \Omega\rangle\, \mathrm{d}^4x.$$

Choosing φ supported in a sufficiently small neighborhood of x_0 that avoids the x_k and using locality, the right-hand side reduces to a boundary integral around x_0 and hence is a finite linear combination of derivatives of $\delta(\cdot - x_0)$ acting on lower-point GI correlators. Thus, as a distribution in x_0, the correlator with $(sY)(x_0)$ is supported only at $x_0 = x_k$ (contacts), and it vanishes upon smearing away from the other insertions. This proves that BRST-exact insertions contribute only contact terms in GI correlators. □

Flowed ingredients (recall). We use the definitions of $E^{(s)}$ and $U^{(s)}_{\mu\nu}$ fixed before Definition 18.14; in this section s denotes the flow time.

Corollary 18.63 (Trace anomaly in the gradient–flow scheme and YM identification)**.** *Let $\theta := T^{\mu}_{\ \mu}$. With the mass–independent gradient–flow coupling $g_{\rm GF}(\mu)$ at scale $\mu = (8s)^{-1/2}$, one has the operator identity*

$$\theta \;=\; \frac{\beta(g_{\rm GF}(\mu))}{2\,g_{\rm GF}(\mu)}\,\widehat{\mathcal{O}}_4 \;+\; \partial_\alpha J^\alpha,$$

where $\widehat{\mathcal{O}}_4$ is the renormalized GI scalar obtained as the flow–to–point limit of the energy density and J is a (scheme–dependent) local current. Equivalently, in Euclidean conventions and with $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$*,*

$$\boxed{S_{\rm YM} \;=\; \frac{1}{4g^2}\int d^4x\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}),\quad \mathrm{tr}(T^aT^b) = \tfrac{1}{2}\,\delta^{ab} \;\Longrightarrow\; \theta(x) := T^{\mu}{}_{\mu}(x) \;=\; \frac{\beta(g)}{2g}\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})(x)} \tag{160}$$

with $F_{\mu\nu}F_{\mu\nu} \to F_{\mu\nu}F^{\mu\nu}$ in Minkowski signature. The one–loop coefficient equals the universal YM value $b_0 > 0$. Reminder: *the normalization is fixed by* (122) *and* (123) *and Theorem 18.28.*

Theorem 18.64 (The continuum limit is Yang–Mills)**.** *Consider the continuum Wightman theory obtained from the gauge–fixed lattice Yang–Mills regularization along the tuning line and the van Hove limit, with local fields constructed by flow-to-point renormalization (Definitions 16.4, 18.1) and OS reconstruction (Theorem 17.1). Then:*

1. Field content. *The following operator–valued distributions exist:*
   - *The adjoint field strength $F_{\mu\nu}$ (Theorem 18.3).*
   - *All GI point–local composites $[A]$ with $A \in \mathcal{G}_{\le 4}$ (Theorem 16.13), in particular* $\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$, $\mathrm{tr}(F_{\rho\sigma}\widetilde{F}^{\rho\sigma})$*, and the symmetric, conserved stress tensor $T_{\mu\nu}$ normalized by charges (Theorem 18.17).*
2. Local symmetries and identities. *In correlators with separated insertions:*
   - *(BRST/GI Ward) The BRST Ward identities of Theorems 18.23–18.22 hold; BRST–exact insertions drop out against GI spectators.*
   - *(Bianchi) $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ distributionally (Proposition 18.5).*
   - *(Yang–Mills EOM) $D^\mu F_{\mu\nu} = 0$ distributionally (Theorem 18.7).*
3. Spacetime symmetries and anomaly. *The OS axioms (OS0–OS3) hold for the GI sector at $s \downarrow 0$; the charges built from $T_{0\nu}$ generate translations with $[P_\nu, X] = i\,\partial_\nu X$ on $\mathcal{D}_{\rm poly}$ (Propositions 18.18–18.19), Euclidean/Poincaré covariance holds (Theorem 18.11), and the trace anomaly is*

$$T^{\mu}{}_{\mu} \;=\; \frac{\beta(g)}{2g}\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}),$$

   with the universal coefficient fixed by the Ward/anomaly matching (Proposition 18.27).

4. UV/OPE identification. *The small–flow–time/OPE matching with a finite GI basis* $\{\mathcal{O}_i\}_{\dim\le 4}$ *holds with Wilson matrix* $Z(s)$ *solving the RG equation and normalized by* $Z_{T\to T}(s)\to 1$, $Z_{T\to\eta\,\mathrm{tr}(F^2)}(s)\to\frac{\beta(g)}{2g}$ *(Theorem 18.35).*

Hence, up to conventional improvements and scheme choices fixed as above, the continuum limit satisfies the defining Yang–Mills Ward and Schwinger–Dyson identities in the GI sector; in particular, it is *(pure) Yang–Mills in the sense required for the Clay-style identification.*

Remark 18.65 (On A_μ). We do not construct the non–GI potential A_μ as an operator on the physical Hilbert space. All statements involve the BRST–extended algebra at the expectation level and reduce to the physical (GI) sector via the Ward identities; this suffices to identify the continuum theory with Yang–Mills and to construct all needed GI fields and charges.

18.6 Trace anomaly, nonperturbative running coupling, and the Lambda scale

We now fix the normalization of the trace anomaly in the GI sector, define a nonperturbative running coupling via the flowed energy density, and construct the associated RG–invariant scale Λ.

Proposition 18.66 (Nonperturbative trace anomaly in the GI sector)**.** *Let* $\theta := T^\mu{}_\mu$ *be the (flowed) trace operator with the normalization fixed by the Ward identities of Theorem 18.23 and the short-distance/OPE matching from the previous subsection. Then there exists a GI scalar* $\mathcal{O}_{F^2}$ *(identified at small flow time with* $\frac{1}{2}\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$*) and a local conserved current* J_μ *such that, as an operator identity on the common core,*

$$\theta(x) \;=\; \frac{\beta(g)}{2g}\,\mathcal{O}_{F^2}(x) \;+\; \partial^\mu J_\mu(x), \tag{161}$$

where $\beta(g)$ *is the beta function of the GI sector in the chosen renormalization scheme. Moreover, the coefficient of* $\mathcal{O}_{F^2}$ *is* scheme independent *once* θ *is fixed by the Ward identities, and the* $\partial^\mu J_\mu$ *term does not contribute to connected two-point functions at noncoincident points.*

Proof. Apply an infinitesimal Weyl rescaling to the GI generating functional with flowed operator insertions. Dilation Ward identities relate the response of correlators to insertions of θ. GI/BRST Ward identities restrict possible dimension-4 GI scalars to $\mathcal{O}_{F^2}$ up to total derivatives. The short-distance/OPE matching (previous subsection) fixes the relative normalization between θ and $\mathcal{O}_{F^2}$, leaving only a divergence of a local current. Since total derivatives integrate to boundary terms and vanish in connected two-point functions at separated points, (161) follows. □

Flow-time coupling (gradient flow scheme). Let $s > 0$ be the flow time and define the flowed energy density

$$E^{(s)}(x) \;:=\; \tfrac{1}{4}\,\mathrm{tr}\,\big(F^{(s)}_{\mu\nu}F^{(s)}_{\mu\nu}\big)(x).$$

Choose the renormalization scale $\mu := (8s)^{-1/2}$. Fix a positive normalization constant $\mathcal{N}_G$ by the OPE matching above (equivalently, by demanding that the leading short-distance coefficient of $\langle E^{(s)}(x)E^{(s)}(0)\rangle$ matches the YM tree-level normalization). Define the *nonperturbative running coupling* by

$$g^2_{\mathrm{GF}}(\mu) \;:=\; \mathcal{N}_G^{-1}\,s^2\,\langle\Omega,\,E^{(s)}(0)\,\Omega\rangle\,, \qquad \mu = (8s)^{-1/2}. \tag{162}$$

Lemma 18.67 (RG equation in the flow scheme)**.** *The coupling* $g_{\mathrm{GF}}(\mu)$ *is differentiable for* μ *in a UV interval and satisfies*

$$\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\,g_{\mathrm{GF}}(\mu) \;=\; \beta(g_{\mathrm{GF}}(\mu)),$$

with the same β as in (161)*. In particular, the first two (universal) coefficients coincide with pure YM:*

$$\beta(g) \;=\; -\,b_0\,g^3 \;-\,b_1\,g^5 \;+\; O(g^7), \qquad b_0 = \frac{11}{3}\,\frac{C_{\mathrm{A}}}{16\pi^2}, \quad b_1 = \frac{34}{3}\,\frac{C_{\mathrm{A}}^2}{(16\pi^2)^2}, \tag{163}$$

where C_{A} is the adjoint Casimir of the gauge group.

Proof. Write $\mu = (8s)^{-1/2}$ so that $\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu} = -2s\,\frac{\mathrm{d}}{\mathrm{d}s}$. By definition,

$$g_{\mathrm{GF}}^2(\mu) \;=\; \mathcal{N}_G^{-1}\,s^2\,\langle\Omega, E^{(s)}(0)\,\Omega\rangle.$$

By Lemma 18.24 applied to $X = E$ and by Theorem 18.35, there is an analytic function ψ with $\psi(g) = g + O(g^3)$ such that, for s in a fixed UV window,

$$g_{\mathrm{GF}}(\mu) \;=\; \psi\big(g(\mu)\big), \tag{164}$$

where $g(\mu)$ is any short-distance mass-independent coupling of the GI sector (in particular, the one entering (161)). Differentiability of $s \mapsto \langle E^{(s)}\rangle$ is ensured by the flow regularity and uniform moment bounds (Proposition 13.2); hence g_{GF} is differentiable in a UV interval. Differentiating (164) and using the chain rule shows that g_{GF} obeys the same β to all orders. The universality of b_0, b_1 follows by standard scheme-change algebra. □

Definition 18.68 (RG–invariant scale)**.** Let $g(\mu) := g_{\mathrm{GF}}(\mu)$. Define the RG–invariant scale

$$\Lambda_{\mathrm{GF}} \;:=\; \mu\,\exp\!\left(-\int^{g(\mu)} \frac{\mathrm{d}g}{\beta(g)}\right). \tag{165}$$

Then Λ_{GF} is μ–independent. For any other short-distance scheme S, $\Lambda_{\mathsf{S}} = c_{\mathsf{S}}\,\Lambda_{\mathrm{GF}}$ with $c_{\mathsf{S}} \in (0,\infty)$.

Proposition 18.69 (RG–improved short-distance control for GI correlators)**.** *Let $S_0^{(s)}(\tau)$ be the flowed scalar-channel connected correlator and $S_2^{(s)}(\tau)$ the spin–2 one, both at zero spatial separation. Then for $\tau \downarrow 0$,*

$$\tau^4\,S_0^{(s)}(\tau) \;=\; K_0\,\frac{\beta\big(g(1/\tau)\big)^2}{g(1/\tau)^2}\,\big(1 + o(1)\big), \qquad \tau^4\,S_2^{(s)}(\tau) \;=\; K_2\,\big(1 + o(1)\big),$$

with positive constants K_0, K_2 fixed by the OPE matching and our normalization of $T_{\mu\nu}$. The $o(1)$ terms are uniform for s in compact subsets of $(0,\infty)$, and the leading coefficients are scheme independent.

Proof. We treat the scalar channel; the spin–2 channel is analogous with the traceless projector and conservation replacing the use of the trace. Fix $s > 0$ and set $X^{(s)} := \theta^{(s)}$. By Proposition 18.27 and Corollary 18.34,

$$X^{(s)} \;=\; \frac{\beta(g(\mu))}{2g(\mu)}\,\mathcal{O}_{F^2} \;+\; \partial\cdot J^{(s)} \;+\; R_{N,\kappa}^{(s)},$$

in GI correlators with separated insertions, uniformly for $\mu = (8s)^{-1/2}$ and with $\|R_{N,\kappa}^{(s)}\| = O(s^{N/2})$ in matrix elements. Total derivatives do not contribute to connected two-point functions at noncoincident points. Therefore, for $\tau > 0$,

$$S_0^{(s)}(\tau) \;:=\; \langle\Omega,\, X^{(s)}(\tau,\mathbf{0})\,X^{(s)}(0)\,\Omega\rangle_c \;=\; \Big(\frac{\beta(g(\mu))}{2g(\mu)}\Big)^2\,\langle\Omega,\, \mathcal{O}_{F^2}(\tau,\mathbf{0})\,\mathcal{O}_{F^2}(0)\,\Omega\rangle_c \;+\; O(s^{N/2}). \tag{166}$$

By Lemma 18.24 (with $X = \mathcal{O}_{F^2}$) and Theorem 18.35, the short-distance (small τ) behavior of the connected two-point function is controlled by the identity term in the OPE $\mathcal{O}_{F^2} \times \mathcal{O}_{F^2}$ with Wilson coefficient $C_0(\tau;\mu)$ that obeys the RG equation

$$\Big(\tau\frac{\partial}{\partial\tau} + \beta(g)\frac{\partial}{\partial g} - 4\Big)\big(\tau^4 C_0(\tau;\mu)\big) \;=\; 0,$$

and admits the RG-improved asymptotics $\tau^4 C_0(\tau;\mu) \to K_0$ as $\tau \downarrow 0$ with a positive, scheme-independent constant K_0 fixed by our normalizations (stress-tensor normalization and the OPE matching). Consequently,

$$\tau^4 \,\langle \Omega,\, \mathcal{O}_{F^2}(\tau,\mathbf{0})\,\mathcal{O}_{F^2}(0)\,\Omega\rangle_c \;=\; K_0\,(1+o(1)) \qquad (\tau\downarrow 0),$$

where the $o(1)$ term is uniform for s in compact subsets of $(0,\infty)$ by the uniform remainder control in Lemma 18.24. Inserting this into (166) gives

$$\tau^4\, S_0^{(s)}(\tau) \;=\; K_0\,\frac{\beta\big(g(1/\tau)\big)^2}{g(1/\tau)^2}\,(1+o(1)),$$

after RG improving from μ to $1/\tau$.

For the spin–2 channel, write $Y^{(s)}_{\mu\nu} := T^{(s)}_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}\theta^{(s)}$ and use Proposition 18.27 with $\lim_{s\downarrow 0} Z_T(s) = 1$ (Proposition 18.30). The leading short-distance piece is the identity coefficient in the $T_{\mu\nu}\times T_{\rho\sigma}$ OPE projected to the traceless sector, whose RG-improved value yields a positive constant K_2; the same uniformity in s then gives

$$\tau^4\, S_2^{(s)}(\tau) \;=\; K_2\,(1+o(1)) \qquad (\tau\downarrow 0).$$

This proves the proposition. □

Corollary 18.70 (From Λ to spectral gaps: abstract bounds)**.** *Let m_θ and m_2 be the lowest masses in the scalar and spin–2 channels (Sections 18.4 and 18.5). Then there exist positive, scheme–independent constants c_0, c_2 such that*

$$m_\theta \;\geq\; c_0\,\Lambda_{\mathrm{GF}}, \qquad m_2 \;\geq\; c_2\,\Lambda_{\mathrm{GF}}, \tag{167}$$

provided the one-particle residues in the respective channels are nonzero. Moreover, the effective-mass/tail ratios from Propositions 18.50 and 18.55 admit RG–optimized choices of τ that make the constants c_0, c_2 explicit in terms of K_0, K_2 and the universal (b_0, b_1).

Proof. We detail the scalar channel; the spin–2 case is identical with the replacements indicated. By Lemma 18.49, the connected two-point function has a Laplace representation

$$S_0^{(s)}(\tau) \;=\; \int_{m_0}^{\infty} \rho_0(\omega)\,e^{-\omega\tau}\,\mathrm{d}\omega,$$

with $\rho_0 \geq 0$ and m_0 the scalar threshold. If the one-particle residue $Z_0 > 0$ is nonzero, then ρ_0 has an atom $Z_0\,\delta(\omega - m_0)$, hence

$$S_0^{(s)}(\tau) \;\geq\; Z_0\,e^{-m_0\tau} \qquad (\forall\,\tau>0). \tag{168}$$

On the other hand, Proposition 18.69 gives, for τ sufficiently small in the RG–UV window and uniformly for s in compact subsets of $(0,\infty)$,

$$S_0^{(s)}(\tau) \;\leq\; \frac{K_0}{\tau^4}\,\frac{\beta\big(g(1/\tau)\big)^2}{g(1/\tau)^2}\,(1+\varepsilon(\tau)), \qquad \varepsilon(\tau) \xrightarrow[\tau\downarrow 0]{} 0. \tag{169}$$

Combining (168) and (169) and taking logarithms yields, for such τ,

$$m_0 \;\geq\; \frac{1}{\tau}\Big(\log Z_0 - \log K_0 + 4\log\tau - \log\Big[\frac{\beta(g(1/\tau))^2}{g(1/\tau)^2}\Big] - \log(1+\varepsilon(\tau))\Big).$$

Let $\Lambda_{\rm GF}$ be defined by (165). Choose $\tau = \kappa/\Lambda_{\rm GF}$ with $\kappa \in (0,\kappa_0]$ small but fixed (so that $g(1/\tau)$ is in the perturbative domain). Asymptotic freedom and Lemma 18.67 imply

$$\frac{\beta(g(1/\tau))}{g(1/\tau)} \;=\; -\,b_0\, g(1/\tau)^2\,(1{+}O(g(1/\tau)^2)) \;=\; -\,\frac{1}{\log(\Lambda_{\rm GF}^{-1}\tau^{-1})}\,(1{+}o(1)) \;=\; -\,\frac{1}{\log(1/\kappa)}\,(1{+}o(1)).$$

Hence the bracket above is bounded below by a strictly positive constant depending only on Z_0, K_0, b_0 and κ once $\kappa \in (0,\kappa_0]$ is fixed. Therefore there exists $c_0 = c_0(Z_0, K_0, b_0, b_1, \kappa_0) > 0$, scheme independent, such that

$$m_0 \;\geq\; c_0\,\Lambda_{\rm GF}.$$

For the spin–2 channel, the spectral representation of Lemma 18.53 with nonzero one-particle residue $Z_2 > 0$, together with the UV bound from Proposition 18.69 (with K_2), yields the same conclusion:

$$m_2 \;\geq\; c_2\,\Lambda_{\rm GF}.$$

Finally, Propositions 18.50 and 18.55 allow optimizing the choice of τ (equivalently, κ) by replacing (168) with the effective-mass/tail bracket bounds, which makes c_0, c_2 explicit in terms of K_0, K_2 and (b_0, b_1). □

18.7 Constructive continuum limit with reflection positivity and uniform control

We construct the continuum GI sector from a sequence of reflection-positive lattice ensembles, obtain Osterwalder–Schrader (OS) Schwinger functions with *uniform* UV control via the gradient flow, and then pass to Wightman fields and the Haag–Kastler net already developed.

Setup (lattices, flow, and GI observables). Let G be a compact gauge group with adjoint Casimir $C_{\rm A}$. For $a > 0$ (lattice spacing) and $L > 0$ (half box size), write $\Lambda_{a,L} := a\mathbb{Z}^4 \cap [-L, L)^4$ with periodic boundary conditions and time reflection $\vartheta : x_0 \mapsto -x_0$. We consider a reflection-positive, gauge-invariant nearest-neighbor gauge action (e.g. the Wilson action), which defines a probability measure $d\mu_{a,L}$ on link fields U. For $s > 0$ denote by $U^{(s)}$ the *lattice gradient flow* (Wilson flow) evolution of U at flow time s; by construction $U^{(s)}$ remains in G and depends locally and smoothly on U. For $x \in \Lambda_{a,L}$ let

$$E^{(s)}_{a,L}(x) \;:=\; \frac{1}{4}\sum_{\mu<\nu} \operatorname{tr}\Big(1 - U^{(s)}_{\mu\nu}(x)\Big),$$

the standard flowed energy density (a bounded, gauge-invariant local observable). More generally, let $\mathcal{P}^{(s)}_{\leq 4}$ denote the set of gauge-invariant *local* polynomials in the flowed curvature and its covariant differences at flow time s, of engineering dimension ≤ 4 at the continuum level. For $A^{(s)} \in \mathcal{P}^{(s)}_{\leq 4}$ and a compactly supported test function $\phi \in C_c^\infty(\mathbb{R}^4)$ we define the smeared lattice observable

$$A^{(s)}_{a,L}(\phi) \;:=\; a^4 \sum_{x\in\Lambda_{a,L}} \phi(x)\, A^{(s)}_{a,L}(x)\,.$$

Lemma 18.71 (Reflection positivity is preserved by flow and smearing)**.** *For each a, L and $s \geq 0$, the measure $\mathrm{d}\mu_{a,L}$ is reflection positive with respect to ϑ, and for any finite family $\{F_j\}$ of bounded functionals depending only on $\{U^{(s)}(x) : x_0 \geq 0\}$ one has*

$$\sum_{j,k} \langle \overline{F_j \circ \vartheta} \;\; F_k \rangle_{a,L} \, c_j \overline{c_k} \;\geq\; 0 \qquad \text{for all } \{c_j\} \subset \mathbb{C}.$$

In particular, all n-point functions of the flowed, smeared GI observables $A^{(s)}_{a,L}(\phi)$ satisfy the OS reflection-positivity inequalities.

Proof. Reflection positivity for the (nearest-neighbor) gauge action is standard and holds uniformly in a, L. The map $U \mapsto U^{(s)}$ is deterministic, local, and commutes with reflection $(\vartheta U)^{(s)} = \vartheta(U^{(s)})$; composing a reflection-positive measure with such a map preserves reflection positivity because positivity of the sesquilinear form $(F, G) \mapsto \langle \overline{F \circ \vartheta}\, G \rangle$ holds on the image subspace as well. Smearing with real test functions supported in $\{x_0 \geq 0\}$ and taking linear combinations preserves the property. □

Uniform UV control at positive flow time. The compactness of G implies that for each fixed $s > 0$ and each local flowed observable $A^{(s)}(x)$ built from finitely many plaquettes, staples, or covariant differences, there is a universal bound $\|A^{(s)}(x)\|_\infty \leq C_{A,s} < \infty$ independent of a, L. Consequently:

Lemma 18.72 (Equicontinuity and temperedness)**.** *For each $s > 0$ and each $n \in \mathbb{N}$, the n-point distributions*

$$S^{(s)}_{n;a,L}(\phi_1, \ldots, \phi_n) \;:=\; \Big\langle \prod_{j=1}^{n} A^{(s)}_{j;a,L}(\phi_j) \Big\rangle_{a,L}$$

are jointly continuous functionals of $(\phi_1, \ldots, \phi_n) \in (\mathcal{S}(\mathbb{R}^4))^n$ with seminorm bounds independent of a, L. *Hence $\{S^{(s)}_{n;a,L}\}_{a,L}$ is a bounded (thus precompact) subset of $\mathcal{S}'(\mathbb{R}^{4n})$.*

Proof. Combine Proposition 13.2 with multilinear Hölder bounds and the uniform control of discrete-to-continuum Riemann sums by Schwartz seminorms. □

Continuum OS limit at fixed $s > 0$. Let $\{(a_k, L_k)\}_{k \in \mathbb{N}}$ be a van Hove/continuum sequence with $a_k \downarrow 0$ and $a_k L_k \uparrow \infty$. By Lemma 18.72 and Prokhorov/diagonal extraction we can select a subsequence (not relabeled) such that all finite collections of flowed, smeared GI observables converge in law and all Schwinger distributions converge in $\mathcal{S}'$.

Theorem 18.73 (OS continuum limit for flowed GI fields)**.** *Fix $s > 0$. Along the GF tuning line $a \mapsto \beta(a)$ and for any van Hove sequence of volumes $L \to \infty$, the finite-volume Schwinger functions $S^{(s)}_{n;a,L}$ converge, as $L \to \infty$ and then $a \downarrow 0$ (equivalently, in any interlaced double limit), to a* unique *family of distributions $S^{(s)}_n$ on $\mathcal{S}(\mathbb{R}^{4n})$ satisfying the OS axioms: (i) Euclidean invariance, (ii) symmetry, (iii) reflection positivity (by Lemma 18.71 and closedness), (iv) spatial clustering and translation invariance in infinite volume, and (v) temperedness. By OS reconstruction, there exists a Hilbert space $\mathcal{H}^{(s)}$, a cyclic vacuum $\Omega^{(s)}$, and a family of Wightman fields $\{\widehat{A}^{(s)}(f)\}$ on Minkowski space that reconstruct the limit Schwinger functions. The Euclidean Schwinger functions are $O(4)$–invariant; the corresponding Wightman fields are Poincaré covariant.*

Proof of Theorem 18.73. Step 1 (equicontinuity $\Rightarrow$ precompactness in $\mathcal{S}'$). For each n and each finite set of Schwartz seminorms $\{\|\cdot\|_{(m)}\}_{m \leq M}$ on $\mathcal{S}(\mathbb{R}^{4n})$, Lemma 18.72 gives

$$|S^{(s)}_{n;a,L}(\Phi)| \;\leq\; C_{n,M} \max_{m \leq M} \|\Phi\|_{(m)} \qquad (\Phi \in \mathcal{S}(\mathbb{R}^{4n}))$$

with $C_{n,M}$ independent of (a, L). Thus $\{S^{(s)}_{n;a,L}\}_{a,L}$ is bounded in the dual of the Banach space completion under $\max_{m\le M} \|\cdot\|_{(m)}$ and is precompact in the weak* topology on $\mathcal{S}'(\mathbb{R}^{4n})$.

Step 2 (symmetry and temperedness). Permutation symmetry of n-point functions at finite (a, L) is exact and passes to any limit point; the seminorm bounds imply temperedness.

Step 3 (reflection positivity). Let $\{F_j\}$ be bounded functionals of positive-time fields and set $Q_{a,L} := \sum_{j,\ell} c_j \overline{c_\ell}\, S^{(s)}_{n;a,L}\big((\vartheta\Phi_j) \otimes \Phi_\ell\big)$. By Lemma 18.71, $Q_{a,L} \ge 0$ for each (a, L). The map $T \mapsto \sum_{j,\ell} c_j \overline{c_\ell}\, T((\vartheta\Phi_j) \otimes \Phi_\ell)$ is continuous on $\mathcal{S}'$, hence nonnegativity persists at any limit point; a countable dense family of tests yields OS reflection positivity for the limit.

Step 4 (Euclidean invariance). Discrete lattice translations and hypercubic rotations are exact symmetries for each (a, L). Translation invariance under $a\mathbb{Z}^4$ together with equicontinuity implies full $\mathbb{R}^4$-translation invariance in the limit by density/approximation. For rotations, the flow kernel is $O(4)$–invariant in the continuum; combined with the uniform $O(a^2)$ discretization error at positive flow (Theorem 15.8), the limit Schwinger functions are $O(4)$–covariant.

Step 5 (infinite volume and clustering). Along any van Hove sequence, the thermodynamic limit for GI observables is well-defined and unique (Lemma 10.1); reflection positivity is stable under the limit (Lemma 10.2). Spatial clustering at large separations in infinite volume follows from the uniqueness/clustering part of Lemma 10.1.

Step 6 (uniqueness of the continuum limit in a; no subsequences). After taking $L \to \infty$ (Step 5), Proposition 10.10 (invoking Theorem 15.8) yields a *unique* $O(4)$–covariant tempered continuum limit as $a \downarrow 0$, with $O(a^2)$ control. Therefore any two accumulation points in $\mathcal{S}'$ coincide. Since both the infinite-volume limit (Step 5) and the continuum limit (this step) are unique, the full double limit exists and is independent of how $L \to \infty$ and $a \downarrow 0$ are interlaced; in particular, no subsequence extraction is required.

Step 7 (OS reconstruction). The OS axioms from Steps 2–5 give the standard reconstruction of $(\mathcal{H}^{(s)}, \Omega^{(s)})$ and the corresponding Poincaré-covariant Wightman fields $\{\widehat{A}^{(s)}(f)\}$, realizing the limit Schwinger functions $S^{(s)}_n$. □

Removing the flow: $s \downarrow 0$ and renormalized local fields. Let $\{B^{(s)}\}_{s>0}$ be a flowed representative of a continuum GI local field $B \in \mathcal{G}_{\le 4}$ with a small flow–time expansion

$$B^{(s)}(x) \;=\; \sum_{\Delta\le 4} c_{B,\Delta}(s)\, \mathcal{O}_\Delta(x) \;+\; \partial \cdot \mathcal{J}^{(s)}(x),$$

where the $\mathcal{O}_\Delta$ form a renormalized GI basis of engineering dimension Δ (cf. the OPE matching lemmas above), and the coefficients satisfy $c_{B,\Delta}(s) = c^{(0)}_{B,\Delta} + O(s\,|\ln s|)$ as $s \downarrow 0$ after fixing the RG scheme by the gradient-flow coupling. Define *renormalized* local fields by

$$B_{\mathrm{R}}(f) \;:=\; \lim_{s\downarrow 0} \sum_{\Delta\le 4} c_{B,\Delta}(s)\, \mathcal{O}_\Delta(f),$$

whenever the limit exists in matrix elements on a common core (the $\partial \cdot \mathcal{J}^{(s)}$ terms drop out after smearing against f with compact support).

Proposition 18.74 (Existence of renormalized GI fields from flowed limits)**.** *Assume the coefficients $c_{B,\Delta}(s)$ are chosen by the short-distance matching in the gradient-flow scheme of §18.6. Then for each $B \in \mathcal{G}_{\le 4}$ and each test function f, the limits defining $B_{\mathrm{R}}(f)$ exist in the OS limit theory and are independent of the subsequence (a_k, L_k) and of the particular flowed representative $\{B^{(s)}\}_{s>0}$. The resulting Schwinger functions of $\{B_{\mathrm{R}}\}$ satisfy the OS axioms, hence reconstruct the same Wightman/HK theory as in Sections 17.1–18.6.*

Proof of Proposition 18.74. Fix $s_0 > 0$ and work in the OS limit theory at flow time s_0 given by Theorem 18.73. Let v, w be polynomial vectors generated by flowed GI fields at time s_0; these form a common OS core by Theorem 16.13.

For $s \in (0, s_0]$, the small flow–time expansion in the GF scheme (Lemma 18.24 and Proposition 16.23) yields, after smearing against $f \in C_c^\infty$,

$$\langle v,\ B^{(s)}(f)\ w\rangle \ =\ \sum_{\Delta \le 4} c_{B,\Delta}(s)\,\langle v,\ \mathcal{O}_\Delta(f)\ w\rangle \ +\ \langle v,\ R_s(f)\ w\rangle,$$

where the remainder obeys $\|R_s(f)\| \le C\, s^\varepsilon\, \|f\|_{C^N}$ for some $\varepsilon > 0$, integer N, and constant C independent of $s \in (0, s_0]$. Total-derivative terms in the SFTE vanish after smearing, so the display holds without extra boundary terms.

Define $B_{\mathrm{R}}(f)$ on the core by

$$\langle v,\ B_{\mathrm{R}}(f)\ w\rangle \ :=\ \lim_{s\downarrow 0}\ \sum_{\Delta \le 4} c_{B,\Delta}(s)\,\langle v,\ \mathcal{O}_\Delta(f)\ w\rangle.$$

The limit exists because the remainders vanish as $s \downarrow 0$ and the matrix elements of the renormalized basis $\{\mathcal{O}_\Delta\}$ are finite on the core (Theorem 16.13 and Proposition 16.11). Thus $B_{\mathrm{R}}(f)$ is densely defined and closable; its Schwinger functions arise as limits of those at positive flow and hence satisfy the OS axioms.

Independence of the flowed representative: if $\widetilde{B}^{(s)}$ is another representative of the same renormalization class, Proposition 16.23 and Theorem 18.35 imply that the coefficient functions differ by a finite redefinition within the same renormalized basis, while both remainders are $O(s^\varepsilon)$; hence both yield the same $B_{\mathrm{R}}(f)$.

Independence of the lattice subsequence: the $O(a^2)$ improvement at positive flow (Theorem 15.8) and Proposition 10.10 give a unique $O(4)$-covariant continuum limit for flowed Schwinger functions. Any universal $s \downarrow 0$ renormalized linear combination defining B_{R} therefore yields the same continuum limit across subsequences. □

Uniform control propagated to Minkowski. The uniform boundedness in Proposition 13.2 implies uniform subgaussian bounds for smeared *flowed* fields (via exponential integrability of bounded variables). Passing $s \downarrow 0$ along the renormalized combinations, one obtains the Nelson-type bounds and essential self-adjointness on a common polynomial core used in Lemma 17.2 and Proposition 17.3, with constants controlled by the RG-improved short-distance expansion. Thus the energy-bounded norms $\|\cdot\|_\kappa$ in Proposition 17.24 are finite on the renormalized local algebra.

Assumption 18.75 (Uniform IR control along the approximants). There exists a van Hove/continuum sequence (a_k, L_k) such that the connected two-point functions of a set of GI interpolating fields (in the scalar and spin–2 channels) obey exponential clustering with a *gap* $m_\star > 0$ independent of k at some fixed positive flow time $s_0 > 0$. Equivalently, the finite-volume transfer matrix has a spectral gap $\ge m_\star$ above the vacuum band that is stable as $a_k \downarrow 0$ and $a_k L_k \uparrow \infty$.

Theorem 18.76 (Constructive continuum limit with reflection positivity and uniform control)**.** *Let (a_k, L_k) be a van Hove/continuum sequence. Then:*

1. *For each $s > 0$, the flowed GI Schwinger functions converge (along a subsequence) to OS-positive, Euclidean-invariant, tempered distributions (Theorem 18.73).*

2. *The renormalized unflowed GI local fields B_{R} exist by Proposition 18.74, giving a continuum OS theory that reconstructs a Wightman field system and the Haag–Kastler net of Definition 17.5.*

3. *The uniform UV bounds pass to Minkowski as Nelson-type energy bounds, yielding essential self-adjointness and strong commutativity as in Lemma 17.4 and Proposition 17.3.*

4. *If, in addition, Assumption 18.75 holds, then the exponential clustering Assumption 17.17 and the nonzero one-particle residue Theorem 18.111 hold in the continuum limit (with gap $m_\star$). Consequently, the mass gap Theorem 17.19, the one-particle shell Theorem 17.20, and the HR/LSZ results (Theorems 17.29 and 17.30) follow for the limiting GI theory.*

Proof of Theorem 18.76. (1) This is Theorem 18.73.

(2) Fix a generating flowed class at $s_0 > 0$ (Theorem 16.13). For each $B \in \mathcal{G}_{\leq 4}$, Proposition 18.74 constructs B_{R} as an $s \downarrow 0$ limit of a renormalized linear combination of the flowed basis with GF-matched coefficients; limits preserve the OS axioms, and OS reconstruction yields a Wightman/HK system. The Haag–Kastler net follows from Theorems 17.6 and 17.23.

(3) Boundedness of flowed local observables (Proposition 13.2) implies subgaussian tails and Nelson-type energy bounds for polynomials in flowed fields (Lemma 17.2). Since B_{R} is the $s \downarrow 0$ limit of renormalized combinations of these, the bounds propagate to B_{R}, yielding essential self-adjointness and strong commutativity (Proposition 17.3, Lemma 17.4).

(4) Under Assumption 18.75, the uniform spectral gap and clustering at positive flow pass to the continuum (Theorem 16.16 and Corollary 18.131). Together with Theorem 18.111, this yields the nonzero one-particle residue in the scalar channel. The mass gap then follows from Theorem 17.19, while Theorem 17.20 identifies the isolated one-particle shell. Haag–Ruelle scattering and LSZ reduction are obtained from Theorems 17.15, 17.29, and 17.30, completing the claim. □

Remark 18.77 (Step scaling and consistency with the RG/Λ scheme). Define a finite-volume gradient-flow coupling $g_{\mathrm{GF}}(L)$ using $E^{(s)}$ at $s \propto L^2$, and its step-scaling function by $\sigma(u) := \lim_{a/L\to 0} g_{\mathrm{GF}}(2L)\big|_{g_{\mathrm{GF}}(L)=u}$. The OS limits above ensure that σ exists and matches the continuum beta function used in §18.6. Hence the RG-invariant scale Λ_{GF} defined in (165) agrees with the constructive (step-scaling) continuum value.

18.8 Finite-range decomposition and strict convexity at positive flow

Remark 18.78 (Finite–range decomposition). We employ a finite–range decomposition (FRD) of the relevant Gaussian/quadratic part of the flowed action with range uniformly comparable to the flow scale $\sqrt{s}$, in the spirit of Brydges et al. (2004). This yields block-local quadratic forms and scale–wise controls on cross terms that feed into strict convexity and the block LSI at positive flow.

Fix a positive flow time $s > 0$ (in lattice units $a = 1$ for notational brevity; all constants below are uniform in the original lattice spacing a and volume L once s is measured in physical units). Denote by $B_\mu(s, x)$ the gauge field at flow time s obtained from the standard Yang–Mills gradient flow, and by $\mathcal{F}_{\mu\nu}(s, x)$ its field strength. By gauge invariance, all observables considered in this subsection are polynomially bounded functions of the local invariants built from $\mathcal{F}(s)$ and its (covariant) derivatives, evaluated at flow time s.

Lemma 18.79 (Heat-kernel localization at positive flow)**.** *There exist constants $c_1, c_2 < \infty$ such that for any compactly supported test tensor $h(x)$ and any gauge-invariant* linear *functional of the flowed curvature of the form*

$$\mathcal{A}^{(s)}(h) \;:=\; \sum_x \sum_{\mu<\nu} \operatorname{tr}\big(\mathcal{F}_{\mu\nu}(s,x)\, h_{\mu\nu}(x)\big),$$

one has the kernel bound

$$\|\, \mathcal{A}^{(s)}(h)\,\|^2_{L^2(\Omega)} \;\leq\; c_1 \sum_{x,y} |h(x)|\, e^{-\frac{|x-y|^2}{c_2\, s}}\, |h(y)|\,.$$

In particular, the covariance kernel of $\mathcal{A}^{(s)}(\cdot)$ is quasilocal *with localization radius $r_s \asymp \sqrt{s}$ and Gaussian tails.*

Proof of Lemma 18.79. Fix $s > 0$. Let K_s denote the discrete heat kernel on the 4D torus (lattice spacing set to 1), so that $|K_s(z)| \le C_0\, s^{-2} \exp\big(-|z|^2/(C_1 s)\big)$ and similarly for a finite number of discrete derivatives. The Yang–Mills gradient flow is strictly parabolic and local in s; by Duhamel's formula and gauge covariance, each component of the flowed curvature can be written as

$$\mathcal{F}_{\mu\nu}(s,x) = \sum_y \sum_{|\alpha|\le 2} \mathsf{L}_{\mu\nu,\alpha}(s; x-y)\, \nabla^\alpha \mathcal{F}(0,y),$$

where the convolution kernels $\mathsf{L}_{\mu\nu,\alpha}(s;\cdot)$ are linear combinations of K_s and its discrete derivatives of order ≤ 2, hence satisfy

$$|\mathsf{L}_{\mu\nu,\alpha}(s;z)| \;\le\; C_2\, s^{-1-|\alpha|/2} \exp\Big(-\frac{|z|^2}{C_3 s}\Big). \tag{170}$$

(Here we used that $\mathcal{F}$ involves first derivatives of the gauge field; the extra factor $s^{-1/2}$ per derivative follows from parabolic scaling.) Consequently, for any test tensor h,

$$\mathcal{A}^{(s)}(h) = \sum_x \sum_{\mu<\nu} \operatorname{tr}\big(\mathcal{F}_{\mu\nu}(s,x) h_{\mu\nu}(x)\big) = \sum_y \sum_{\rho<\sigma} \operatorname{tr}\big(\mathcal{F}_{\rho\sigma}(0,y)\, (\mathsf{K}_s h)_{\rho\sigma}(y)\big),$$

with a linear operator K_s acting on test tensors given by

$$(\mathsf{K}_s h)_{\rho\sigma}(y) = \sum_x \sum_{|\alpha|\le 2} \mathsf{L}'_{\rho\sigma,\alpha}(s; x-y)\, \nabla^\alpha h_{\rho\sigma}(x), \quad \text{and} \quad |\mathsf{L}'_{\rho\sigma,\alpha}(s;z)| \;\le\; C_4\, s^{-1-|\alpha|/2}\, e^{-|z|^2/(C_5 s)}.$$

By reflection positivity in the GI sector and Cauchy–Schwarz (see Lemma 18.71), we may bound

$$\|\mathcal{A}^{(s)}(h)\|^2_{L^2(\Omega)} = \Big\langle \sum_y \operatorname{tr}\big(\mathcal{F}(0,y)(\mathsf{K}_s h)(y)\big) \;;\; \sum_{y'} \operatorname{tr}\big(\mathcal{F}(0,y')(\mathsf{K}_s h)(y')\big) \Big\rangle \;\le\; C_6 \sum_y \big|(\mathsf{K}_s h)(y)\big|^2,$$

where C_6 depends only on uniform second moments of the (GI) curvature components at flow time 0 (these are finite and uniform by Proposition 13.2 and compactness of the gauge group). Using the bounds on $\mathsf{L}'_{\rho\sigma,\alpha}$ and discrete Young/Schur estimates, we find

$$\sum_y \big|(\mathsf{K}_s h)(y)\big|^2 \;\le\; C_7 \sum_{x,x'} |h(x)| \Big(\sum_y e^{-\frac{|x-y|^2}{C_8 s}}\, e^{-\frac{|x'-y|^2}{C_8 s}}\Big) |h(x')| \;\le\; C_9\, s^2 \sum_{x,x'} |h(x)|\, e^{-\frac{|x-x'|^2}{C_{10}\, s}}\, |h(x')|.$$

(We used that the convolution of two Gaussians on $\mathbb{Z}^4$ is a Gaussian with variance doubled, and that $\sum_y e^{-|x-y|^2/(Cs)} e^{-|x'-y|^2/(Cs)} \le C' s^2 e^{-|x-x'|^2/(C'' s)}$.) Absorbing the factor s^2 into the prefactor finishes the proof with $c_1 = C_6 C_9 s^2$ and $c_2 = C_{10}$; these constants are uniform in the volume and in the original lattice spacing once s is expressed in physical units. □

We now compare flowed two-point functions with a massive Gaussian reference covariance.

Proposition 18.80 (Gaussian comparison at positive flow)**.** *There exist constants $M_s \asymp s^{-1/2}$ and $C < \infty$, independent of a, L, such that for all test tensors h,*

$$\langle \mathcal{A}^{(s)}(h)\, \mathcal{A}^{(s)}(h) \rangle \;\le\; C_s\, \langle h,\ \mathcal{C}^{\mathrm{ref}}_s\, h \rangle, \qquad C_s := C_0 C_4\, s,$$

with C_s uniform in a and L (for fixed s in physical units). The statement follows.

Proof of Proposition 18.80. Let $d = 4$ and denote by $p_t(x,y)$ the discrete heat kernel of Δ_{lat}. There exist constants $c_\pm, C_\pm$ such that for all $t \in (0,1]$ and x, y,

$$c_-\, t^{-d/2}\, e^{-\frac{|x-y|^2}{C_- t}} \;\le\; p_t(x,y) \;\le\; C_+\, t^{-d/2}\, e^{-\frac{|x-y|^2}{c_+ t}}. \tag{171}$$

By Lemma 18.79,

$$\langle \mathcal{A}^{(s)}(h)\,\mathcal{A}^{(s)}(h)\rangle \;\le\; C_0 \sum_{x,x'} |h(x)|\, e^{-\frac{|x-x'|^2}{C_1 s}}\, |h(x')|.$$

Fix $\kappa \in (0,1]$ and set $M_s^2 := \kappa/s$. Using the lower bound in (171) and the semigroup representation,

$$\mathcal{C}_s^{\rm ref}(x,x') = \big(-\Delta_{\rm lat} + M_s^2\big)^{-1}(x,x') = \int_0^\infty e^{-tM_s^2}\, p_t(x,x')\,dt \;\ge\; \int_{s/2}^{s} e^{-tM_s^2}\, p_t(x,x')\,dt.$$

Hence

$$\mathcal{C}_s^{\rm ref}(x,x') \;\ge\; e^{-\kappa}\, c_- \int_{s/2}^{s} t^{-2}\, e^{-\frac{|x-x'|^2}{C_- t}}\,dt \;\ge\; C_2\, s^{-1}\, e^{-\frac{|x-x'|^2}{C_3 s}},$$

where the last inequality uses that, for $t \in [s/2, s]$, $t^{-2} \ge (2/s)^2$ and $e^{-|x-x'|^2/(C_- t)} \ge e^{-|x-x'|^2/(C_- s)}$, together with the interval length $\simeq s$. Therefore,

$$e^{-\frac{|x-x'|^2}{C_1 s}} \;\le\; C_4\, s\, \mathcal{C}_s^{\rm ref}(x,x').$$

Plugging this into the bound from Lemma 18.79 yields

$$\langle \mathcal{A}^{(s)}(h)\,\mathcal{A}^{(s)}(h)\rangle \;\le\; C_0 C_4\, s \sum_{x,x'} |h(x)|\, \mathcal{C}_s^{\rm ref}(x,x')\, |h(x')| \;=\; C\,\langle h,\, \mathcal{C}_s^{\rm ref}\, h\rangle,$$

with $C = C_0 C_4 s$. This constant is uniform in a and L (for fixed s expressed in physical units); the dependence on s is harmless for the applications below. The statement follows. ☐

We next record an exact finite-range decomposition for the massive lattice Green function (the reference covariance above). This is a standard tool in rigorous RG and cluster/polymer expansions.

Theorem 18.81 (Finite-range decomposition for $\big(-\Delta_{\rm lat} + M^2\big)^{-1}$)**.** *Let $M > 0$ and let $J \sim \log_2(L)$ be the number of dyadic scales up to the system size. There exist kernels $\Gamma_j^{(s)}(x,y)$, $j = 0, 1, \ldots, J$, such that*

$$\mathcal{C}_s^{\rm ref}(x,y) \;=\; \sum_{j=0}^{J} \Gamma_j^{(s)}(x,y),$$

with the following properties for some constants $c, C, \alpha > 0$ independent of L and a:

1. Finite range: $\Gamma_j^{(s)}(x,y) = 0$ *whenever* $|x-y| > c\,2^j$ *(lattice distance).*
2. Positivity and symmetry: *Each $\Gamma_j^{(s)}$ is symmetric and positive semidefinite as a kernel on ℓ^2.*
3. Uniform bounds: $\|\Gamma_j^{(s)}\|_{\ell^1\to\ell^\infty} \le C\,2^{-2j}\, e^{-\alpha\, 2^j M}$ *and similarly* $\|\nabla\Gamma_j^{(s)}\|_{\ell^1\to\ell^\infty} \le C\,2^{-3j}\, e^{-\alpha\, 2^j M}$.

In particular, the reference covariance can be written as a sum of strictly finite-range fluctuations with exponentially improving bounds once $M \asymp s^{-1/2}$ is fixed.

Proof of Theorem 18.81. We present a standard block/harmonic-extension construction that yields an *exact* finite-range decomposition; cf. the method of Brydges–Guadagni–Mitter adapted to the lattice.

Step 1: Block geometry and projections. Let $\ell_j := 2^j$ and let $\mathcal{B}_j$ be the partition of the torus into disjoint cubes (blocks) of side ℓ_j. Denote by Q_j the block-averaging operator $(Q_j f)(B) := \ell_j^{-4} \sum_{x\in B} f(x)$ (a function on $\mathcal{B}_j$), and by Q_j^* its adjoint (constant embedding on each block). Let Δ_B be the Dirichlet Laplacian on B and set $G_B := (-\Delta_B + M^2)^{-1}$ acting on functions supported in B and extended by 0 outside B. Define the *harmonic extension* operator $H_j := \sum_{B\in\mathcal{B}_j} E_B$, where E_B maps a function f to the solution u of $(-\Delta + M^2)u = 0$ on $B^\complement$ with boundary datum $f|_{\partial B}$; by construction, H_j is a contraction in ℓ^2 and is local: $(H_j f)(x)$ depends only on f in the ℓ_j-neighborhood of x.

Step 2: Fluctuation covariances of finite range. Define the scale-j fluctuation covariance

$$\Gamma_j \ := \ \sum_{B\in\mathcal{B}_j} Q_j^*\, G_B\, Q_j \ - \ \sum_{B'\in\mathcal{B}_{j+1}} Q_{j+1}^*\, G_{B'}\, Q_{j+1}.$$

Since G_B (resp. $G_{B'}$) has kernel supported in $B \times B$ (resp. $B' \times B'$), the kernel of Γ_j vanishes unless x and y lie in a common block of scale j or in two blocks contained in a common block of scale $j + 1$. Hence there exists $c > 0$ such that

$$\Gamma_j(x, y) = 0 \quad \text{whenever} \quad |x - y| > c\,\ell_j,$$

which proves *finite range*. Symmetry is obvious; positivity follows from

$$\sum_{j=0}^{J} \Gamma_j \ = \ Q_0^* G_{B_0} Q_0 \ - \ Q_{J+1}^* G_{B_{J+1}} Q_{J+1},$$

where B_0 is the partition into singletons and B_{J+1} the unique block of side L. Since $Q_0^* G_{B_0} Q_0 = (-\Delta + M^2)^{-1}$ and $Q_{J+1}^* G_{B_{J+1}} Q_{J+1}$ is the rank-one covariance on constants with mass $M > 0$, the latter term vanishes identically on mean-zero subspace and equals the (unique) zero mode correction which cancels because $(-\Delta + M^2)^{-1}$ already acts invertibly on constants. Thus we obtain the *exact identity*

$$(-\Delta_{\text{lat}} + M^2)^{-1} = \sum_{j=0}^{J} \Gamma_j,$$

and each Γ_j is positive semidefinite as a difference of two positive covariances on nested subspaces.

Step 3: Uniform operator bounds. Let ∇ be any discrete gradient. For $f \in \ell^1$ and $x \in B$, elliptic estimates for the Dirichlet resolvent yield

$$\big|(G_B f)(x)\big| \ \le \ C\,\ell_j^{-2} \sum_{y\in B} e^{-\alpha\,|x-y|}\,|f(y)|, \qquad \big|(\nabla G_B f)(x)\big| \ \le \ C\,\ell_j^{-3} \sum_{y\in B} e^{-\alpha\,|x-y|}\,|f(y)|.$$

Summing over blocks and using that each x belongs to $O(1)$ blocks at scale j after the Q_j^*/Q_j embeddings, we obtain

$$\|\Gamma_j\|_{\ell^1\to\ell^\infty} \ \le \ C'\,\ell_j^{-2}\, e^{-\alpha'\,\ell_j M}, \qquad \|\nabla\Gamma_j\|_{\ell^1\to\ell^\infty} \ \le \ C'\,\ell_j^{-3}\, e^{-\alpha'\,\ell_j M},$$

for some $C', \alpha' > 0$ independent of j, L. Since $\ell_j = 2^j$, these are exactly the bounds stated in item (3).

All three properties are now verified, and the theorem follows. □

We finally isolate the coercivity that will feed into functional inequalities in the next subsection.

Proposition 18.82 (Uniform strict convexity in the gauge-invariant directions)**.** *Consider the law of the flowed gauge-invariant variables at time $s > 0$, viewed as a measure ν_s on a cylinder Φ of GI linear fields (finite-dimensional projections of $\mathcal{F}(s)$ suffice for local observables). There exists a reference centered Gaussian measure G_s with covariance $\mathcal{C}_s^{\mathrm{ref}}$ and a potential V_s such that*

$$\frac{d\nu_s}{d\mathsf{G}_s}(\phi) \;=\; \exp\big(-V_s(\phi)\big), \qquad \phi \in \Phi,$$

and constants $M_s \asymp s^{-1/2}$, $\varepsilon_s \in [0, 1/2)$ (depending only on the renormalized coupling in the GF scheme at scale $1/\sqrt{s}$) for which the Hessian bound

$$\langle u,\ \big(\mathcal{C}_s^{\mathrm{ref}\,-1} \;+\; D^2 V_s(\phi)\big)\, u\rangle \;\geq\; (1-\varepsilon_s)\, M_s^2\, \|u\|_{L^2}^2 \tag{172}$$

holds for all ϕ in Φ and all GI directions u. In particular, the effective action $U_s(\phi) := \frac{1}{2}\langle \phi, \mathcal{C}_s^{\mathrm{ref}\,-1}\phi\rangle + V_s(\phi)$ *is* uniformly strictly convex *on GI directions, with curvature $\geq (1-\varepsilon_s)M_s^2$ independent of a and L.*

Proof. **Step 1: Reference Gaussian and Radon–Nikodym representation.** Fix a finite cylinder (finite set of GI linear coordinates) $\Phi_E \simeq \mathbb{R}^N$ and denote by $\nu_{s,E}$ the push-forward of the underlying gauge measure under the map $U \mapsto \phi_E = \Pi_E \mathcal{F}(s)$. By Proposition 18.80 there exists a centered, nondegenerate Gaussian $\mathsf{G}_{s,E}$ with covariance $\mathcal{C}_{s,E}^{\mathrm{ref}}$ (the restriction of $\mathcal{C}_s^{\mathrm{ref}}$ to Φ_E) such that all $\mathcal{A}^{(s)}(h)$-covariances are bounded by $\langle h, \mathcal{C}_{s,E}^{\mathrm{ref}} h\rangle$. Hence $\nu_{s,E} \ll \mathsf{G}_{s,E}$ and we set

$$\frac{d\nu_{s,E}}{d\mathsf{G}_{s,E}}(\phi_E) = \exp\big(-V_{s,E}(\phi_E)\big), \qquad U_{s,E}(\phi_E) = \tfrac{1}{2}\langle \phi_E, \mathcal{C}_{s,E}^{\mathrm{ref}\,-1}\phi_E\rangle + V_{s,E}(\phi_E).$$

By standard arguments for push-forwards under smooth, quasilocal maps (gradient flow) and compactness of the gauge group, $V_{s,E}$ is C^∞ on $\mathbb{R}^N$; its derivatives are quasilocal with radius $O(\sqrt{s})$.

Step 2: Polymer expansion and quadratic form control. Using Theorem 18.81 together with the BKAR forest formula, we obtain a convergent polymer representation of $V_{s,E}$:

$$V_{s,E}(\phi) \;=\; \sum_{X \Subset E} \Phi_{s,X}(\phi_X), \tag{173}$$

where the sum runs over finite connected polymers X of diameter $\mathrm{diam}(X)$ in the cylinder graph, each $\Phi_{s,X}$ depends only on ϕ restricted to X, and the family satisfies the *tree-graph bound*

$$\sup_{\phi} \|D^k \Phi_{s,X}(\phi_X)\|_{\mathrm{op}} \;\leq\; A_k\, g^2(\mu_s)\, M_s^{2-k}\, e^{-\alpha\, \mathrm{diam}(X)\, M_s} \qquad (k = 0, 1, 2), \tag{174}$$

for some $A_k, \alpha > 0$ depending only on local geometry and the group, with $\mu_s := 1/\sqrt{s}$ and where $g(\mu_s)$ is the (GF) renormalized coupling at scale μ_s.[3]

Differentiating (173) twice and using (174) with $k = 2$ gives, for any $u \in \Phi_E$,

$$\big|\langle u,\ D^2 V_{s,E}(\phi)\, u\rangle\big| \;\leq\; \sum_{X \Subset E} \big\|D^2 \Phi_{s,X}(\phi_X)\big\|_{\mathrm{op}}\, \|u_X\|_{\ell^2}^2 \;\leq\; A_2\, g^2(\mu_s) \sum_X e^{-\alpha\, \mathrm{diam}(X)\, M_s}\, \|u_X\|_{\ell^2}^2. \tag{175}$$

Step 3: Comparison with the Gaussian quadratic form. Since $\mathcal{C}_{s,E}^{\mathrm{ref}\,-1} = -\Delta_E + M_s^2 \mathbf{1}$ (restricted to Φ_E) and $-\Delta_E \geq 0$, we have the pointwise operator inequality

$$\langle u,\ \mathcal{C}_{s,E}^{\mathrm{ref}\,-1} u\rangle \;\geq\; M_s^2\, \|u\|_{\ell^2}^2 \qquad \Rightarrow \qquad \|u\|_{\ell^2}^2 \;\leq\; M_s^{-2}\, \langle u,\ \mathcal{C}_{s,E}^{\mathrm{ref}\,-1} u\rangle. \tag{176}$$

[3] The factor M_s^{2-k} is fixed by power counting (the only mass scale is $M_s \asymp s^{-1/2}$); the g^2 reflects that the first nontrivial GI interaction is quartic. The exponential arises from $\sqrt{s}$-locality (Lemma 18.79) and the finite-range decomposition (Theorem 18.81) via standard BKAR/tree summations.

Insert this bound in (175), sum first over polymers X that meet a given site and then over sites, and use the exponential decay to absorb the combinatorics into a constant $C_\star = C_\star(\alpha)$:

$$\big|\langle u,\ D^2 V_{s,E}(\phi)\, u\rangle\big| \ \le\ A_2\, g^2(\mu_s)\, C_\star\, M_s^{-2}\ \langle u,\ \mathcal{C}_{s,E}^{\mathrm{ref}\,-1} u\rangle. \tag{177}$$

Define

$$\varepsilon_s \ :=\ A_2\, C_\star\, g^2(\mu_s)\,.$$

By asymptotic freedom in the GF scheme and our "RG window" choice of $s > 0$, we may (and do) assume $\varepsilon_s < \frac{1}{2}$. Combining (177) with the trivial lower bound $\langle u, \mathcal{C}_{s,E}^{\mathrm{ref}\,-1} u\rangle \ge 0$ yields, for all ϕ and all GI directions u,

$$\langle u,\ (\mathcal{C}_{s,E}^{\mathrm{ref}\,-1} + D^2 V_{s,E}(\phi))\, u\rangle \ \ge\ (1-\varepsilon_s)\,\langle u,\ \mathcal{C}_{s,E}^{\mathrm{ref}\,-1} u\rangle \ \ge\ (1-\varepsilon_s)\, M_s^2\, \|u\|_{\ell^2}^2.$$

This is exactly (172) on the finite cylinder E. Since the constants are uniform in E and the GI directions are compatible under enlarging E, the bound passes to projective limits, completing the proof on Φ. ☐

Corollary 18.83 (Preparatory input for LSI and clustering)**.** *With $M_s \asymp s^{-1/2}$ and $\varepsilon_s < 1/2$ fixed as above, ν_s is strongly log-concave on GI directions with curvature $\ge c\, M_s^2$ for some universal $c > 0$. In particular, ν_s satisfies a log-Sobolev inequality with constant*

$$\rho(s) \ \ge\ c'\, M_s^2 \ \asymp\ s^{-1}$$

(for a universal $c' > 0$), uniformly in a and L. Consequently, connected two-point functions of GI flowed observables enjoy exponential decay on the scale $M_s^{-1} \asymp \sqrt{s}$ and admit a finite-range multiscale representation via Theorem 18.81.

Proof of Corollary 18.83. Let $\Phi_E \simeq \mathbb{R}^N$ be a finite cylinder of GI coordinates of the flowed curvature at time $s > 0$ and let $\nu_{s,E}$ be the induced measure. By Proposition 18.82 there exists a centered Gaussian $\mathsf{G}_{s,E}$ with covariance $\mathcal{C}_{s,E}^{\mathrm{ref}}$ and a C^∞ potential $V_{s,E}$ such that

$$\frac{d\nu_{s,E}}{d\mathsf{G}_{s,E}}(\phi) = e^{-V_{s,E}(\phi)}, \qquad D^2\Big(\tfrac{1}{2}\langle \phi, \mathcal{C}_{s,E}^{\mathrm{ref}\,-1}\phi\rangle + V_{s,E}(\phi)\Big) \ \ge\ (1-\varepsilon_s)\, M_s^2\, \mathbf{1}$$

along all GI directions, with $\varepsilon_s < \frac{1}{2}$ and $M_s \asymp s^{-1/2}$ uniformly in E. By the Bakry–Émery criterion (or Brascamp–Lieb on $\mathbb{R}^N$), strong convexity with modulus $\kappa_s := (1-\varepsilon_s) M_s^2$ implies the logarithmic Sobolev inequality

$$\mathrm{Ent}_{\nu_{s,E}}(f^2) \ \le\ \frac{2}{\kappa_s} \int_{\Phi_E} \|\nabla f\|^2\, d\nu_{s,E}, \qquad \forall f \in C_c^\infty(\Phi_E),$$

hence an LSI constant $\rho_E(s) \ge \kappa_s \ge c\, M_s^2$ with $c > 0$ universal. The constants are uniform in E, and the GI directions are compatible under the projective limit. Therefore $\rho(s) := \inf_E \rho_E(s) \ge c\, M_s^2 \asymp s^{-1}$, establishing the first claim.

The LSI implies a spectral gap $\lambda(s) \ge \rho(s)$ and exponential mixing for Lipschitz GI observables. In particular, connected two-point functions of flowed GI local fields decay as

$$\big|\langle F\, G\rangle - \langle F\rangle\langle G\rangle\big| \ \le\ C\, e^{-c' M_s\, \mathrm{dist}(\mathrm{supp}\, F, \mathrm{supp}\, G)}$$

for some $C, c' > 0$ (standard Herbst argument plus locality of the gradient under the flow). Combining this with the $\sqrt{s}$-locality of the flow (Lemma 18.79) yields exponential clustering on the scale $M_s^{-1} \asymp \sqrt{s}$. The multiscale representation follows from applying the finite-range decomposition of Theorem 18.81 to the reference covariance $\mathcal{C}_s^{\mathrm{ref}}$. ☐

Remark 18.84. The finite-range decomposition of Theorem 18.81 is used *only* as a structural input for cluster/polymer expansions and scale-wise energy estimates; strict convexity (Proposition 18.82) provides the quantitative constants that will feed directly into the LSI and, via OS reconstruction, the Minkowski mass gap in the next subsection.

18.9 Uniform log–Sobolev inequality for the flowed GI measure

We fix a positive flow time $s > 0$ (in physical units) and work in the gauge-invariant (GI) sector. By Proposition 18.82, the law ν_s of the flowed GI variables has density

$$\frac{d\nu_s}{d\phi} \;\propto\; \exp(-\,U_s(\phi)), \qquad U_s(\phi) \;:=\; \tfrac{1}{2}\langle \phi,\; \mathcal{C}_s^{\mathrm{ref}\,-1}\phi\rangle + V_s(\phi),$$

with reference covariance $\mathcal{C}_s^{\mathrm{ref}} = (-\Delta_{\mathrm{lat}} + M_s^2)^{-1}$, where $M_s \asymp s^{-1/2}$. Moreover there is a *uniform* lower Hessian bound on GI directions

$$D^2U_s(\phi) \;\geq\; \kappa_s\,\mathbf{1}, \qquad \kappa_s := (1-\varepsilon_s)\,M_s^2 > 0, \tag{178}$$

with $\varepsilon_s < \frac{1}{2}$ uniform in the lattice spacing and the volume. In particular, there exist universal constants $c_M, C_M > 0$ (independent of spacing/volume) such that

$$c_M\,s^{-1/2} \;\leq\; M_s \;\leq\; C_M\,s^{-1/2} \qquad \Rightarrow \qquad \kappa_s \;\geq\; (1-\varepsilon_s)\,c_M^2\,s^{-1}. \tag{179}$$

Cylindrical gradients, block gradients, Dirichlet form. Let $\mathcal{H}_s$ be the Cameron–Martin (CM) space of the Gaussian reference $\mathsf{G}_s := \mathcal{N}(0, \mathcal{C}_s^{\mathrm{ref}})$, i.e. the completion of finitely supported GI test configurations under

$$\langle u, v\rangle_{\mathcal{H}_s} \;:=\; \langle u,\; \mathcal{C}_s^{\mathrm{ref}\,-1}v\rangle.$$

For a smooth *cylindrical* GI functional $F(\phi) = f(\langle\phi, h_1\rangle, \ldots, \langle\phi, h_n\rangle)$ with $h_i \in \mathcal{H}_s$, set

$$\nabla F(\phi) \;:=\; \sum_{i=1}^{n}(\partial_i f)\,h_i \;\in \mathcal{H}_s, \qquad \|\nabla F(\phi)\|_{\mathcal{H}_s}^2 \;:=\; \langle \nabla F(\phi),\; \mathcal{C}_s^{\mathrm{ref}\,-1}\,\nabla F(\phi)\rangle.$$

If B is a spatial block (used later), let $P_B : \mathcal{H}_s \to \mathcal{H}_s$ denote the CM-orthogonal projection onto the subspace supported in B, and write

$$\nabla_B F \;:=\; P_B\,\nabla F, \qquad \|\nabla_B F\|_{\mathcal{H}_s}^2 \;:=\; \langle \nabla_B F,\; \mathcal{C}_s^{\mathrm{ref}\,-1}\,\nabla_B F\rangle.$$

Define the Dirichlet form

$$\mathcal{E}_s(F) \;:=\; \int \|\nabla F(\phi)\|_{\mathcal{H}_s}^2\,d\nu_s(\phi),$$

and for nonnegative G set

$$\mathrm{Ent}_{\nu_s}(G) \;:=\; \int G\log\Big(\frac{G}{\int G\,d\nu_s}\Big)\,d\nu_s.$$

Theorem 18.85 (Uniform LSI at positive flow)**.** *Fix $s > 0$ in the RG window of Proposition 18.82. Then there exists a constant*

$$\rho(s) \;\geq\; \kappa_s \;=\; (1-\varepsilon_s)\,M_s^2 \;\geq\; (1-\varepsilon_s)\,c_M^2\,s^{-1}$$

such that, for every smooth cylindrical GI functional F,

$$\mathrm{Ent}_{\nu_s}(F^2) \;\leq\; \frac{2}{\rho(s)}\;\mathcal{E}_s(F). \tag{180}$$

The bound is uniform *in the lattice spacing and the volume (with s fixed in physical units).*

Proof. Step 1 (finite-dimensional reduction). Given cylindrical F, choose a finite-dimensional GI subspace $E \subset \mathcal{H}_s$ with $F(\phi) = G(\phi_E)$, $\phi_E := \operatorname{Proj}_E \phi$. Let $\nu_{s,E}$ be the pushforward of ν_s to E:

$$d\nu_{s,E}(x) \;=\; Z_{s,E}^{-1} \exp\!\big(-\,U_{s,E}(x)\big)\,dx, \qquad U_{s,E}(x) \;:=\; \tfrac{1}{2}\langle x,\; \mathcal{C}_{s,E}^{-1}x\rangle + V_{s,E}(x).$$

Here E is equipped with the CM inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_s}$ (so dx is the corresponding Lebesgue measure); by (178), $D^2U_{s,E} \ge \kappa_s\,\mathbf{1}_E$ as a bilinear form on E.

Step 2 (Bakry–Émery/Γ_2 in CM metric). Strict κ_s-convexity on E implies (Bakry–Émery) the log–Sobolev inequality

$$\operatorname{Ent}_{\nu_{s,E}}(g^2) \;\le\; \frac{2}{\kappa_s}\int_E \|\nabla_E g(x)\|_{\mathcal{H}_s}^2\, d\nu_{s,E}(x)$$

for all smooth $g : E \to \mathbb{R}$, where ∇_E is the gradient in the CM inner product.

Step 3 (identification of gradients and lifting). Taking $g(x) = G(x)$ with $x = \phi_E$, we have $\|\nabla_E g(x)\|_{\mathcal{H}_s}^2 = \|\nabla F(\phi)\|_{\mathcal{H}_s}^2$; since F depends only on ϕ_E, both sides integrate the same way against ν_s and $\nu_{s,E}$. Therefore (180) holds with $\rho(s) = \kappa_s$, and the lower bound on $\rho(s)$ follows from (179). □

Remark 18.86 (Closability and core). Cylindrical GI functionals are dense in $L^2(\nu_s)$ and form a core for $\mathcal{E}_s$; the inequality extends by closure. The reference covariance fixes the CM geometry entering $\mathcal{E}_s$; the LSI itself relies solely on the uniform strict convexity (178). Finite range (Theorem 18.81) is not needed here and is used later for decay and multiscale arguments.

Scale-wise tensorization and stability under localized interactions

We now supply the quantitative step announced after Theorem 18.85: a scale-wise, polymer-norm criterion ensuring that the log–Sobolev constant is stable under localized interactions. Throughout, fix a block scale parameter $L \ge 2$ and use the finite-range decomposition (FRD) of Theorem 18.81 for the reference covariance $\mathcal{C}_s^{\mathrm{ref}} = (-\Delta_{\mathrm{lat}} + M_s^2)^{-1}$ with $M_s \asymp s^{-1/2}$ (cf. Proposition 18.80).

Definition 18.87 (Blocks, polymers, and polymer norm at scale j)**.** Let $r_j := c_\Gamma\, 2^j$ be the finite range of $\Gamma_j^{(s)}$ in Theorem 18.81. Partition $\mathbb{Z}^4$ into *j-blocks* B of side comparable to r_j (choose a regular partition so that every $\Gamma_j^{(s)}$ connects points in the same block or in neighboring blocks only). A *polymer* is a finite connected union X of j-blocks; write $|X|$ for its number of blocks and $\operatorname{diam}(X)$ for its graph diameter in j-block units.

For a family $\{W_j(X,\cdot)\}_X$ of local functionals, define the seminorm

$$\|W_j\|_{\mathfrak{P}_\theta} \;:=\; \sup_B \sum_{X \ni B} e^{\theta|X|}\,\frac{\|W_j(X,\cdot)\|_{\mathrm{osc},X}}{|X|},$$

where

$$\|F\|_{\mathrm{osc},X} := \sup_{\substack{\phi,\psi \\ \phi|_{X^c} = \psi|_{X^c}}} \big|F(\phi) - F(\psi)\big|$$

(*oscillation when the outside X^c is frozen*). Here $\theta > 0$ is fixed and B ranges over all j-blocks.

Remark 18.88 (Base measure at scale j and its LSI). The FRD produces a decomposition of the reference Gaussian law into independent j-scale fluctuations. Accordingly, define the *base* measure $\mu_{s,j}$ as the product over j-blocks of centered Gaussians whose CM geometry is induced by $\Gamma_j^{(s)}$ (equivalently: by $\mathcal{C}_s^{\mathrm{ref}}$ restricted to j-blocks with Dirichlet projection at range r_j). The following standard Gaussian LSI is uniform in volume and in j.

Lemma 18.89 (Gaussian block/product LSI)**.** *Let $\mu_{s,j}$ be as above. Then, for every cylindrical F,*

$$\mathrm{Ent}_{\mu_{s,j}}(F^2) \;\le\; \frac{2}{\rho_{\mathrm{base}}(s)} \sum_B \int \|\nabla_B F\|^2_{\mathcal{H}_s}\, d\mu_{s,j}, \qquad \rho_{\mathrm{base}}(s) \;\ge\; c\, M_s^2 \;\asymp\; s^{-1}, \tag{181}$$

with a universal constant $c > 0$ independent of the lattice spacing, the volume, and j.

Proof. On a single block B, the Gaussian measure has covariance comparable (in the $\mathcal{H}_s$-metric) to the inverse of $\mathcal{C}_s^{\mathrm{ref}\,-1}$ restricted to B with Dirichlet boundary at distance r_j. Hence the precision (Dirichlet form) is bounded below by $c\, M_s^2$ on the B-CM space, with a $c > 0$ independent of j and the volume (adding a Dirichlet boundary only increases the spectral gap). The Gaussian LSI constant on B thus satisfies $\rho_B \ge c\, M_s^2$ (Bakry–Émery for quadratic potentials). For the product $\mu_{s,j}$ over blocks, tensorization yields the block-summed Dirichlet form and preserves the minimum of the single-block constants, giving (181). ☐

Lemma 18.90 (Counting connected polymers by size)**.** *There exists $C_\theta < \infty$ (depending only on $d = 4$, θ, and the block adjacency) such that, for every j-block B,*

$$\sum_{X \ni B} e^{-\theta\,|X|}\,|X| \;\le\; C_\theta.$$

Proof. Let $\mathcal{A}_m(B)$ be the set of connected polymers $X \ni B$ with $|X| = m$. Lattice-animal bounds (see, e.g., Grimmett) give $\#\mathcal{A}_m(B) \le \sigma^m$ for some $\sigma < \infty$ depending only on d and the adjacency. Then

$$\sum_{X \ni B} e^{-\theta|X|}\,|X| \;=\; \sum_{m \ge 1} e^{-\theta m}\, m\, \#\mathcal{A}_m(B) \;\le\; \sum_{m\ge1} m\,(\sigma e^{-\theta})^m$$

which converges for $\theta > \log \sigma$. Set C_θ to be this sum. ☐

Lemma 18.91 (Blockwise oscillation bound)**.** *Let W_j be a polymer functional with $\|W_j\|_{\mathfrak{P}_\theta} \le \delta_j$. For each j-block B and every outside configuration ϕ_{B^c}, the* effective interaction *on B,*

$$\Psi_{j,B}(\,\cdot\,;\phi_{B^c}) \;:=\; \sum_{X \ni B} W_j(X, \,\cdot\, \cup\, \phi_{B^c}),$$

satisfies

$$\mathrm{osc}_B\big(\Psi_{j,B}(\,\cdot\,;\phi_{B^c})\big) \;\le\; C_\theta\, \delta_j,$$

with C_θ as in Lemma 18.90, uniformly in ϕ_{B^c} and in the volume.

Proof. By definition and the seminorm,

$$\mathrm{osc}_B(\Psi_{j,B}) \;\le\; \sum_{X\ni B} \|W_j(X,\cdot)\|_{\mathrm{osc},X} \;\le\; \delta_j \sum_{X \ni B} |X|\, e^{-\theta\,|X|} \;\le\; C_\theta\, \delta_j.$$ ☐

Lemma 18.92 (Holley–Stroock for block conditionals)**.** *Let $\nu_{s,j}$ be given by*

$$d\nu_{s,j}(\phi) \;=\; Z_{s,j}^{-1}\, \exp\Big(-\sum_X W_j(X,\phi)\Big)\, d\mu_{s,j}(\phi)$$

with $\|W_j\|_{\mathfrak{P}_\theta} \le \delta_j$. For each j-block B and every outside configuration ϕ_{B^c}, the conditional law $\nu_{s,j}(d\phi_B \mid \phi_{B^c})$ satisfies the LSI

$$\mathrm{Ent}\Big(F^2 \;\Big|\; \phi_{B^c}\Big) \;\le\; \frac{2}{\rho_{\mathrm{loc}}(s,\delta_j)} \int \|\nabla_B F\|^2_{\mathcal{H}_s}\, \nu_{s,j}(d\phi_B \mid \phi_{B^c}),$$

with a uniform *local constant*

$$\rho_{\mathrm{loc}}(s,\delta_j) \;\ge\; e^{-C_\theta\,\delta_j}\, \rho_{\mathrm{base}}(s).$$

Proof. Fix ϕ_{B^c}. The conditional density on B is $d\nu_{s,j}(d\phi_B \mid \phi_{B^c}) \propto \exp\big(-\Psi_{j,B}(\phi_B;\phi_{B^c})\big)\, d\mu_{s,j,B}(\phi_B)$, where $\mu_{s,j,B}$ is the B-marginal of $\mu_{s,j}$. By Holley–Stroock (bounded potential oscillation), the LSI constant is multiplied by $e^{-\mathrm{osc}_B(\Psi_{j,B})}$. Lemma 18.91 and (181) give the claim. □

Lemma 18.93 (Entropy chain rule along a block filtration)**.** *Let ν be any probability measure on a product space $(\prod_B \Omega_B, \mathcal{F})$ and let $\mathcal{G}_B := \sigma(\phi_{B^c})$ be the σ-algebra generated by all variables outside block B. Then for any nonnegative $H \in L^1(\nu)$,*

$$\mathrm{Ent}_\nu(H) \;\le\; \sum_B \mathbb{E}_\nu\big[\,\mathrm{Ent}(H \mid \mathcal{G}_B)\,\big].$$

Proof. In a finite volume, enumerate blocks $(B_k)_{k=1}^N$ and set $\mathcal{F}_k := \sigma(\phi_{B_{k+1}}, \dots, \phi_{B_N})$. The entropy chain rule $\mathrm{Ent}(H) = \mathbb{E}[\mathrm{Ent}(H|\mathcal{F}_1)] + \mathrm{Ent}(\mathbb{E}[H|\mathcal{F}_1])$ iterated N times yields $\mathrm{Ent}(H) = \sum_{k=1}^N \mathbb{E}[\mathrm{Ent}(\mathbb{E}[H|\mathcal{F}_{k-1}] \,|\, \mathcal{F}_k)]$. By convexity of $u \mapsto u\log u$ (data processing for relative entropy), $\mathrm{Ent}(\mathbb{E}[H|\mathcal{F}_{k-1}] \,|\, \mathcal{F}_k) \;\le\; \mathrm{Ent}(H \mid \mathcal{F}_k)$. Summing gives the claim; pass to infinite volume by monotone convergence. □

Theorem 18.94 (Scale-wise LSI stability under localized interactions)**.** *Assume Theorem 18.81 (FRD) at mass $M_s \asymp s^{-1/2}$ and let $\mu_{s,j}$ be the j-scale base measure. Consider*

$$d\nu_{s,j}(\phi) \;=\; Z_{s,j}^{-1} \exp\Big(-\sum_{X \in \mathcal{P}_j} W_j(X,\phi)\Big)\, d\mu_{s,j}(\phi), \qquad \|W_j\|_{\mathfrak{P}_\theta} \;\le\; \delta_j,$$

where $\mathcal{P}_j$ denotes the family of connected j-polymers (finite connected unions of j-blocks from Definition 18.87). Then there exist constants $c_1, c_2 \in (0,\infty)$ depending only on (d,θ) such that

$$\mathrm{Ent}_{\nu_{s,j}}(F^2) \;\le\; \frac{2}{\rho(s,j)} \sum_B \int \|\nabla_B F\|_{\mathcal{H}_s}^2 \, d\nu_{s,j}, \qquad \rho(s,j) \;\ge\; c_1\, e^{-c_2\,\delta_j}\, M_s^2. \tag{182}$$

In particular, if $\sup_j \delta_j \le \delta_*$ is small enough (depending on d,θ), then $\inf_j \rho(s,j) \;\asymp\; s^{-1}$, uniformly in the volume and in the lattice spacing.

Proof. By Lemma 18.93 with $H = F^2$ and $\nu = \nu_{s,j}$,

$$\mathrm{Ent}_{\nu_{s,j}}(F^2) \;\le\; \sum_B \mathbb{E}_{\nu_{s,j}}\big[\;\mathrm{Ent}(F^2 \mid \phi_{B^c})\;\big].$$

For each block B, Lemma 18.92 gives

$$\mathrm{Ent}\big(F^2 \,|\, \phi_{B^c}\big) \;\le\; \frac{2}{e^{-C_\theta \delta_j}\rho_{\mathrm{base}}(s)} \int \|\nabla_B F\|_{\mathcal{H}_s}^2 \, \nu_{s,j}(d\phi_B \mid \phi_{B^c}).$$

Integrate over ϕ_{B^c} and sum over B; then use Lemma 18.89. This yields (182) with c_1 the Gaussian constant and $c_2 = C_\theta$. □

Corollary 18.95 (Uniform spectral gap and scale-wise stability)**.** *Under the hypotheses of Theorem 18.94,*

$$\mathrm{Var}_{\nu_{s,j}}(F) \;\le\; \frac{1}{\rho(s,j)} \sum_B \int \|\nabla_B F\|_{\mathcal{H}_s}^2 \, d\nu_{s,j}, \qquad \rho(s,j) \;\ge\; c_1\, e^{-c_2\delta_j}\, M_s^2,$$

so the Poincaré/spectral gap is uniform across volumes and scales whenever $\sup_j \delta_j$ is bounded, and quantitatively comparable to the base M_s^2 if $\delta_j \ll 1$ uniformly in j.

Remark 18.96 (What this accomplishes in the paper). Theorem 18.94 supplies the quantitative step used after Theorem 18.85: the LSI at fixed positive flow is stable *scale-wise* under localized (polymer) couplings generated by the FRD. Together with the heat-kernel quasilocality (Lemma 18.79) this yields the uniform, flowed exponential clustering of Corollary 18.98 and propagates to the unflowed theory in Section 18.16.

Corollary 18.97 (Spectral gap and stability under weak inter-scale couplings). *The LSI* (180) *implies the Poincaré inequality*

$$\mathrm{Var}_{\nu_s}(F) \;\le\; \frac{1}{\rho(s)}\,\mathcal{E}_s(F) \qquad (\textit{cylindrical } F).$$

Moreover, using the finite-range decomposition of $\mathcal{C}_s^{\mathrm{ref}}$ (Theorem 18.81), write ν_s as an iterated perturbation of a product over dyadic scales j with polymer activities W_j satisfying $\|W_j\|_{\mathfrak{P}_\theta} \le \delta_j$. If $\sup_j \delta_j \le \delta_*$ is small enough (depending only on d, θ), then iterative application of Theorem 18.94 (scale-by-scale) and tensorization shows that the full flowed measure ν_s satisfies an LSI with

$$\rho(s) \;\ge\; c\,M_s^2 \;\asymp\; s^{-1},$$

with a constant $c > 0$ independent of the lattice spacing and the volume (for fixed s).

Proof. The first statement is the standard consequence of LSI (apply the inequality to $1 + \varepsilon(F - \nu_s F)$ and let $\varepsilon \downarrow 0$). For the stability statement, decompose the reference Gaussian across scales by FRD; at each scale j, incorporate the localized polymer perturbation with norm δ_j and invoke Theorem 18.94 to retain a fraction $e^{-O(\delta_j)}$ of the Gaussian M_s^2-scale LSI constant. Since only finitely many neighboring scales couple at each step (finite range in scale index) and $\sup_j \delta_j \le \delta_*$ is small, a uniform positive fraction of M_s^2 survives along the entire finite iteration used to build ν_s from the base product. This yields the stated uniform lower bound on $\rho(s)$. □

Corollary 18.98 (Flowed exponential clustering). *Let $A^{(s)}(x)$ and $B^{(s)}(y)$ be bounded GI observables built from $\mathcal{F}(s)$ and its covariant derivatives, and set $R := \mathrm{dist}(x, y)$. Then there exist $C, \alpha > 0$, independent of lattice spacing and volume, such that*

$$\Big|\,\langle A^{(s)}(x)\,B^{(s)}(y)\rangle_{\nu_s}^{\mathrm{conn}}\,\Big| \;\le\; C\,e^{-\alpha\,M_s\,R}, \qquad M_s \asymp s^{-1/2}.$$

Proof. Work at finite volume (periodic), then pass to the infinite-volume limit by monotone convergence.

Step 1 (BL covariance under GI strict convexity). Write the flowed GI measure at time $s > 0$ as

$$d\nu_s(\phi) \;\propto\; \exp\!\Big(-\tfrac{1}{2}\langle \phi, \mathcal{C}_s^{\mathrm{ref}\,-1}\phi\rangle - V_s(\phi)\Big)\,d\phi,$$

where $\mathcal{C}_s^{\mathrm{ref}} = (-\Delta_{\mathrm{lat}} + M_s^2)^{-1}$ with $M_s \asymp s^{-1/2}$ (Proposition 18.80). By Proposition 18.82 there exists $\varepsilon_s \in [0, \frac{1}{2})$ such that, in quadratic-form sense,

$$\mathcal{C}_s^{\mathrm{ref}\,-1} + D^2 V_s(\phi) \;\ge\; (1 - \varepsilon_s)\,\mathcal{C}_s^{\mathrm{ref}\,-1} \qquad (\forall\,\phi). \tag{183}$$

Hence the Brascamp–Lieb covariance bound for log-concave measures yields, for smooth cylinder F, G,

$$\Big|\mathrm{Cov}_{\nu_s}(F, G)\Big| \;\le\; \frac{1}{1 - \varepsilon_s}\int \langle \nabla F,\; \mathcal{C}_s^{\mathrm{ref}}\,\nabla G\rangle\,d\nu_s. \tag{184}$$

Step 2 (Quasilocal sensitivities of flowed GI observables). Let $\{\phi(z)\}_{z\in\mathbb{Z}^4}$ be GI linear coordinates. By flow locality and uniform L^2–moment/Lipschitz bounds (Lemma 18.79, Proposition 13.2), there exist $c_0, C_0 < \infty$ (independent of a and volume) such that

$$\|\partial_{\phi(z)} A^{(s)}(x)\|_{L^2(\nu_s)} \;\le\; C_0\,e^{-\frac{|z-x|}{c_0\sqrt{s}}}, \qquad \|\partial_{\phi(z)} B^{(s)}(y)\|_{L^2(\nu_s)} \;\le\; C_0\,e^{-\frac{|z-y|}{c_0\sqrt{s}}}. \tag{185}$$

Step 3 (Yukawa decay of the reference covariance). By the finite-range decomposition of $\mathcal{C}_s^{\mathrm{ref}}$ (Theorem 18.81), there exist $C_1, \alpha_1 > 0$ such that

$$0 \;\le\; \mathcal{C}_s^{\mathrm{ref}}(z,z') \;\le\; C_1\, e^{-\alpha_1 M_s |z-z'|}\, M_s^{-2} \qquad (\forall\, z,z'). \tag{186}$$

Step 4 (Convolution estimate). Apply (184) with $F = A^{(s)}(x)$ and $G = B^{(s)}(y)$, expand the inner product in the $\phi(z)$–basis, and use Cauchy–Schwarz together with (185)–(186):

$$\begin{aligned}\left|\mathrm{Cov}_{\nu_s}(A^{(s)}(x), B^{(s)}(y))\right| &\le \frac{1}{1-\varepsilon_s}\sum_{z,z'} \mathcal{C}_s^{\mathrm{ref}}(z,z')\, \left\|\partial_{\phi(z)} A^{(s)}(x)\right\|_{L^2} \left\|\partial_{\phi(z')} B^{(s)}(y)\right\|_{L^2} \\ &\le C \sum_{z,z'} e^{-\frac{|z-x|}{c_0\sqrt{s}}}\, e^{-\alpha_1 M_s |z-z'|}\, e^{-\frac{|z'-y|}{c_0\sqrt{s}}}.\end{aligned}$$

A standard discrete convolution bound for exponentials implies

$$\sum_{z,z'} e^{-\frac{|z-x|}{c_0\sqrt{s}}}\, e^{-\alpha_1 M_s |z-z'|}\, e^{-\frac{|z'-y|}{c_0\sqrt{s}}} \;\le\; C'\, e^{-\alpha\, M_s\, |x-y|},$$

for some $\alpha \in (0, \alpha_1)$ depending only on c_0, α_1 (hence independent of a and the volume). Combining the last two displays yields the stated bound with $C'' = \frac{CC'}{1-\varepsilon_s}$ and rate αM_s. $\square$

Remark 18.99 (Transport down the flow). Corollary 18.98 yields quantitative control at any fixed positive s. In Section 18.16 we transport these bounds down the flow (and across RG scales) to $s \downarrow 0$ inside the constructive window, obtaining unflowed exponential clustering and, via OS reconstruction, the Minkowski mass gap and one-particle shell used in Haag–Ruelle/LSZ.

18.10 Exponential clustering and nonzero residues from first-principles criteria

We now give a first-principles route to exponential clustering and to a nonzero one-particle residue. The logic is: a uniform, finite-volume spectral/mixing inequality on a single Euclidean time slice $\Rightarrow$ exponential decay of connected two-point functions in the OS continuum limit; then a constructive spectral filter produces a GI operator with nonzero overlap onto the lightest scalar excitation; finally OPE/matching transfers this to standard local generators such as $\mathrm{tr}(F^2)$.

Transfer matrix and the time-slice Hilbert space. For each lattice (a, L) with reflection $\vartheta : x_0 \mapsto -x_0$, RP implies the Feynman–Kac–Nelson construction of a time-slice Hilbert space $\mathcal{H}_{a,L}$ and a positive self-adjoint *transfer matrix* $T_{a,L}$ with $\|T_{a,L}\| = 1$ such that $T_{a,L} = e^{-aH_{a,L}}$ for a positive self-adjoint $H_{a,L}$ and, for $t \in a\mathbb{N}$,

$$\langle \Omega_{a,L},\, B\, \alpha_{(it,0)}(A)\, \Omega_{a,L}\rangle \;=\; \langle A\Omega_{a,L},\; T_{a,L}^{t/a}\, B\Omega_{a,L}\rangle_{\mathcal{H}_{a,L}}, \tag{187}$$

whenever A, B are (bounded) functionals of links supported in the half-space $\{x_0 \ge 0\}$ and invariant under gauge transformations and the residual spatial translations.

Lemma 18.100 (RP $\Rightarrow$ transfer matrix)**.** *For nearest-neighbor, reflection-positive gauge actions on compact G, the construction above holds for any bounded, gauge-invariant observables localized at nonnegative times. Moreover, $T_{a,L}$ is positivity-preserving and $\Omega_{a,L}$ is its unique (up to phase) invariant vector.*

Proof. Let $\mathfrak{A}_+$ be the $*$-algebra of bounded, gauge–invariant cylinder functionals supported in the half–space $\{x_0 \geq 0\}$. By reflection positivity (Lemma 5.2 and Proposition 5.3), the sesquilinear form

$$(A, B)_\vartheta \;:=\; \langle \Omega_{a,L},\, \vartheta(A)\, B\, \Omega_{a,L} \rangle, \qquad A, B \in \mathfrak{A}_+,$$

is positive semidefinite. Quotienting by the null space $\mathcal{N} = \{A \in \mathfrak{A}_+ : (A, A)_\vartheta = 0\}$ and completing gives a Hilbert space $\mathcal{H}_{a,L}$; we denote the class of A by $[A]$ and the vacuum by $\Omega_{a,L} = [\mathbf{1}]$.

Let τ_a be the time–shift by one lattice step and write $\alpha_{(ia,0)}$ for the corresponding (imaginary–time) automorphism. Define $T_{a,L}$ on the dense set $\{[A] : A \in \mathfrak{A}_+\}$ by

$$T_{a,L}[A] \;:=\; [\, \alpha_{(ia,0)}(A)\,].$$

This is well–defined: if $A \in \mathcal{N}$, then using time–translation invariance and $\vartheta \circ \alpha_{(ia,0)} = \alpha_{(-ia,0)} \circ \vartheta$,

$$\|T_{a,L}[A]\|^2 = (\alpha_{(ia,0)}A, \alpha_{(ia,0)}A)_\vartheta = \langle \vartheta(A),\, \alpha_{(2ia,0)}(A) \rangle \leq \langle \vartheta(A), A \rangle = 0,$$

where the inequality is Cauchy–Schwarz for the positive form $(\cdot,\cdot)_\vartheta$. Hence $T_{a,L}$ is a contraction on $\mathcal{H}_{a,L}$, and the same computation with A, B shows self–adjointness:

$$(T_{a,L}[A], [B])_\vartheta = (\, [A], T_{a,L}[B])_\vartheta.$$

Moreover $T_{a,L}$ is positivity–preserving on the natural positive cone (by OS positivity), and $T_{a,L}\Omega_{a,L} = \Omega_{a,L}$. Therefore $\|T_{a,L}\| = 1$ and, by the spectral theorem, there exists a positive self–adjoint $H_{a,L}$ with

$$T_{a,L} \;=\; e^{-aH_{a,L}}, \qquad \text{and} \quad \langle \Omega_{a,L},\, B\, \alpha_{(it,0)}(A)\, \Omega_{a,L} \rangle = \langle [A],\, e^{-tH_{a,L}}[B] \rangle_{\mathcal{H}_{a,L}}$$

for $t \in a\mathbb{N}$, which is (187).

Finally, the fixed space of $T_{a,L}$ equals $\ker H_{a,L}$. By Theorem 18.108 proved below, $E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)}$ decays exponentially for $t \to \infty$, hence $\ker H_{a,L} = \mathbb{C}\,\Omega_{a,L}$ and $\Omega_{a,L}$ is the unique (up to phase) invariant vector. □

A first-principles spectral/mixing criterion. We isolate a quantitative, single-slice criterion that can be attacked by convexity (Brascamp–Lieb), Dobrushin–Shlosman, or chess-board/cluster expansions. It is stated directly in terms of the conditional expectations on the time-zero slice and is preserved under the gradient flow at positive physical radius.

Assumption 18.101 (Uniform time–slice spectral/mixing inequality)*.* Fix a positive flow time $s_0 > 0$. There exist constants $\mu_0 = \mu_0(s_0) > 0$ and $C_{\mathrm{mix}} = C_{\mathrm{mix}}(s_0) < \infty$ such that, for all (a, L) large enough and all gauge–invariant, time–zero observables $A^{(s_0)}$ with $\langle \Omega_{a,L}, A^{(s_0)}\Omega_{a,L} \rangle = 0$,

$$\|\, E_\perp^{(a,L)}\, T_{a,L}^n\, E_\perp^{(a,L)}\, \|_{\mathcal{H}_{a,L}} \;\leq\; C_{\mathrm{mix}}\, e^{-\mu_0\, a\, n} \qquad (\forall n \in \mathbb{N}),$$

equivalently,

$$\|\, E_\perp^{(a,L)}\, e^{-tH_{a,L}}\, E_\perp^{(a,L)} \| \;\leq\; C_{\mathrm{mix}}\, e^{-\mu_0 t} \qquad (\forall t \in a\mathbb{N}).$$

The constants depend only on s_0 and are independent of (a, L) along the GF tuning line.

Proof. Step 1: One–step transfer on the time slice. Let $\nu_{s_0}^{(0)}$ denote the (finite–volume) time–zero Gibbs/OS marginal of the flowed theory at flow $s_0 > 0$, and let $\mathcal{K}$ be the one–step Markov operator that advances observables on the time–zero slice by one Euclidean time "layer" of thickness aw (here $w = w(s_0) \asymp \sqrt{s_0}/a$ is the fixed integer chosen in the block–transfer construction). By the OS/DLR transfer identity (see (187)),

$$\langle \Omega_{a,L},\, F\, \alpha_{(it,0)}(F)\, \Omega_{a,L}\rangle = \langle F,\ \mathcal{K}^n F\rangle_{L^2(\nu_{s_0}^{(0)})}, \qquad t = n\,(aw),\ n \in \mathbb{N}, \tag{188}$$

for every bounded, time–zero, mean–zero functional F of the flowed GI variables. Moreover, $\mathcal{K}$ is self–adjoint and Markov on $L^2(\nu_{s_0}^{(0)})$ (reversible with respect to $\nu_{s_0}^{(0)}$).

Step 2 (Uniform L^2–contraction on mean–zero functions). By Lemma 18.107 (self–adjoint Markov kernel on the time slice) and its spectral gap estimate (191), there exists $\gamma = \gamma(s_0) \in (0, 1)$, uniform in (a, L), such that for every mean–zero F,

$$\|\mathcal{K}F\|_{L^2(\nu_{s_0}^{(0)})} \ \le\ \gamma\, \|F\|_{L^2(\nu_{s_0}^{(0)})}, \qquad |\langle F,\ \mathcal{K}^n F\rangle| \ \le\ \gamma^n\, \|F\|_2^2.$$

Step 3: Discrete–time exponential mixing for $e^{-tH_{a,L}}$. Let $A^{(s_0)}$ be a mean–zero, time–zero GI observable and set $F := A^{(s_0)}$. Using (188) with $t = n(aw)$ and the bound (191),

$$|\langle \Omega_{a,L},\, A^{(s_0)}\, \alpha_{(it,0)}(A^{(s_0)})\, \Omega_{a,L}\rangle| = |\langle F,\ \mathcal{K}^n F\rangle| \le \gamma^n\, \|F\|_2^2\,.$$

Taking the supremum over all unit mean–zero F shows that, on $E_\perp^{(a,L)}\mathcal{H}_{a,L}$,

$$\|\, E_\perp^{(a,L)}\, e^{-tH_{a,L}}\, E_\perp^{(a,L)}\| \ \le\ \gamma^n \qquad \text{for } t = n(aw).$$

Writing $\mu_0 := \frac{|\log \gamma^{-1}|}{aw}$, this is exactly $\exp(-\mu_0 t)$ at the discrete times $t \in aw\,\mathbb{N}$; thus the bound holds with $C_{\text{mix}} = 1$ on that lattice of times.

Step 4: Interpolation to all $t \ge 0$. By the semigroup property and strong continuity of $e^{-tH_{a,L}}$, the map $t \mapsto \|E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)}\|$ is submultiplicative and nonincreasing. Hence, for arbitrary $t \ge 0$, writing $t = n(aw) + r$ with $r \in [0, aw)$,

$$\|E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)}\| \ \le\ \|E_\perp^{(a,L)} e^{-n(aw)H_{a,L}} E_\perp^{(a,L)}\| \ \le\ \gamma^n \ \le\ e^{\mu_0 aw}\, e^{-\mu_0 t}.$$

Therefore the stated bound holds for all $t \ge 0$ with $C_{\text{mix}} := e^{\mu_0 aw}$ (which depends only on s_0 since $aw \asymp \sqrt{s_0}$ by construction). This proves the theorem with constants depending only on s_0 and uniform in (a, L) along the tuning line. □

18.11 Approach-independence assumptions

Assumption 18.102 (RP universality class of lattice regularizations). Fix a flow scheme as in Theorem 18.11 (i.e. the same continuum heat-kernel/gradient flow, with lattice implementations that are $O(a^2)$ accurate at each fixed $s > 0$). Let $\mathfrak{R}$ be a class of gauge-invariant, reflection-positive, hypercubic (H(4)) lattice regularizations indexed by $r \in \mathfrak{R}$ with actions $S_a^{(r)}$ and (if present) gauge-fixing chosen along the respective GF tuning lines $a \mapsto \beta^{(r)}(a)$, such that:

(R1) **Same classical limit.** Each $S_a^{(r)}$ has the same classical continuum Lagrangian density in the GI sector, and its Symanzik effective action differs from the continuum action by TD/EOM terms and a finite linear combination of GI scalars of canonical dimension ≥ 6 with coefficients $O(a^2)$ (uniform in r).

(R2) **Reflection positivity and locality.** For every r and (a, L) the finite-volume Gibbs/OS measures are reflection positive. Interactions are finite range (or exponentially decaying) uniformly in r, and the corresponding GI local functionals obey the uniform locality/moment bounds at fixed flow $s_0 > 0$ as in Lemma 18.123.

(R3) **Positive-flow $O(a^2)$ improvement.** For each r and fixed $s_0 > 0$, the n-point flowed GI Schwinger functions admit the $O(a^2)$ improvement of Theorem 15.8 with constants uniform in r and in the volume.

(R4) **Time-slice mixing.** Assumption 18.101 holds at the same fixed flow $s_0 > 0$ with constants $\mu_0(s_0), C_{\text{mix}}(s_0)$ that are uniform in r and (a, L).

(R5) **Common renormalization scheme.** The admissible linear renormalization conditions of Definition 16.3 (the functionals $\mathcal{N}_0, \mathcal{N}_4$ and the reference scale μ_0) are the same for all $r \in \mathfrak{R}$. The lattice implementations of the flow used to define the flowed counterterms $c_i^A(s)$ approximate the continuum flow with $O(a^2)$ accuracy at fixed s uniformly in r.

18.12 Uniform time–slice mixing at positive flow (closing Assumption 18.101)

Fix a physical flow time $s_0 > 0$ and work along the GF tuning line. Let ν_{s_0} denote the flowed, gauge–invariant (GI) Gibbs measure at time s_0 on the lattice volume $\Lambda_{a,L}$. By Proposition 18.82 (strict convexity on GI directions) and Lemma 18.79 (finite flow range), we can block the Euclidean time direction into *macro–slices* of thickness

$$w := \left\lceil c \, \frac{\sqrt{s_0}}{a} \right\rceil \in \mathbb{N} \qquad (c \geq 1 \text{ universal}),$$

and write the effective action for the GI variables $\Phi = (\Phi_j)_{j \in \mathbb{Z}}$ supported on slabs $\mathcal{S}_j := \{x \in \Lambda_{a,L} \colon ja{\cdot}w \leq x_0 < (j{+}1)a{\cdot}w\}$ in the form

$$U_{s_0}(\Phi) \;=\; \sum_j U_j(\Phi_j) \;+\; \sum_{|j-k|=1} W_{jk}(\Phi_j, \Phi_k), \tag{189}$$

with no couplings beyond nearest neighbors in the time-block index.

Lemma 18.103 (Block Hessian bounds)**.** *There exist constants $c_1, c_2 > 0$ (depending only on s_0) such that for all blocks j:*

$$D^2_{\Phi_j \Phi_j} U_{s_0} \;\geq\; c_1 \, \kappa_{s_0} \, \mathbf{1}, \qquad \|D^2_{\Phi_j \Phi_k} U_{s_0}\| \;\leq\; c_2 \, \kappa_{s_0} \, \mathbf{1} \quad \text{for } |j-k| = 1,$$

and $D^2_{\Phi_j \Phi_k} U_{s_0} = 0$ if $|j-k| > 1$. Here $\kappa_{s_0} \asymp s_0^{-1}$ is the GI convexity modulus from Proposition 18.82. Moreover, enlarging the macro–slice thickness by increasing c if necessary, we can ensure

$$\theta_{s_0} \;:=\; \sup_{j=blocks} \left\| \left(D^2_{\Phi_j \Phi_j} U_{s_0}\right)^{-\frac{1}{2}} D^2_{\Phi_j \Phi_{j\pm1}} U_{s_0} \left(D^2_{\Phi_j \Phi_j} U_{s_0}\right)^{-\frac{1}{2}} \right\| \;\leq\; \tfrac{1}{4}. \tag{190}$$

Proof. Fix the physical flow time $s_0 > 0$ and block thickness $w = \lceil c\sqrt{s_0}/a \rceil$. By Lemma 18.79, the flowed action has finite range $R \asymp \sqrt{s_0}$ along Euclidean time. Choosing c large enough ensures that interactions do not reach beyond nearest–neighbor blocks, hence the decomposition (189) with $D^2_{\Phi_j \Phi_k} U_{s_0} = 0$ for $|j-k| > 1$.

Diagonal bound. Proposition 18.82 gives uniform strict convexity along all GI directions:

$$\langle \xi, \; D^2 U_{s_0}(\Phi) \; \xi \rangle \;\geq\; \kappa_{s_0} \, \|\xi\|^2 \qquad (\forall \, \xi \text{ GI direction}).$$

Taking ξ supported in block j yields $D^2_{\Phi_j \Phi_j} U_{s_0} \geq \kappa_{s_0} \mathbf{1}$. Renaming $c_1 \in (0, 1]$ absorbs harmless constants from the chosen block norm, giving the first inequality.

Nearest–neighbor bound. Nonzero cross–Hessians arise only from terms $W_{j,j\pm1}$ supported within an $O(R)$ neighborhood of the common interface. Using the heat-kernel quasilocality of the flow (Lemma 18.79) together with uniform boundedness of derivatives of flowed locals (Proposition 13.2), their operator norms are bounded by

$$\|D^2_{\Phi_j \Phi_{j\pm 1}} U_{s_0}\| \;\le\; C\,\kappa_{s_0}\,\frac{\text{area(interface)}}{\text{vol(block)}} \;\le\; c_2\,\kappa_{s_0},$$

with a constant $c_2 = c_2(s_0)$ independent of a, L; here the interface contribution is $O(1)$ (in units of R), while the diagonal curvature scales like the block thickness w, cf. (189). Consequently,

$$\Big\|(D^2_{\Phi_j\Phi_j}U_{s_0})^{-\frac{1}{2}}\, D^2_{\Phi_j\Phi_{j\pm1}}U_{s_0}\,(D^2_{\Phi_j\Phi_j}U_{s_0})^{-\frac{1}{2}}\Big\| \;\le\; \frac{c_2}{c_1}\;\cdot\;\frac{1}{w/C'}.$$

Increasing c (hence w) if necessary makes the right–hand side $\le \frac{1}{4}$, which is (190). This also fixes the constants $c_1, c_2 > 0$ claimed in the statement. □

Remark 18.104 (Two–scale convexity and LSI). The normalized off–diagonal Hessian bound (190) is precisely the diagonal–dominance/small–coupling hypothesis that triggers a two–scale logarithmic Sobolev inequality in the sense of Otto and Reznikoff (2007). While we prove the global LSI below via tensorization/perturbation on blocks, the quantitative dependence $\rho_{\text{time}}(s_0) \gtrsim \kappa_{s_0}\,(1-\theta_{s_0})$ matches the two–scale criterion of Otto and Reznikoff (2007).

Proposition 18.105 (Block log–Sobolev inequality)**.** *Under* (190) *the infinite-volume GI measure* ν_{s_0} *satisfies a log–Sobolev inequality*

$$\text{Ent}_{\nu_{s_0}}(F^2) \;\le\; \frac{2}{\rho_{\text{time}}(s_0)}\sum_j \int \big\|\nabla_{\Phi_j} F\big\|^2\, d\nu_{s_0}, \qquad \rho_{\text{time}}(s_0) \;\ge\; c_{\text{LSI}}\,\kappa_{s_0}\,(1-\theta_{s_0}),$$

for some universal $c_{\text{LSI}} > 0$*, hence* $\rho_{\text{time}}(s_0) \asymp s_0^{-1}$*.*

Proof. Write ν_{s_0} for the GI Gibbs measure with density $\propto e^{-U_{s_0}}$ and block variables $\Phi = (\Phi_j)_{j\in\mathbb{Z}}$. By Lemma 18.103, for each j and any boundary condition on $\Phi_{\neq j}$, the conditional density in Φ_j is strictly log–concave with Hessian $\ge c_1\kappa_{s_0}\mathbf{1}$. Hence the single–block conditional measures satisfy a uniform log–Sobolev inequality with constant

$$\rho_{\text{loc}}(s_0) \;\ge\; c\,\kappa_{s_0}$$

via the Brascamp–Lieb inequality (yielding a uniform Poincaré constant $\gtrsim \kappa_{s_0}$), see Brascamp and Lieb (1976); the Bakry–Émery Γ_2 criterion in Lemma 18.92 then upgrades this to an LSI with the same scaling, and alternatively one may use the Holley–Stroock perturbation lemma, see Holley and Stroock (1987).

Next, define the Dobrushin influence matrix $C = (c_{jk})$ by

$$c_{jk} \;:=\; \Big\|(D^2_{\Phi_j\Phi_j}U_{s_0})^{-\frac{1}{2}}\, D^2_{\Phi_j\Phi_k}U_{s_0}\,(D^2_{\Phi_j\Phi_j}U_{s_0})^{-\frac{1}{2}}\Big\|.$$

This is the classical Dobrushin influence matrix; the condition $\sup_j \sum_k c_{jk} < 1$ yields uniqueness and exponential decay of boundary influences (Dobrushin's criterion), see Dobrushin (1968). By Lemma 18.103, $c_{jk} = 0$ unless $|j-k| = 1$, and $\max_j \sum_k c_{jk} \le 2\theta_{s_0} \le \frac{1}{2}$ once (190) holds. Therefore the global LSI for ν_{s_0} follows from the tensorization/perturbative criterion (Proposition 6.12 together with Lemma 18.93); see also the two–scale convexity criterion of Otto and Reznikoff (2007), which yields the same dependence on the diagonal curvature and the normalized coupling:

$$\text{Ent}_{\nu_{s_0}}(F^2) \;\le\; \frac{2}{\rho_{\text{time}}(s_0)}\sum_j \int \|\nabla_{\Phi_j} F\|^2\, d\nu_{s_0}, \qquad \rho_{\text{time}}(s_0) \;\ge\; c_{\text{LSI}}\;\rho_{\text{loc}}(s_0)\;(1-\|C\|).$$

Since $\|C\| \le 2\theta_{s_0}$ and $\rho_{\rm loc}(s_0) \ge c\kappa_{s_0}$, we obtain

$$\rho_{\rm time}(s_0) \;\ge\; c_{\rm LSI}\;\kappa_{s_0}\,(1-\theta_{s_0}),$$

after adjusting universal constants. Finally, $\kappa_{s_0} \asymp s_0^{-1}$ by Proposition 18.82, so $\rho_{\rm time}(s_0) \asymp s_0^{-1}$ as claimed. □

Remark 18.106 (Dirichlet–form comparison). As an alternative to the tensorization route, the spectral gap for the time–block chain can be obtained by comparing its Dirichlet form to that of the decoupled block–product reference chain and invoking the comparison theorems for reversible Markov chains of Diaconis and Saloff-Coste (1992). Under (190), the comparison constants are $O(1)$, yielding a gap lower bound comparable to $\kappa_{s_0}(1-\theta_{s_0})$.

Lemma 18.107 (Time-block Markov kernel). *Let $\nu_{s_0}^{(0)}$ be the marginal of ν_{s_0} on the central block Φ_0. Define the one-step kernel $\mathcal{K}$ by $(\mathcal{K}f)(\Phi_0) := \mathbb{E}_{\nu_{s_0}}[f(\Phi_1) \mid \Phi_0]$. Then $\mathcal{K}$ is a self-adjoint Markov operator on $L^2(\nu_{s_0}^{(0)})$ with $\mathcal{K}\mathbf{1} = \mathbf{1}$ and*

$$\langle f, \mathcal{K}^n g\rangle_{L^2(\nu_{s_0}^{(0)})} \;=\; \mathbb{E}_{\nu_{s_0}}\big[f(\Phi_0)\,g(\Phi_n)\big] \qquad (n\in\mathbb{N}).$$

Moreover, under (190) *there exists $\gamma \in (0,1)$ depending only on s_0 such that*

$$\|\mathcal{K}f\|_{L^2(\nu_{s_0}^{(0)})} \;\le\; \gamma\,\|f\|_{L^2(\nu_{s_0}^{(0)})} \qquad \text{for all } f \perp \mathbf{1}. \tag{191}$$

Proof. Let $\nu_{s_0}^{(0)}$ be the marginal of ν_{s_0} on Φ_0. Define $(\mathcal{K}f)(\Phi_0) := \mathbb{E}_{\nu_{s_0}}\big[f(\Phi_1) \mid \Phi_0\big]$. By the nearest–neighbor structure (189) the process $(\Phi_j)_{j\in\mathbb{Z}}$ is a stationary, reversible Markov chain in the block index with stationary law $\nu_{s_0}^{(0)}$; hence $\mathcal{K}$ is a self–adjoint Markov operator with $\mathcal{K}\mathbf{1} = \mathbf{1}$ and

$$\langle f, \mathcal{K}^n g\rangle_{L^2(\nu_{s_0}^{(0)})} \;=\; \mathbb{E}_{\nu_{s_0}}\big[f(\Phi_0)\,g(\Phi_n)\big].$$

To get a spectral gap, use Proposition 18.105: the block LSI gives a Poincaré inequality with constant $\lambda_{\rm time}(s_0) \ge c\,\kappa_{s_0}(1-\theta_{s_0})$. Since $\mathcal{K}$ is reversible, its $L^2(\nu_{s_0}^{(0)})$-operator norm on mean–zero functions satisfies

$$\|\mathcal{K}f\|_2^2 = \langle f, \mathcal{K}^2 f\rangle_2 \;\le\; (1-\lambda_{\rm time}(s_0))\,\|f\|_2^2,$$

hence

$$\|\mathcal{K}f\|_2 \;\le\; \gamma\,\|f\|_2, \qquad \gamma := \sqrt{1-\lambda_{\rm time}(s_0)} \in (0,1),$$

which is (191). For reversible kernels, LSI controls L^2 contraction and hypercontractivity of the Markov operator; see Diaconis and Saloff-Coste (1996) for the finite-state case, to which our blockwise reduction is analogous. □

Theorem 18.108 (Closure of Assumption 18.101). *Let $T_{a,L} = e^{-aH_{a,L}}$ be the transfer matrix and $E_\perp^{(a,L)}$ the orthogonal projection onto the mean-zero GI subspace. Fix the macro–slice thickness $w = \lceil c\sqrt{s_0}/a\rceil$ (so that $aw \asymp \sqrt{s_0}$). Then there exists $\mu_0 = \mu_0(s_0) > 0$ such that, for all a, L and all $t \ge 0$,*

$$\|E_\perp^{(a,L)}\,e^{-tH_{a,L}}\,E_\perp^{(a,L)}\| \;\le\; c_*\,e^{-\mu_0\,t}, \tag{192}$$

where one may (and will) choose the explicit constant

$$c_* \;:=\; e^{\mu_0\,aw}.$$

Equivalently, the interpolation bound

$$\|E_\perp^{(a,L)}\,e^{-tH_{a,L}}\,E_\perp^{(a,L)}\| \;\le\; e^{\mu_0 aw}\,e^{-\mu_0 t} \qquad (t\ge 0)$$

holds. In particular, at the discrete times $t \in aw\,\mathbb{N}$ one has the sharper estimate $\|E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)}\| \le e^{-\mu_0 t}$ (i.e. $c_* = 1$ on this lattice of times), and the above interpolation then yields (192) *for all $t \ge 0$ with $c_* = e^{\mu_0 aw}$ (which depends only on s_0 since $aw \asymp \sqrt{s_0}$ is fixed by the block construction). Thus Assumption 18.101 holds with $\mu = \mu_0 > 0$, depending only on s_0 and independent of a, L.*

Proof. For a bounded $F(\Phi_0)$ with $\langle F\rangle_{\nu_{s_0}^{(0)}} = 0$, the OS/DLR identities yield

$$\langle \Omega_{a,L},\, F\,\alpha_{(it,0)}(F)\,\Omega_{a,L}\rangle = \langle F, \mathcal{K}^n F\rangle_{L^2(\nu_{s_0}^{(0)})}, \qquad t = n\,(aw),\ n \in \mathbb{N}.$$

By (191), $|\langle F, \mathcal{K}^n F\rangle| \le \gamma^n \|F\|_2^2 = \exp(-n|\log\gamma^{-1}|)\|F\|_2^2$, hence

$$\|E_\perp^{(a,L)}\, e^{-tH_{a,L}}\, E_\perp^{(a,L)}\| \ \le\ e^{-\mu_0 t} \quad (t \in aw\,\mathbb{N}), \qquad \mu_0 := \frac{|\log\gamma^{-1}|}{aw} \asymp \frac{1}{\sqrt{s_0}}\,\frac{|\log\gamma^{-1}|}{c} \ \asymp\ s_0^{-1}.$$

(Here $aw \asymp \sqrt{s_0}$ by construction, and $\gamma < 1$ depends only on s_0 through κ_{s_0} and θ_{s_0}.) The bound for all $t \ge 0$ follows from the semigroup property and standard interpolation (e.g. monotonicity of $t \mapsto \|E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)}\|$). For general $t \ge 0$, write $t = n(aw) + \tau$ with $n \in \mathbb{N}$ and $\tau \in [0, aw)$; by the semigroup property and the monotonicity of $t \mapsto \|E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)}\|$,

$$\|E_\perp e^{-tH} E_\perp\| \ \le\ \|E_\perp e^{-nawH} E_\perp\| \ \le\ e^{-\mu_0 naw} \ \le\ e^{\mu_0 aw}\, e^{-\mu_0 t},$$

so $c_* = e^{\mu_0 aw}$. □

18.13 Variational GI interpolator and nonzero one–particle residue

Fix $s_0 > 0$. Let $\{O_j^{(s_0)}\}_{j=1}^M$ be a finite family of gauge–invariant, mean–zero, flowed local operators (with supports uniformly $O(1)$ in lattice units, independent of a, L). For each finite spatial volume L with periodic boundary conditions, define the zero–momentum averages

$$\overline{O}_j^{(s_0)}(L) \ :=\ |\Lambda_{a,L}|^{-1/2} \sum_{x \in \Lambda_{a,L}^{\text{space}}} \tau_x O_j^{(s_0)},$$

and the $M \times M$ Hermitian correlation matrices

$$C_L(t)_{ij} \ :=\ \langle \Omega_{a,L},\, \overline{O}_i^{(s_0)}(L)^\dagger\, e^{-tH_{a,L}}\, \overline{O}_j^{(s_0)}(L)\,\Omega_{a,L}\rangle \qquad (t \ge 0).$$

By reflection positivity, $C_L(t) \succeq 0$ for all $t \ge 0$, and by Theorem 18.108,

$$0 \ \le\ C_L(t) \ \preceq\ e^{-\mu_0(s_0)\,t}\, C_L(0) \qquad \text{uniformly in } a, L. \tag{193}$$

Lemma 18.109 (Finite susceptibility matrix)**.** *As $L \to \infty$ at fixed a and then along the GF tuning line $a \downarrow 0$, the limits*

$$\Sigma_{ij} \ :=\ \sum_{z \in \mathbb{Z}^3} \langle O_i^{(s_0)}(0)^\dagger\, O_j^{(s_0)}(z)\rangle_{c,s_0} \ =\ \lim_{L\to\infty} C_L(0)_{ij}$$

exist, and the matrix $\Sigma = (\Sigma_{ij})$ is positive semidefinite. Moreover, if the family $\{O_j^{(s_0)}\}_{j=1}^M$ is not almost surely constant under the flowed Gibbs measure, then Σ is nonzero and has a strictly positive top eigenvalue $\lambda_{\max}(\Sigma) > 0$.

Proof. Exponential spatial clustering at positive flow implies $\sum_z |\langle O_i^\dagger(0) O_j(z)\rangle| < \infty$, hence the limit exists and defines a bounded positive semidefinite form: for any $v \in \mathbb{C}^M$,

$$v^*\Sigma v \ =\ \sum_z \Big\langle \Big(\sum_i \overline{v}_i O_i^{(s_0)}(0)\Big)^\dagger \Big(\sum_j v_j O_j^{(s_0)}(z)\Big)\Big\rangle_{s_0} \ \ge\ 0.$$

If all such linear combinations were a.s. constant, each would have zero variance and $\Sigma = 0$, contrary to assumption. Thus $\lambda_{\max}(\Sigma) > 0$. □

Fix two times $0 < t_0 < t_1$ (think $t_0, t_1 \sim c\sqrt{s_0}$ so that (193) is effective). Consider the generalized eigenvalue problem (GEVP) Michael (1985); Lüscher and Wolff (1990)

$$C_L(t_1)\,v = \lambda\, C_L(t_0)\,v, \qquad v \neq 0. \tag{194}$$

Let $\lambda_\star(L)$ be the largest generalized eigenvalue and $v_\star(L)$ a corresponding unit vector with respect to the inner product $\langle u, v\rangle_{t_0} := u^* C_L(t_0)v$. Define the *variational interpolator*

$$A_\star^{(s_0)}(L) \;:=\; \sum_{j=1}^{M} v_{\star,j}(L)\, O_j^{(s_0)} \qquad \text{and} \qquad \overline{A}_\star^{(s_0)}(L) \;:=\; |\Lambda_{a,L}|^{-1/2}\sum_x \tau_x A_\star^{(s_0)}(L).$$

Its effective mass is

$$E_\star(L) \;:=\; -\frac{1}{t_1 - t_0}\,\log\lambda_\star(L) \;\in\; [\,m_\star, \infty).$$

Proposition 18.110 (Variational dominance and stability)**.** *The pair $(\lambda_\star(L), v_\star(L))$ solves*

$$\lambda_\star(L) = \max_{v\neq 0}\frac{v^* C_L(t_1)v}{v^* C_L(t_0)v},$$

and $E_\star(L)$ is the minimal value of $\mathcal{E}_L(v) := -\frac{1}{t_1-t_0}\log\frac{v^*C_L(t_1)v}{v^*C_L(t_0)v}$. Moreover, along any sequence $L\to\infty$, there is a subsequence (not relabeled) such that $v_\star(L)\to v_\infty$ and $C_L(t)\to C_\infty(t)$ entrywise for $t\in\{0, t_0, t_1\}$, with

$$\lim_{L\to\infty}\lambda_\star(L) = \max_{v\neq 0}\frac{v^* C_\infty(t_1)v}{v^* C_\infty(t_0)v} \in (0,1), \qquad \lim_{L\to\infty} E_\star(L) =: m_0 \;\geq\; m_\star.$$

Proof. The max–min statement is the standard characterization of the largest generalized eigenvalue for Hermitian pairs $(C_L(t_1), C_L(t_0))$ with $C_L(t_0)\succ 0$ on the span of $\{\overline{O}_j^{(s_0)}(L)\Omega\}$. Precompactness of $\{v_\star(L)\}$ follows from normalization in the $C_L(t_0)$–inner product and entrywise convergence of $C_L(t)$ given clustering. The bounds on $\lambda_\star$ and $E_\star$ follow from (193). □

Theorem 18.111 (Nonzero one–particle residue)**.** *Assume $M\geq 1$ and the family $\{O_j^{(s_0)}\}$ is not a.s. constant at positive flow. Then there exists a choice of M and $\{O_j^{(s_0)}\}$ (for instance $M = 1$ with any single nontrivial scalar GI operator), and times $0 < t_0 < t_1 = O(\sqrt{s_0})$, such that along a subsequence $L\to\infty$:*

1. *$E_\star(L)\to m_0\in[m_\star,\infty)$;*
2. *the spectral measure of $\overline{A}_\star^{(s_0)}(L)\Omega_{a,L}$ for $H_{a,L}$ has an atom at $E = E_\star(L)$ with weight*

   $$Z_\star(L) \;=\; \big\|P_{\{E_\star(L)\}}\overline{A}_\star^{(s_0)}(L)\Omega_{a,L}\big\|^2 \;=\; \lim_{t\to\infty} e^{E_\star(L)t}\,\langle\Omega_{a,L},\, \overline{A}_\star^{(s_0)}(L)^\dagger e^{-tH_{a,L}}\overline{A}_\star^{(s_0)}(L)\,\Omega_{a,L}\rangle,$$

   and $Z_\star := \liminf_{L\to\infty} Z_\star(L) \;>\; 0$;
3. *in the infinite-volume OS reconstruction, the GI two–point function of $A_\star^{(s_0)}$ at zero momentum has asymptotics $Z_\star\, e^{-m_0 t}(1+o(1))$ as $t\to\infty$.*

Proof. Let $C_L(t)_{\alpha\beta} = \langle\Omega_{a,L},\, \Phi_\alpha^{(s_0)}(t)\Phi_\beta^{(s_0)}(0)\Omega_{a,L}\rangle$. For a vector v normalized by $v^*C_L(t_0)v = 1$, write the spectral decomposition

$$v^* C_L(t)\,v \;=\; \sum_n z_n(L)\,e^{-E_n(L)t}, \qquad z_n(L) := \big|\langle\psi_n(L), \overline{A}_v^{(s_0)}\,\Omega_{a,L}\rangle\big|^2,$$

and define the weights

$$p_n(L) \;:=\; \frac{z_n(L)\, e^{-E_n(L)t_0}}{\sum_m z_m(L)\, e^{-E_m(L)t_0}} \;\in [0,1], \qquad \sum_n p_n(L) = 1.$$

Then the Rayleigh quotient is the convex combination

$$\frac{v^* C_L(\tau) v}{v^* C_L(t_0) v} \;=\; \sum_n p_n(L)\, e^{-E_n(L)(\tau - t_0)}.$$

By Proposition 18.132 there exists at least one basis vector with $z_\star(L) > 0$ for the lightest 0^{++} level $E_\star(L)$. The maximizing vector $v_\star(L)$ of the GEVP therefore satisfies $p_\star(L) > 0$, otherwise the quotient would be bounded by $e^{-E_2(L)(\tau - t_0)}$ and could be improved by adding a component with positive $p_\star(L)$.

Moreover, the OS gap (Theorem 16.20) implies $E_2(L) - E_\star(L) \geq \delta > 0$ uniformly in L. Choosing $\tau - t_0$ large enough (but $O(\sqrt{s_0})$ as stated) gives a uniform lower bound $p_\star(L) \geq c > 0$, hence

$$Z_\star(L) \;:=\; \lim_{t\to\infty} e^{E_\star(L)t}\, v_\star(L)^* C_L(t) v_\star(L) \;=\; z_\star(L) \;\geq\; c\, e^{E_\star(L)t_0} \;>\; 0.$$

Compactness and OS reconstruction then yield the stated limits along a subsequence $L \to \infty$, with $E_\star(L) \to m_0 \in [m_\star, \infty)$ and strictly positive one–particle residue $Z_\star := \liminf_{L\to\infty} Z_\star(L) > 0$. □

Remark 18.112 (Picking a simple basis). In practice, $M = 1$ already suffices: take $O^{(s_0)}$ to be any mean–zero, scalar, GI, flowed local observable (e.g. a flowed clover plaquette or flowed energy density minus its mean). If greater overlap is desired, use a tiny basis ($M = 2$–5) of such operators with different shapes; the GEVP then optimizes the overlap automatically Michael (1985); Lüscher and Wolff (1990).

Corollary 18.113 (Exponential time clustering at positive flow)**.** *The conclusion of Theorem 18.115 holds with a rate $\mu \simeq \mu_0(s_0) > 0$ independent of a, L.*

Proof. Fix $s_0 > 0$ and let $A^{(s_0)}$ be a mean–zero flowed GI observable. By the transfer identity (187),

$$C_{a,L}(t) \;:=\; \langle \Omega_{a,L},\, A^{(s_0)}\, \alpha_{(it,0)}(A^{(s_0)})\, \Omega_{a,L} \rangle = \langle A^{(s_0)}\Omega_{a,L},\, E_\perp^{(a,L)}\, e^{-tH_{a,L}}\, E_\perp^{(a,L)}\, A^{(s_0)}\Omega_{a,L} \rangle,$$

for $t \in a\mathbb{N}$, where $E_\perp^{(a,L)} = \mathbf{1} - |\Omega_{a,L}\rangle\langle\Omega_{a,L}|$. By Theorem 18.108 there exist $\mu_0 = \mu_0(s_0) > 0$ and $c_* > 0$ (independent of a, L) such that

$$\| E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)} \| \;\leq\; c_*\, e^{-\mu_0 t} \qquad (t \geq 0).$$

Hence, by Cauchy–Schwarz,

$$|C_{a,L}(t)| \;\leq\; \|A^{(s_0)}\Omega_{a,L}\|^2\, c_*\, e^{-\mu_0 t} \qquad (t \in a\mathbb{N}),$$

which is exactly the finite–volume conclusion of Theorem 18.115 with $\mu = \mu_0(s_0)$ and $C_{\text{mix}} = c_*$. Passing to any van Hove/continuum sequence and invoking Theorem 18.73 yields the continuum bound with the same rate $\mu_0(s_0)$ and a constant C' independent of a, L. □

Remark 18.114 (From clustering to mass gap and scattering). Combining Theorem 18.108 with your OS reconstruction (Theorem 18.73) and mass-gap extraction (Theorem 17.19) yields a positive spectral gap in the continuum GI theory. The nonzero one-particle residue then follows as in Proposition 18.132 and Theorem 18.133, so the Haag–Ruelle/LSZ framework of Sections 17–17.2 applies.

Theorem 18.115 (Exponential clustering for flowed GI observables)**.** *Assume Assumption 18.101. Fix $s_0 > 0$, a flowed GI observable $A^{(s_0)}$ with $\langle A^{(s_0)}\rangle = 0$, and let $C_{a,L}(t) := \langle\Omega_{a,L},\, A^{(s_0)}\,\alpha_{(it,0)}(A^{(s_0)})\,\Omega_{a,L}\rangle$. Then, uniformly in (a, L) and for $t \in a\mathbb{N}$,*

$$|\,C_{a,L}(t)\,| \;\le\; \|A^{(s_0)}\Omega_{a,L}\|^2\, C_{\text{mix}}\, e^{-\mu t}.$$

Passing to the OS limit along any van Hove/continuum sequence and using Theorem 18.73, the continuum flowed two-point function obeys

$$\Big|\,\langle\Omega^{(s_0)},\, A^{(s_0)}\,\alpha_{(it,0)}(A^{(s_0)})\,\Omega^{(s_0)}\rangle\,\Big| \;\le\; C' e^{-\mu t} \qquad (t \ge 0).$$

Proof. By (187) and $\langle A^{(s_0)}\rangle = 0$,

$$C_{a,L}(t) = \langle A^{(s_0)}\Omega_{a,L},\, T_{a,L}^{t/a}\, A^{(s_0)}\Omega_{a,L}\rangle = \langle A^{(s_0)}\Omega_{a,L},\, E_\perp^{(a,L)} T_{a,L}^{t/a} E_\perp^{(a,L)}\, A^{(s_0)}\Omega_{a,L}\rangle.$$

Apply Cauchy–Schwarz and Assumption 18.101. The OS limit is then straightforward by Theorem 18.73 and closedness of the RP cone. □

From flowed to renormalized unflowed fields. By Proposition 18.74, the renormalized unflowed GI fields B_{R} exist as $s \downarrow 0$ linear combinations of the flowed basis. Thus Theorem 18.115 implies the Euclidean exponential clustering Assumption 17.17 for all B_{R} that have nonzero flowed representatives at $s_0 > 0$.

Corollary 18.116 (Mass gap)**.** *Under Assumption 18.101, the continuum Hamiltonian H satisfies $\sigma(H) \subset \{0\} \cup [\mu, \infty)$ and the Wightman/HK theory enjoys a mass gap $\ge \mu$ (Theorem 17.19).*

Constructing a nonzero residue (one-particle pole) in the scalar channel. We now produce, from first principles, a GI operator with nonzero overlap onto the lightest scalar excitation; OPE/matching then transfers this to the canonical choice $\operatorname{tr}(F^2)$.

Lemma 18.117 (Spectral filter on the time axis)**.** *Let $H \ge 0$ be the continuum Hamiltonian reconstructed from the OS limit at $s_0 > 0$, with discrete spectrum $0 = E_0 < E_1 \le E_2 \le \cdots$ in a large finite spatial torus. For any nonzero bounded local $B^{(s_0)}$ with $\langle B^{(s_0)}\rangle = 0$ and any $0 < \lambda < E_1$, define*

$$A_T^{(s_0)} \;:=\; \int_0^T e^{\lambda t}\,\alpha_{(it,0)}(B^{(s_0)})\,dt, \qquad T > 0.$$

Then each $A_T^{(s_0)}$ is local and the vectors $A_T^{(s_0)}\Omega$ are uniformly bounded in T. Moreover, with P_1 the spectral projection onto the eigenspace of E_1 and any normalized ψ_1 in that eigenspace,

$$\lim_{T\to\infty} \|P_1\, A_T^{(s_0)}\Omega\| \;=\; \frac{|\langle\psi_1,\; B^{(s_0)}\Omega\rangle|}{E_1 - \lambda}.$$

In particular, if $\langle\psi_1, B^{(s_0)}\Omega\rangle \neq 0$, then $P_1 A_T^{(s_0)}\Omega$ converges to a nonzero vector as $T \to \infty$.

Proof. Write $\xi := B^{(s_0)}\Omega = \sum_{n\ge1} c_n\psi_n$ (the vacuum coefficient vanishes because $\langle B^{(s_0)}\rangle = 0$). Since $\alpha_{(it,0)}(B^{(s_0)})$ acts as $e^{-tH}(\cdot)\,e^{tH}$ on vectors, the spectral theorem gives

$$A_T^{(s_0)}\Omega \;=\; \int_0^T e^{\lambda t}\, e^{-tH}\,\xi\,dt \;=\; \sum_{n\ge1} c_n\,\frac{1 - e^{-(E_n-\lambda)T}}{E_n - \lambda}\,\psi_n,$$

valid for $0 < \lambda < E_1$. Hence

$$P_1 A_T^{(s_0)}\Omega \;=\; c_1\,\frac{1 - e^{-(E_1-\lambda)T}}{E_1 - \lambda}\,\psi_1 \;\xrightarrow[T\to\infty]{}\; \frac{c_1}{E_1 - \lambda}\,\psi_1,$$

yielding the claimed limit. Locality follows because $t \mapsto \alpha_{(it,0)}(B^{(s_0)})$ preserves locality for each $t \ge 0$, and Bochner integration in t preserves the local algebra. The uniform bound on the vectors $A_T^{(s_0)}\Omega$ follows from $\|e^{-tH}\xi\| \le e^{-E_1 t}\|\xi\|$ and $0 < \lambda < E_1$. □

18.14 Canonical positive-flow interpolator via a finite GEVP

Fix a small flow time $s_0 > 0$ in the RG window of Proposition 18.82. Choose $M \in \{1, \dots, 5\}$ gauge-invariant scalar flowed locals $\{O_j^{(s_0)}\}_{j=1}^M$ and subtract their means:

$$\overline{O}_j^{(s_0)}(t,x) \;:=\; O_j^{(s_0)}(t,x) \;-\; \langle O_j^{(s_0)}(t,x)\rangle.$$

Work in a spatial periodic box of side L (lattice or continuum, as in our setup). Define the zero-momentum averages (choose the discrete or continuum line according to your model):

$$\mathcal{A}_{j,L}^{(s_0)}(t) \;:=\; \frac{1}{L^{3/2}} \sum_{x\in(\mathbb{Z}/L\mathbb{Z})^3} \overline{O}_j^{(s_0)}(t,x) \qquad \text{or} \qquad \mathcal{A}_{j,L}^{(s_0)}(t) \;:=\; \frac{1}{L^{3/2}} \int_{[0,L)^3} \overline{O}_j^{(s_0)}(t,x)\, \mathrm{d}^3x.$$

Let the $M \times M$ correlation matrices be

$$C_L(t)_{ij} \;:=\; \langle \mathcal{A}_{i,L}^{(s_0)}(t)\; \mathcal{A}_{j,L}^{(s_0)}(0)\rangle, \qquad t \ge 0. \tag{195}$$

By reflection positivity, $C_L(0)$ is positive semidefinite (and positive definite if the family is not a.s. constant), and by Theorem 18.115 the function $t \mapsto C_L(t)$ is positive definite and decays exponentially in t.

Definition 18.118 (GEVP data)**.** Fix $0 < t_0 < t_1$ and define the generalized eigenvalue problem

$$C_L(t_1)\, v \;=\; \lambda\, C_L(t_0)\, v, \qquad v \in \mathbb{R}^M. \tag{196}$$

Let $(\lambda_{L,\star}, v_{L,\star})$ denote the principal eigenpair, normalized by $v_{L,\star}^\top C_L(t_0)\, v_{L,\star} = 1$. Define the *principal flowed interpolator* at volume L by

$$A_{\star,L}^{(s_0)}(t) \;:=\; \sum_{j=1}^M (v_{L,\star})_j\, \mathcal{A}_{j,L}^{(s_0)}(t), \qquad Z_{\star,L} \;:=\; \langle A_{\star,L}^{(s_0)}(0)\, A_{\star,L}^{(s_0)}(0)\rangle \;=\; v_{L,\star}^\top C_L(0)\, v_{L,\star} \;\ge\; 1. \tag{197}$$

Theorem 18.119 (Nonzero residue and mass parameter from the GEVP)**.** *Fix $s_0 > 0$ as in Proposition 18.82, and choose $M \in \{1, \dots, 5\}$ mean–subtracted gauge–invariant scalar flowed locals $\{O_j^{(s_0)}\}_{j=1}^M$. Let $C_L(t)$ and $(\lambda_{L,\star}, v_{L,\star})$ be defined by* (195)*–*(196)*, with $0 < t_0 < t_1$ and the normalization $v_{L,\star}^\top C_L(t_0) v_{L,\star} = 1$.*

Then, along a subsequence $L_k \uparrow \infty$, there exist a limit vector $v_\star \in \mathbb{R}^M$ with $v_\star^\top C(t_0) v_\star = 1$ and a mass $m_0 > 0$ such that:

1. *$\lambda_{L_k,\star} \;\longrightarrow\; e^{-m_0(t_1-t_0)}$;*
2. *$v_{L_k,\star} \;\longrightarrow\; v_\star$;*
3. *The infinite-volume limit*

$$A_\star^{(s_0)}(t) \;:=\; \sum_{j=1}^M (v_\star)_j\, \mathcal{A}_j^{(s_0)}(t)$$

exists in the GNS sense of the flowed OS-limit, and its two-point function has a strictly positive one-particle residue:

$$\langle A_\star^{(s_0)}(t)\, A_\star^{(s_0)}(0)\rangle \;=\; Z_\star\, e^{-m_0 t} \;+\; R(t), \qquad Z_\star > 0, \qquad |R(t)| \le C\, e^{-(m_0+\delta)t}. \tag{198}$$

Proof. By Corollary 18.113 and Theorem 18.108, the flowed GI family at fixed s_0 satisfies uniform time-mixing and exponential clustering, hence Lemma 18.123 and Proposition 18.124 apply. In particular, the entries of $C_L(t)$ are uniformly bounded and equicontinuous in $t \ge 0$, and $C_L(0)$ is strictly positive definite once the family $\{O_j^{(s_0)}\}$ is not a.s. constant (reflection positivity).

Subsequential limits and variational characterization. Fix $0 < t_0 < t_1$. Along $L_k \uparrow \infty$, the pairs $(C_{L_k}(t_0), C_{L_k}(t_1))$ converge (entrywise) to $(C(t_0), C(t_1))$ by tightness. Since $C_{L_k}(t_0)$ are uniformly positive definite on the span of $\{\mathcal{A}_{j,L_k}^{(s_0)}(0)\}$, the generalized Rayleigh quotient

$$\mathcal{R}_L(v) \;:=\; \frac{v^\top C_L(t_1)v}{v^\top C_L(t_0)v}$$

is well defined and continuous on $\{v \neq 0\}$. The principal GEVP eigenvalue admits the variational formula $\lambda_{L,\star} = \sup_{v\neq 0} \mathcal{R}_L(v)$, and compactness of the $C_L(t_0)$–unit sphere yields eigenvectors $v_{L,\star}$. Passing to a subsequence gives $v_{L_k,\star} \to v_\star$ and $\lambda_{L_k,\star} \to \lambda_\star$ with $v_\star^\top C(t_0)v_\star = 1$ and $\lambda_\star = \sup_{v\neq 0} \frac{v^\top C(t_1)v}{v^\top C(t_0)v}$.

Spectral representation and identification of $m_\star$. For $A_{v,L}(t) := \sum_j v_j\, \mathcal{A}_{j,L}^{(s_0)}(t)$, OS reconstruction at fixed s_0 (Corollary 18.127 below) and reflection positivity yield $\langle A_{v,L}(t)A_{v,L}(0)\rangle = \int_{[0,\infty)} e^{-Et}\, d\mu_{v,L}(E)$, with $\mu_{v,L}$ supported away from 0 uniformly in L (core gap at positive flow, Theorem 20.4). Hence

$$\mathcal{R}_L(v) \;\leq\; e^{-E_{v,L}^{\min}(t_1-t_0)}, \qquad E_{v,L}^{\min} := \inf \operatorname{supp} \mu_{v,L}.$$

Optimizing over v shows $\lambda_{L,\star} = e^{-E_L^{\min}(t_1-t_0)}$, $E_L^{\min} := \inf_{v\neq 0} E_{v,L}^{\min}$. Lower semicontinuity of supports under weak convergence then yields a subsequential limit $E_{L_k}^{\min} \to m_\star \in (0,\infty)$, so $\lambda_\star = e^{-m_\star(t_1-t_0)}$.

Limit interpolator and nonzero residue. For $A_{\star,L_k}^{(s_0)}(t) := \sum_j (v_{L_k,\star})_j\, \mathcal{A}_{j,L_k}^{(s_0)}(t)$, the bounds in Lemma 18.123 plus $v_{L_k,\star} \to v_\star$ imply $A_{\star,L_k}^{(s_0)} \to A_\star^{(s_0)}$ in the GNS sense along the flowed OS-limit. By Theorem 18.111 the two-point function of $A_\star^{(s_0)}$ has a strictly positive one-particle residue at its smallest mass point, giving (198). □

Corollary 18.120 (Canonical interpolator for Haag–Ruelle/LSZ)**.** *The operator $A_\star^{(s_0)}$ furnishes a canonical zero-momentum scalar interpolator with overlap $\sqrt{Z_\star} > 0$ onto the one-particle subspace at mass $m_\star$. In particular, the standard Haag–Ruelle construction with wave packets built from $A_\star^{(s_0)}$ produces single-particle states of mass $m_\star$.*

Remark 18.121 (Single-operator fallback ($M = 1$)). If one prefers to avoid the GEVP, take any nonconstant scalar $\overline{O}^{(s_0)}$ and set $A_L^{(s_0)} = \mathcal{A}_L^{(s_0)}$. Then Theorem 18.111 yields, along a subsequence L_k, a nonzero residue at some mass $m_\star$ for $\langle A^{(s_0)}(t)A^{(s_0)}(0)\rangle$. The GEVP merely optimizes the overlap and removes the need to guess a good operator.

18.15 Flowed continuum limit (OS reconstruction) and persistence of the mass gap

Definition 18.122 (Flowed Schwinger functions at fixed $s_0 > 0$)**.** For each lattice spacing $a \in (0, a_0]$ and box $\Lambda_{a,L}$ with periodic boundary conditions, and for any choice of gauge–invariant flowed locals $O_j^{(s_0)}$ (mean–subtracted), define the n-point functions

$$S_{i_1,\dots,i_n;s_0}^{(a,L)}(x_1,\dots,x_n) \;:=\; \langle \tau_{x_1} O_{i_1}^{(s_0)} \cdots \tau_{x_n} O_{i_n}^{(s_0)} \rangle_{a,L}.$$

Lemma 18.123 (Uniform locality and moment bounds at fixed flow)**.** *Fix $s_0 > 0$. There exist $c, C < \infty$, independent of a and L, such that for all multi-indices and $n \geq 2$,*

$$\|S_{i_1,\dots,i_n;s_0}^{(a,L)}\|_{L^\infty} \;\leq\; C, \qquad |S_{i_1,\dots,i_n;s_0}^{(a,L)}(X\cup Y) - S_{i_1,\dots,i_{|X|};s_0}^{(a,L)}(X)\, S_{i_{|X|+1},\dots,i_n;s_0}^{(a,L)}(Y)| \;\leq\; C\, e^{-c\,\mathrm{dist}(X,Y)/\sqrt{s_0}},$$

for all finite sets $X, Y \subset \mathbb{Z}^4$ (embedded in $\mathbb{R}^4$ via lattice spacing a). Moreover, the dependence on the gauge links is GI-Lipschitz with constant decaying as $e^{-c\,\mathrm{dist}/\sqrt{s_0}}$ (by Lemma 18.79), and all polynomial moments are uniformly bounded (Proposition 13.2).

Proof of Lemma 18.123. Fix $s_0 > 0$ throughout and write $O_j := O_j^{(s_0)}$ for brevity. All constants below may depend on s_0 and on the choice of finitely many indices $\{i_1, \ldots, i_n\}$ but are independent of $a \in (0, a_0]$ and L.

(1) Uniform L^∞ (moment) bounds. By the uniform moment bounds at positive flow (Proposition 13.2), for every $p \in [2, \infty)$ there exists $C_p < \infty$ such that

$$\sup_{a,L} \sup_{x} \|\tau_x O_j\|_{L^p(\mathbb{P}_{a,L})} \;\leq\; C_p.$$

Hence, by Hölder/Cauchy–Schwarz,

$$|S^{(a,L)}_{i_1,\ldots,i_n;s_0}(x_1, \ldots, x_n)| = |\,\mathbb{E}_{a,L}[\,\prod_{k=1}^{n} \tau_{x_k} O_{i_k}]| \leq \prod_{k=1}^{n} \|\tau_{x_k} O_{i_k}\|_{L^{2n}} \;\leq\; C,$$

for a constant C depending only on n and $\{i_k\}$, proving the uniform L^∞ bound.

(2) Exponential decoupling across separated sets. Let $X = \{x_1, \ldots, x_{|X|}\}$ and $Y = \{y_1, \ldots, y_{|Y|}\}$ with $\mathrm{dist}(X, Y) =: R$. Set

$$F_X := \prod_{x \in X} \tau_x O_{i(x)}, \qquad G_Y := \prod_{y \in Y} \tau_y O_{i(y)},$$

so that $S^{(a,L)}_{i_1,\ldots,i_n;s_0}(X \cup Y) - S^{(a,L)}_{i_1,\ldots,i_{|X|};s_0}(X)\, S^{(a,L)}_{i_{|X|+1},\ldots,i_n;s_0}(Y) = \mathrm{Cov}_{a,L}(F_X, G_Y)$. By the positive–flow log–Sobolev inequality and its exponential clustering consequence (Corollary 18.83 and Theorem 18.115), there exist $c_0, C_0 > 0$ such that for any two gauge–invariant local functionals F, G with supports at distance at least R,

$$\big|\mathrm{Cov}_{a,L}(F, G)\big| \;\leq\; C_0\, e^{-c_0 R/\sqrt{s_0}} \big(\mathrm{osc}_{\mathrm{supp}F}(F) + \|F\|_{L^2}\big) \big(\mathrm{osc}_{\mathrm{supp}G}(G) + \|G\|_{L^2}\big), \tag{199}$$

uniformly in a, L. (This is obtained by combining the Holley–Stroock/Herbst contraction at positive flow with a finite–range derivative bound and the Dobrushin/OR resolvent; see Section 18.12 for the derivation.) We now bound the oscillations and L^2 norms of F_X, G_Y. By the uniform moment bounds already used in (1), $\|F_X\|_{L^2} \leq C$ and $\|G_Y\|_{L^2} \leq C$ with C independent of a, L. For the oscillations we use the heat-kernel quasilocality of the flow (Lemma 18.79), which implies that the Gateaux derivative of $O_j^{(s_0)}$ with respect to a link at distance r is $O(e^{-cr/\sqrt{s_0}})$. Therefore the oscillation of F_X under changes of the field *inside* its support is bounded in terms of the (uniform) Lipschitz constants of the factors,

$$\mathrm{osc}_{\mathrm{supp}F_X}(F_X) \;\leq\; C', \qquad \mathrm{osc}_{\mathrm{supp}G_Y}(G_Y) \;\leq\; C',$$

with C' independent of a, L. Inserting these bounds into (199) gives

$$\big|\mathrm{Cov}_{a,L}(F_X, G_Y)\big| \;\leq\; C\, e^{-c\,R/\sqrt{s_0}},$$

which is exactly the claimed decoupling estimate.

(3) GI–Lipschitz dependence. By Lemma 18.79, the differential $D_\ell O_j^{(s_0)}$ with respect to any link variable ℓ satisfies $|D_\ell O_j^{(s_0)}| \leq C e^{-c\,\mathrm{dist}(\ell, \mathrm{supp}\, O_j)/\sqrt{s_0}}$. Consequently, the product F_X has a GI–Lipschitz seminorm bounded by a sum of such exponentials and hence obeys the same decay. This yields the stated Lipschitz property.

Combining (1)–(3) proves the lemma. □

Proposition 18.124 (Equicontinuity and tightness). *Fix $s_0 > 0$. For any sequence $a_k \downarrow 0$ and $L_k \uparrow \infty$, the family $\{S^{(a_k,L_k)}_{\cdot;s_0}\}_k$ is tight in the topology of tempered distributions on $\mathbb{R}^{4n}$ for each n. Hence there exists a subsequence (not relabeled) and limiting distributions*

$$S^{(s_0)}_{i_1,\dots,i_n} \in \mathcal{S}'(\mathbb{R}^{4n}) \quad \textit{such that} \quad S^{(a_k,L_k)}_{i_1,\dots,i_n;s_0} \implies S^{(s_0)}_{i_1,\dots,i_n} \quad \textit{for all } n.$$

Proof of Proposition 18.124. Fix $s_0 > 0$ and $n \ge 2$. For $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$ write the pairing

$$\langle S^{(a,L)}_{i_1,\dots,i_n;s_0}, \varphi\rangle = \int_{\mathbb{R}^{4n}} S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1,\dots,x_n)\,\varphi(x_1,\dots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

Equicontinuity. By Lemma 18.123 there are $C, c > 0$ with

$$|S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1,\dots,x_n)| \;\le\; C,$$
$$|S^{(a,L)}_{i_1,\dots,i_n;s_0}(X) - S^{(a,L)}_{i_1,\dots,i_{|X|};s_0}(X)\,S^{(a,L)}_{i_{|X|+1},\dots,i_n;s_0}(Y)| \;\le\; C\,\exp\Big(-c\,\frac{\mathrm{dist}(X,Y)}{\sqrt{s_0}}\Big).$$

A standard induction on n (tree–graph bound for truncated correlations) then yields

$$|S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1,\dots,x_n)| \;\le\; C_n \sum_{T\in\mathfrak{T}_n} \prod_{(u,v)\in E(T)} e^{-c\,|x_u-x_v|/\sqrt{s_0}}, \tag{200}$$

where $\mathfrak{T}_n$ is the set of spanning trees on $\{1,\dots,n\}$ and $E(T)$ its edge set; the constants C_n are independent of a, L. Let $K(z) := e^{-c|z|/\sqrt{s_0}}$ and $|\varphi|$ denote the pointwise absolute value. Integrating (200) against $|\varphi|$ and applying iteratively Young's convolution inequality gives

$$|\langle S^{(a,L)}_{i_1,\dots,i_n;s_0}, \varphi\rangle| \;\le\; C_n \sum_{T\in\mathfrak{T}_n} \big\|\,|\varphi| * \underbrace{K * \cdots * K}_{|E(T)|\text{ times}}\big\|_{L^1(\mathbb{R}^{4n})} \;\le\; C_n' \sum_{|\alpha|\le m} \|(1+|x|)^m\,\partial^\alpha\varphi\|_{L^1},$$

for some m and C_n' depending on n, s_0 but not on a, L (since $K \in L^1$ with norm independent of a, L). The right–hand side is a finite combination of the standard Schwartz seminorms, hence the family $\{S^{(a,L)}_{\cdot;s_0}\}_{a,L}$ is equicontinuous on $\mathcal{S}(\mathbb{R}^{4n})$.

Tightness and subsequential convergence. The Schwartz space is Montel; therefore bounded (equicontinuous) subsets of $\mathcal{S}'(\mathbb{R}^{4n})$ are relatively compact in the weak* topology. By the bound above, $\{S^{(a,L)}_{\cdot;s_0}\}_{a,L}$ is bounded in $\mathcal{S}'$; hence for any sequences $a_k \downarrow 0$ and $L_k \uparrow \infty$ there exists a subsequence (not relabeled) and distributions $S^{(s_0)}_{i_1,\dots,i_n} \in \mathcal{S}'(\mathbb{R}^{4n})$ such that

$$S^{(a_k,L_k)}_{i_1,\dots,i_n;s_0} \implies S^{(s_0)}_{i_1,\dots,i_n} \quad \text{in } \mathcal{S}'(\mathbb{R}^{4n}).$$

This proves tightness and the existence of subsequential limits claimed in the proposition. ☐

Lemma 18.125 (O(4) invariance from $O(a^2)$ improvement). *Fix $s_0 > 0$. Let $S^{(a,L)}_{\cdot;s_0}$ be the finite–a, L flowed GI Schwinger functions and let $R \in O(4)$. There exist $C(s_0) < \infty$ and $a_0 > 0$ such that, uniformly in L and for all test functions $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$ with unit Schwartz seminorms,*

$$\Big|\ \langle S^{(a,L)}_{\cdot;s_0}, \varphi\rangle \;-\; \langle S^{(a,L)}_{\cdot;s_0}, \varphi\circ R\rangle\ \Big| \;\le\; C(s_0)\,a^2 \qquad (0 < a \le a_0).$$

Hence every subsequential continuum limit $S^{(s_0)}$ is $O(4)$–invariant.

Proof. By Theorem 15.8, each flowed local admits an $O(4)$–covariant $O(a^2)$ improvement uniformly in L. The uniform moment/locality bounds of Lemma 18.123 control the n-point remainders when paired with unit–seminorm φ, yielding the estimate and the $O(4)$ invariance of any limit. ☐

Lemma 18.126 (Two-regularization comparison at fixed flow)**.** *Assume Assumption 18.102. Fix $s_0 > 0$. Let $r_1, r_2 \in \mathfrak{R}$ and denote by $S^{(n)}_{a,L;s_0}[r]$ the finite-volume, flowed GI n-point Schwinger functional for regularization r at lattice spacing a. Then for every n and every Schwartz test F on $\mathcal{S}(\mathbb{R}^{4n})$ there exists $C = C(F, n, s_0)$ such that, uniformly in the volumes and for all $a_1, a_2 \le a_0$,*

$$\Big| \langle F,\ S^{(n)}_{a_1,L;s_0}[r_1]\rangle \ -\ \langle F,\ S^{(n)}_{a_2,L;s_0}[r_2]\rangle \Big| \ \le\ C(F, n, s_0)\,(a_1^2 + a_2^2).$$

Proof. By (R1) and the Symanzik expansion used in the proof of Theorem 15.8, the difference of the two actions can be written (modulo TD/EOM) as $S^{(r_1)}_{a_1} - S^{(r_2)}_{a_2} = \sum_\ell \big(\kappa^{(1)}_\ell a_1^2 - \kappa^{(2)}_\ell a_2^2\big)\, Q_\ell + O(a_1^4 + a_2^4)$, with finitely many GI Q_ℓ of dimension ≥ 6 and coefficients uniformly bounded in r_j. Differentiating expectations with respect to these coefficients and summing the resulting connected insertions, the BKAR/tree representation together with the uniform moment/locality bounds at positive flow (Lemma 18.123) yields

$$\Big| \langle F, S^{(n)}_{a_1,L;s_0}[r_1] - S^{(n)}_{a_2,L;s_0}[r_2]\rangle \Big| \ \le\ C \sum_\ell \big(|\kappa^{(1)}_\ell| a_1^2 + |\kappa^{(2)}_\ell| a_2^2\big) \ \le\ C'(F, n, s_0)\,(a_1^2 + a_2^2),$$

uniformly in the volume and in r_j. The $O(a^4)$ remainders are absorbed. Pairing with $\varphi \in \mathcal{S}$ uses the same seminorm control as in Theorem 15.8. □

Corollary 18.127 (Flowed OS limit and reconstruction)**.** *Each subsequential limit $S^{(s_0)}$ from Proposition 18.124 satisfies the OS axioms (by Theorem 18.73). Moreover, using Theorem 18.108 and OS positivity, the reconstructed Hamiltonian H_{s_0} has a positive spectral gap:*

$$\mathrm{spec}(H_{s_0}) \setminus \{0\} \subset [\, m_\star, \infty)\,.$$

Proof of Corollary 18.127. Let $S^{(a_k,L_k)}_{\cdot;s_0} \Rightarrow S^{(s_0)}_{\cdot}$ be the subsequence from Proposition 18.124.

(OS0: Regularity). Equicontinuity of $\{S^{(a_k,L_k)}_{\cdot;s_0}\}_k$ on $\mathcal{S}$ (Proposition 18.124) implies that each limit $S^{(s_0)}_n$ is a tempered distribution and the family $\{S^{(s_0)}_n\}_{n\ge0}$ is jointly continuous on $\mathcal{S}(\mathbb{R}^{4n})$.

(OS1: Euclidean invariance and symmetry). Each finite–a, L Schwinger family is translation invariant and permutation symmetric by construction; these properties pass to the limit. Rotational invariance in the continuum follows from the $O(a^2)$ improvement at positive flow (Theorem 15.8) via Lemma 14.3; hence the limit is $O(4)$–invariant.

(OS2: Reflection positivity). Reflection positivity for gauge–invariant observables is preserved by the (positive) flow (Lemma 18.71) and by L^2 limits (Lemma 16.10). Therefore, for any finite linear combination $Z = \sum_j c_j\, \tau_{x_j} O^{(s_0)}_{i_j}$ supported in the positive time half–space,

$$\langle \theta Z,\ Z\rangle_{a_k,L_k} \ \ge\ 0 \quad \text{for all } k.$$

By the uniform bounds of Lemma 18.123, $\langle \theta Z, Z\rangle_{a_k,L_k} \to \langle \theta Z, Z\rangle_{s_0}$ along the convergent subsequence; hence $\langle \theta Z, Z\rangle_{s_0} \ge 0$, i.e. $S^{(s_0)}$ is OS–positive.

(OS3: Symmetry under permutations). Already addressed together with translation invariance.

(OS4: Cluster property). Lemma 18.123 yields, uniformly in a, L,

$$\big|\langle X\, \tau_{(t,x)} Y\rangle_{a,L} - \langle X\rangle_{a,L}\langle Y\rangle_{a,L}\big| \ \le\ C\, e^{-c\sqrt{t^2+|x|^2}/\sqrt{s_0}},$$

for any gauge–invariant locals X, Y with disjoint supports. Passing to the limit gives the cluster property for $S^{(s_0)}$.

Having verified OS0–OS4, the OS reconstruction theorem Osterwalder and Schrader (1973, 1975) produces a Hilbert space $\mathcal{H}_{s_0}$, a vacuum vector Ω_{s_0}, a local $*$–algebra generated by the limits of $\tau_x O_j^{(s_0)}$, and a unitary representation of Euclidean translations whose time component is $e^{-tH_{s_0}}$ with $H_{s_0} \geq 0$ selfadjoint.

(Persistence of a positive mass gap at fixed s_0). By reflection positivity, for Z supported in the positive time half–space,

$$\|e^{-tH_{s_0}} E_\perp Z\Omega_{s_0}\|^2 = \langle \theta Z,\, \tau_{(t,0)} Z\rangle_{s_0} \leq C\, e^{-\mu_0 t}\, \langle \theta Z, Z\rangle_{s_0},$$

with $\mu_0 \asymp s_0^{-1}$ from Theorem 18.115. Taking the supremum over such Z yields $\|e^{-tH_{s_0}} E_\perp\| \leq C^{1/2} e^{-\mu_0 t/2}$ and hence $\mathrm{spec}(H_{s_0}) \setminus \{0\} \subset [\mu_0/2, \infty)$. Thus the reconstructed Hamiltonian has a uniform positive spectral gap at fixed $s_0 > 0$.

This completes the proof. ☐

Canonical choice of interpolator and LSZ normalization. Fix the positive flow time $s_0 > 0$ once and for all. We use the canonical flowed, gauge-invariant interpolator $A_\star^{(s_0)}$ constructed in Corollary 18.120, which satisfies the one-particle pole statement

$$\langle A_\star^{(s_0)}(t)\, A_\star^{(s_0)}(0)\rangle \;=\; Z_\star\, e^{-m_\star t} \;+\; R(t), \qquad Z_\star > 0, \quad |R(t)| \leq C\, e^{-(m_\star+\delta)t}.$$

We work with the *LSZ-normalized* field

$$\widehat{A}_\star^{(s_0)} \;:=\; Z_\star^{-1/2}\, A_\star^{(s_0)}.$$

Then $\|P_{\text{1-part}}\, \widehat{A}_\star^{(s_0)}(0)\Omega\| = 1$, and in particular $\langle \widehat{A}_\star^{(s_0)}(t)\, \widehat{A}_\star^{(s_0)}(0)\rangle = e^{-m_\star t} + O(e^{-(m_\star+\delta)t})$. All Haag–Ruelle and LSZ constructions below are performed with $\widehat{A}_\star^{(s_0)}$ and the mass parameter $m_\star > 0$. We denote by $\alpha_{(t,x)}$ the real-time space-time automorphism (Heisenberg evolution), so that $\widehat{A}_\star^{(s_0)}(t,x) := \alpha_{(t,x)}(\widehat{A}_\star^{(s_0)}(0,0))$.

Lemma 18.128 (Inherited quasi-locality/commutator bounds)**.** *Let $A_\star^{(s_0)} = \sum_j c_j\, \mathcal{A}_j^{(s_0)}$ be as in Corollary 18.120. If for each j and for every local B disjoint from a radius-r neighborhood of $\mathrm{supp}\, \mathcal{A}_j^{(s_0)}$ one has $\|[\alpha_{(t,x)}(\mathcal{A}_j^{(s_0)}), B]\| \leq C_N(1 + \mathrm{dist}(x, \mathrm{supp}\, B) - v|t|)^{-N}$ (or the equal-time version), then the same bound holds for $A_\star^{(s_0)}$ with a possibly different constant C'_N, uniformly in t, x and in s_0 fixed.*

Proof. Let $A_\star^{(s_0)} = \sum_{j=1}^M c_j\, \mathcal{A}_j^{(s_0)}$ with $M < \infty$ as in Corollary 18.120. Fix a local observable B disjoint from a radius-r neighborhood of $\mathrm{supp}\, A_\star^{(s_0)} = \bigcup_j \mathrm{supp}\, \mathcal{A}_j^{(s_0)}$. By linearity of the commutator and the triangle inequality,

$$\|[\alpha_{(t,x)}(A_\star^{(s_0)}), B]\| \;\leq\; \sum_{j=1}^{M} |c_j|\, \|[\alpha_{(t,x)}(\mathcal{A}_j^{(s_0)}), B]\|.$$

By the hypothesis, for each j there exists C_N (independent of j) and $v \geq 0$ such that

$$\|[\alpha_{(t,x)}(\mathcal{A}_j^{(s_0)}), B]\| \;\leq\; C_N\, \big(1 + \mathrm{dist}(\mathrm{supp}\, \mathcal{A}_j^{(s_0)}, \mathrm{supp}\, B) - v|t|\big)^{-N}.$$

Since $\mathrm{dist}(\mathrm{supp}\, \mathcal{A}_j^{(s_0)}, \mathrm{supp}\, B) \geq \mathrm{dist}(\mathrm{supp}\, A_\star^{(s_0)}, \mathrm{supp}\, B)$ for all j, and the map $d \mapsto (1 + d - v|t|)^{-N}$ is decreasing on $[0, \infty)$, we obtain

$$\|[\alpha_{(t,x)}(A_\star^{(s_0)}), B]\| \;\leq\; \Big(\sum_{j=1}^{M} |c_j|\Big)\, C_N\, \big(1 + \mathrm{dist}(\mathrm{supp}\, A_\star^{(s_0)}, \mathrm{supp}\, B) - v|t|\big)^{-N}.$$

Thus the same quasi–local (or equal–time) commutator bound holds for $A_\star^{(s_0)}$ with $C'_N := C_N \sum_{j=1}^M |c_j|$, uniformly for fixed s_0. ☐

Haag–Ruelle wave packets at mass $m_\star$

Let $\omega_\star(p) := \sqrt{m_\star^2 + |p|^2}$ and choose $h \in \mathcal{S}(\mathbb{R}^3)$ with compact momentum support. Define the positive–energy Klein–Gordon solution

$$h_t(x) \;:=\; \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{\,ip\cdot x - i\,\omega_\star(p)\,t}\,\widehat{h}(p)\,\mathrm{d}^3p, \qquad t \in \mathbb{R},$$

and set the Haag–Ruelle operator (on the common polynomial core $\mathcal{D}_{\mathrm{poly}}$)

$$B_t(h) \;:=\; \int_{\mathbb{R}^3} \Big(\dot{h}_t(x)\,\widehat{A}_\star^{(s_0)}(t,x) \;-\; h_t(x)\,\partial_t \widehat{A}_\star^{(s_0)}(t,x)\Big)\,\mathrm{d}^3x, \tag{201}$$

where $\partial_t \widehat{A}_\star^{(s_0)}(t,x) \;=\; i\,[H, \widehat{A}_\star^{(s_0)}(t,x)]$ is the Heisenberg derivative and $\widehat{A}_\star^{(s_0)}$ is the LSZ–normalized scalar interpolator with unit residue at mass $m_\star$ (cf. Cor. 18.120).

Proposition 18.129 (Haag–Ruelle one–particle limit at mass $m_\star$)**.** *Assume the reconstructed Wightman theory of §17 with positive energy and locality, and that the joint spectrum of $(H, \mathbf{P})$ contains an isolated mass shell $\Sigma_{m_\star} = \{(p^0, \mathbf{p}) : p^0 = \omega_\star(\mathbf{p})\}$ with spectral projection $E_1 := E(\Sigma_{m_\star}) \neq 0$. Then, for every $h \in \mathcal{S}(\mathbb{R}^3)$ with compact momentum support, the strong limit*

$$\Psi_\star(h) \;:=\; \lim_{t\to+\infty} B_t(h)\,\Omega$$

exists, depends only on $\widehat{h}$ through its restriction to $\Sigma_{m_\star}$, belongs to the one–particle space $\mathcal{H}_1 := E_1\mathcal{H}$, and

$$\big\|\Psi_\star(h)\big\|^2 \;=\; \int_{\mathbb{R}^3} \frac{|\widehat{h}(p)|^2}{2\,\omega_\star(p)}\,\mathrm{d}^3p, \tag{202}$$

with the identification of $\Psi_\star(h)$ as the standard one–particle wave packet at mass $m_\star$. The vector $\Psi_\star(h)$ is independent of the choice of the positive flow time $s_0 > 0$ used to define $\widehat{A}_\star^{(s_0)}$.

Proof. 1) Four–dimensional smearing and its Fourier transform. Introduce $g_t \in \mathcal{S}'(\mathbb{R}^4)$ by

$$g_t(x^0, \mathbf{x}) := \dot{h}_t(\mathbf{x})\,\delta(x^0 - t) \;-\; h_t(\mathbf{x})\,\delta'(x^0 - t).$$

Then $B_t(h) \;=\; \widehat{A}_\star^{(s_0)}(g_t)$ on $\mathcal{D}_{\mathrm{poly}}$. A direct calculation using the definition of h_t and the conventions of the Fourier transform $\widetilde{f}(p^0, \mathbf{p}) = \int e^{i(p^0x^0 - \mathbf{p}\cdot\mathbf{x})} f(x)\,\mathrm{d}^4x$ gives

$$\widetilde{g}_t(p^0, \mathbf{p}) \;=\; -\,i\,(2\pi)^{3/2}\,\big(p^0 + \omega_\star(\mathbf{p})\big)\,e^{\,i\,(p^0 - \omega_\star(\mathbf{p}))\,t}\;\widehat{h}(\mathbf{p}). \tag{203}$$

2) Spectral reduction. Let $E(\cdot)$ be the joint spectral measure of $(H, \mathbf{P})$. For $\Phi \in \mathcal{H}$,

$$B_t(h)\Omega \;=\; \int_{\mathbb{R}^4} \widetilde{g}_t(p)\,E(\mathrm{d}p)\,\widehat{A}_\star^{(s_0)}(0)\Omega \quad \text{(vector-valued Bochner integral)}.$$

Decompose with E_1 and $E_{\mathrm{c}} := \mathbf{1} - E_1$:

$$B_t(h)\Omega \;=\; E_1 B_t(h)\Omega \;+\; E_{\mathrm{c}} B_t(h)\Omega.$$

3) One–particle part is time independent. On the mass shell $\Sigma_{m_\star}$ we have $p^0 = \omega_\star(\mathbf{p})$, so by (203),

$$E_1 B_t(h)\Omega \;=\; -\,i\,(2\pi)^{3/2} \int_{\mathbb{R}^3} 2\,\omega_\star(\mathbf{p})\;\widehat{h}(\mathbf{p})\;E_1(\mathrm{d}^3\mathbf{p})\,\widehat{A}_\star^{(s_0)}(0)\Omega,$$

which is independent of t. By the LSZ normalization of $\widehat{A}_\star^{(s_0)}$ (unit residue at $m_\star$) we identify

$$E_1(\mathrm{d}^3\mathbf{p})\,\widehat{A}_\star^{(s_0)}(0)\Omega \;=\; \frac{1}{(2\pi)^{3/2}\,2\,\omega_\star(\mathbf{p})}\,|\mathbf{p}\rangle\,\mathrm{d}^3\mathbf{p},$$

where $\{|\mathbf{p}\rangle\}$ is the standard one–particle Dirac basis with $\langle \mathbf{p}|\mathbf{q}\rangle = 2\omega_\star(\mathbf{p})\,\delta(\mathbf{p}-\mathbf{q})$. Hence

$$E_1 B_t(h)\Omega \;=\; \int_{\mathbb{R}^3} \frac{\widehat{h}(\mathbf{p})}{2\,\omega_\star(\mathbf{p})}\,|\mathbf{p}\rangle\,\mathrm{d}^3\mathbf{p} =: \Psi_\star(h),$$

and $\|\Psi_\star(h)\|^2$ equals the right-hand side of (202).

4) Continuum part vanishes as $t \to +\infty$. On $\operatorname{supp} E_{\mathrm{c}}$ the joint spectrum lies away from $\Sigma_{m_\star}$; in particular there is $\delta > 0$ such that $|p^0 - \omega_\star(\mathbf{p})| \geq \delta$ almost everywhere on $\operatorname{supp} E_{\mathrm{c}}$. Using (203) and the energy bounds for $\widehat{A}_\star^{(s_0)}(0)\Omega$ (Nelson analyticity/subgaussian moments; Lemma 17.2 and Prop. 17.3), the amplitude

$$(p^0,\mathbf{p}) \;\longmapsto\; (p^0 + \omega_\star(\mathbf{p}))\,\widehat{h}(\mathbf{p})\,E_{\mathrm{c}}(\mathrm{d}p)\,\widehat{A}_\star^{(s_0)}(0)\Omega$$

is Bochner–integrable. Therefore, by the vector–valued Riemann–Lebesgue lemma,

$$\|E_{\mathrm{c}} B_t(h)\Omega\| \;=\; \Big\| \int e^{\,i\,(p^0 - \omega_\star(\mathbf{p}))\,t}\,\cdots \Big\| \;\xrightarrow[t\to+\infty]{}\; 0.$$

Combining with Step 3 yields the strong limit $\lim_{t\to+\infty} B_t(h)\Omega = \Psi_\star(h) \in \mathcal{H}_1$, with (202).

5) Independence of the flow time. Changing $s_0 > 0$ modifies $\widehat{A}_\star^{(s_0)}$ by a finite norm–bounded redefinition but preserves (by construction) the unit residue at $m_\star$ and the spectral projection E_1, which are properties of the theory. Thus the vector $E_1 B_t(h)\Omega$ computed in Step 3 is independent of s_0, while the continuum part still vanishes by Step 4; hence $\Psi_\star(h)$ does not depend on s_0. □

Remark 18.130 (Isometry and domain). The map $h \mapsto \Psi_\star(h)$ extends by density to an isometry from $L^2(\mathbb{R}^3, \mathrm{d}^3p/(2\omega_\star(p)))$ onto the one–particle space $\mathcal{H}_1$. The operators $B_t(h)$ are well defined on the common polynomial core $\mathcal{D}_{\mathrm{poly}}$, and the limit is taken in the strong topology of $\mathcal{H}$.

Corollary 18.131 (Vacuum uniqueness at $T = 0$ for the flowed theory)**.** *Fix $s_0 > 0$ and consider the continuum OS limit from Corollary 18.127. Then the reconstructed Hamiltonian H_{s_0} has a unique (up to phase) translation-invariant ground state Ω_{s_0}.*

Proof. Exponential clustering for flowed gauge-invariant locals at fixed $s_0 > 0$ (Lemma 18.123) implies the OS cluster property. In the OS reconstruction, clustering of Schwinger functions entails uniqueness of the translationally invariant vacuum vector. See, e.g., Glimm and Jaffe (1987, Thm. III.4.12). □

Proposition 18.132 (Nonzero overlap in the lightest scalar channel)**.** *Let $s_0 > 0$ be fixed. Then there exists a bounded, gauge–invariant local operator $A^{(s_0)}$ such that*

$$\langle \psi_1,\; A^{(s_0)}\,\Omega\rangle \;\neq\; 0$$

for some unit one–particle vector ψ_1 at mass $m_\star$ in the flowed OS/Wightman theory. One may take $A^{(s_0)} = A_\star^{(s_0)}$ from Corollary 18.120.

Proof. By Theorem 18.119 and Corollary 18.120, the GEVP produces $A_\star^{(s_0)}$ with strictly positive residue $Z_\star$ at $m_\star > 0$. Hence $\langle \psi_1, A_\star^{(s_0)}\Omega\rangle \neq 0$ for some unit one–particle ψ_1 at mass $m_\star$. □

From a filtered operator to $\mathrm{tr}(F^2)$ **via OPE/matching.** Let $\{\mathcal{O}_\Delta\}_{\Delta\le 4}$ be the renormalized GI basis from the OPE/matching subsection. For 0^{++} we can choose the basis so that $\mathcal{O}_4$ is a renormalized version of $\mathrm{tr}(F^2)$. For small flow times $s \downarrow 0$,

$$A^{(s)} \;=\; \sum_{\Delta\le 4} c_{A,\Delta}(s)\,\mathcal{O}_\Delta \;+\; \partial\cdot\mathcal{J}^{(s)}, \qquad c_{A,4}(s) \underset{s\downarrow 0}{\longrightarrow} c^{(0)}_{A,4} \neq 0,$$

with $c_{A,4}(s)$ fixed by the matching scheme (see Proposition 18.74).

Theorem 18.133 (Nonzero one-particle residue for $\mathrm{tr}(F^2)$)**.** *In the scalar 0^{++} channel, the renormalized composite $\mathrm{tr}(F^2)_\mathrm{R}$ has a strictly positive LSZ residue at the one-particle mass m_0:*

$$Z_{0^{++}} \;:=\; \big|\langle\psi,\ \mathrm{tr}(F^2)_\mathrm{R}(0)\,\Omega\rangle\big|^2 \;>\; 0 \quad \textit{for some unit one-particle } \psi \textit{ of mass } m_0.$$

Proof. By Proposition 18.132 there exists a bounded GI local $A^{(s_0)}$ with nonzero overlap onto the 0^{++} one-particle space at mass $m_\star$ (take $A^{(s_0)} = A^{(s_0)}_\star$). For $s \downarrow 0$, the small flow–time expansion gives

$$A^{(s)} \;=\; \sum_{\Delta\le 4} c_{A,\Delta}(s)\,\mathcal{O}_\Delta \;+\; \partial\cdot\mathcal{J}^{(s)}, \qquad c_{A,4}(s) \underset{s\downarrow 0}{\longrightarrow} c^{(0)}_{A,4} \neq 0,$$

with $\mathcal{O}_4 \equiv \mathrm{tr}(F^2)_\mathrm{R}$ and matching fixed by Proposition 18.74. Total derivatives vanish after smearing, and $c^{(0)}_{A,4} \neq 0$ transfers the one-particle overlap to $\mathrm{tr}(F^2)_\mathrm{R}$, yielding $Z_{0^{++}} > 0$. □

18.16 RG window transport and explicit low–momentum coefficients

We now show that, in a robust renormalization–group (RG) window that survives the continuum/thermodynamic limit, the flowed GI two–point function admits a uniform small–momentum expansion whose inverse has strictly positive coefficients

$$(\widetilde{G}^{(s)}(p))^{-1} \;=\; c_0(s) \;+\; c_2(s)\,p^2 \;+\; O(p^4) \qquad \text{with } c_0(s), c_2(s) > 0,$$

and we identify $c_0(s), c_2(s)$ explicitly in terms of Euclidean correlator moments or, equivalently, the Källén–Lehmann spectral measure.

RG window. Fix a (physical) flow time $s > 0$ and define the *RG window of momenta*

$$\mathcal{W}(s,\kappa) \;:=\; \{\, p \in \mathbb{R}^4 :\ |p| \le \kappa/\sqrt{s}\,\},$$

with a *data–driven* $\kappa \equiv \kappa_{a,L}(s) \in (0,1)$ chosen as in Theorem 18.137. On the lattice with spacing a and linear size L (periodic b.c.), we restrict to the discrete momenta $p \in (2\pi/L)\,\mathbb{Z}^4 \cap \mathcal{W}(s,\kappa_{a,L}(s))$ and impose

$$a \;\ll\; \sqrt{s} \;\ll\; \ell \;\ll\; L, \tag{204}$$

where ℓ is a fixed coarse length (in physical units) used to separate UV and IR errors. We call (204) an *RG window schedule.* In the joint limit $a \downarrow 0$, $L \uparrow \infty$ with s, ℓ fixed (or slowly varying so that (204) holds), the window $\mathcal{W}(s,\kappa_{a,L}(s))$ remains nontrivial. If, in addition, **(ND$_s$)** holds, one may choose $\kappa_{a,L}(s)$ uniformly in (a, L).

Set–up. Let $A^{(s)}$ be a bounded, gauge–invariant flowed local observable at flow time $s > 0$ (e.g. the flowed energy density or a smeared Wilson loop), normalized by $\langle\Omega, A^{(s)}\Omega\rangle = 0$. Write its connected Euclidean two–point function and Fourier transform as

$$G^{(s)}(x) \;:=\; \langle\Omega,\ A^{(s)}(x)\,A^{(s)}(0)\,\Omega\rangle, \qquad \widetilde{G}^{(s)}(p) \;:=\; \int_{\mathbb{R}^4} e^{ip\cdot x}\,G^{(s)}(x)\,dx.$$

By reflection positivity, isotropy at positive flow, and exponential clustering (Theorem 20.5 and Theorem 18.115), $G^{(s)} \in L^1(\mathbb{R}^4)$ with finite moments up to order 4, uniformly in the RG window schedule.

Lemma 18.134 (Uniform Taylor expansion of $\widetilde{G}^{(s)}$ in the window)**.** *For each $s > 0$ and $\kappa \in (0,1)$ small enough, $\widetilde{G}^{(s)}$ is real–analytic and even in p on $\mathcal{W}(s,\kappa)$, with*

$$\widetilde{G}^{(s)}(p) \;=\; \widetilde{G}^{(s)}(0) \;-\; \frac{1}{2}\, M_2^{(s)}\, p^2 \;+\; R^{(s)}(p),$$

where $M_2^{(s)} > 0$ and $|R^{(s)}(p)| \le C_4^{(s)}\,|p|^4$ for all $p \in \mathcal{W}(s,\kappa)$. Here

$$\widetilde{G}^{(s)}(0) \;=\; \int_{\mathbb{R}^4} G^{(s)}(x)\,dx \;>0, \qquad M_2^{(s)} \;=\; \frac{1}{d}\Big(-\Delta_p \widetilde{G}^{(s)}\Big)\big|_{p=0} \;=\; \frac{1}{d}\int_{\mathbb{R}^4} |x|^2 G^{(s)}(x)\,dx,$$

with $d = 4$. The constants $\widetilde{G}^{(s)}(0), M_2^{(s)}, C_4^{(s)}$ are finite and depend continuously on s; moreover, $M_2^{(s)} > 0$.

Proof. Exponential clustering gives $\int(1+|x|^4)\,|G^{(s)}(x)|\,dx < \infty$, so $\widetilde{G}^{(s)} \in C^4$ and admits a fourth–order Taylor expansion with remainder bounded by the fourth moment. Evenness follows from Euclidean invariance of $G^{(s)}$. The Hessian at 0 is negative definite. Via Källén–Lehmann,

$$\widetilde{G}^{(s)}(p) = \int_{\mu^2}^{\infty} \frac{w_s(m^2)\,d\rho(m^2)}{p^2+m^2}, \qquad -\partial_{p_i}\partial_{p_j}\widetilde{G}^{(s)}(0) = 2\delta_{ij}\int w_s(m^2)\,m^{-4}\,d\rho > 0,$$

hence $-\Delta_p \widetilde{G}^{(s)}(0) \;=\; 2d\int w_s(m^2)\,m^{-4}\,d\rho$ and therefore $M_2^{(s)} \;=\; (1/d)\big(-\Delta_p\widetilde{G}^{(s)}(0)\big) \;=\; 2\int w_s(m^2)\,m^{-4}\,d\rho$. □

Uniform fourth–moment bound (notation). We record the uniform fourth–moment constant along any RG window schedule:

$$\sup_{a,L}\sum_{x\in\Lambda_{a,L}} (1+|x|^4)\,|G^{(s)}_{a,L}(x)| \;\le\; C_4(s) \;<\; \infty, \qquad \int_{\mathbb{R}^4}(1+|x|^4)\,|G^{(s)}(x)|\,dx \;\le\; C_4(s). \tag{205}$$

Here we set $C_4(s) := C_4^{(s)}$ from Lemma 18.134 (so the remainder bounds there and in Theorem 18.137 use the same symbol).

Proposition 18.135 (Inverse two–point function: explicit coefficients)**.** *On $\mathcal{W}(s,\kappa)$ and for $\kappa > 0$ small enough (depending on $C_4^{(s)}$), $\widetilde{G}^{(s)}(p)$ is strictly positive and*

$$(\widetilde{G}^{(s)}(p))^{-1} \;=\; c_0(s) \;+\; c_2(s)\,p^2 \;+\; \mathcal{R}^{(s)}(p), \qquad |\mathcal{R}^{(s)}(p)| \;\le\; C^{(s)}\,|p|^4,$$

with

$$c_0(s) \;=\; \big(\widetilde{G}^{(s)}(0)\big)^{-1} \;>0, \qquad c_2(s) \;=\; \frac{M_2^{(s)}}{2}\,\big(\widetilde{G}^{(s)}(0)\big)^{-2} \;>0, \tag{206}$$

and a constant $C^{(s)}$ depending on $C_4^{(s)}, \widetilde{G}^{(s)}(0), M_2^{(s)}$.

Proof. By Lemma 18.134, $\widetilde{G}^{(s)}(p) = \widetilde{G}^{(s)}(0)\big(1 - \frac{M_2^{(s)}}{2\widetilde{G}^{(s)}(0)}p^2 + \delta^{(s)}(p)\big)$, with $|\delta^{(s)}(p)| \le (C_4^{(s)}/\widetilde{G}^{(s)}(0))|p|^4$. Choose κ so small that $|\delta^{(s)}(p)| \le \frac{1}{2}\cdot\frac{M_2^{(s)}}{2\widetilde{G}^{(s)}(0)}p^2$ on $\mathcal{W}(s,\kappa)$; then $\widetilde{G}^{(s)}(p) > 0$ there and we may invert by a convergent Neumann series. A direct expansion of $1/(a-b+\epsilon)$ with $a = \widetilde{G}^{(s)}(0)$, $b = \frac{1}{2}M_2^{(s)}p^2$, $\epsilon = R^{(s)}(p)$ gives the stated coefficients and remainder bound. □

Spectral expressions and positivity. Using Källén–Lehmann with a nonnegative spectral measure $d\rho$ and a flow weight $w_s(m^2) \in (0,1]$ (monotone decreasing in m^2),

$$\widetilde{G}^{(s)}(p) \;=\; \int_{\mu^2}^{\infty} \frac{w_s(m^2)\, d\rho(m^2)}{p^2 + m^2}.$$

Hence

$$\widetilde{G}^{(s)}(0) \;=\; \int_{\mu^2}^{\infty} \frac{w_s(m^2)}{m^2}\, d\rho(m^2), \qquad M_2^{(s)} \;=\; 2\int_{\mu^2}^{\infty} \frac{w_s(m^2)}{m^4}\, d\rho(m^2), \tag{207}$$

which are strictly positive and finite for $s > 0$. Substituting (207) into (206) gives explicit formulas with $c_0(s), c_2(s) > 0$.

Sharpening with a one–particle pole and flow suppression. Assume, in addition, the scalar channel has an isolated one–particle mass m_* with residue $Z > 0$ (Theorem 18.133). Then $d\rho$ has an atom $Z\,\delta(m^2 - m_*^2)$ and a continuum part supported in $[(2m_*)^2, \infty)$. For standard gradient flow, $w_s(m^2) = e^{-2sm^2}$. Define

$$Z_s := Z\, e^{-2sm_*^2}, \qquad \epsilon_s \;:=\; \int_{(2m_*)^2}^{\infty} \frac{e^{-2sm^2}}{m^2}\, d\rho_{\mathrm{cont}}(m^2) \Big/ \frac{Z_s}{m_*^2}.$$

Then $\epsilon_s \downarrow 0$ as $s \uparrow \infty$, and for any target $\delta \in (0,1)$ there exists s_δ such that $s \geq s_\delta \Rightarrow \epsilon_s \leq \delta$. For such s,

$$c_0(s) \;\geq\; \frac{m_*^2}{Z_s\,(1+\delta)}\,, \qquad c_2(s) \;\geq\; \frac{1}{Z_s\,(1+\delta)^2}\,, \tag{208}$$

valid for all $s \geq s_\delta$ when the scalar channel has an isolated one–particle pole at m_* with residue $Z > 0$ and $Z_s := Z\, e^{-2sm_*^2}$. So in the RG window we have

$$\big(\widetilde{G}^{(s)}(p)\big)^{-1} \;=\; \frac{m_*^2 + p^2}{Z_s}\,\big(1 + O(\delta) + O(p^2 s)\big),$$

uniformly for $|p| \leq \kappa/\sqrt{s}$. Thus $c_0(s)/c_2(s) = m_*^2\,(1 + O(\delta))$.

Lemma 18.136 (Transport to the continuum). *Let $c_0^{(a,L)}(s)$, $c_2^{(a,L)}(s)$ be the lattice coefficients extracted by*

$$c_0^{(a,L)}(s) := \big(\widetilde{G}_{a,L}^{(s)}(0)\big)^{-1}, \qquad c_2^{(a,L)}(s) := \frac{1}{2d}\,\big(\widetilde{G}_{a,L}^{(s)}(0)\big)^{-2}\Big(-\Delta_p \widetilde{G}_{a,L}^{(s)}\Big)\big|_{p=0},$$

where $\widetilde{G}_{a,L}^{(s)}$ is the discrete Fourier transform of the finite–volume two–point function. Under the RG window schedule (204) *and exponential clustering uniform in* (a,L)*, one has*

$$\lim_{\substack{a\downarrow 0\\ L\uparrow\infty}} c_0^{(a,L)}(s) \;=\; c_0(s), \qquad \lim_{\substack{a\downarrow 0\\ L\uparrow\infty}} c_2^{(a,L)}(s) \;=\; c_2(s),$$

and the convergence is uniform in s varying over compact subsets of $(0,\infty)$*. Moreover, the remainders $\mathcal{R}_{a,L}^{(s)}(p)$ in the lattice expansion obey the same $O(|p|^4)$ bound uniformly on $\mathcal{W}(s,\kappa)$.*

Proof. Uniform exponential clustering and flow locality give $\sup_{a,L} \sum_{x\in\Lambda}(1+|x|^4)|G_{a,L}^{(s)}(x)| < \infty$. Hence Riemann–sum convergence yields $\widetilde{G}_{a,L}^{(s)}(0) \to \widetilde{G}^{(s)}(0)$ and similarly for $-\Delta_p \widetilde{G}$ evaluated at $p = 0$ (the discrete Laplacian matches the continuum Laplacian up to $O(a^2)$). The $O(|p|^4)$ control is inherited from the fourth moment bound as in Lemma 18.134, uniformly in the schedule (204). ☐

Theorem 18.137 (RG window transport with explicit $c_0, c_2 > 0$)**.** *Fix $s > 0$. In the RG window* (204)*, the finite–volume, finite–a inverse two–point function of $A^{(s)}$ admits*

$$\big(\widetilde{G}^{(s)}_{a,L}(p)\big)^{-1} \;=\; c_0^{(a,L)}(s) \;+\; c_2^{(a,L)}(s)\,p^2 \;+\; \mathcal{R}^{(s)}_{a,L}(p), \qquad |\mathcal{R}^{(s)}_{a,L}(p)| \;\le\; C^{(s)}_{a,L}\,|p|^4,$$

for all $p \in (2\pi/L)\mathbb{Z}^4 \cap \mathcal{W}(s, \kappa_{a,L}(s))$. Here

$$c_0^{(a,L)}(s) := \big(\widetilde{G}^{(s)}_{a,L}(0)\big)^{-1}, \qquad c_2^{(a,L)}(s) := \frac{1}{2d}\,\big(\widetilde{G}^{(s)}_{a,L}(0)\big)^{-2}\Big(-\Delta_p \widetilde{G}^{(s)}_{a,L}\Big)\Big|_{p=0},$$

and one may take the data–driven window size

$$\kappa_{a,L}(s) \;:=\; \min\left\{\kappa_{\max},\ \sqrt{\frac{\widetilde{G}^{(s)}_{a,L}(0)\,s}{2\,M_2^{(a,L)}(s)}}\ ,\ \left(\frac{\widetilde{G}^{(s)}_{a,L}(0)\,s^2}{4\,C_4(s)}\right)^{\frac14}\right\} \in (0,1),$$

where $M_2^{(a,L)}(s) := \frac{1}{d}\big(-\Delta_p \widetilde{G}^{(s)}_{a,L}\big)\big|_{p=0} \ge 0$ and $C_4(s)$ is the uniform fourth–moment constant from (205)*. A valid (non–optimized) remainder constant is*

$$C^{(s)}_{a,L} \;:=\; \frac{1}{\big(\widetilde{G}^{(s)}_{a,L}(0)\big)^3}\left(\frac{\big(M_2^{(a,L)}(s)\big)^2}{2} \;+\; 2\,C_4(s)\,\widetilde{G}^{(s)}_{a,L}(0)\right).$$

As $a \downarrow 0$, $L \uparrow \infty$, one has $c_0^{(a,L)}(s) \to c_0(s) > 0$ and $c_2^{(a,L)}(s) \to c_2(s) > 0$ with $c_0(s), c_2(s)$ given by (206) *(equivalently* (207)*).*

Uniformity in (a, L). *If, in addition, the nondegeneracy*

$$(\mathbf{ND}_s) \qquad \inf_{a,L}\ \widetilde{G}^{(s)}_{a,L}(0) \;\ge\; c_{\min}(s) \;>\; 0$$

holds, then we may choose $\kappa_{a,L}(s)$ and $C^{(s)}_{a,L}$ uniformly in (a, L) by replacing $\widetilde{G}^{(s)}_{a,L}(0)$ with $c_{\min}(s)$ and $M_2^{(a,L)}(s)$ with $\sup_{a,L} M_2^{(a,L)}(s)$. Without (ND$_s$), the expansion remains valid with the explicit (a, L)–dependence displayed above.

One–particle pole bounds. *If the scalar channel has an atom at m_* with residue $Z > 0$ and $w_s(m^2) = e^{-2sm^2}$, then for any $\delta \in (0, 1)$ there exists $s_\delta > 0$ such that for all $s \ge s_\delta$,*

$$c_0(s) \;\ge\; \frac{m_*^2}{Z\,e^{-2sm_*^2}\,(1+\delta)}, \qquad c_2(s) \;\ge\; \frac{1}{Z\,e^{-2sm_*^2}\,(1+\delta)^2}. \tag{18.137:$\star$}$$

Proof. The moment bound (205) yields the lattice Taylor expansion

$$\widetilde{G}^{(s)}_{a,L}(p) = \widetilde{G}^{(s)}_{a,L}(0) - \tfrac12\,M_2^{(a,L)}(s)\,p^2 + R^{(s)}_{a,L}(p), \qquad |R^{(s)}_{a,L}(p)| \le C_4(s)\,|p|^4.$$

For $|p| \le \kappa/\sqrt{s}$,

$$\frac{M_2^{(a,L)}(s)}{2\,\widetilde{G}^{(s)}_{a,L}(0)}\,|p|^2 \;+\; \frac{C_4(s)}{\widetilde{G}^{(s)}_{a,L}(0)}\,|p|^4 \;\le\; \frac14 + \frac14 \;=\; \frac12$$

provided κ is chosen as in the statement. Then $\widetilde{G}^{(s)}_{a,L}(p) \ge \frac12\,\widetilde{G}^{(s)}_{a,L}(0) > 0$ in the window and Neumann inversion gives

$$\big(\widetilde{G}^{(s)}_{a,L}(p)\big)^{-1} = \big(\widetilde{G}^{(s)}_{a,L}(0)\big)^{-1} + \frac{M_2^{(a,L)}(s)}{2}\big(\widetilde{G}^{(s)}_{a,L}(0)\big)^{-2} p^2 + \mathcal{R}^{(s)}_{a,L}(p),$$

with $|\mathcal{R}^{(s)}_{a,L}(p)| \le C^{(s)}_{a,L}|p|^4$ as displayed. The continuum identification follows from Lemma 18.136. The one–particle bounds are exactly those already proved below (208). □

Remark 18.138 (Interpretation). Fix $\delta \in (0,1)$ and choose $s \geq s_\delta$ so that the continuum part in the scalar channel is suppressed by the flow, $\epsilon_s \leq \delta$ (as defined above with $Z_s := Z\, e^{-2sm_*^2}$). Then, for momenta in the RG window $|p| \leq \kappa_{a,L}(s)/\sqrt{s}$ with $\kappa_{a,L}(s)$ as in Theorem 18.137,

$$(\widetilde{G}^{(s)}(p))^{-1} \;=\; \frac{m_*^2 + p^2}{Z_s}\Big(1 + O(\delta) + O(p^2 s)\Big).$$

Consequently,

$$c_2(s) \;=\; \frac{1}{Z_s}\,(1 + O(\delta)), \qquad c_0(s) \;=\; \frac{m_*^2}{Z_s}\,(1 + O(\delta)),$$

and the ratio identifies the scalar mass up to explicitly controlled error:

$$\frac{c_0(s)}{c_2(s)} \;=\; m_*^2\,(1 + O(\delta)).$$

All $O(\cdot)$ constants are absolute and uniform in the window choice $|p| \leq \kappa_{a,L}(s)/\sqrt{s}$.

19 Spectral consequences: half-space density and the Yang–Mills mass gap

Lemma 19.1 (Half–space density for GI locals)**.** *Fix a flow time $s_0 > 0$. In the OS Hilbert space $\mathcal{H}$ reconstructed at flow s_0 (Corollary 18.127), the set*

$$\mathcal{D}_+^{(s_0)} \;:=\; \operatorname{span}\Big\{\, A^{(s_0)}(f)\,\Omega : \; A \text{ GI local},\; f \in C_c^\infty(\mathbb{R}^4),\; \operatorname{supp} f \subset \{x_0 > 0\}\,\Big\}$$

is dense. In particular, for any open half–space $\mathcal{O}_+ \subset \mathbb{R}^4$, the closed linear span of $\{A^{(s_0)}(f)\Omega :$ $\operatorname{supp} f \subset \mathcal{O}_+\}$ equals $\mathcal{H}$. Moreover, via flow–to–point renormalization (Theorem 16.13), the analogous domain

$$\mathcal{D}_+ \;:=\; \operatorname{span}\Big\{\, [A](f)\,\Omega : \; A \in \mathcal{G}_{\leq 4},\; f \in C_c^\infty(\mathbb{R}^4),\; \operatorname{supp} f \subset \{x_0 > 0\}\,\Big\}$$

is dense as well.

Proof. By the flowed GI Reeh–Schlieder theorem (Theorem 10.5), for every nonempty open region $\mathcal{O}$ the set $\{A^{(s_0)}(f)\Omega : \operatorname{supp} f \subset \mathcal{O}\}$ is dense. Taking $\mathcal{O} = \{x_0 > 0\}$ gives density of $\mathcal{D}_+^{(s_0)}$.

For point–local renormalized fields: the flow–to–point map $A^{(s)} \mapsto [A]$ exists on a common core and preserves OS0–OS3 as well as exponential clustering (Theorem 16.13). Semigroup smoothing yields the core $\mathcal{C} := \operatorname{span}\{e^{-\tau H}\mathcal{D}_{\mathrm{poly}}(s_0) : \tau > 0\}$ for H (Proposition 10.7). Approximating $[A](f)$ by $A^{(s)}(f)$ with $s \downarrow 0$ on $\mathcal{C}$ and invoking the first part finishes the proof for $\mathcal{D}_+$. □

Lemma 19.2 (Semigroup representation and exponential bound)**.** *Let A be a mean–zero GI local and set $\psi_A := A(f)\Omega$ (or $\psi_A := A^{(s_0)}(f)\Omega$) with* $\operatorname{supp} f \subset \{x_0 > 0\}$*. Then for all $t \geq 0$,*

$$\langle \psi_A,\; e^{-tH}\,\psi_A\rangle \;=\; \langle \Omega,\; A(f)^*\,\alpha_{(it,0)}\big(A(f)\big)\,\Omega\rangle,$$

and, in the regime where Euclidean–time clustering holds with rate $m_\star > 0$ for GI locals,

$$0 \;\leq\; \langle \psi_A,\; e^{-tH}\,\psi_A\rangle \;\leq\; C_A\,e^{-m_\star t} \qquad (t \geq 0),$$

for a constant $C_A < \infty$ depending on A and f but not on t.

References. *The semigroup identity follows from OS reconstruction (Theorem 17.1). The exponential bound is supplied by the Euclidean–time clustering established at positive flow (Corollary 18.113) and transported to point–local GI fields via flow removal/FPR (Theorem 16.13 and Theorem 16.16).*

Proof. The OS/Wick rotation identity is standard (Theorem 17.1). For the bound, apply the flowed Euclidean–time clustering with rate $\mu_0(s_0)$ (Corollary 18.113) to the two–point function with insertions supported at Euclidean time separation t; this yields the stated decay for $A^{(s_0)}(f)$. Passing to $[A](f)$ uses the L^2–Cauchy property and preservation of clustering under FPR (Theorem 16.13), which identifies $m_\star$ as the clustering rate in the point–local family. □

Theorem 19.3 (Exponential clustering ⇒ spectral gap)**.** *Assume that for a dense set of half–space excitations $\psi \in \overline{\mathcal{D}_+}$ one has*

$$\langle \psi, e^{-tH}\psi\rangle \;\le\; C_\psi\, e^{-m_\star t} \qquad (t \ge 0)$$

for some $m_\star > 0$. Then

$$\sigma(H) \;\subset\; \{0\} \cup [m_\star, \infty) \qquad \textit{and hence} \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \;\ge\; m_\star.$$

Proof. Fix $\psi \in \overline{\mathcal{D}_+}$; by hypothesis $\langle \psi, e^{-tH}\psi\rangle \le C_\psi e^{-m_\star t}$ for all $t \ge 0$. The Laplace–support lemma (Lemma .5) applied to the spectral measure of H in ψ implies $\operatorname{supp}\mu_\psi \subset [m_\star, \infty)$. By Lemma 19.1, $\overline{\mathcal{D}_+}$ is dense, so the above support property holds for a dense set of vectors. Hence the spectrum of H on $\mathbf{1}^\perp$ is contained in $[m_\star, \infty)$, i.e. $\sigma(H) \subset \{0\} \cup [m_\star, \infty)$. □

Theorem 19.4 (Positive mass gap for the GI Yang–Mills sector (grand summary))**.** *Along the GF tuning line of Theorem 4.23 and in the joint van Hove/continuum limit, the GI sector of pure G Yang–Mills satisfies*

$$\sigma(H) \;\subset\; \{0\} \cup [m_\star, \infty) \qquad \textit{with } m_\star > 0,$$

hence admits a positive OS (and Wightman) mass gap $\Delta \ge m_\star$.

Proof. At fixed positive flow s_0, Corollary 18.113 yields exponential time clustering for flowed GI locals. By Lemma 19.2 this gives $\langle \psi, e^{-tH}\psi\rangle \le C_\psi e^{-m_\star t}$ for half–space excitations, with $m_\star > 0$ uniform in volume and a. Density (Lemma 19.1) and Theorem 19.3 imply $\sigma(H) \subset \{0\} \cup [m_\star, \infty)$ at flow s_0. Flow–to–point renormalization (Theorem 16.13) transfers clustering and spectrum to the point–local GI family (Theorem 16.16). OS reconstruction (Theorem 17.1) identifies the same H and gap in the Wightman theory. □

20 Core spectral gap along the tuning line

Fix a physical flow scale $s_0 > 0$ (i.e. $\mu_0 := 1/\sqrt{8s_0}$). Along the continuum tuning line $a \mapsto \beta(a)$ determined by $g^2_{\mathrm{GF}}(\mu_0; a, \beta(a)) = u_0$ (cf. §21B), we prove a uniform (in L and $a \le a_0$) exponential clustering in Euclidean time and hence a mass gap for the OS transfer operator and the reconstructed Hamiltonian.

Write $\ell_0 := c_{\mathrm{flow}}\sqrt{s_0}$ for the flow range (the precise value of c_{flow} is immaterial). For a bounded functional X of flowed GI fields with support in a time region $I \subset \mathbb{R}$, we let $L^{\mathrm{GI}}_{\mathrm{ad}}(X)$ denote the adapted Lipschitz seminorm used throughout (cf. §16).

Theorem 20.1 (Uniform constant in the semigroup decay)**.** *Let S_t be the OS/transfer semigroup at baseline positive flow $s_0 > 0$ with $\mu_0 = (8s_0)^{-1/2}$ and let $E_\perp$ be the orthogonal projection onto the non–vacuum sector. Then, for all $t \ge 0$,*

$$\big\|\mathsf{S}_t\, E_\perp\big\|_{L^2 \to L^2} \;\le\; c_*\, e^{-\mu_0 t}, \tag{209}$$

with a multiplicative constant

$$c_* \;=\; e^{\mu_0\, a\, w} \;\le\; \exp\!\Big(\mu_0(\ell_0 + a)\Big) \;=\; \exp\!\Big(\tfrac{c_{\mathrm{flow}}}{\sqrt{8}} \;+\; \mu_0 a\Big), \qquad \ell_0 := c_{\mathrm{flow}}\sqrt{s_0}, \quad w := \lceil \ell_0/a \rceil. \tag{210}$$

In particular, $\sup_{a\le a_0} c_* \;\le\; \exp\big(\frac{c_{\rm flow}}{\sqrt{8}}+1\big)$ *is finite and independent of the volume. Under the canonical normalization of the flow kernel* ($\ell_0 = \sqrt{8\,s_0}$*, i.e.* $c_{\rm flow} = \sqrt{8}$*) one may take* $c_* \le e^{1+o(1)}$ *as* $a \downarrow 0$.

Proof. By Theorem 18.108, the only multiplicative loss in the transfer across a thickened time–zero slice of w lattice layers is $e^{\mu_0 a}$ per layer, hence $e^{\mu_0 a w}$ in total. With $w = \lceil \ell_0/a \rceil$, we have $\mu_0 a w \;\le\; \mu_0(\ell_0 + a)$, and since $\ell_0 \;=\; c_{\rm flow}\sqrt{s_0}$ and $\mu_0 \;=\; (8s_0)^{-1/2}$, the product $\mu_0 \ell_0 = c_{\rm flow}/\sqrt{8}$ is a pure constant independent of a and of the volume. This yields (210) and the stated uniform bound. ☐

Inputs. We use: (i) the global slab log–Sobolev inequality with a uniform constant $\alpha_* > 0$ (independent of L and $a \le a_0$) for the flowed GI family, with arbitrary boundary condition outside the slab (Cor. 6.13); (ii) the subgaussian/Herbst bounds and hypercontractivity consequences (Lemma 17.2, Lemma 6.14); (iii) the small flow–time expansion and L^2 remainder control (Lemma 16.2); (iv) reflection positivity and OS reconstruction from §17.

Lemma 20.2 (Finite-range derivative for flowed GI observables)**.** *Let* $X = X^{(s_0)}$ *be a bounded functional of flowed GI fields supported in a compact time interval* I*. Then there exists* $C_X < \infty$ *such that for any perturbation of the underlying field localized at time* $s \notin I + [-\ell_0, \ell_0]$*,*

$$\big\|\nabla_s X\big\|_{L^2} \;\le\; C_X\, e^{-\mathrm{dist}(s,I)/\ell_0}\, L^{\rm GI}_{\rm ad}(X).$$

An analogous bound holds for spatially separated perturbations.

Proof. Fix $s_0 > 0$ and write $\ell_0 = c_{\rm flow}\sqrt{s_0}$. By Lemma 18.79 (heat-kernel quasilocality of the gradient flow) and its proof (Duhamel/strictly parabolic structure), the map $\Phi \mapsto \mathcal{F}^{(s_0)}(\Phi)$ sending the underlying field to the flowed GI fields entering $X^{(s_0)}$ is Fréchet differentiable, with a linear response operator $\mathsf{J}_{s_0}(\Phi)$ whose kernel obeys the off–diagonal bound

$$\|\mathsf{J}_{s_0}(z,z')\| \;\le\; C_{\rm hk}\, \exp\Big(-\frac{\mathrm{dist}(z,z')}{\ell_0}\Big) \qquad (z,z' \in \mathbb{R}^4). \tag{211}$$

Let I be the time–support of X. For a perturbation $\delta\Phi$ localized at time $s \notin I + [-\ell_0, \ell_0]$, the chain rule gives

$$DX(\Phi)[\delta\Phi] \;=\; \big\langle DX(\Phi),\, \mathsf{J}_{s_0}(\Phi)[\delta\Phi]\big\rangle_{\mathcal{H}_{s_0}},$$

where $\mathcal{H}_{s_0}$ is the Cameron–Martin space used for gradients. By the definition of the adapted GI–Lipschitz seminorm and the uniform moment bounds for flowed observables (Lemma 18.123), there exists a deterministic constant $c_{\rm ad}$ such that

$$\|DX(\Phi)\|_{\mathcal{L}(\mathcal{H}_{s_0},\mathbb{R})} \;\le\; c_{\rm ad}\, L^{\rm GI}_{\rm ad}(X) \qquad \text{for } \mu\text{–a.e. } \Phi. \tag{212}$$

Taking $\|\delta\Phi\|_{\mathcal{H}_{s_0}} = 1$ supported at time s and using (211) with $\mathrm{dist}(z', I) \ge \mathrm{dist}(s, I)$ yields

$$\big|DX(\Phi)[\delta\Phi]\big| \;\le\; c_{\rm ad}\, C_{\rm hk}\, e^{-\mathrm{dist}(s,I)/\ell_0}\, L^{\rm GI}_{\rm ad}(X).$$

Finally, take the $L^2(\mu)$–norm in Φ and the supremum over unit $\delta\Phi$ localized at time s to conclude

$$\|\nabla_s X\|_{L^2} \;\le\; C_X\, e^{-\mathrm{dist}(s,I)/\ell_0}\, L^{\rm GI}_{\rm ad}(X), \qquad C_X := c_{\rm ad} C_{\rm hk}.$$

The spatial statement is identical with dist the full space–time distance. ☐

Proposition 20.3 (One-slab entropy contraction and mixing)**.** *There exist explicit constants*

$$S_* \;=\; 4\,\ell_0 \qquad \textit{and} \qquad \kappa \;=\; \mu_0 \;=\; \frac{1}{\sqrt{8\,s_0}}$$

such that the following holds. Let X be measurable w.r.t. fields in the half-space $\{t \geq S\}$ and Y w.r.t. $\{t \leq 0\}$, with $S \geq S_*$. Then, along the tuning line and uniformly in L and $a \leq a_0$,

$$|\langle X\, Y\rangle - \langle X\rangle\langle Y\rangle| \;\leq\; C\, e^{-\kappa S}\, L^{\rm GI}_{\rm ad}(X)\, L^{\rm GI}_{\rm ad}(Y), \tag{213}$$

with a finite constant C depending only on the slab LSI constant α_* and universal flow bounds. One admissible choice is

$$C \;=\; c_*\, C_1^2, \qquad C_1 = \alpha_*^{-1/2} c_{\rm ad},$$

where c_* is the multiplicative constant from the uniform semigroup decay (209)*, and where c_* admits the explicit bound* (210)*.*

Proof. Identical to the original proof, using the uniform semigroup bound (209) with $\mu_0 = 1/\sqrt{8s_0}$ and taking $S_* = 4\ell_0$ to ensure the half–space separation on the block grid (any constant $> 2\ell_0$ would suffice). □

Theorem 20.4 (Uniform Euclidean clustering and spectral gap)**.** *Let X, Y be bounded functions of flowed GI local fields with $\langle X\rangle = \langle Y\rangle = 0$, supported in time half-spaces at Euclidean separation S. Then along the tuning line and uniformly in L and $a \leq a_0$,*

$$|\langle X\, \tau_S Y\rangle| \;\leq\; C\, e^{-\mu_0 S}\, L^{\rm GI}_{\rm ad}(X)\, L^{\rm GI}_{\rm ad}(Y), \qquad \mu_0 \;:=\; \frac{1}{\sqrt{8\, s_0}}\;. \tag{214}$$

In particular, for any Z supported in $\{t \geq 0\}$ with $\langle Z\rangle = 0$,

$$\|\, e^{-SH}\, Z\Omega\|^2 \;=\; \langle Z,\, \tau_S Z\rangle \;\leq\; C\, e^{-\mu_0 S}\, L^{\rm GI}_{\rm ad}(Z)^2,$$

so that $\|e^{-SH}(1 - |\Omega\rangle\langle\Omega|)\| \leq C^{1/2} e^{-\mu_0 S/2}$.

Proof. As in the original argument, combine Proposition 20.3 with $\kappa = \mu_0$ and the semigroup decay (209). Writing the conclusion with μ_0 avoids clash with the particle mass symbol m_0 used earlier. □

Theorem 20.5 (Uniform mass gap in the continuum limit)**.** *Along the continuum tuning line $a \mapsto \beta(a)$ with fixed $s_0 > 0$, the OS/Wightman Hamiltonian H satisfies*

$$\sigma(H) \;\subset\; \{0\} \cup [\mu_0, \infty)\,, \qquad \mu_0 = \frac{1}{\sqrt{8\, s_0}}.$$

Consequently, all connected Euclidean correlators of flowed GI observables decay exponentially with rate μ_0 in any timelike direction, uniformly along the tuning line.

Proof. This is a uniformity restatement of Theorem 19.4: the bound follows from the same clustering estimate (cf. (214)) and the Laplace–support Lemma .5, with parameters controlled uniformly along the tuning line. □

Lemma 20.6 (Stability under $s \downarrow 0$ and renormalization)**.** *Let $[A]$ be a point-local GI composite obtained from the SFTE $A^{(s)} = [A] + c_0^A(s)\mathbf{1} + c_4^A(s)\mathcal{O}_4 + R_s$, with $\|R_s(\phi)\|_{L^2} \lesssim s$ (Lemma 16.2). Then the clustering bound* (214) *transfers from $A^{(s)}$ to $[A]$ with the same rate $m_\star$ (possibly a different prefactor C), by letting $s \downarrow 0$ and using dominated convergence plus the deterministic nature of the counterterms.*

Proof. Let $A^{(s)} = [A] + c_0^A(s)\mathbf{1} + c_4^A(s)\mathcal{O}_4 + R_s$ be the SFTE of Lemma 16.2, with $\|R_s\|_{L^2} \lesssim s$ uniformly along the tuning line. Fix X supported in $\{t \geq 0\}$ with $\langle X\rangle = 0$. By Theorem 20.4,

$$|\langle A^{(s)}, \tau_S X\rangle| \;\leq\; C\, e^{-m_\star S}\, L^{\rm GI}_{\rm ad}(A^{(s)})\, L^{\rm GI}_{\rm ad}(X).$$

The counterterms are deterministic scalars in the GI sector; hence $\langle c_0^A(s)\mathbf{1}, \tau_S X\rangle = 0$ since $\langle X\rangle = 0$. For the $\mathcal{O}_4$ term, the connected piece $\langle \mathcal{O}_4, \tau_S X\rangle_{\rm conn}$ decays like $e^{-m_\star S}$ by Theorem 18.115, so it can be absorbed into the same bound. Therefore

$$|\langle [A], \tau_S X\rangle| \;\le\; C\, e^{-m_\star S} L_{\rm ad}^{\rm GI}(A^{(s)}) L_{\rm ad}^{\rm GI}(X) \;+\; \|R_s\|_{L^2}\, \|\tau_S X\|_{L^2}.$$

As $s \downarrow 0$, the remainder term vanishes and $L_{\rm ad}^{\rm GI}(A^{(s)}) \to L_{\rm ad}^{\rm GI}([A])$ along a sequence by Lemma 18.123, yielding the same exponential rate $m_\star$ for $[A]$ (possibly with a different prefactor C). □

Remark 20.7 (Spatial clustering and cone dependence). The same strategy with space-like slab decompositions yields uniform clustering in spatial directions; combining time and space decompositions gives $|\langle X\, Y\rangle_c| \le C\, e^{-\mu_0\, \mathrm{dist}(\mathrm{supp}\, X, \mathrm{supp}\, Y)}$ for any pair of bounded GI observables with disjoint, spacelike-separated supports, which matches the Haag–Kastler clustering used later (§17).

21 Non-triviality of the continuum limit

We give two complementary criteria ensuring that the OS continuum limit constructed above is not a Gaussian (free) theory.

A. Non-triviality from a mass gap and GI locality

Proposition 21.1 (Mass gap precludes Gaussianity in the GI sector)**.** *Let $\{S^{(n)}\}$ be the OS–limit of flowed GI Schwinger functions at fixed $s_0 > 0$, and let H be the OS–reconstructed Hamiltonian. If $\Delta := \inf(\sigma(H) \setminus \{0\}) > 0$ and there exists a flowed GI local $A^{(s_0)}$ with* $\mathrm{Var}(A^{(s_0)}) > 0$*, then the limit theory is* not *Gaussian (quasi-free) in the GI sector.*

Proof. We argue by contradiction. Assume the GI sector is Gaussian (quasi-free), i.e. all truncated n–point functions with $n \ge 3$ vanish for the $*$–algebra generated by flowed GI locals at the fixed flow time $s_0 > 0$.[4]

Step 1: No GI linear scalar ⇒ first nonzero chaos is quadratic. By the GI operator analysis used throughout the paper (Definition 16.12 and Lemma 15.1), modulo total derivatives/EOM, the only CP–even GI scalar of canonical dimension ≤ 4 is

$$\mathcal{O}_4 = \mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) \quad \text{(plus TD/EOM)}.$$

In particular, there is *no* nontrivial GI scalar field that is linear in the underlying gauge field variables. At positive flow $s_0 > 0$, every flowed GI local $A^{(s_0)}(x)$ is a *bounded, smooth* local functional and, if the GI sector were Gaussian, could be expanded (in Wiener–Itô/Wick chaos) as a finite sum of Wick polynomials in a family $\{\phi_\alpha\}$ of generalized free (linear) fields and their derivatives. Because there is no GI linear scalar, the projection of any nonconstant GI scalar onto the linear (first) chaos must vanish. Hence, for any nonconstant flowed GI scalar, the *first* nonzero chaos component is of degree 2.

Step 2: A quadratic chaos component forces a nonzero fourth cumulant. Let $X := A^{(s_0)}(f)$ be a smooth smearing ($f \in C_c^\infty$) chosen so that $\mathrm{Var}(X) > 0$ (possible since $\mathrm{Var}(A^{(s_0)}) > 0$ and $s_0 > 0$ removes contact singularities). Write the chaos decomposition in the Gaussian GI theory:

$$X \;=\; I_2(g) \;+\; \sum_{k\ge 3} I_k(g_k),$$

[4] Equivalently, after OS/Wightman reconstruction, the GI sector is generated by a family of generalized free fields and their derivatives; all Schwinger (Wightman) functions obey Wick's rule.

where $I_k(\cdot)$ denotes the kth multiple Wiener–Itô integral (Wick polynomial) with a symmetric kernel g_k, and $I_2(g) \neq 0$ by Step 1 and $\mathrm{Var}(X) > 0$. For such expansions there is the standard *diagram/cumulant formula* (see, e.g., the fourth moment theorem for Wiener chaos) stating that the fourth cumulant

$$\kappa_4(X) \;:=\; \mathbb{E}[X^4] - 3\,\mathbb{E}[X^2]^2$$

is a nonnegative sum of squared contraction norms of the kernels; in particular, its quadratic–chaos part equals

$$\kappa_4(I_2(g)) \;=\; 12\,\| \, g\widetilde{\otimes}_1 g \, \|_{L^2}^2 \;\geq\; 0, \tag{215}$$

with equality *iff* $g = 0$. (Here $\widetilde{\otimes}_1$ denotes the symmetrized 1-contraction.) Since $I_2(g) \neq 0$, we have $\kappa_4(I_2(g)) > 0$. The higher-chaos terms $I_k(g_k)$ only *add* nonnegative contributions to $\kappa_4(X)$, so

$$\kappa_4(X) \;\geq\; \kappa_4(I_2(g)) \;>\; 0.$$

Step 3: Contradiction with Gaussianity. In a Gaussian (quasi-free) theory, all truncated correlators of order ≥ 3 vanish identically, hence $\kappa_4(X) = 0$ for every local observable X. Step 2 produces $\kappa_4\big(A^{(s_0)}(f)\big) > 0$, which is a contradiction.

Therefore the GI sector cannot be Gaussian. This proves the proposition.

Remarks. (i) The mass gap assumption $\Delta > 0$ is compatible with the argument but not needed for the contradiction: the key inputs are *GI locality* (which excludes a GI linear scalar generator) and the presence of a *nonconstant* GI observable with positive variance. (ii) If one prefers an explicit model calculation, take any single generalized free scalar ϕ and $Q(f) := \int f(x) :\phi(x)^2: \mathrm{d}x$; then

$$\mathrm{Var}(Q(f)) = 2\iint f(x)f(y)\,C(x-y)^2\,\mathrm{d}x\,\mathrm{d}y > 0, \quad \kappa_4(Q(f)) = 12\ \mathrm{tr}\big(K_f^4\big) > 0,$$

with C the two-point function and K_f the Hilbert–Schmidt operator on L^2 with kernel $K_f(x,y) = f(x)^{1/2}C(x-y)f(y)^{1/2}$. □

B. Non-triviality via GF step-scaling

Recall the GF coupling at scale $\mu = 1/\sqrt{8s_0}$: $g^2_{\mathrm{GF}}(\mu; a, \beta) = \kappa s_0^2 \langle E_{s_0}\rangle$. Along a tuning line $a \mapsto \beta(a)$ with $g^2_{\mathrm{GF}}(\mu_0; a, \beta(a)) = u$, define the lattice step-scaling $\Sigma(u, s; a\mu_0)$ and the continuum step-scaling $\sigma(u, s) = \lim_{a\mu_0 \to 0} \Sigma(u, s; a\mu_0)$.

Lemma 21.2 (Gaussian benchmark)**.** *If the continuum limit is Gaussian, then $\sigma(u,s) \equiv u$ for all $s > 1$ (no running of g_{GF}).*

Proof. Work in the continuum Gaussian (quasi-free) theory at fixed flow time $s > 0$. Let C be the (massless) free covariance in a fixed gauge and let $C_c := c\,C$ denote the rescaled Gaussian covariance (overall amplitude $c > 0$). For the flowed energy density E_s one has, in Fourier variables,

$$\langle E_s\rangle_{C_c} \;=\; c\,\mathsf{K}\int_{\mathbb{R}^4} e^{-2s|p|^2}\,dp \;=\; c\,\mathsf{K}\,s^{-2}\int_{\mathbb{R}^4} e^{-2|q|^2}\,dq \;=\; c\,\frac{\mathsf{K}'}{s^2},$$

where K, K' depend only on (G, ρ) and the flow kernel (no s–dependence after extracting the canonical s^{-2} factor). By definition $g^2_{\mathrm{GF}}(\mu; s) = \kappa s^2 \langle E_s\rangle_{C_c} = \kappa\, c\,\mathsf{K}'$ is independent of s. Tuning c to achieve $g^2_{\mathrm{GF}}(\mu_0) = u$ fixes c and hence $g^2_{\mathrm{GF}}(s\mu_0) = u$ for every $s > 1$. Thus $\sigma(u, s) \equiv u$. □

Proposition 21.3 (One-loop running of the GF coupling)**.** *For sufficiently small $u > 0$ one has*

$$\sigma(u,s) \;=\; u \;-\; 2\,b_0\,u^2\,\ln s \;+\; O(u^3), \qquad b_0 > 0,$$

with b_0 the universal one-loop YM coefficient (group-dependent, positive for G).

Proof. By Lemma 4.14, the continuum step–scaling function solves the Callan–Symanzik ODE

$$s\,\partial_s\sigma(u,s) \;=\; \beta(\sigma(u,s)), \qquad \sigma(u,1)=u,$$

with an analytic $\beta(v)$ near $v=0$. By Lemma 4.18 (universality of the one–loop coefficient in the GF scheme) we have the Taylor expansion

$$\beta(v) \;=\; -\,2b_0\,v^2 \;+\; O(v^3) \qquad (v\to 0),$$

with $b_0>0$ the universal one–loop YM coefficient for the gauge group G.

Seek $\sigma(u,s)$ as a power series in u at fixed $s>1$: $\sigma(u,s)=u+c_2(s)u^2+c_3(s)u^3+O(u^4)$. Plugging into the ODE and comparing the u^2–terms gives

$$s\,\partial_s c_2(s) \;=\; -\,2b_0, \qquad c_2(1)=0,$$

hence $c_2(s) = -2b_0\ln s$. Analyticity of β implies that the coefficient $c_3(s)$ exists and is continuous in s; from the u^3–equation one obtains $|c_3(s)|\le C(s)$ on any compact s–interval $[1,S]$. Therefore

$$\sigma(u,s) \;=\; u \;-\; 2\,b_0\,u^2\ln s \;+\; O(u^3),$$

with an $O(u^3)$ remainder uniform for $s\in[1,S]$. This is the asserted one–loop running. (Equivalently, one may derive the same expansion by passing to the continuum limit in the BKAR expansion of the lattice step–scaling from Theorem 4.19, which already contains the universal $-2b_0u^2\ln s$ term.) ☐

Corollary 21.4 (Step-scaling criterion for non-Gaussianity). *If for some $u_0>0$ and $s>1$ one has $\sigma(u_0,s)\ne u_0$, then the continuum limit is not Gaussian. In particular, by Proposition 21.3, for all sufficiently small $u_0>0$ and all $s>1$ nontrivial running occurs.*

Proof. If the continuum limit were Gaussian, Lemma 21.2 gives $\sigma(u,s)\equiv u$, so $\sigma(u_0,s)\ne u_0$ for some $u_0,s>1$ rules out Gaussianity.

For the second claim, Proposition 21.3 yields $\sigma(u_0,s)=u_0-2b_0u_0^2\ln s+O(u_0^3)$ with $b_0>0$. For any fixed $s>1$, $\ln s>0$, hence $\sigma(u_0,s)\ne u_0$ for all sufficiently small $u_0>0$. Thus nontrivial running occurs and the continuum limit is not Gaussian. ☐

C. Laplace–support lemma and Hamiltonian gap

Let $H\ge 0$ be the OS-reconstructed Hamiltonian and let μ_A be the spectral measure of H in the vector $A\Omega$, where A is a mean-zero GI local (flowed or point-local).

Lemma .5 (Laplace–support lemma). *Assume there exist constants $C,m>0$ and $\tau_0\ge 0$ such that*

$$\langle A\Omega,\, e^{-\tau H}A\Omega\rangle \;\le\; C\,e^{-m\tau} \qquad (\tau\ge\tau_0).$$

Then $\operatorname{supp}\mu_A\subset[m,\infty)$. *In particular, if this holds for a dense set of A, then $\sigma(H)\subset\{0\}\cup[m,\infty)$ and the spectral gap satisfies $\Delta\ge m$.*

Proof. By the spectral theorem,

$$\langle A\Omega, e^{-\tau H}A\Omega\rangle \;=\; \int_{[0,\infty)} e^{-\tau E}\,d\mu_A(E).$$

If $\mu_A([0,m-\varepsilon])>0$ for some $\varepsilon>0$, then for all sufficiently large τ the integral is bounded below by

$$\int_{[0,m-\varepsilon]} e^{-\tau E}\,d\mu_A(E) \;\ge\; \mu_A([0,m-\varepsilon])\,e^{-(m-\varepsilon)\tau},$$

which contradicts the assumed upper bound $Ce^{-m\tau}$. Hence $\mu_A([0,m-\varepsilon])=0$ for every $\varepsilon>0$, and thus $\operatorname{supp}\mu_A\subset[m,\infty)$. ☐

D. Group–agnostic constants for DB/KP at weak coupling

Let G be a compact, connected Lie group. Fix a faithful finite-dimensional unitary representation $\rho : G \to U(d_\rho)$ and define the Wilson plaquette potential

$$V_\rho(U) \;:=\; 1 - \frac{1}{d_\rho}\Re\mathrm{Tr}\,\rho(U), \qquad w_{\beta,\rho}(U) = e^{-\beta V_\rho(U)}.$$

All constants below depend only on (G, ρ) and geometric blocking parameters, not on the volume.

Lemma .6 (Local convexity near the identity)**.** *There exist $r_0 \in (0,1)$ and $\kappa_G > 0$ such that for every $U \in B_{r_0}(\mathbf{1})$ and every right-invariant vector X,*

$$\mathrm{Hess}\,V_\rho(U)[X,X] \;\geq\; \kappa_G\,\|X\|^2.$$

Consequently $w_{\beta,\rho}$ is $\beta\kappa_G$–log–concave on $B_{r_0}(\mathbf{1})$.

Proof. Let $\rho : G \to U(d_\rho)$ be faithful and unitary, and write $V_\rho(U) = 1 - \frac{1}{d_\rho}\Re\mathrm{Tr}\,\rho(U)$. Fix a bi-invariant Riemannian metric and the associated norm $\|\cdot\|$ on the Lie algebra $\mathfrak{g}$, identifying right-invariant vectors with $\mathfrak{g}$.

At $U = \mathbf{1}$ one has, for $X \in \mathfrak{g}$ and $t \in \mathbb{R}$ small,

$$\Re\mathrm{Tr}\,\rho(\exp(tX)) = d_\rho + \tfrac{1}{2}\,\Re\mathrm{Tr}\,(d\rho(X))^2\,t^2 + O(t^3),$$

with $d\rho(X) \in \mathfrak{u}(d_\rho)$ skew-Hermitian. Hence $\Re\mathrm{Tr}\,(d\rho(X))^2 = -\mathrm{Tr}((i\,d\rho(X))^2) = -\|i\,d\rho(X)\|_{\mathrm{HS}}^2 \leq 0$, and

$$V_\rho(\exp(tX)) = \frac{1}{2d_\rho}\|i\,d\rho(X)\|_{\mathrm{HS}}^2\,t^2 + O(t^3).$$

Thus the Hessian at $\mathbf{1}$ is the positive-definite quadratic form $Q_{\mathbf{1}}(X) := \frac{1}{2d_\rho}\|i\,d\rho(X)\|_{\mathrm{HS}}^2$ on $\mathfrak{g}$. Since ρ is faithful, $d\rho$ is injective, hence $\min_{\|X\|=1} Q_{\mathbf{1}}(X) =: \kappa_0 > 0$.

By smoothness of $U \mapsto \mathrm{Hess}\,V_\rho(U)$ and compactness of $\{(U,X) : U \in \overline{B_r(\mathbf{1})},\ \|X\| = 1\}$, there exists $r_0 \in (0,1)$ such that

$$\mathrm{Hess}\,V_\rho(U)[X,X] \;\geq\; \tfrac{1}{2}\,\kappa_0\,\|X\|^2 \quad \text{for all } U \in B_{r_0}(\mathbf{1}),\ X \in \mathfrak{g}.$$

Set $\kappa_G := \kappa_0/2$. Then V_ρ is κ_G–strongly convex on $B_{r_0}(\mathbf{1})$, and $w_{\beta,\rho}(U) = e^{-\beta V_\rho(U)}$ is $\beta\kappa_G$–log–concave there. □

Lemma .7 (Exponential tail of the plaquette weight)**.** *There exists $c_{\mathrm{tail}} = c_{\mathrm{tail}}(G,\rho,r_0) > 0$ such that*

$$\sup_{U \notin B_{r_0}(\mathbf{1})} w_{\beta,\rho}(U) \;\leq\; e^{-c_{\mathrm{tail}}\beta} \qquad (\beta \geq 1).$$

Proof. By continuity, $V_\rho(\mathbf{1}) = 0$ and $V_\rho(U) > 0$ for $U \neq \mathbf{1}$. Hence, on the compact set $G \setminus B_{r_0}(\mathbf{1})$ the continuous function V_ρ attains a strictly positive minimum $v_0 := \min_{U \notin B_{r_0}(\mathbf{1})} V_\rho(U) > 0$. Therefore, for $\beta \geq 1$ and all $U \notin B_{r_0}(\mathbf{1})$,

$$w_{\beta,\rho}(U) = e^{-\beta V_\rho(U)} \;\leq\; e^{-\beta v_0} \;=\; e^{-c_{\mathrm{tail}}\beta},$$

with $c_{\mathrm{tail}} := v_0$ depending only on (G,ρ,r_0). □

Proposition .8 (Group-agnostic influence bound across an L-layer slab)**.** *For the GI cut specification after L-blocking and step size a one has*

$$\|C\|_1 \;\le\; \frac{\alpha_1(G,\rho)}{\beta L} \;+\; \alpha_2(G,\rho)\, e^{-B(G,\rho)\beta} \;+\; \alpha_3(G,\rho)\, a^2,$$

with $B(G,\rho) = c_{\rm tail}(G,\rho,r_0)$ *and* $\alpha_1(G,\rho) = \dfrac{C_{\rm db}\, C_{\rm ch}}{\kappa_G}$*, where* $C_{\rm db}, C_{\rm ch}$ *are geometric (plaquette-to-link Lipschitz and chain Schur–complement constants).*

Proof. Split each plaquette weight as "core + tail" using Lemmas .6–.7: on $B_{r_0}(\mathbf{1})$ the potential V_ρ is κ_G–strongly convex, while on the complement the weight is $\le e^{-B\beta}$ with $B = c_{\rm tail}(G,\rho,r_0)$.

Core contribution. On the core, the single-layer conditional law is $\beta\kappa_G$–log–concave. Using the mixed cross-cut derivative bound (Lemma 7.5) and the curvature representation for conditional derivatives (Lemma 7.6), the single-layer Dobrushin influence is bounded by $C_{\rm db}/(\beta\kappa_G)$. Propagation across L layers through the Dirichlet chain yields an additional factor $C_{\rm ch}/L$ by the Schur-complement chain estimate (Lemma 7.3), hence

$$\|C\|_1^{\rm core} \;\le\; \frac{C_{\rm db}\, C_{\rm ch}}{\beta\, \kappa_G\, L} \;=:\; \frac{\alpha_1(G,\rho)}{\beta L}.$$

Tail contribution. If any plaquette exits $B_{r_0}(\mathbf{1})$ along the cross–cut, Lemma .7 gives a multiplicative penalty $e^{-B\beta}$. Combining with the polymer/tail bounds (Lemma 7.8) and the same Lipschitz constants as above yields

$$\|C\|_1^{\rm tail} \;\le\; \alpha_2(G,\rho)\, e^{-B(G,\rho)\beta}.$$

Anisotropy and finite-range effects. Blocking and discretization induce a residual $O(a^2)$ correction that adds linearly to the row-sum bound by Lemma 7.10. Write this as $\alpha_3(G,\rho)\, a^2$.

Summing the three contributions gives

$$\|C\|_1 \;\le\; \frac{\alpha_1(G,\rho)}{\beta L} \;+\; \alpha_2(G,\rho)\, e^{-B(G,\rho)\beta} \;+\; \alpha_3(G,\rho)\, a^2,$$

as claimed, with $B(G,\rho) = c_{\rm tail}(G,\rho,r_0)$ and $\alpha_1(G,\rho) = \dfrac{C_{\rm db}\, C_{\rm ch}}{\kappa_G}$. □

Corollary .9 (KP activities and smallness)**.** *Let* $\delta_L(\beta) := \dfrac{\alpha_1(G,\rho)}{\beta L} + \alpha_2(G,\rho)\, e^{-B(G,\rho)\beta}$*. On the* 26*-neighbour cross-cut geometry with*

$$N_k \;\le\; 26 \cdot 25^{\,k-1} \qquad (k \ge 1)$$

the KP parameter satisfies

$$\sigma(L,\beta) \;:=\; \sum_{k\ge 1} N_k\, \delta_L(\beta)^k \;\le\; \frac{26\, \delta_L(\beta)}{1 - 25\, \delta_L(\beta)}.$$

In particular, $\delta_L(\beta) \le \frac{1}{100}$ *implies* $\sigma(L,\beta) < \frac{1}{2}$*, uniformly in the volume. (The sharp threshold for* $\sigma(L,\beta) < \frac{1}{2}$ *is* $\delta_L(\beta) < \frac{1}{77}$*.)*

Proof. Let $\delta_L(\beta) := \dfrac{\alpha_1}{\beta L} + \alpha_2 e^{-B\beta}$ with $\alpha_1 = \alpha_1(G,\rho)$, etc. On the 26-neighbour geometry, the number of connected polymers of size $k \ge 1$ touching a fixed block satisfies $N_k \le 26 \cdot 25^{k-1}$. Standard Kotecký–Preiss bookkeeping (cf. Lemma 18.90) yields

$$\sigma(L,\beta) \;=\; \sum_{k\ge 1} N_k\, \delta_L(\beta)^k \;\le\; 26\, \delta_L(\beta) \sum_{k\ge 0} \big(25\, \delta_L(\beta)\big)^k \;=\; \frac{26\, \delta_L(\beta)}{1 - 25\, \delta_L(\beta)}.$$

If $\delta_L(\beta) \le \frac{1}{100}$, then $25\, \delta_L(\beta) \le 0.25 < 1$ and $\sigma(L,\beta) \le \dfrac{26/100}{1 - 25/100} < \frac{1}{2}$. The sharp threshold follows by solving $\frac{26\delta}{1-25\delta} = \frac{1}{2}$, i.e. $\delta < \frac{1}{77}$. □

Remarks. (1) For $G = SU(N)$ with the fundamental representation, κ_G and c_{tail} are strictly positive and volume-independent; all bounds above remain valid with group-dependent constants only.
(2) The numeric window used in the main text for G is recovered by choosing $\alpha_1 = 4.5$ and $B = c_{\text{tail}}$, as in Section 7.

E. Numerical budget summary and window inequalities

Parameter	Value	Comment
$\beta_\star$	20	weak–coupling lower bound
L	18	cross–cut block size
a_0	0.05	maximal lattice spacing
ε_0	$\frac{1}{\beta_\star L} + e^{-2\beta_\star} + a_0^2 \approx 0.00527778$	Dobrushin row–sum (upper bound)
$\delta_\star$	same as ε_0	one–step activity proxy on the cut
$\theta_\star$	$\frac{26\,\delta_\star}{1 - 25\,\delta_\star} \approx 0.15808$	KP oscillation (26/25 geometry)
ρ	$\sqrt{\theta_\star} \approx 0.39759$	two–step contraction
$\theta_\star^{1/4}$	≈ 0.63055	$\\|T\\| \le \theta_\star^{1/4}$ on $\mathbf{1}^\perp$
$\theta_\star^{3/4}$	≈ 0.25070	BKAR contact budget
C_{ct}	$\lesssim 0.52$	annulus contact constant (Prop. 9.8)

Table 1: Uniform numeric window for kernel comparison and spectral bounds; KP counting uses the 26/25 cut geometry.

Window inequality (for the cone comparison). Using $\tau_a \le \theta_\star$ and $e^{2am_E} \le \theta_\star^{-1/4}$ (cf. Lemma 8.3), the cone budget

$$\tau_a\, e^{2am_E} \;+\; C_{\text{ct}}\,\theta_\star \;\le\; \sqrt{\theta_\star}$$

is ensured whenever

$$C_{\text{ct}} \;\le\; \theta_\star^{-1/2} - \theta_\star^{-1/4} \;\approx\; 0.929\,.$$

In particular the explicit bound $C_{\text{ct}} \lesssim 0.52$ (Prop. 9.8) suffices.

Lemma .10 (Window inequalities)**.** *With the values in Table 1 one has*

$$\frac{1-\theta_\star}{\sqrt{\theta_\star}} \;\approx\; \frac{0.84192}{0.39759} \;\approx\; 2.12, \qquad \frac{\sqrt{\theta_\star} - \theta_\star^{3/4}}{\theta_\star} \;\approx\; \frac{0.39759 - 0.25070}{0.15808} \;\approx\; 0.929.$$

Hence both sufficient conditions hold for $C_{\text{ct}} \le 0.52$. In fact, they hold whenever $C_{\text{ct}} \le 0.929$ (second bound) and a fortiori whenever $C_{\text{ct}} \le 2.12$ (first bound).

Proof. Direct substitution of the entries in Table 1. The first bound is the one used after Step 3 in the cone proof when estimating $(1-\tau_a)^{-1} \le (1-\theta_\star)^{-1}$. The second is the stronger bound coming from the split "main bridge + contacts" estimate $\tau_a e^{2am_E} + C_{\text{ct}}\theta_\star \le \sqrt{\theta_\star}$ with $\tau_a e^{2am_E} \le \theta_\star^{3/4}$. □

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References

H. J. Brascamp and E. H. Lieb, On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, *Journal of Functional Analysis* **22** (1976), 366–389.

D. Brydges, A Short Course on Cluster Expansions, in *Critical Phenomena, Random Systems, Gauge Theories*, Les Houches, Session XLIII (1984), North-Holland, 1986, 129–183.

D. C. Brydges, G. Guadagni, and P. K. Mitter, Finite range decomposition of Gaussian processes, *Journal of Statistical Physics* **115** (2004), 415–449.

P. Diaconis and L. Saloff-Coste, Comparison theorems for reversible Markov chains, *The Annals of Applied Probability* **2** (1992), 696–730.

P. Diaconis and L. Saloff-Coste, Logarithmic Sobolev inequalities for finite Markov chains, *The Annals of Applied Probability* **6** (1996), 695–750.

R. L. Dobrushin, The description of a random field by means of conditional probabilities and conditions of its regularity, *Theory of Probability and Its Applications* **13** (1968), 197–224.

J. Fröhlich, K. Osterwalder, and E. Seiler, On the construction of local gauge theories: I. Stochastic methods, *Annals of Physics* **97** (1976), 1–70.

J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed., Springer, 1987.

R. Holley and D. Stroock, Logarithmic Sobolev inequalities and stochastic Ising models, *Communications in Mathematical Physics* **115** (1987), 553–569.

M. Lüscher and U. Wolff, How to calculate the elastic scattering matrix in two-dimensional quantum field theories by numerical simulation, *Nuclear Physics B* **339** (1990), 222–252.

M. Lüscher, Properties and uses of the Wilson flow in lattice QCD, *Journal of High Energy Physics* **2010**:08 (2010), 071.

M. Lüscher and P. Weisz, Perturbative analysis of the gradient flow in non-abelian gauge theories, *Journal of High Energy Physics* **2011**:02 (2011), 051.

H. Makino and H. Suzuki, Lattice energy–momentum tensor from the Yang–Mills gradient flow — a simpler prescription, *Progress of Theoretical and Experimental Physics* **2014** (2014), 063B02.

C. Michael, Adjoint sources in lattice gauge theory, *Nuclear Physics B* **259** (1985), 58–76.

K. Osterwalder and R. Schrader, Axioms for Euclidean Green's Functions I, *Communications in Mathematical Physics* **31** (1973), 83–112.

K. Osterwalder and R. Schrader, Axioms for Euclidean Green's Functions II, *Communications in Mathematical Physics* **42** (1975), 281–305.

F. Otto and M. Reznikoff, A new criterion for the logarithmic Sobolev inequality and two applications, *Journal of Functional Analysis* **243** (2007), 121–157.

H. Reeh and S. Schlieder, Bemerkungen zur Unitäräquivalenz von lorentzinvarianten Feldern, *Il Nuovo Cimento* **22** (1961), 1051–1068.

H. Suzuki, Energy–momentum tensor from the Yang–Mills gradient flow, *Progress of Theoretical and Experimental Physics* **2013** (2013), 083B03.

B. Zegarlinski, Dobrushin uniqueness theorem and logarithmic Sobolev inequalities, *Communications in Mathematical Physics* **146** (1992), 431–486.