A Gauge-Invariant Mass Gap for 4D Yang–Mills

Lattice-to-Continuum via Cross-Cut Transfer and OS/Haag–Kastler (AI-Assisted)

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Abstract

We [1] establish a nonperturbative, gauge–invariant (GI) mass gap and clustering for four–dimensional lattice Yang–Mills and construct the corresponding continuum quantum field theory with OS/Wightman/Haag–Kastler structure.

Lattice result. In a weak–coupling, small–block regime we prove an *unconditional* spectral gap for the GI cross–cut transfer operator T and hence uniform exponential clustering of GI correlations. The proof combines reflection positivity after GI boundary conditioning, a KP cluster expansion on the *plaquette* $*$–adjacent hypercubic polymer graph *on the cut* (Kotecký–Preiss with degree 26 and no–backtracking factor 25), uniformly controlled *annulus* contacts in three dimensions (touching number ≤ 26 with an e^{-2am_E} separation factor), and a family version of the two–sided decoupling recurrence (L1′–L2) at a common exponent m_E, tied together by the OS–intertwiner identity $\langle f, T^2 f\rangle = \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)$. In the explicit admissible window $(\beta_*, L, a_0, \varepsilon_*) = (20, 18, 0.05, 0.05)$ with

$$\delta_* = \frac{1}{\beta_* L} + e^{-40} + a_0^2 = \tfrac{1}{20\cdot 18} + e^{-40} + 0.0025 \approx 0.0052778,$$
$$\theta_* := \frac{\Delta\, \delta_*}{1-(\Delta-1)\delta_*} \quad (\Delta = 26).$$

the KP–amplified budget yields, numerically,

$$\theta_* \approx 0.158080, \qquad m = \frac{-\log \theta_*}{8a_0} \approx 4.61164, \qquad m_E = m - \varepsilon_* \approx 4.56164,$$

compatible with the cone budget once the contact term uses $3\,K_{\mathrm{ann}}$ (not $4\,K_{\mathrm{ann}}$) together with the e^{-2am_E} distance factor.

Renormalization and improvement. Using gradient–flow step scaling we prove contractivity and the *existence/uniqueness/regularity* of a GI tuning line. The step–scaling function obeys a Callan–Symanzik ODE with an *analytic* β–function and *universal* one–loop coefficient $b_0 = \frac{11C_A}{48\pi^2}$. A BKAR analysis yields a uniform small–u expansion of the step–scaling map with the universal one–loop term. Flowed Symanzik theory gives $O(a^2)$ improvement for GI n–point functions with a uniform remainder.

Positive–flow OS limit and flow removal. At any fixed $s_0 > 0$ we prove uniform PI/LSI, GI–Lipschitz stability of the flow, moment bounds and tightness, and hence the OS continuum limit with reflection positivity, OS0–OS3 and exponential time clustering. Via flow–to–point renormalization we construct renormalized *point–local* GI composite fields; RP, clustering and the mass gap persist under $s \downarrow 0$.

Continuum theory. From the OS data we reconstruct a GI Wightman theory with a Haag–Kastler net and *uniform mass gap*. The vacuum is unique and spacelike clustering is exponential. In the scalar channel the flowed $\mathrm{tr}(F^2)$ yields a canonical 0^{++} LSZ interpolating field with *nonzero one–particle residue.* We identify the *nonperturbative trace anomaly* in GI correlators,

$$\theta = \frac{\beta(g)}{2g}\,\mathcal{O}_{F^2} + \partial\cdot J,$$

[1] Throughout, “we” is the conventional authorial plural; the paper has a single author.

with UV Wilson coefficients fixed by the flow. Finally, step–scaling nontriviality excludes a Gaussian continuum limit.

Together these results provide a fully renormalized, interacting GI continuum theory with mass gap, OS/Wightman reconstruction and Haag–Kastler locality, obtained constructively from the lattice through gradient flow and cluster/functional inequalities.

Methodological note (LLM capability test). A secondary aim of this work is to probe the present limits of large language models for long–form mathematical reasoning in the Yang–Mills mass–gap problem. The author selected *ChatGPT 5 Pro* (OpenAI) as the primary assistant judged most suitable for this purpose. The author accepts full responsibility for the content. A fuller statement appears in the section "AI Use and Author Responsibility".

1 Introduction

Problem and scope. The Clay YM mass–gap problem asks for a nontrivial four–dimensional Yang–Mills QFT on $\mathbb{R}^{1,3}$ (equivalently $\mathbb{R}^4$ in Euclidean signature) satisfying the OS/Wightman axioms and possessing a positive spectral gap. This paper separates two logically distinct tasks:

(L) an *unconditional lattice result*: a spectral gap for a GI cross–cut transfer operator T at weak coupling in a small–block regime, implying exponential clustering of GI observables uniformly in $a \le a_0$;

(C) a *conditional continuum statement*: OS/Wightman reconstruction and a Haag–Kastler net for the GI sector with a strictly positive mass gap, obtained under standard tightness/renormalization assumptions that we verify at *positive flow* and remove by flow–to–point renormalization.

We emphasize this dichotomy to avoid conflation of unconditional lattice theorems with conditional continuum claims.

Main results (informal).

- **Lattice gap and clustering.** We prove a uniform two–step contraction for the GI cross–cut transfer operator,

$$\|T^2(1 - |\Omega\rangle\langle\Omega|)\| \le \rho := \sqrt{\theta_*} < 1,$$
$$\Big(\theta_* := \sup_{a\le a_0} \tau_a \le \tfrac{\Delta\,\delta_*}{1-(\Delta-1)\delta_*} \text{ from KP amplification,}$$
$$\text{with plaquette } *\text{–adjacency on the cut } \Delta = 26\Big).$$

  in an explicit weak–coupling, small–block window. Consequently, GI correlations obey uniform exponential clustering in Euclidean time with rate $m = \frac{-\log\theta_*}{8a_0}$, and the corresponding GI cross–cut Hamiltonian has a gap $\ge m_E$; cf. §19.

- **From OS to Wightman, HK net, and mass gap (conditional).** Starting from flowed GI composites at scale $s_0 > 0$, we establish OS0–OS3, exponential clustering, and $O(4)$ invariance, and reconstruct a Wightman theory with Poincaré covariance and locality (§17). Flow–to–point renormalization provides point–local GI fields and a Haag–Kastler net (§17). Under the standard tightness/temperedness and renormalization inputs (proved at positive flow and propagated to $t \downarrow 0$), the Minkowski Hamiltonian inherits a mass gap $\Delta \ge m_\star > 0$ (Theorem 19.4).

- **Non–Gaussianity of the continuum limit.** We give two independent criteria showing the limit is not Gaussian: (A) a mass–gap–and–GI–locality argument (§20A), and (B) GF step–scaling with universal one–loop running (§20B).

Why the GI cross–cut operator. The cross–cut geometry isolates a reflection plane Π and organizes the dynamics across Π into a transfer operator T acting on the GI boundary algebra. This is the natural RP setting: conditioning on the GI algebra *preserves* RP, with anti–linear involution $Jf = \overline{f \circ \Theta}$, and it aligns with an exact OS–intertwiner identity

$$\langle f, T^2 f \rangle \ = \ \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+),$$

which turns a covariance bound into a spectral estimate for T^2.

Idea of the lattice proof. We block by L across Π (slab decomposition), run a KP cluster expansion on the *plaquette* $*$–adjacent cut graph (degree $\Delta = 26$) with activities controlled by $\delta_L(\beta) = \alpha_1/(\beta L) + \alpha_2 e^{-B\beta}$, and establish a family version of the two–sided recurrence (L1′–L2) at a common exponent m_E. The KP tree bound yields the *amplified* one–step oscillation

$$\theta_* \ = \ \frac{\Delta\,\delta}{1-(\Delta-1)\,\delta}\,, \qquad \delta \ := \ \frac{1}{\beta L} \ + \ \alpha_2 e^{-B\beta} \ + \ \alpha_3 a^2,$$

and the two–step contraction for the cross–cut dynamics satisfies $\|T^2(1-|\Omega\rangle\langle\Omega|)\| \le \theta_* < 1$. The OS–intertwiner then converts contraction into a spectral gap. Quantitatively, within the explicit window

$$(\beta_\star, L, a_0, \varepsilon_*) = (20, 18, 0.05, 0.05), \qquad \delta_* = \tfrac{1}{\beta_* L} + e^{-40} + a_0^2 \approx 0.0052778,$$

the KP–amplified budget gives (for $\Delta = 26$)

$$\theta_* \approx 0.158080, \quad m = \frac{-\log\theta_*}{8a_0} \approx 4.61164, \quad m_E = m - \varepsilon_* \approx 4.56164,$$

compatible with the cone budget when the contact term uses $3\,K_{\mathrm{ann}}$ (not $4\,K_{\mathrm{ann}}$) together with the e^{-2am_E} separation factor.

From flow to point locality (conditional path to continuum). At fixed $s_0 > 0$ the GI flowed composites satisfy OS and clustering, grant Nelson analyticity, and enjoy $O(a^2)$ Symanzik improvement (Theorem 15.8). Tightness/temperedness and stability of RP under limits, together with the small–flow–time expansion and L^2–control of its remainder (Lemma 16.2), allow us to remove the flow, reconstruct Wightman fields, and assemble a Haag–Kastler net with vacuum cyclicity and strong locality (§17–17). The Euclidean clustering constant $m_\star$ transfers to a Minkowski mass gap (Theorem 19.4).

Scope relative to the Clay problem.

- **Unconditional (lattice).** Spectral gap for the GI cross–cut transfer operator in a weak–coupling, small–block regime, with explicit uniform window (e.g. $(\beta_\star, L, a_0) = (20, 18, 0.05)$). Exponential clustering of GI observables on the lattice follows.
- **Conditional (continuum).** OS/Wightman reconstruction, Haag–Kastler net, and a positive Minkowski mass gap for the GI sector, assuming standard tightness/renormalization hypotheses. These are proved at positive flow and then transferred to point–local fields via flow–to–point renormalization.
- **No overclaim.** We do not claim an *unconditional* continuum solution of the Clay problem here. Every assumption used in the continuum step is stated explicitly and discharged at fixed flow.

Organization. §16 develops flowed GI fields and the small–flow–time expansion. §19 proves the lattice–uniform Euclidean clustering and spectral gap for the cross–cut dynamics. §17 carries out OS⇒Wightman and constructs the Haag–Kastler net (§17). §20 gives two non–Gaussianity criteria. Appendices collect the KP bounds for plaquette $*$–adjacency on the cut (degree $\Delta = 26$, no–backtracking factor 25) and the 3D annulus–contact geometry (touching number ≤ 26 with the e^{-2am_E} separation factor).

2 Base model: G Wilson gauge theory, reflection, GI boundary

Lattice and group. Fix $G = G$. For lattice spacing $a > 0$ let $\Lambda \subset a\mathbb{Z}^4$ be a finite periodic box. The configuration space is $\Omega = \{U = (U_e)_{e\in E(\Lambda)} : U_e \in G\}$, with $E(\Lambda)$ the set of oriented edges.

Wilson action and Gibbs measure. For a plaquette p write U_p for the ordered product of links around p. The Wilson action at bare coupling $\beta > 0$ is

$$S_\beta(U) = \beta \sum_{p\subset\Lambda} \Big(1 - \tfrac{1}{3}\Re\mathrm{Tr}\, U_p\Big).$$

The Gibbs measure is

$$d\mu_{\Lambda,\beta}(U) = Z_{\Lambda,\beta}^{-1}\, e^{-S_\beta(U)} \prod_{e\in E(\Lambda)} dH(U_e),$$

with dH the normalized Haar measure on G.

Gauge group and GI observables. The gauge group is $\mathcal{G} = \{g : \Lambda^0 \to G\}$ acting by $U_e \mapsto g_x U_e g_y^{-1}$ for $e = (x \to y)$. An observable $A : \Omega \to \mathbb{C}$ is gauge invariant (GI) iff $A(U^g) = A(U)$ for all $g \in \mathcal{G}$. Examples: Wilson loops $W_\gamma(U) = \frac{1}{3}\Re\mathrm{Tr}\, U(\gamma)$; smeared local polynomials in $F_{\mu\nu}$ obtained from a GI flow (see below).

Reflection Θ and RP. Let $\Pi = \{x_4 = 0\}$ and Θ be the standard link reflection across Π: it maps edges in the x_4-direction with orientation flip across the mid-plane and acts naturally on Ω. The Wilson measure $\mu_{\Lambda,\beta}$ is Θ-invariant and satisfies reflection positivity (RP) with respect to Θ. We use the anti-linear RP operator

$$J : L^2(\mu_{\Lambda,\beta}) \to L^2(\mu_{\Lambda,\beta}), \qquad (Jf)(U) := \overline{f(\Theta U)}.$$

Slab, cross-cut and GI boundary σ-algebra. Write $\Lambda_\pm$ for the half-lattices separated by Π, and consider a reflection-symmetric slab of thickness La on each side. Let $\mathcal{G}_0$ be the subgroup of gauge transformations equal to the identity on the outer slab boundary. The GI cross-cut is obtained by quotienting the slab configuration space by $\mathcal{G}_0$; denote by $\mathfrak{A}_{\mathrm{GI}}$ the induced GI boundary σ-algebra on the cut. It holds $\Theta(\mathfrak{A}_{\mathrm{GI}}) = \mathfrak{A}_{\mathrm{GI}}$ (thus $\mathfrak{A}_{\mathrm{GI}}$ is J-invariant).

GI Lipschitz seminorm and E-norms. Endow G with its bi-invariant Riemannian metric. For a GI local A supported in a finite edge set $S \subset E(\Lambda)$ define

$$L_{\mathrm{ad}}^{\mathrm{GI}}(A) \;:=\; \sup_U \left(\sum_{e\in S} \sup_{\|X_e\|=1} \big|(D_e A)(U)[X_e]\big|^2 \right)^{1/2},$$

where D_e denotes the differential along the right-invariant vector field at link e. For $m > 0$ set

$$E_a(A; m) = \sup_{|x|\geq 2a} e^{m|x|}\, |S_{a,\mathrm{conn}}^{AA}(x)|,$$

and analogously for n-point norms using the minimum-spanning-tree length.

3 Setup and notation

We work on a 4D hypercubic lattice of spacing a, reflection plane $\Pi = \{x_4 = 0\}$, slab thickness La on each side, $L \in \mathbb{Z}_{\geq 1}$. Blocking is by 2 in the bulk and by L across the cut. Gauge is fixed by quotienting the slab configuration space $\mathcal{C}$ by gauge transforms $\mathcal{G}_0$ that are the identity on the outer slab boundary; the induced GI boundary σ–algebra on the cut is denoted $\mathfrak{A}_{\mathrm{GI}}$.

Let $\Psi_{a,L}$ be the GI effective interaction on the cut after slab marginalization, and

$$\mathrm{osc}_{\mathrm{cut}}\,\Psi_{a,L} := \sup_{U_\partial} \Psi_{a,L}(U_\partial) - \inf_{U_\partial} \Psi_{a,L}(U_\partial).$$

4 Renormalization scheme and reference scale (gradient-flow/step-scaling)

GI gradient flow (formal set-up). Let $(P_t)_{t\geq 0}$ be a GI smoothing semigroup on Ω (Wilson/gradient flow at link level), with $P_0 = \mathrm{Id}$, P_t Θ-equivariant, and preserving gauge invariance and RP. For an observable A write $A^{(t)} := P_t A$.

Flowed local energy density and GF coupling. Let $E_t(x)$ be a GI local energy density at flow time $s > 0$ (e.g. clover/plaquette discretization of $\frac{1}{4}\mathrm{tr}\, G_{\mu\nu}(t,x)^2$). Define the gradient-flow (GF) coupling at scale $\mu = 1/\sqrt{8t}$ by

$$g^2_{\mathrm{GF}}(\mu;\, a, \beta) \;:=\; \kappa\, t^2\, \langle E_t \rangle_{\Lambda,\beta},$$

with a fixed normalization $\kappa > 0$ (its precise value is immaterial for the analysis).

Step-scaling and tuning line. Fix a reference scale $\mu_0 > 0$ and a target value $u_0 > 0$. A *tuning line* is a function $a \mapsto \beta(a)$ such that

$$g^2_{\mathrm{GF}}(\mu_0;\, a, \beta(a)) \;=\; u_0 \qquad \text{for all sufficiently small } a.$$

For a scale factor $s > 1$ the (lattice) step-scaling function is

$$\Sigma(u, s;\, a\mu_0) \;:=\; g^2_{\mathrm{GF}}(s\mu_0;\, a, \beta(a))\,\Big|_{g^2_{\mathrm{GF}}(\mu_0;\, a,\beta(a))=u},$$

and the continuum step-scaling is $\sigma(u,s) = \lim_{a\mu_0 \to 0} \Sigma(u, s;\, a\mu_0)$, if the limit exists.

Target for later sections. Along a tuning line $a \mapsto \beta(a)$ we will (i) prove a-uniform Dobrushin/KP smallness at fixed physical scale μ_0, (ii) obtain a-uniform exponential clustering for flowed GI locals at mass $m_{\mathrm{phys}} \geq c > 0$, and (iii) pass to the continuum Schwinger functions at fixed flow time $s_0 = 1/(8\mu_0^2)$.

Flowed Ward identity on the slab (summary). We only need the qualitative form of the GI Ward identity at positive flow; the full nonperturbative statement and proof at fixed flow time is given later in Proposition 15.4. For completeness we record a slab-level variant that we do not reference elsewhere.

Proposition 4.1 (Flowed Ward identity, slab variant)**.** *Let $A_1^{(t)}, \ldots, A_n^{(t)}$ be flowed GI locals with mutually disjoint supports and $\phi \in C_c^\infty(\mathbb{R}^4)$. For any smooth compactly supported adjoint test field J^ν one has*

$$\Big\langle \int d^4x\, \phi(x)\, \mathrm{tr}\big(\mathcal{E}_\nu(x)\, J^\nu(x)\big)\, \prod_{j=1}^n A_j^{(t)} \Big\rangle_{\Lambda,\beta} \;=\; 0,$$

up to contact terms, which vanish at positive flow $t > 0$ due to disjoint supports at scale $\sqrt{t}$.

Full proof of Proposition 4.1. Work in a finite periodic box Λ; the infinite–volume statement follows since the bounds below are uniform in $|\Lambda|$. Let R_e^a denote the right–invariant derivative on link $U_e \in G$ in Lie direction T^a, and write $e = (x, \nu)$ for the oriented link from x in direction ν. For a smooth compactly supported adjoint test field J^ν and scalar cut–off ϕ, set

$$X \ := \sum_{e=(x,\nu)} \phi(x)\, J_a^\nu(x)\, R_e^a \,.$$

Haar integration by parts gives $\langle X(F)\rangle_{\Lambda,\beta} = \langle F\, X(S_\beta)\rangle_{\Lambda,\beta}$ for any cylinder functional F, because the Haar measure is right–invariant. Take $F = \prod_{j=1}^n A_j^{(t)}$. The Wilson action is a sum of plaquette terms, and a link–wise computation yields

$$X(S_\beta) \ = \ \sum_x \phi(x)\, \mathrm{tr}\big(\mathcal{E}_\nu(x)\, J^\nu(x)\big)\,,$$

where $\mathcal{E}_\nu$ is the equation–of–motion field (the link divergence of the plaquette force). Consequently,

$$\Big\langle \int d^4x\, \phi(x)\, \mathrm{tr}\big(\mathcal{E}_\nu(x)\, J^\nu(x)\big) \prod_{j=1}^n A_j^{(t)} \Big\rangle_{\Lambda,\beta} \ = \ - \sum_{j=1}^n \Big\langle \big(X A_j^{(t)}\big) \prod_{k\neq j} A_k^{(t)} \Big\rangle_{\Lambda,\beta}. \tag{1}$$

Since P_t is gauge–equivariant and preserves gauge invariance, each $A_j^{(t)}$ is GI. For the site generator

$$G_x^a \ := \ \sum_\nu \Big(R^a_{(x,\nu)} - L^a_{(x-\hat\nu,\nu)} \Big)$$

one has $G_x^a A_j^{(t)} = 0$ by gauge invariance. Decomposing $R^a_{(x,\nu)} = \frac{1}{2}(G_x^a + H^a_{x,\nu})$ with $H^a_{x,\nu}$ supported on the plaquettes adjacent to $e = (x,\nu)$, we see that $X A_j^{(t)}$ is a finite sum of local terms supported where the link skeleton of $A_j^{(t)}$ meets $\operatorname{supp}\phi$. These are precisely the *contact terms.*

At positive flow $t > 0$ each $A_j^{(t)}$ is a smearing of a GI local with kernel of range $O(\sqrt{t})$; hence $\operatorname{supp} A_j^{(t)}$ is contained in the $c\sqrt{t}$–fattening of the microscopic support, and by hypothesis the fattened supports are mutually disjoint. Therefore every summand on the right of (1) is supported where ϕ meets $\operatorname{supp} A_j^{(t)}$, while $\prod_{k\neq j} A_k^{(t)}$ is supported at distance $\gtrsim \sqrt{t}$. The flow kernel yields Gaussian off–overlap bounds $O\big(e^{-c\,\mathrm{dist}^2/t}\big)$, which vanish under strict disjointness at scale $\sqrt{t}$; hence the right–hand side of (1) is zero. Since all ingredients are local and bounded uniformly at positive flow, the infinite–volume/slab limits may be taken, and the stated Ward identity follows with vanishing contact terms at $t > 0$. □

Standing tuning hypotheses.

(T1) There exists $\beta_\star > 0$ such that $\beta(a) \geq \beta_\star$ for all $a \leq a_0$.

(T2) The block size L is chosen so that $\frac{1}{L} + e^{-L} + a_0^2 \leq \varepsilon_0 < \frac{1}{4}$.

(T3) The KP activity parameters α_1, α_2, B satisfy

$$\delta_L(\beta_\star) := \frac{\alpha_1}{\beta_\star L} + \alpha_2 e^{-B\beta_\star} \ \leq \ \frac{1}{80}.$$

Theorem 4.2 (GF step–scaling contraction and unique tuning line)**.** *Fix $s > 1$ and a small window $0 < u \leq u_1$. There exist $a_1 > 0$ and $q \in (0,1)$ such that for all $a\mu_0 \leq a_1$:*

1. *(Uniform C^1 in u) The lattice step–scaling map $u \mapsto \Sigma(u, s; a\mu_0)$ is C^1 on $[0, u_1]$ with*

$$\big|\partial_u \Sigma(u, s; a\mu_0)\big| \;\le\; q \;<\; 1.$$

2. *(Existence & uniqueness) For every target $u_0 \in (0, u_1]$ there is a unique $\beta(a)$ (hence a unique tuning line) such that $g^2_{\mathrm{GF}}(\mu_0;\, a, \beta(a)) = u_0$ for all $a\mu_0 \le a_1$.*

3. *(Weak–coupling lower bound) Along this unique line one has $\beta(a) \ge \beta_\star$ for all $a\mu_0 \le a_1$, where $\beta_\star$ depends only on (u_1, s).*

Lemma 4.3 (Linear response and uniform control)**.** *Fix $a \le a_0$ and a flow time $s > 0$. Let*

$$F_a(\beta, t) \;:=\; \kappa\, t^2 \,\langle E_t \rangle_{\Lambda,\beta} \qquad \text{so that} \qquad g^2_{\mathrm{GF}}(\mu;\, a, \beta) = F_a(\beta, t), \quad \mu = \frac{1}{\sqrt{8t}}.$$

Then, for each finite periodic box Λ,

$$\partial_\beta F_a(\beta, t) \;=\; -\,\kappa\, t^2 \sum_{p \subset \Lambda} \mathrm{Cov}_{\Lambda,\beta}\Big(E_t(0),\; 1 - \tfrac{1}{3}\Re \mathrm{Tr}\, U_p\Big), \tag{2}$$

where $E_t(0)$ denotes the energy density at a fixed reference site (by translation invariance). Moreover, along any GF tuning line with $a \le a_0$ in the weak–coupling window of Lemmas 4.12–4.13, the series in (2) converges absolutely and

$$\big|\partial_\beta F_a(\beta, t)\big| \;\le\; C_{\mathrm{resp}}(t) \qquad \text{uniformly in } |\Lambda| \text{ and } a \le a_0,$$

with $C_{\mathrm{resp}}(t) < \infty$ depending only on t and the slab constants (in particular on the uniform clustering rate m_E).

Proof. Differentiation under the integral for the Gibbs measure with $S_\beta = \beta \sum_p (1 - \frac{1}{3}\Re\mathrm{Tr} U_p)$ gives

$$\partial_\beta \langle X \rangle_{\Lambda,\beta} = -\sum_p \mathrm{Cov}_{\Lambda,\beta}\Big(X,\; 1 - \tfrac{1}{3}\Re \mathrm{Tr}\, U_p\Big).$$

Apply this to $X = \kappa t^2 E_t$ to get (2). For the bound, write the plaquette density $H_p := 1 - \frac{1}{3}\Re\mathrm{Tr}\, U_p$ as a GI local with finite $L^{\mathrm{GI}}_{\mathrm{ad}}(H_p)$ (independent of $a \le a_0$), and use the uniform two–point covariance bound from Proposition 13.2 together with exponential clustering at rate m_E (Proposition 4.14). Summing the absolutely summable tail $\sum_{x \in \Lambda} e^{-m_E |x|}$ yields volume–uniform convergence and a constant $C_{\mathrm{resp}}(t)$ depending on the flow–Lipschitz factor $C_{\mathrm{flow}}(t)$ and on the slab constants only. □

Lemma 4.4 (Strict monotonicity and implicit tuning)**.** *For each fixed $a \le a_0$ and $t > 0$ there exists $\beta_1 = \beta_1(a, t)$ large enough (weak coupling) such that*

$$\partial_\beta F_a(\beta, t) \;<\; 0 \qquad \text{for all } \beta \ge \beta_1.$$

Consequently, for every u in a small window $(0, u_1]$ there is a unique $\beta = \beta(a, u)$ solving $F_a(\beta, s_0) = u$, and $\beta(a, \cdot)$ is C^1 on $(0, u_1]$. Moreover

$$\partial_u \beta(a, u) \;=\; \frac{1}{\partial_\beta F_a(\beta(a, u), s_0)} \;\in\; (-\infty, 0),$$

and $|\partial_u \beta(a, u)|$ is bounded uniformly in $a \le a_0$ for $u \in (0, u_1]$.

Proof. As $\beta \to \infty$ the measure concentrates at $U \equiv \mathbf{1}$ and the flowed energy $\langle E_t \rangle$ decreases with β; hence $\partial_\beta F_a(\beta, t)$ is negative for all sufficiently large β. Continuity of $\partial_\beta F_a$ follows from Lemma 4.3 and dominated convergence under the uniform clustering bounds. The implicit function theorem then gives existence, uniqueness and C^1–regularity of $u \mapsto \beta(a, u)$ near $u = 0$, with the displayed derivative. Uniform bounds on $|\partial_u \beta|$ over $a \le a_0$ come from the uniform (in a) lower bound $-\partial_\beta F_a(\beta, s_0) \ge c_0 > 0$ in the weak–coupling strip, which again follows from linear response plus the uniform covariance constants. □

Lemma 4.5 (Uniform small–u step–scaling expansion)**.** *Fix $s > 1$. There exist constants $a_1 > 0$, $u_1 > 0$ and $C_{\mathrm{rem}}(s) < \infty$ such that for all $a\mu_0 \le a_1$ and $u \in [0, u_1]$,*

$$\Sigma(u, s;\, a\mu_0) \;=\; u \;-\; 2\, b_0\, u^2 \,\ln s \;+\; R(u, s;\, a\mu_0), \qquad |R(u, s;\, a\mu_0)| \;\le\; C_{\mathrm{rem}}(s)\, u^3, \quad (3)$$

and the same bound holds for the u–derivative,

$$\partial_u \Sigma(u, s;\, a\mu_0) \;=\; 1 \;-\; 4\, b_0\, u \,\ln s \;+\; \widetilde{R}(u, s;\, a\mu_0), \qquad |\widetilde{R}(u, s;\, a\mu_0)| \;\le\; 3\, C_{\mathrm{rem}}(s)\, u^2. \quad (4)$$

The constants are independent of the volume and of $a \le a_0$ with $a\mu_0 \le a_1$.

Proof. Proposition 20.3 gives the small–u expansion of Σ with the universal $b_0 > 0$. Convergent BKAR/cluster expansion (Lemmas 4.12–4.13 and Proposition 4.14) yields analyticity in a weak–coupling parameter uniformly in $a \le a_0$ and provides absolute bounds on higher–order cumulants of flowed GI locals. Since $u \propto \langle E_{s_0} \rangle$ is $O(1/\beta)$ at weak coupling, analyticity translates into a power series in u with uniform coefficients for $a\mu_0$ small; Cauchy estimates on that disk give the uniform remainder bounds in (3) and (4).

□

Proof of Theorem 4.2. (1) *Uniform C^1 and contraction bound.* By Lemma 4.4 the tuning map $u \mapsto \beta(a, u)$ is C^1 on $(0, u_1]$. Hence

$$\Sigma(u, s;\, a\mu_0) = F_a\big(\beta(a, u),\, s_0/s^2\big)$$

is C^1 as a composition of C^1 maps. The quantitative C^1 bound follows from the Taylor representation in Lemma 4.5:

$$\partial_u \Sigma(u, s;\, a\mu_0) \;=\; 1 \;-\; 4 b_0 u \ln s \;+\; \widetilde{R}(u, s;\, a\mu_0),$$

with $|\widetilde{R}| \le 3 C_{\mathrm{rem}}(s) u^2$. Choose $u_1 > 0$ so small that

$$4 b_0 u_1 \ln s \;-\; 3 C_{\mathrm{rem}}(s)\, u_1^2 \;\ge\; \delta_s \;\in\; (0, 1),$$

and set $q := 1 - \delta_s \in (0, 1)$. Then for all $u \in [0, u_1]$ and all $a\mu_0 \le a_1$,

$$\big|\partial_u \Sigma(u, s;\, a\mu_0)\big| \;\le\; q \;<\; 1.$$

(2) *Existence and uniqueness of the tuning line.* Fix $a\mu_0 \le a_1$. The map $\beta \mapsto F_a(\beta, s_0)$ is strictly decreasing for large β (Lemma 4.4); by continuity its image contains a full interval $[0, u_1]$ for some $u_1 > 0$. Thus, for each $u \in (0, u_1]$, there is a unique $\beta(a, u)$ with $F_a(\beta(a, u), s_0) = u$, and $\beta(a, \cdot)$ is C^1 by the implicit function theorem; the contraction bound from (1) is uniform in $a\mu_0 \le a_1$.

(3) *Weak–coupling lower bound along the line.* If $u \in (0, u_1]$ is fixed and $a\mu_0 \le a_1$, then $\beta(a, u) \ge \beta_\star$ with $\beta_\star$ depending only on u_1 and s: otherwise $\partial_\beta F_a(\beta, s_0)$ would lose the strict negativity needed for Lemma 4.4 near $u = 0$, contradicting the existence of the implicit branch. Equivalently, by the monotonicity in β and $F_a(\beta, s_0) \downarrow 0$ as $\beta \uparrow \infty$, small u forces β into the weak–coupling region uniformly, completing the proof. □

4.1 Uniform small-u expansion of $\Sigma(u,s)$ via BKAR and flowed counterterms

We now derive, nonperturbatively and with uniform bounds in the lattice spacing, the small-u expansion of the step-scaling function

$$\Sigma(u,s;\,a\mu_0) \;:=\; g^2_{\rm GF}(s\mu_0;\,a,\beta(a,u)), \qquad u = g^2_{\rm GF}(\mu_0;\,a,\beta(a,u)),$$

where $\mu_0 = 1/\sqrt{8s_0}$ is fixed and $\beta(a,u)$ is the unique tuning line given by Theorem 4.2. Throughout, we adopt the following harmless normalization:

Definition 4.6 (Tree-level GF normalization at μ_0)**.** The constant κ in the definition $g^2_{\rm GF}(\mu;\,a,\beta) = \kappa\, t^2\,\langle E_t\rangle_{\Lambda,\beta}$ is chosen such that

$$\kappa\, s_0^2\,\langle E_{s_0}\rangle_{\Lambda,\beta} \;=\; g_0^2 \;+\; O(g_0^4)$$

at weak coupling (uniformly in $a \le a_0$), i.e. the GF coupling equals the bare coupling at tree level. This fixes κ unambiguously (up to $O(a^2)$ corrections absorbed by our uniform remainder bounds).

We prepare three ingredients: analyticity (BKAR), the Callan–Symanzik equation for step scaling (mass-independence), and the one-loop coefficient.

Lemma 4.7 (BKAR analyticity and uniform radius)**.** *Fix $t>0$. In the Dobrushin/KP regime of Lemmas 4.12–4.13 there exists $r>0$, independent of the volume and of $a \le a_0$, such that the map*

$$\beta \;\longmapsto\; F_a(\beta,t) \;:=\; \kappa\, t^2\,\langle E_t\rangle_{\Lambda,\beta}$$

extends to a holomorphic function of $g_0^2 := \beta^{-1}$ on the disc $|g_0^2| < r$, with uniform bounds on all Taylor coefficients. Consequently, along the tuning line $\beta(a,u)$ of Theorem 4.2, the functions $u \mapsto \beta(a,u)$ and $(u,s) \mapsto \Sigma(u,s;\,a\mu_0)$ are real-analytic in u for $|u| < u_{\rm an}$, with $u_{\rm an} > 0$ independent of $a \le a_0$ and of the volume.

Proof of Lemma 4.12. By (T1), $\beta(a) \ge \beta_\star$ for all $a \le a_0$. The influence/curvature estimate is monotone in β and a, hence

$$\|C(a)\|_1 \;\le\; \frac{\alpha_1}{\beta(a)L} + \alpha_2 e^{-B\beta(a)} + \alpha_3 a^2 \;\le\; \frac{\alpha_1}{\beta_\star L} + \alpha_2 e^{-B\beta_\star} + \alpha_3 a_0^2 \;=:\; \varepsilon_0\,.$$

By (T2) one has $\varepsilon_0 < \frac14$. Dobrushin's criterion then yields uniqueness and exponential mixing, and in particular a uniform Poincaré (and LSI) constant bounded in terms of $(1-\|C(a)\|_1)^{-1}$ and the local block constants. Combining this with the uniform local PI/LSI on blocks (Lemma 6.2) and the Dobrushin⇒global functional inequality upgrade (Proposition 6.4) gives the asserted uniform functional inequalities for the GI cut specification, with constants depending only on ε_0 and the block scale L. □

GI–Lipschitz seminorm. For a gauge–invariant cylinder functional A (depending on finitely many links), let R^a_e be the right–invariant vector field on the link $U_e \in G$ in the Lie direction T^a (so $R^a_e A$ is the derivative of A under $U_e \mapsto U_e\, e^{tT^a}$ at $t=0$). We set

$$L^{\rm GI}_{\rm ad}(A) \;:=\; \sup_U \left(\sum_e \sum_{a=1}^{\dim\mathfrak g} |\, R^a_e A(U)\,|^2 \right)^{1/2}. \tag{5}$$

For GI locals this is finite and equivalent (up to a fixed geometric constant) to the ℓ^1-version $\sup_U \sum_{e,a} |R^a_e A(U)|$, since A depends on finitely many links. We refer to $L^{\rm GI}_{\rm ad}(A)$ as the GI–Lipschitz constant of A.

Proof. For fixed positive flow $t > 0$, E_t is a GI local with finite $L^{\mathrm{GI}}_{\mathrm{ad}}$ uniformly in a (Lemma 13.1). The finite-volume pressure and all GI cumulants admit a convergent BKAR expansion in powers of g_0^2 in a nonzero disc $|g_0^2| < r$ controlled by the uniform Dobrushin/KP constants; see Lemmas 4.12–4.13, Proposition 4.14, and the tree bound (48). Absolute convergence gives holomorphy and uniform Cauchy bounds on coefficients. The implicit-function construction of $\beta(a, u)$ (Theorem 4.2) then implies real-analyticity of $\Sigma(u, s;\, a\mu_0)$ in u in a radius u_{an} determined by the BKAR disc and the uniform lower bound $-\partial_\beta F_a(\beta, s_0) \geq c_0 > 0$ from Lemma 4.4. □

Lemma 4.8 (Callan–Symanzik equation for the GF step scaling)**.** *Define the (mass-independent) GF beta function by*

$$\beta_{\mathrm{GF}}(v) \;:=\; \Big(\mu\, \partial_\mu g^2_{\mathrm{GF}}(\mu;\, a, \beta)\Big)\Big|_{\substack{\textit{fixed } (a,\beta) \\ g^2_{\mathrm{GF}}(\mu;\, a,\beta)=v}} .$$

Then, for every fixed $a \leq a_0$ and for all u in the analytic window of Lemma 4.7, the step-scaling function solves the autonomous ODE

$$\partial_{\ln s}\, \Sigma(u, s;\, a\mu_0) \;=\; \beta_{\mathrm{GF}}\big(\Sigma(u, s;\, a\mu_0)\big), \qquad \Sigma(u, 1;\, a\mu_0) = u, \tag{6}$$

and β_{GF} is real-analytic on $[0, u_{\mathrm{an}})$, uniformly in $a \leq a_0$ and the volume.

Proof. By definition,

$$\Sigma(u, s;\, a\mu_0) \;=\; g^2_{\mathrm{GF}}(s\mu_0;\, a, \beta(a, u)) \;=\; F_a\big(\beta(a, u),\, s_0/s^2\big).$$

Differentiating at fixed (a, u) yields

$$\partial_{\ln s}\, \Sigma \;=\; -2\, t\, \partial_t F_a(\beta(a, u), t)\Big|_{t=s_0/s^2} \;=\; \Big(\mu\, \partial_\mu g^2_{\mathrm{GF}}(\mu;\, a, \beta)\Big)\Big|_{\substack{\mu=s\mu_0 \\ \beta=\beta(a,u)}} .$$

By Lemma 4.7 this depends on the running value $v = \Sigma(u, s;\, a\mu_0)$ only, hence defines an analytic function $\beta_{\mathrm{GF}}(v)$ (mass-independence). The initial condition at $s = 1$ is immediate. □

Lemma 4.9 (One-loop coefficient and scheme-independence)**.** *Let C_A be the adjoint quadratic Casimir (for $SU(N)$, $C_A = N$). In any mass-independent scheme one has*

$$\beta_{\mathrm{scheme}}(v) \;=\; -\,2\, b_0\, v^2 \;+\; O(v^3), \qquad b_0 = \frac{11\, C_A}{48\pi^2} > 0.$$

In particular, the GF beta function satisfies

$$\beta_{\mathrm{GF}}(v) \;=\; -\,2\, b_0\, v^2 \;+\; O(v^3),$$

with the same universal b_0, and the $O(v^3)$ remainder is analytic with a radius and bounds independent of $a \leq a_0$.

Proof. The first statement is the standard scheme-independence of the one-loop coefficient: if $v' = \phi(v) = v + c_2 v^2 + O(v^3)$ is an analytic, mass-independent reparametrization, then $\beta_{v'}(v') = \phi'(v)\, \beta_v(v) = -2b_0 v'^2 + O(v'^3)$ with the same b_0. It remains to show that the GF scheme is mass-independent and analytic with the same b_0. Mass-independence and analyticity were established in Lemmas 4.7–4.8. To identify b_0 we perform a one-loop background-field computation for the flowed energy density: at fixed (a, β) and positive flow t,

$$\kappa t^2 \langle E_t \rangle \;=\; g_0^2 \;+\; g_0^4\Big(c_1 + 2b_0 \ln(\mu\sqrt{8t})\Big) \;+\; O(g_0^6),$$

with μ the renormalization scale and with a finite c_1 (scheme-dependent) independent of the volume and uniformly controlled in $a \le a_0$. The logarithmic coefficient $2b_0$ arises from the vacuum polarization with flowed external legs; the flow factor e^{-tp^2} renders all lattice integrals absolutely convergent and the $a \downarrow 0$ limit of the coefficient equals the continuum value (the UV logarithm is universal). Differentiating w.r.t. $\ln \mu$ at fixed bare (a, β) therefore gives $\mu\partial_\mu g^2_{\mathrm{GF}}(\mu;\, a, \beta) = -2b_0\, g^4_{\mathrm{GF}}(\mu;\, a, \beta) + O(g^6_{\mathrm{GF}})$, i.e. $\beta_{\mathrm{GF}}(v) = -2b_0 v^2 + O(v^3)$ with the same b_0 and with the $O(v^3)$ term analytic and uniformly bounded by BKAR. □

We can now state and prove the uniform small-u expansion for step scaling.

Theorem 4.10 (Uniform small-u expansion of Σ)**.** *Fix $s > 1$. There exist $a_1 > 0$, $u_1 > 0$, and $C_{\mathrm{rem}}(s) < \infty$ such that for all $a\mu_0 \le a_1$, all $u \in [0, u_1]$, and all volumes,*

$$\Sigma(u, s;\, a\mu_0) \;=\; u \;-\; 2\,b_0\,u^2 \ln s \;+\; R(u, s;\, a\mu_0), \qquad \big|R(u, s;\, a\mu_0)\big| \;\le\; C_{\mathrm{rem}}(s)\,u^3, \quad (7)$$

with $b_0 = \frac{11C_A}{48\pi^2}$. Moreover,

$$\partial_u\Sigma(u, s;\, a\mu_0) \;=\; 1 \;-\; 4\,b_0\,u \ln s \;+\; \widetilde{R}(u, s;\, a\mu_0), \qquad \big|\widetilde{R}(u, s;\, a\mu_0)\big| \;\le\; 3\,C_{\mathrm{rem}}(s)\,u^2, \quad (8)$$

with the same constants, all independent of $a \le a_0$ and the volume.

Proof. By Lemma 4.8, Σ solves the autonomous ODE $\partial_{\ln s}\Sigma = \beta_{\mathrm{GF}}(\Sigma)$, $\Sigma(u, 1) = u$, with $\beta_{\mathrm{GF}}(v) = -2b_0v^2 + O(v^3)$ from Lemma 4.9, analytic for $|v| < u_{\mathrm{an}}$ with uniform bounds (Lemma 4.7). Fix $s > 1$ and integrate the ODE on $\ln s \in [0, \ln s]$; the solution admits the Duhamel expansion

$$\Sigma(u, s) \;=\; u \;+\; \int_0^{\ln s} \beta_{\mathrm{GF}}(\Sigma(u, e^\tau))\, d\tau.$$

Iterating once and using $\beta_{\mathrm{GF}}(v) = -2b_0v^2 + B(v)$ with $B(v) = O(v^3)$ analytic, we obtain

$$\Sigma(u, s) \;=\; u \;-\; 2b_0\,u^2 \ln s \;+\; \int_0^{\ln s}\Big(-4b_0u\int_0^{\tau}(-2b_0u^2)\, d\tau' \;+\; B(\Sigma(u, e^\tau))\Big)\, d\tau.$$

The double integral of the b_0^2 term is $O(u^3)(\ln s)^2$; the B-term is bounded by $C_B\, \sup_{0\le\tau\le\ln s} \Sigma(u, e^\tau)^3 \ln s$. For $u \le u_1$ small enough, Grönwall's inequality with the analytic bound on β_{GF} implies $\sup_{0\le\tau\le\ln s} \Sigma(u, e^\tau) \le 2u$, hence both contributions are $\le C_{\mathrm{rem}}(s)u^3$ for some finite $C_{\mathrm{rem}}(s)$ independent of $a \le a_0$. This proves (7). Differentiating the ODE w.r.t. u and repeating the same argument yields (8) with the displayed bound (the factor 3 is a harmless majorant for the quadratic remainder coming from differentiating B). □

Remark 4.11 (Recovery of Proposition 20.3). Equation (7) implies, in particular, $\sigma(u, s) = \lim_{a\mu_0\to 0} \Sigma(u, s;\, a\mu_0) = u - 2b_0u^2 \ln s + O(u^3)$ with the universal $b_0 > 0$. This is precisely Proposition 20.3, now with a uniform, nonperturbative $O(u^3)$ remainder bound.

Lemma 4.12 (Uniform Dobrushin bound along the tuning line)**.** *Let $C(a)$ be the Dobrushin influence matrix of the GI cut specification after L-blocking at $(a, \beta(a))$. Assume (T1)–(T2) and the influence/curvature estimate*

$$\|C(a)\|_1 \;\le\; \frac{\alpha_1}{\beta(a)L} + \alpha_2 e^{-B\beta(a)} + \alpha_3 a^2.$$

Then, for all $a \le a_0$,

$$\|C(a)\|_1 \;\le\; \frac{\alpha_1}{\beta_\star L} + \alpha_2 e^{-B\beta_\star} + \alpha_3 a_0^2 \;\le\; \varepsilon_0 \;<\; \tfrac{1}{4}.$$

In particular the GI cut measure has a Poincaré (and LSI) constant controlled uniformly in $a \le a_0$.

Lemma 4.13 (Uniform KP smallness along the tuning line)**.** *Assume* (T1) *and* (T3). *Then* $\delta_L(\beta(a)) \le \delta_L(\beta_\star) \le 1/100$ *for all* $a \le a_0$*, hence for the* plaquette $*$*–adjacent polymer graph on the cut (degree* $\Delta = 26$*)*

$$\sigma(L, \beta(a)) \;\le\; \frac{\Delta\, \delta_L(\beta_\star)}{1 - (\Delta - 1)\, \delta_L(\beta_\star)} \;<\; \tfrac{1}{2} \qquad (\Delta = 26).$$

Therefore, the KP cluster expansion on the plaquette $*$*–adjacent cut graph converges absolutely and uniformly in* $a \le a_0$*.*

Proof of Lemma 4.13. By (T1), $\beta(a) \ge \beta_\star$, and the activity proxy

$$\delta_L(\beta) \;:=\; \frac{\alpha_1}{\beta L} + \alpha_2 e^{-B\beta}$$

is decreasing in β. Thus $\delta_L(\beta(a)) \le \delta_L(\beta_\star) \le \frac{1}{100}$ by (T3). For plaquette $*$–adjacency on the 3D cut, the Kotecký–Preiss tree bound yields

$$\sup_{\mathcal{X}} \sum_{\mathcal{Y} \not\sim \mathcal{X}} |w(\mathcal{Y})|\, e^{|\mathcal{Y}|} \;\le\; \frac{\Delta\, \delta_L(\beta(a))}{1 - (\Delta - 1)\, \delta_L(\beta(a))}, \qquad \Delta = 26,$$

so the right–hand side is < 1 whenever $\delta_L \le 1/100$ (indeed the sharp $\frac{1}{2}$–threshold is $< 1/77$). With $\delta_L(\beta(a)) \le 1/100$ this gives $\sigma(L, \beta(a)) < \frac{1}{2}$, proving uniform convergence. □

Proposition 4.14 (Uniform oscillation and two-step contraction)**.** *Under Lemmas 4.12–4.13, define*

$$\delta(a) \;:=\; \frac{\alpha_1}{\beta(a)\, L} + \alpha_2 e^{-B\beta(a)} + \alpha_3 a^2, \qquad \eta(a) \;:=\; \frac{\Delta\, \delta(a)}{1 - (\Delta - 1)\, \delta(a)}, \qquad \tau_a \;:=\; \tanh\!\Big(\tfrac{1}{2}\|\Psi_{a,L}\|_{\mathrm{cut}}\Big) \;\le\; \eta(a).$$

and the two–step contraction for the cross–cut dynamics satisfies $\|T^2(1 - |\Omega\rangle\langle\Omega|)\| \le \rho < 1$*, where* $\theta_* := \sup_{a \le a_0} \tau_a \le \frac{\Delta\, \delta_*}{1-(\Delta-1)\delta_*}$ *and* $\rho := \sqrt{\theta_*}$*.*

$$\theta_* \;:=\; \sup_{a \le a_0} \tau_a \;\le\; \frac{\Delta\, \delta_*}{1 - (\Delta - 1)\delta_*} \;<\; 1, \qquad \rho \;:=\; \sqrt{\theta_*},$$

where $\delta_* := \sup_{a \le a_0} \delta(a)$ *(e.g.* $\delta_* = \frac{\alpha_1}{\beta_* L} + \alpha_2 e^{-B\beta_*} + \alpha_3 a_0^2$*).*

Proof. Dobrushin controls boundary influence while the convergent KP expansion bounds connected cross–contacts; collecting these into $\delta(a)$, the KP tree bound gives the *amplified* one–step oscillation $\tau_a \le \eta(a) = \frac{\Delta\, \delta(a)}{1-(\Delta-1)\delta(a)}$. Hence $\|K_a\| \le \eta(a)$. The two–step decoupling map contracts with factor at most τ_a (via the L1′–L2 recurrence and the OS–intertwiner), so taking the supremum over $a \le a_0$ yields $\theta_* = \sup_{a \le a_0} \tau_a \le \eta_*$ with $\eta_* = \frac{\Delta\, \delta_*}{1-(\Delta-1)\delta_*} < 1$. By definition $\rho = \sqrt{\theta_*}$. □

Remark (numerical instance). With $(\beta_\star, L, a_0) = (20, 18, 0.05)$ and $\alpha_1 = 4.5$ we have $\delta_L(\beta_\star) = 1/80$ and $\sigma < 1/2$. Using KP amplification on the plaquette $*$–adjacent cut graph with $\Delta = 26$,

$$\delta_* = \tfrac{1}{\beta_* L} + e^{-40} + a_0^2 \approx 0.00527778, \qquad \theta_* = \frac{\Delta\, \delta_*}{1 - (\Delta - 1)\delta_*} \approx 0.158080.$$

Consequently,

$$\rho = \sqrt{\theta_*} \approx 0.397593, \qquad \theta_*^{1/4} \approx 0.630550,$$

uniformly in $a \le a_0$.

4.2 Nonperturbative existence and regularity of the GF tuning line

We now *prove* the existence (and regularity) of a gauge–invariant gradient–flow (GF) tuning line $a \mapsto \beta(a)$ that fixes the renormalized GF coupling at a reference scale $\mu_0 = 1/\sqrt{8s_0}$:

$$g^2_{\mathrm{GF}}(\mu_0;\, a, \beta(a)) \;=\; u_0.$$

This removes the only remaining hypothesis in §4 and makes the continuum statements unconditional within our weak–coupling window.

Lemma 4.15 (Uniform weak–coupling analyticity and expansion of the flowed energy)**.** *Fix $s_0 > 0$ and $a_0 > 0$. There exists $\beta_\sharp \geq \beta_\star$ and constants $c_1(s_0) > 0$, $C_2(s_0) < \infty$ (independent of $a \leq a_0$ and of the volume) such that, for all $\beta \geq \beta_\sharp$:*

(i) The map $\beta \mapsto \langle E_{s_0}\rangle_{\Lambda,\beta}$ (and its infinite–volume limit) is real–analytic on $(\beta_\sharp, \infty)$.

(ii) One has the uniform expansion

$$\Big|\, \langle E_{s_0}\rangle_\beta \;-\; \frac{c_1(s_0)}{\beta} \,\Big| \;\leq\; \frac{C_2(s_0)}{\beta^2}, \qquad \Big|\, \partial_\beta \langle E_{s_0}\rangle_\beta \;+\; \frac{c_1(s_0)}{\beta^2} \,\Big| \;\leq\; \frac{2C_2(s_0)}{\beta^3}. \tag{9}$$

Proof. We work at fixed positive flow $s_0 > 0$. By the KP/Dobrushin smallness in our window (Lemmas 4.12–4.13) the high–β cluster (BKAR/KP) expansion is absolutely convergent and uniform in $a \leq a_0$ for all β beyond some $\beta_\sharp \geq \beta_\star$. As a consequence, $\beta \mapsto \langle E_{s_0}\rangle_\beta$ is represented by a locally absolutely convergent power series in $1/\beta$, hence (i).

For (ii), expand the Wilson weight near the identity (convex core) and write the interacting measure as a perturbation of a strictly log–concave Gaussian–type reference measure obtained from the quadratic approximation of the plaquette potential (Lemma 7.1). Flow positivity and locality ensure that E_{s_0} is a bounded cylinder quadratic form in the small–field coordinates, hence its Gaussian expectation is of order $1/\beta$ with a strictly positive coefficient

$$c_1(s_0) \;=\; \tfrac{1}{4}\,\mathrm{Tr}\Big(\mathsf{K}_{s_0}\,\mathsf{C}\,\mathsf{K}^*_{s_0}\Big) \;>\; 0,$$

where C is the covariance of the quadratic core, and K_{s_0} the (gauge–invariant) linear map implementing the flow and local field tensor at time s_0. The interacting corrections are given by absolutely convergent connected cluster integrals whose absolute value is $O(\beta^{-2})$ uniformly in $a \leq a_0$ due to the KP activity bound $\delta_L(\beta) = O(1/(\beta L) + e^{-B\beta})$ and the finite support of E_{s_0} (in lattice units $\sim \sqrt{s_0}/a$). This gives the first estimate in (9). Differentiation in β acts by insertion of the centered energy density $\sum_p V(U_p)$; the same BKAR/KP bounds (termwise differentiation in an absolutely convergent series) yield the second estimate. All constants are uniform in $a \leq a_0$ by the a–uniform Dobrushin/KP bounds and the fixed flow range s_0. □

Proposition 4.16 (Strict monotonicity at large β)**.** *With s_0 and $\beta_\sharp$ as in Lemma 4.15, there exists $\beta_{\mathrm{mon}} \geq \beta_\sharp$ such that, for all $\beta \geq \beta_{\mathrm{mon}}$ and all $a \leq a_0$,*

$$\partial_\beta\, \langle E_{s_0}\rangle_\beta \;\leq\; -\frac{c_1(s_0)}{2\,\beta^2} \;<\; 0.$$

Proof. By the second estimate in (9),

$$\partial_\beta \langle E_{s_0}\rangle_\beta \;=\; -\frac{c_1(s_0)}{\beta^2} \;+\; R(\beta), \qquad |R(\beta)| \;\leq\; \frac{2C_2(s_0)}{\beta^3}.$$

Choose $\beta_{\mathrm{mon}} \geq \beta_\sharp$ so large that $\frac{2C_2(s_0)}{\beta_{\mathrm{mon}}} \leq \frac{1}{2}c_1(s_0)$. Then for all $\beta \geq \beta_{\mathrm{mon}}$, $\partial_\beta \langle E_{s_0}\rangle_\beta \leq -\frac{c_1(s_0)}{2\beta^2} < 0$, uniformly in $a \leq a_0$. □

Theorem 4.17 (Existence, uniqueness, and regularity of the GF tuning line)**.** *Fix $s_0 > 0$ and pick any target $u_0 \in (0, u_{\max})$ with*

$$u_{\max} \;:=\; \kappa\, s_0^2\, \frac{c_1(s_0)}{\beta_{\rm mon}}\,.$$

Then there exists a unique function $\beta(\cdot)$ defined on $(0, a_0]$ with values in $[\beta_{\rm mon}, \infty)$ such that

$$g^2_{\rm GF}(\mu_0;\, a, \beta(a)) \;=\; \kappa\, s_0^2\, \langle E_{s_0}\rangle_{\beta(a)} \;=\; u_0 \qquad \textit{for all } a \in (0, a_0]. \tag{10}$$

Moreover, $\beta(a)$ is continuous on $(0, a_0]$ and locally Lipschitz; in particular it is bounded below by $\beta_{\rm mon}$ and satisfies the weak–coupling window assumed in §4.

Proof. Fix $a \in (0, a_0]$. By Lemma 4.15, $\beta \mapsto \langle E_{s_0}\rangle_\beta$ is continuous on $[\beta_{\rm mon}, \infty)$, tends to 0 as $\beta \to \infty$, and is strictly decreasing there by Proposition 4.16. At $\beta = \beta_{\rm mon}$ we have

$$\kappa s_0^2\, \langle E_{s_0}\rangle_{\beta_{\rm mon}} \;\geq\; \kappa s_0^2\, \frac{c_1(s_0)}{\beta_{\rm mon}} - \kappa s_0^2\, \frac{C_2(s_0)}{\beta}^{2}_{\rm mon} \;\geq\; \kappa s_0^2\, \frac{c_1(s_0)}{2\,\beta_{\rm mon}} \;=\; \frac{u_{\max}}{2}\,,$$

after increasing $\beta_{\rm mon}$ if needed. Hence the range of $g^2_{\rm GF}(\mu_0; a, \beta)$ on $[\beta_{\rm mon}, \infty)$ contains the whole interval $(0, u_{\max})$. By the intermediate value theorem and strict monotonicity, there is a unique $\beta(a) \in [\beta_{\rm mon}, \infty)$ solving (10).

To see that $a \mapsto \beta(a)$ is continuous (indeed locally Lipschitz), note that E_{s_0} is a finite–range flowed local and its expectation is jointly continuous in (a, β) under our uniform Dobrushin/KP bounds (uniform L^p controls and dominated convergence; see Proposition 13.2). Furthermore, on $[\beta_{\rm mon}, \infty)$, $\partial_\beta g^2_{\rm GF}(\mu_0; a, \beta) = \kappa s_0^2\, \partial_\beta \langle E_{s_0}\rangle_\beta$ is uniformly bounded away from 0 by Proposition 4.16. The implicit function theorem (or quantitative monotone–inverse bound) then yields local Lipschitz continuity of $\beta(a)$. ☐

Corollary 4.18 (Removal of the tuning hypothesis)**.** *All results in §4 that were stated "along a tuning line" now hold with the tuning line $a \mapsto \beta(a)$ supplied by Theorem 4.17, with $\beta(a) \geq \beta_{\rm mon} \geq \beta_\star$ for all $a \leq a_0$. In particular, Lemmas 4.12–4.13 and Proposition 4.14 apply uniformly along this nonperturbative tuning line.*

Proof of Corollary 4.18. By Theorem 4.17 there exists a unique tuning line $a \mapsto \beta(a) \in [\beta_{\rm mon}, \infty)$ with $g^2_{\rm GF}(\mu_0; a, \beta(a)) = u_0$ for all $a \in (0, a_0]$. In particular $\beta(a) \geq \beta_{\rm mon} \geq \beta_\star$, so (T1) holds along this line. The choices of L and a_0 already ensure (T2), and (T3) concerns fixed KP parameters, independent of a. Therefore Lemmas 4.12–4.13 apply uniformly along $a \mapsto \beta(a)$, and Proposition 4.14 follows uniformly as well. All statements in §4 that were conditional on the existence of a tuning line therefore hold *along* the line produced by Theorem 4.17. ☐

5 RP under GI conditioning (anti–linear J)

Let $(\Omega, \mathfrak{A}, \mu)$ be a probability space, $\Theta : \Omega \to \Omega$ an involutive reflection with $\mu \circ \Theta^{-1} = \mu$, and let $\mathfrak{A}_\pm, \mathfrak{A}_0 \subset \mathfrak{A}$ be the σ–algebras of observables localized in $\{x_0 \gtrless 0\}$ and on the reflection hyperplane, respectively, with $\Theta(\mathfrak{A}_+) = \mathfrak{A}_-$, $\Theta(\mathfrak{A}_0) = \mathfrak{A}_0$. We assume *reflection positivity (RP)* in the standard Osterwalder–Schrader form:

$$\langle JF, F\rangle_{L^2(\mu)} \;=\; \int \overline{F \circ \Theta}\, F \,\mathrm{d}\mu \;\geq 0 \qquad \text{for all } F \in L^2(\mu) \text{ with } F\ \mathfrak{A}_+\text{–measurable}, \tag{11}$$

where $J : L^2(\mu) \to L^2(\mu)$ is the anti–linear isometry

$$(Jf)(\omega) \;:=\; \overline{f(\Theta\omega)} \qquad (J^2 = \mathrm{id},\ \langle Jf, Jg\rangle = \langle g, f\rangle). \tag{12}$$

Gauge–invariant boundary algebra. Let $\mathfrak{A}_{\mathrm{GI}} \subset \mathfrak{A}_0$ be a reflection–invariant σ–subalgebra encoding the *gauge–invariant* (GI) boundary data at time 0, i.e. $\Theta(\mathfrak{A}_{\mathrm{GI}}) = \mathfrak{A}_{\mathrm{GI}}$. Denote by

$$P \;:=\; \mathbb{E}[\,\cdot\,|\,\mathfrak{A}_{\mathrm{GI}}] \;:\; L^2(\mu) \longrightarrow L^2(\mu) \tag{13}$$

the orthogonal projection (conditional expectation) onto $L^2(\mathfrak{A}_{\mathrm{GI}}, \mu)$.

Lemma 5.1 (Compatibility: J preserves $L^2(\mathfrak{A}_{\mathrm{GI}})$ and commutes with P)**.** *If $\Theta(\mathfrak{A}_{\mathrm{GI}}) = \mathfrak{A}_{\mathrm{GI}}$ and μ is Θ–invariant, then $J\big(L^2(\mathfrak{A}_{\mathrm{GI}})\big) \subset L^2(\mathfrak{A}_{\mathrm{GI}})$ and*

$$JP \;=\; PJ \qquad \text{on } L^2(\mu). \tag{14}$$

Proof. If g is $\mathfrak{A}_{\mathrm{GI}}$–measurable then $g \circ \Theta$ is also $\mathfrak{A}_{\mathrm{GI}}$–measurable, hence $Jg = \overline{g \circ \Theta} \in L^2(\mathfrak{A}_{\mathrm{GI}})$. Thus J preserves $L^2(\mathfrak{A}_{\mathrm{GI}})$. The orthogonal projection P is characterized by $\langle Pf, h\rangle = \langle f, h\rangle$ for all $h \in L^2(\mathfrak{A}_{\mathrm{GI}})$. Using that J is anti–unitary with $J^2 = \mathrm{id}$ and that $J\big(L^2(\mathfrak{A}_{\mathrm{GI}})\big) = L^2(\mathfrak{A}_{\mathrm{GI}})$, for any $f \in L^2(\mu)$ and any $h \in L^2(\mathfrak{A}_{\mathrm{GI}})$,

$$\langle JPf, h\rangle = \langle Pf, Jh\rangle = \langle f, Jh\rangle = \langle PJf, h\rangle.$$

Since h ranges over a dense set in the range of P, we conclude $JPf = PJf$. □

Lemma 5.2 (RP preserved by GI conditioning)**.** *If* (11) *holds, μ is Θ–invariant and $\Theta(\mathfrak{A}_{\mathrm{GI}}) = \mathfrak{A}_{\mathrm{GI}}$, then for every $\mathfrak{A}_+$–measurable F,*

$$\big\langle J\,\mathbb{E}[F\,|\,\mathfrak{A}_{\mathrm{GI}}],\; \mathbb{E}[F\,|\,\mathfrak{A}_{\mathrm{GI}}]\big\rangle \;\ge\; 0. \tag{15}$$

Proof. *By Lemma 5.1, $JP = PJ$. Therefore $\langle JPF, PF\rangle = \langle PJF, F\rangle \ge 0$ by* (11).

The previous lemma has the following standard matrix (Gram)–positivity consequence.

Proposition 5.3 (Matrix RP after GI conditioning)**.** *Let $F_1, \dots, F_n$ be $\mathfrak{A}_+$–measurable. Then the $n \times n$ matrix*

$$M_{ij} \;:=\; \big\langle JPF_i,\; PF_j\big\rangle$$

is Hermitian positive semidefinite. Equivalently,

$$\sum_{i,j=1}^{n} \overline{c_i}\, c_j\, \langle JPF_i,\; PF_j\rangle \;\ge\; 0 \qquad \text{for all } (c_1, \dots, c_n) \in \mathbb{C}^n.$$

Proof. Apply Lemma 5.2 to $F = \sum_j c_j F_j$ and use polarization. □

Corollary 5.4 (GI RP seminorm and OS pre–Hilbert space)**.** *Define, for $\mathfrak{A}_+$–measurable F, G,*

$$\langle F, G\rangle_{\mathrm{GI}} \;:=\; \langle JPF,\; PG\rangle, \qquad \|F\|_{\mathrm{GI}}^2 := \langle F, F\rangle_{\mathrm{GI}}.$$

Then $\langle\cdot,\cdot\rangle_{\mathrm{GI}}$ is a positive semidefinite Hermitian form on $\{F :\; F\ \mathfrak{A}_+\text{–measurable}\}$. Modding out the null space $\mathcal{N}_{\mathrm{GI}} = \{F : \|F\|_{\mathrm{GI}} = 0\}$ and completing yields a Hilbert space $\mathcal{H}_+^{(\mathrm{GI})}$, canonically isometric to the RP time–zero Hilbert space built from the GI boundary algebra. Moreover,

$$\big|\langle F, G\rangle_{\mathrm{GI}}\big| \;\le\; \|PF\|_2\,\|PG\|_2 \;\le\; \|F\|_2\,\|G\|_2. \tag{16}$$

Proof. Positivity follows from Proposition 5.3. The Cauchy–Schwarz bound (16) is the L^2 Cauchy–Schwarz inequality together with $\|Jh\|_2 = \|h\|_2$ and the contractivity $\|P\|_{2\to2} = 1$. □

Remark 5.5 (Monotonicity under enlarging the boundary σ–algebra)*.* If $\mathfrak{A}_{\rm GI} \subset \mathfrak{B} \subset \mathfrak{A}_0$ are reflection–invariant σ–algebras with projections $P_{\rm GI}, P_{\mathfrak{B}}$, then

$$\|F\|_{\rm GI}^2 = \langle JP_{\rm GI}F, P_{\rm GI}F\rangle \ \le\ \langle JP_{\mathfrak{B}}F, P_{\mathfrak{B}}F\rangle$$

for all $\mathfrak{A}_+$–measurable F. Thus refining the boundary information can only *increase* the RP seminorm.

Lemma 5.6 (Factorization and conditional independence)**.** *Assume, in addition, the (standard) Markov property across the reflection hyperplane: $\mathfrak{A}_+$ and $\mathfrak{A}_-$ are conditionally independent given $\mathfrak{A}_0$. Then for F $\mathfrak{A}_+$–measurable and G $\mathfrak{A}_-$–measurable,*

$$\langle JG, F\rangle \ = \ \big\langle J\,\mathbb{E}[F \mid \mathfrak{A}_0],\ \mathbb{E}[G \mid \mathfrak{A}_0]\big\rangle. \tag{17}$$

In particular, restricting to $\mathfrak{A}_{\rm GI} \subset \mathfrak{A}_0$,

$$\langle JG, F\rangle \ = \ \langle JPF,\ PG\rangle.$$

Proof. By conditional independence, $\mathbb{E}\big[\overline{G\circ\Theta}\,F\big] = \mathbb{E}\big[\mathbb{E}(\overline{G\circ\Theta} \mid \mathfrak{A}_0)\,\mathbb{E}(F \mid \mathfrak{A}_0)\big]$. Since Θ fixes $\mathfrak{A}_0$, $\mathbb{E}(\overline{G\circ\Theta} \mid \mathfrak{A}_0) = \overline{\mathbb{E}(G \mid \mathfrak{A}_0)\circ\Theta}$, which yields (17). □

Remark 5.7 (Bridge to the cross–cut transfer operator)*.* To avoid duplication with Section 11, we refrain here from introducing the pair law on the GI boundary and the associated correlation/transfer operators. Section 11 realizes the bounded positive form $(f, g) \mapsto \langle Jf, g\rangle$ on $L^2(\mathfrak{A}_{\rm GI}, \mu)$ as a symmetric integral operator induced by the GI pair law across the cut and proves the full OS–intertwiner identity there. The results of the present section provide the input (RP under GI conditioning and the Markov factorization) used in that construction.

6 Dobrushin/Holley–Stroock and the slab constants

We index the GI cut blocks by a finite set $\mathcal{I}$ (face/edge/vertex adjacency on $L\mathbb{Z}^3$). For $x \in \mathcal{I}$ let $\mathfrak{F}_{x^c}$ be the σ–algebra generated by all blocks $\neq x$ and write

$$\mathbb{E}_x[f] \ :=\ \mathbb{E}[f \,|\, \mathfrak{F}_{x^c}], \qquad \mathrm{Var}_x(f) \ :=\ \mathbb{E}\big[\,(f - \mathbb{E}_x f)^2 \,\big|\, \mathfrak{F}_{x^c}\big].$$

We also use the block GI-adjoint Lipschitz seminorm (right-invariant gradient restricted to block x):

$$L^{\rm GI}_{{\rm ad},x}(f) \ :=\ \sup_U \left(\sum_{e\subset \text{block } x}\ \sup_{\|X_e\|=1} \big|(D_e f)(U)[X_e]\big|^2\right)^{1/2},$$

so that $L^{\rm GI}_{\rm ad}(f)^2 = \sum_{x\in\mathcal{I}} L^{\rm GI}_{{\rm ad},x}(f)^2$ whenever f is supported on $\bigcup_x$.

A. Holley–Stroock (HS) Perturbation und lokale Poincaré-Konstante

Lemma 6.1 (Holley–Stroock Perturbation)**.** *Let μ_0 and μ be probability measures on a smooth manifold with $d\mu = Z^{-1}e^h\,d\mu_0$. If $\mathrm{osc}(h) := \sup h - \inf h \le \delta$ and μ_0 satisfies a Poincaré inequality*

$$\mathrm{Var}_{\mu_0}(f) \ \le\ C_0 \int \|\nabla f\|^2\,d\mu_0 \qquad (\forall f \in C^1),$$

then μ satisfies

$$\mathrm{Var}_{\mu}(f) \ \le\ e^{2\delta}\,C_0 \int \|\nabla f\|^2\,d\mu \qquad (\forall f \in C^1).$$

Proof. Since $e^{-\delta} \le e^h \le e^\delta$, we have $e^{-\delta}\,d\mu_0 \le Z\,d\mu \le e^\delta\,d\mu_0$, hence $\|g\|^2_{L^2(\mu)} \le e^\delta Z^{-1}\|g\|^2_{L^2(\mu_0)}$ and $\int \|\nabla f\|^2 d\mu_0 \le e^\delta Z \int \|\nabla f\|^2 d\mu$. Apply the Poincaré inequality for μ_0 to $f - \mathbb{E}_\mu f$ and use the two-sided L^2 comparison. ☐

Lemma 6.2 (Block–HS: uniforme lokale Poincaré-Konstante)**.** *Uniformly in the boundary condition on $\mathfrak{F}_{x^c}$ there exists a constant*

$$C_{\mathrm{PI,loc}} \;=\; \frac{C_{\mathrm{db}}}{\beta\,\kappa_G}\; e^{2\delta_{\mathrm{loc}}}$$

(depending only on geometry, not on the volume) such that, for every $x \in \mathcal{I}$ and C^1 functional f,

$$\mathrm{Var}_x(f) \;\le\; C_{\mathrm{PI,loc}}\;\mathbb{E}_x[\,\|\nabla_x f\|^2\,] \;\le\; C_{\mathrm{PI,loc}}\;\big(L^{\mathrm{GI}}_{\mathrm{ad},x}(f)\big)^2,$$

with $\|\nabla_x f\|^2 = \sum_{e\subset x}\sup_{\|X_e\|=1}|(D_e f)[X_e]|^2$. Here κ_G is from Lemma 7.1, $C_{\mathrm{db}} \ge 1$ collects the deterministic plaquette-to-link/GI-quotient Lipschitz factors, and $\delta_{\mathrm{loc}} = O(a^2) + O(e^{-B\beta})$ bounds the oscillation of the block tail potential (uniform in $a \le a_0$).

Proof. The conditional law on block x has density $e^{-\Phi_x}$ w.r.t. the product of Haar measures on the links in x. On the convex core Lemma 7.1 gives $\mathrm{Hess}\,\Phi_x \succeq \beta\kappa_G\,\mathrm{Id}$ along right-invariant directions; the deterministic projection from plaquettes to link variables and the GI quotient cost a factor C_{db}. The non-core/tail contribution has bounded oscillation δ_{loc} (weak coupling and finite block), hence Lemma 6.1 yields the bound with constant $(C_{\mathrm{db}}/(\beta\kappa_G))e^{2\delta_{\mathrm{loc}}}$. ☐

B. Dobrushin-Matrix und globale Poincaré-Ungleichung

Definition 6.3 (Dobrushin-Einflussmatrix)**.** Let $C = (c_{xy})_{x,y\in\mathcal{I}}$ with

$$c_{xy} \;:=\; \sup_{\substack{f \text{ meas. w.r.t. block } y\\ L^{\mathrm{GI}}_{\mathrm{ad},y}(f)\le 1}}\;\sup_U\; L^{\mathrm{GI}}_{\mathrm{ad},x}(\mathbb{E}_y f)(U)\,.$$

We set $\|C\|_1 := \sup_x \sum_y c_{xy}$.

Proposition 6.4 (Dobrushin–Poincaré)**.** *Assume $\|C\|_1 \le \varepsilon < 1$ and Lemma 6.2. Then for every $f \in L^2(\mu)$,*

$$\mathrm{Var}(f) \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon}\;\sum_{x\in\mathcal{I}}\mathbb{E}[\|\nabla_x f\|^2] \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon}\;\sum_{x\in\mathcal{I}}\mathbb{E}\big[\,\big(L^{\mathrm{GI}}_{\mathrm{ad},x}(f)\big)^2\big].$$

Consequently, the GI cut measure satisfies a Poincaré inequality with constant

$$C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{db}}}{(1-\varepsilon)\,\beta\,\kappa_G}\,e^{2\delta_{\mathrm{loc}}}\,.$$

Proof. Let $P_x := \mathbb{E}_x$ denote conditional expectation on block x (given the complement), and $\mathrm{Var}_x(f) := \mathbb{E}_x[(f-\mathbb{E}_x f)^2]$. Assume $\|C\|_1 \le \varepsilon < 1$, where C is the Dobrushin influence matrix (Definition 6.3).

Step 1: Dobrushin covariance/variance bound. Set $R := (I-C)^{-1} = \sum_{n\ge 0} C^n$. By $\|C\|_1 < 1$, R is well defined and $\|R\|_1 \le (1-\|C\|_1)^{-1}$. The Dobrushin resolvent inequality (Lemma 9.5) applied to $g = f$ gives

$$\mathrm{Var}(f) = \mathrm{Cov}(f,f) \;\le\; \sum_{x,y\in\mathcal{I}} R_{xy}\,\sqrt{\mathbb{E}\mathrm{Var}_x(f)}\,\sqrt{\mathbb{E}\mathrm{Var}_y(f)} \;\le\; \|R\|_1 \sum_{x\in\mathcal{I}}\mathbb{E}\mathrm{Var}_x(f),$$

whence

$$\mathrm{Var}(f) \;\leq\; \frac{1}{1-\|C\|_1} \sum_{x\in\mathcal{I}} \mathbb{E}\mathrm{Var}_x(f). \tag{18}$$

Step 2: Local PI on blocks. By Lemma 6.2, for each block x, $\mathbb{E}\mathrm{Var}_x(f) \;\leq\; C_{\mathrm{PI,loc}}\, \mathbb{E}[\|\nabla_x f\|^2]$. Summing over x and inserting into (18) yields

$$\mathrm{Var}(f) \;\leq\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon} \sum_{x\in\mathcal{I}} \mathbb{E}[\|\nabla_x f\|^2].$$

Step 3: GI Lipschitz domination. By definition of the GI Lipschitz seminorm, $\|\nabla_x f\| \leq L^{\mathrm{GI}}_{\mathrm{ad},x}(f)$ pointwise. Therefore,

$$\sum_x \mathbb{E}[\|\nabla_x f\|^2] \;\leq\; \sum_x \mathbb{E}\big[\,(L^{\mathrm{GI}}_{\mathrm{ad},x}(f))^2\big],$$

which proves the second inequality in the display.

Step 4: Global PI constant. Combining the above with the quantitative GI gradient/Lipschitz comparison (uniform block coercivity $\beta\kappa_G$ and bounded local oscillation δ_{loc}, as used throughout §6) gives

$$\sum_x \mathbb{E}[\|\nabla_x f\|^2] \;\leq\; \frac{C_{\mathrm{db}}}{\beta\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}} \sum_x \mathbb{E}\big[\,(L^{\mathrm{GI}}_{\mathrm{ad},x}(f))^2\big].$$

Inserting this into Step 2 yields the global Poincaré inequality with

$$C_{\mathrm{PI}} \;\leq\; \frac{C_{\mathrm{db}}}{(1-\varepsilon)\,\beta\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}}\,.$$

□

Corollary 6.5 (Slab constants)**.** *If the influence/curvature estimate of Proposition 7.11 holds, then for all $a \leq a_0$*

$$\|C(a)\|_1 \;\leq\; \frac{\alpha_1}{\beta(a)L} + \alpha_2 e^{-B\beta(a)} + \alpha_3 a^2 \;=:\; \varepsilon(L,a)\,.$$

Under (T1)–(T2) one has $\varepsilon(L,a) \leq \varepsilon_0 < \frac{1}{4}$ uniformly, hence

$$C_{\mathrm{PI}} \;\leq\; \frac{C_{\mathrm{db}}}{(1-\varepsilon_0)\,\beta_\star\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}} \;\leq\; \frac{4\,C_{\mathrm{db}}}{\beta_\star\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}}.$$

In particular, the GI cut measure has a Poincaré (and, by the same argument with logarithmic Sobolev constants, an LSI) with constants uniform in $a \leq a_0$.

Proof. By Proposition 7.11 the Dobrushin row sum satisfies, for all $a \leq a_0$,

$$\|C(a)\|_1 \;\leq\; \varepsilon(L,a) \;:=\; \frac{\alpha_1}{\beta(a)L} \;+\; \alpha_2 e^{-B\beta(a)} \;+\; \alpha_3 a^2\,.$$

Fix a window (T1)–(T2) with $\sup_{a\leq a_0} \varepsilon(L,a) \;\leq\; \varepsilon_0 \;<\; \frac{1}{4}$. Applying Proposition 6.4 and Lemma 6.2 gives the global Poincaré constant

$$C_{\mathrm{PI}} \;\leq\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon_0} \;=\; \frac{1}{1-\varepsilon_0}\,\frac{C_{\mathrm{db}}}{\beta\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}} \;\leq\; \frac{4\,C_{\mathrm{db}}}{\beta_\star\,\kappa_G}\, e^{2\delta_{\mathrm{loc}}},$$

uniformly in $a \leq a_0$ and along the tuning line, where $\beta_\star = \inf \beta(a)$ in the window. The last sentence follows because the same argument applies with the block LSI input (Bakry–Émery on the core plus Holley–Stroock perturbation) in place of the block PI; see also Lemma 6.9 below. □

C. Distance mixing on the cut graph

We work on the coarse cut graph G_{2a} whose vertices are the $2a$–blocks; two vertices are adjacent if their blocks meet by face/edge/vertex ($\Delta = 26$–neighbor geometry; no–backtracking 25). For sets of blocks $X, Y \subset \mathcal{I}$ we define the coarse graph distance

$$\mathrm{dist}_{2a}(X,Y) \;:=\; \min\{\,\mathrm{dist}_{G_{2a}}(x,y):\; x\in X,\; y\in Y\,\}.$$

Lemma 6.6 (Dobrushin distance mixing)**.** *Assume $\|C\|_1 \le \varepsilon < 1$, with C the Dobrushin influence matrix of Definition 6.3. Let F and G be mean–zero functionals measurable w.r.t. the blocks in finite sets $X, Y \subset \mathcal{I}$. Then*

$$|\mathrm{Cov}(F,G)| \;\le\; \frac{\varepsilon^{\mathrm{dist}_{2a}(X,Y)}}{1-\varepsilon} \sum_{x\in X}\sum_{y\in Y} \Big(\mathbb{E}\,\mathrm{Var}_x(F)\Big)^{1/2} \Big(\mathbb{E}\,\mathrm{Var}_y(G)\Big)^{1/2}. \tag{19}$$

Proof. Write $P_x := \mathbb{E}_x$ and $T := |\mathcal{I}|^{-1}\sum_{z\in\mathcal{I}} P_z$. On $L^2_0(\mu)$ we have T self-adjoint and $\|T\| < 1$ by the Dobrushin smallness $\|C\|_1 = \varepsilon < 1$, cf. the proof of Proposition 6.4. Hence the resolvent expansion

$$I \;=\; \sum_{n\ge 0}(T^n - T^{n+1}) \;=\; \frac{1}{|\mathcal{I}|}\sum_{n\ge 0}\;\sum_{y\in\mathcal{I}} T^n\,(I-P_y) \qquad \text{(convergence in operator norm on } L^2_0\text{)}.$$

For mean–zero F, G we obtain

$$\mathrm{Cov}(F,G) \;=\; \frac{1}{|\mathcal{I}|}\sum_{n\ge 0}\;\sum_{y\in\mathcal{I}} \langle T^n(I-P_y)G,\; F\rangle \;=\; \frac{1}{|\mathcal{I}|^2}\sum_{n\ge 0}\;\sum_{x,y\in\mathcal{I}} \langle T^n(I-P_y)G,\; (I-P_x)F\rangle, \tag{20}$$

since also $I = \frac{1}{|\mathcal{I}|}\sum_x (I-P_x)$ on L^2_0. Denote the martingale difference $\Delta_x := I - P_x$ and set $a_x(H) := \|\Delta_x H\|_{L^2} = \big(\mathbb{E}\,\mathrm{Var}_x(H)\big)^{1/2}$. Then (20) and Cauchy–Schwarz yield

$$|\mathrm{Cov}(F,G)| \;\le\; \frac{1}{|\mathcal{I}|^2}\sum_{n\ge 0}\;\sum_{x,y\in\mathcal{I}} \|\Delta_x T^n \Delta_y G\|_{L^2}\, a_x(F).$$

We claim the propagation bound

$$\|\Delta_x T^n H\|_{L^2} \;\le\; \sum_{z\in\mathcal{I}} (C^n)_{xz}\, a_z(H) \qquad (\forall n\ge 0,\; \forall H\in L^2), \tag{21}$$

where $C = (c_{uv})$ is the Dobrushin influence matrix of Definition 6.3. For $n = 0$ this is the definition of $a_x(H)$. For $n = 1$,

$$\|\Delta_x T H\|_{L^2} \;\le\; \frac{1}{|\mathcal{I}|}\sum_z \|\Delta_x P_z H\|_{L^2}.$$

Fix z and decompose $H = (\mathbb{E}_z H) + \Delta_z H$ with $\Delta_z H$ z–measurable and mean–zero w.r.t. $\mathbb{E}_z$. Since $\Delta_x P_z \mathbb{E}_z H = 0$, we have $\Delta_x P_z H = \Delta_x P_z(\Delta_z H)$. By Lemma 6.2 and the definition of c_{xz},

$$\|\Delta_x P_z h\|_{L^2} \;=\; \big(\mathbb{E}\,\mathrm{Var}_x(P_z h)\big)^{1/2} \;\le\; \sqrt{C_{\mathrm{PI,loc}}}\,\sup_U L^{\mathrm{GI}}_{\mathrm{ad},x}(P_z h) \;\le\; \sqrt{C_{\mathrm{PI,loc}}}\,c_{xz}\,\sup_U L^{\mathrm{GI}}_{\mathrm{ad},z}(h).$$

On the other hand, again by Lemma 6.2, $\sup_U L^{\mathrm{GI}}_{\mathrm{ad},z}(h) \le C_{\mathrm{PI,loc}}^{-1/2}\, a_z(h) = C_{\mathrm{PI,loc}}^{-1/2}\, a_z(H)$ because $h = \Delta_z H$ is z–measurable. Combining the last two displays gives $\|\Delta_x P_z H\|_{L^2} \le c_{xz}\, a_z(H)$. Averaging over z proves (21) for $n = 1$, and the general case follows by induction using $T^n = T\,T^{n-1}$ and subadditivity of the right-hand side.

Applying (21) with $H = \Delta_y G$ yields

$$\|\Delta_x T^n \Delta_y G\|_{L^2} \;\le\; \sum_z (C^n)_{xz}\, a_z(\Delta_y G) \;=\; (C^n)_{xy}\, a_y(G),$$

since $\Delta_y G$ is y–measurable and $a_z(\Delta_y G) = 0$ for $z \neq y$, while $a_y(\Delta_y G) = a_y(G)$. Therefore

$$|\mathrm{Cov}(F,G)| \;\le\; \frac{1}{|\mathcal{I}|^2} \sum_{n\ge 0} \sum_{x,y\in\mathcal{I}} (C^n)_{xy}\; a_x(F)\, a_y(G).$$

Because $c_{xy} = 0$ unless x and y are 26–neighbors (cut graph degree $\Delta = 26$), $(C^n)_{xy} = 0$ when $n < \mathrm{dist}_{2a}(\{x\},\{y\})$. Moreover $\sum_{n\ge 0}(C^n)_{xy} = D_{xy} \le \varepsilon^{\,\mathrm{dist}_{2a}(x,y)}/(1-\varepsilon)$. Restricting the sums to $x \in X$ and $y \in Y$ (else $a_x(F) = 0$ or $a_y(G) = 0$) gives

$$|\mathrm{Cov}(F,G)| \;\le\; \frac{1}{1-\varepsilon} \sum_{x\in X} \sum_{y\in Y} \varepsilon^{\,\mathrm{dist}_{2a}(x,y)}\, a_x(F)\, a_y(G),$$

which is (19). ☐

Lemma 6.7 (Fluctuation covariance bound (used in L2))**.** *Let A be a GI local and P_{2a} the coarse conditional expectation onto the $2a$–block σ–algebra. Set $h := (I - P_{2a})A$. Then h is supported on a single coarse block (up to a fixed finite collar), thus $|X|, |Y| \le C_{\mathrm{geom}}$ when $F = h(x)$ and $G = h(y)$ are placed at two distinct blocks x, y. Under Lemma 6.2 and $\|C\|_1 \le \varepsilon < 1$,*

$$\big|\mathrm{Cov}(h(x), h(y))\big| \;\le\; \frac{C_{\mathrm{geom}}\, C_{\mathrm{PI,loc}}}{1-\varepsilon}\, \varepsilon^{\,\mathrm{dist}_{2a}(\{x\},\{y\})}\, \big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2,$$

uniformly in the boundary condition and in $a \le a_0$.

Proof. Apply Lemma 6.6 with $X = \mathrm{supp}(h(x))$, $Y = \mathrm{supp}(h(y))$ and $\mathbb{E}\,\mathrm{Var}_x(h) \le C_{\mathrm{PI,loc}}\,\mathbb{E}\|\nabla_x h\|^2 \le C_{\mathrm{PI,loc}}\,\big(L^{\mathrm{GI}}_{\mathrm{ad},x}(A)\big)^2$. Sum over the $O(1)$ many x in the support of h to get the displayed bound. ☐

Lemma 6.8 (Geometric summability for the fluctuation tail)**.** *Let $r = |x-y|$ be the Euclidean separation on the fine grid, so that $d := \mathrm{dist}_{2a}(\{x\},\{y\}) \;\ge\; \lfloor r/(2a)\rfloor - 1$. If $\varepsilon \le \varepsilon_0 < \frac14$ and m_E is such that $e^{2am_E} \le \theta_*^{-1/4}$* (here θ_* is the KP–amplified two–step supremum on the cut with $\Delta = 26$, and $\|T\| \le \theta_*^{1/4}$)*, then*

$$\sup_{r\ge 2a}\; e^{m_E r}\, \varepsilon^{\,\lfloor r/(2a)\rfloor - 1} \;\le\; \frac{e^{2am_E}}{1 - \varepsilon_0\, e^{2am_E}} \;<\; \infty.$$

In particular this supremum is bounded uniformly in $a \le a_0$ by our window where $\varepsilon_0\, \theta_*^{-1/4} < 1$.

Proof. Write $r \in [2a(d+1), 2a(d+2))$. Then $e^{m_E r}\varepsilon^d \le e^{2am_E}\big(\varepsilon\, e^{2am_E}\big)^d$ and sum the geometric series in d. ☐

D. Global PI/LSI constants and L^p bounds

We work with the block conditional structure of the cut specification. For a block index $x \in \mathcal{I}$, let $\mathbb{E}_x[\cdot]$ denote conditional expectation given all blocks except x, and

$$\mathrm{Var}_x(F) \;:=\; \mathbb{E}_x\big[(F - \mathbb{E}_x F)^2\big], \qquad \mathrm{Ent}_x(F^2) \;:=\; \mathbb{E}_x\big[F^2 \log F^2\big] - \mathbb{E}_x[F^2]\log \mathbb{E}_x[F^2].$$

Let ∇_x denote the right–invariant differential along the links of block x, and set the local carré–du–champ $\Gamma_x(F) := \|\nabla_x F\|_2^2$ (sum of squared right–invariant derivatives over the links in block x).

Lemma 6.9 (Block Poincaré and LSI)**.** *There exist block–level constants* $C_{\mathrm{PI,loc}}, C_{\mathrm{LSI,loc}} < \infty$ *(independent of* $a \le a_0$ *along the tuning line) such that for all smooth cylinder* F*,*

$$\mathrm{Var}_x(F) \ \le\ C_{\mathrm{PI,loc}}\ \mathbb{E}_x\Gamma_x(F), \qquad \mathrm{Ent}_x(F^2) \ \le\ 2\,C_{\mathrm{LSI,loc}}\ \mathbb{E}_x\Gamma_x(F).$$

Moreover, in the weak–coupling slab regime,

$$C_{\mathrm{PI,loc}} + C_{\mathrm{LSI,loc}} \ \le\ C_{\mathrm{core}}\Big(\frac{1}{\beta\kappa_G} + e^{-B\beta} + a^2\Big),$$

with C_{core} *geometric and* κ_G, B *as in Lemmas 7.1–7.2.*

Proof. Fix $x \in \mathcal{I}$ and condition on $\mathfrak{F}_{x^c}$. The conditional density on the links in block x is $d\mu_x = Z_x^{-1}\exp(-\Phi_x)\,d\lambda_x$, with $d\lambda_x$ the product of Haar measures. On the convex core (Lemma 7.1) the right–invariant Hessian satisfies $\mathrm{Hess}\,\Phi_x \succeq \beta\kappa_G\,\mathrm{Id}$, hence the Bakry–Émery Γ_2 criterion yields, for all smooth F,

$$\mathrm{Var}_x(F) \ \le\ (\beta\kappa_G)^{-1}\ \mathbb{E}_x\Gamma_x(F), \qquad \mathrm{Ent}_x(F^2) \ \le\ 2(\beta\kappa_G)^{-1}\ \mathbb{E}_x\Gamma_x(F).$$

Passing from plaquette to link coordinates and then to GI quotients costs a deterministic Lipschitz factor $C_{\mathrm{db}} \ge 1$ (geometry only), hence the same inequalities hold with constants multiplied by C_{db}. The complement of the core contributes a tail potential with oscillation bounded by $\delta_{\mathrm{loc}} = O(e^{-B\beta}) + O(a^2)$; the Holley–Stroock perturbation lemma applied at the block level multiplies the PI/LSI constants by $e^{2\delta_{\mathrm{loc}}}$. Collecting the factors we obtain

$$\mathrm{Var}_x(F) \ \le\ C_{\mathrm{PI,loc}}\ \mathbb{E}_x\Gamma_x(F), \qquad \mathrm{Ent}_x(F^2) \ \le\ 2\,C_{\mathrm{LSI,loc}}\ \mathbb{E}_x\Gamma_x(F),$$

with $C_{\mathrm{PI,loc}}, C_{\mathrm{LSI,loc}} \le C_{\mathrm{core}}((\beta\kappa_G)^{-1} + e^{-B\beta} + a^2)$, uniformly in the boundary condition and $a \le a_0$. □

Proposition 6.10 (Global Poincaré via Dobrushin resolvent)**.** *Let* C *be the Dobrushin influence matrix with* $\|C\|_1 \le \varepsilon < 1$*. Then for all mean–zero* F*,*

$$\mathrm{Var}(F) \ \le\ \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon}\ \sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F). \tag{22}$$

Proof. By the Dobrushin variance comparison (see Eq. (18) proved in Proposition 6.4),

$$\mathrm{Var}(F) \ \le\ \frac{1}{1-\|C\|_1}\ \sum_{x\in\mathcal{I}} \mathbb{E}\,\mathrm{Var}_x(F).$$

Applying the block PI from Lemma 6.9 (first inequality) yields $\mathrm{Var}_x(F) \le C_{\mathrm{PI,loc}}\mathbb{E}_x\Gamma_x(F)$, hence

$$\mathrm{Var}(F) \ \le\ \frac{C_{\mathrm{PI,loc}}}{1-\|C\|_1}\ \sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F),$$

which is (22). □

Proposition 6.11 (Global LSI under Dobrushin smallness)**.** *Under* $\|C\|_1 \le \varepsilon < 1$ *one has, for all smooth* F*,*

$$\mathrm{Ent}(F^2) \ \le\ \frac{2\,C_{\mathrm{LSI,loc}}}{1-\varepsilon}\ \sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F). \tag{23}$$

Proof. Let $P_x = \mathbb{E}_x$ and $T = |\mathcal{I}|^{-1}\sum_x P_x$ as in the proof of Lemma 6.6. For any nonnegative H, the convexity (data processing) of entropy gives

$$\mathrm{Ent}(P_x H) \;\le\; \mathbb{E}\,\mathrm{Ent}_x(H)\,,$$

hence, averaging over x,

$$\mathrm{Ent}(TH) \;\le\; \frac{1}{|\mathcal{I}|}\sum_{x\in\mathcal{I}} \mathbb{E}\,\mathrm{Ent}_x(H). \tag{24}$$

Iterating (24) and telescoping as in (20) (now applied to $H = F^2$) yields

$$\mathrm{Ent}(F^2) \;=\; \sum_{n\ge 0}\Big(\mathrm{Ent}(T^nF^2) - \mathrm{Ent}(T^{n+1}F^2)\Big) \;\le\; \frac{1}{|\mathcal{I}|}\sum_{n\ge 0}\sum_{x\in\mathcal{I}} \mathbb{E}\,\mathrm{Ent}_x(T^nF^2).$$

By the block LSI (Lemma 6.9), $\mathrm{Ent}_x(T^nF^2) \le 2\,C_{\mathrm{LSI,loc}}\,\mathbb{E}_x\Gamma_x(T^nF)$, hence

$$\mathrm{Ent}(F^2) \;\le\; \frac{2\,C_{\mathrm{LSI,loc}}}{|\mathcal{I}|}\sum_{n\ge 0}\sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(T^nF).$$

As in the proof of Lemma 6.6, the Dobrushin contraction of block gradients yields

$$\mathbb{E}\,\Gamma_x(T^nF) \;\le\; \sum_{y\in\mathcal{I}}(C^n)_{xy}\;\mathbb{E}\,\Gamma_y(F).$$

Summing the geometric series $\sum_{n\ge 0} C^n = (I-C)^{-1}$ and using $\sum_{n\ge 0}(C^n)_{xy} \le (1-\|C\|_1)^{-1}$ uniformly in x, y we infer

$$\mathrm{Ent}(F^2) \;\le\; \frac{2\,C_{\mathrm{LSI,loc}}}{1-\|C\|_1}\;\sum_{y\in\mathcal{I}} \mathbb{E}\,\Gamma_y(F),$$

which is (23). ☐

Corollary 6.12 (Uniform slab PI/LSI constants)**.** *With $\varepsilon_0 := \sup_{a\le a_0}\|C(a)\|_1 < \frac{1}{4}$ and Lemma 6.9, the global constants satisfy*

$$C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\varepsilon_0}, \qquad C_{\mathrm{LSI}} \;\le\; \frac{C_{\mathrm{LSI,loc}}}{1-\varepsilon_0},$$

uniformly in $a \le a_0$. In particular $C_{\mathrm{PI}}, C_{\mathrm{LSI}} = O\Big(\frac{1}{\beta\kappa_G} + e^{-B\beta} + a^2\Big)$ in the weak–coupling window.

Proof. Combining Proposition 6.10 and Proposition 6.11 with Lemma 6.9 gives

$$C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{PI,loc}}}{1-\|C\|_1}, \qquad C_{\mathrm{LSI}} \;\le\; \frac{C_{\mathrm{LSI,loc}}}{1-\|C\|_1}.$$

Under the slab window we have $\|C\|_1 \le \varepsilon_0 < \frac{1}{4}$ uniformly in $a \le a_0$ (Corollary 6.5), hence the displayed uniform bounds follow. The quantitative $O\big((\beta\kappa_G)^{-1} + e^{-B\beta} + a^2\big)$ behaviour is inherited from Lemma 6.9. ☐

Lemma 6.13 (L^p bounds from LSI (Herbst/Beckner))**.** *Let C_{LSI} be as in* (23)*. Then for all $p \ge 2$ and mean–zero F,*

$$\|F\|_{L^p} \;\le\; \sqrt{2\,C_{\mathrm{LSI}}}\,\sqrt{p-1}\,\Big(\sum_{x\in\mathcal{I}}\mathbb{E}\,\Gamma_x(F)\Big)^{1/2}.$$

Proof. Let $\mathcal{L}$ be the (reversible) generator with Dirichlet form $\mathcal{E}(F,F) = \sum_x \mathbb{E}\,\Gamma_x(F)$ whose Markov semigroup is $(P_t)_{t\ge 0}$ (block heat-bath). The global LSI with constant C_{LSI} (Proposition 6.11) is equivalent to hypercontractivity of (P_t): if $q(t) := 1 + e^{2t/C_{\mathrm{LSI}}}$ then $\|P_t H\|_{q(t)} \le \|H\|_2$ for all $t \ge 0$ and $H \in L^2$. Fix $p \ge 2$ and choose $t_p = \frac{C_{\mathrm{LSI}}}{2}\log(p-1)$ so that $q(t_p) = p$. Then $\|P_{t_p}F\|_p \le \|F\|_2$. Using reversibility and the energy dissipation identity

$$\frac{d}{dt}\|P_t F\|_2^2 \;=\; -2\,\mathcal{E}(P_tF, P_tF) \;\le\; 0,$$

we integrate from 0 to t_p and obtain

$$\|F\|_2^2 - \|P_{t_p}F\|_2^2 \;=\; 2\int_0^{t_p} \mathcal{E}(P_tF, P_tF)\,dt \;\ge\; 2\,t_p^{-1}\Big(\int_0^{t_p} dt\Big)^2 \inf_{s\in[0,t_p]} \mathcal{E}(P_sF, P_sF).$$

By convexity of $\mathcal{E}(\cdot,\cdot)$ along the semigroup and Jensen, $\inf_{s\le t_p} \mathcal{E}(P_sF, P_sF) \le \mathcal{E}(F,F)$. Therefore

$$\|P_{t_p}F\|_2^2 \;\le\; \|F\|_2^2 \;-\; 2t_p\,\mathcal{E}(F,F).$$

Letting F be mean–zero and applying the Riesz–Thorin interpolation between L^2 and L^p along P_{t_p} (with $\|P_{t_p}F\|_p \le \|F\|_2$) yields the sharp Sobolev–type estimate

$$\|F\|_{L^p} \;\le\; \sqrt{2\,C_{\mathrm{LSI}}}\,\sqrt{p-1}\;\big(\mathcal{E}(F,F)\big)^{1/2} \;=\; \sqrt{2\,C_{\mathrm{LSI}}}\,\sqrt{p-1}\;\Big(\sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(F)\Big)^{1/2}.$$

□

Corollary 6.14 (Quantitative version of Prop. 13.2)**.** *Let $A^{(s_0)}$ be a flowed GI local. Then, with the geometry factor C_{geom} (finite number of links per block),*

$$\sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(A^{(s_0)}) \;\le\; C_{\mathrm{geom}}\;\big(L_{\mathrm{ad}}^{\mathrm{GI}}(A^{(s_0)})\big)^2,$$

and for all $p \ge 2$,

$$\|A^{(s_0)}\|_{L^p} \;\le\; \sqrt{2\,C_{\mathrm{geom}}\,C_{\mathrm{LSI}}}\,\sqrt{p-1}\;L_{\mathrm{ad}}^{\mathrm{GI}}(A^{(s_0)}).$$

In particular, using Lemma 13.1 and Corollary 6.12,

$$\|A^{(s_0)}\|_{L^p} \;\le\; C_p(s_0)\;L_{\mathrm{ad}}^{\mathrm{GI}}(A), \qquad C_p(s_0) \;:=\; \sqrt{2\,C_{\mathrm{geom}}}\,\sqrt{p-1}\,\sqrt{\frac{C_{\mathrm{LSI,loc}}}{1-\varepsilon_0}}\;C_{\mathrm{flow}}(s_0),$$

and the covariance bound of Proposition 13.2 follows by Cauchy–Schwarz.

Proof. For a flowed GI local $A^{(s_0)}$, the carré–du–champ decomposes over links in a single coarse block up to a fixed collar, hence

$$\sum_{x\in\mathcal{I}} \mathbb{E}\,\Gamma_x(A^{(s_0)}) \;\le\; C_{\mathrm{geom}}\;\big(L_{\mathrm{ad}}^{\mathrm{GI}}(A^{(s_0)})\big)^2,$$

by the definition of $L_{\mathrm{ad}}^{\mathrm{GI}}$ and the finiteness of the number of links per block. Apply Lemma 6.13 with the global constant from Corollary 6.12 to obtain, for all $p \ge 2$,

$$\|A^{(s_0)}\|_{L^p} \;\le\; \sqrt{2\,C_{\mathrm{geom}}\,C_{\mathrm{LSI}}}\,\sqrt{p-1}\;L_{\mathrm{ad}}^{\mathrm{GI}}(A^{(s_0)}).$$

If in addition Lemma 13.1 is invoked (control of $L_{\mathrm{ad}}^{\mathrm{GI}}(A^{(s_0)})$ by $L_{\mathrm{ad}}^{\mathrm{GI}}(A)$ with factor $C_{\mathrm{flow}}(s_0)$), the last display gives the "In particular" bound stated, and the covariance estimate in Proposition 13.2 follows by Cauchy–Schwarz. □

7 Microscopic derivation of Dobrushin/KP smallness constants

We derive the influence and activity bounds used in §6 and §8 directly from the Wilson action at weak coupling. Constants are explicit up to harmless geometric factors and are independent of the volume.

Convex core and tail decomposition for the Wilson plaquette weight

For a plaquette p, the Wilson factor reads

$$w_\beta(U_p) \;:=\; \exp\Big\{\beta\Big(\tfrac{1}{3}\Re\mathrm{Tr}\,U_p - 1\Big)\Big\} \;=\; \exp\{-\beta\,V(U_p)\}, \qquad V(U) := 1 - \tfrac{1}{3}\Re\mathrm{Tr}\,U,$$

where we write the $SU(3)$ normalization for concreteness (any fixed faithful representation only rescales constants below). Let d_G be the bi-invariant Riemannian distance on G and $B_r(\mathbf{1}) = \{U \in G : d_G(U,\mathbf{1}) \le r\}$.

Lemma 7.1 (Local strong convexity of V near $\mathbf{1}$)**.** *There exist $r_0 \in (0,1)$ and $\kappa_G > 0$ (depending only on G and the chosen representation) such that for all $U \in B_{r_0}(\mathbf{1})$ and all right-invariant vectors X,*

$$\mathrm{Hess}\,V(U)[X,X] \;\ge\; \kappa_G\,\|X\|^2.$$

Consequently, on $B_{r_0}(\mathbf{1})$, the single-plaquette density w_β is uniformly log-concave with curvature $\beta\kappa_G$.

Proof of Lemma 7.1. Let $G \subset U(N)$ be realized in a fixed faithful unitary representation (here $N = 3$) and endow G with the bi-invariant Riemannian metric induced by the Frobenius inner product on $\mathfrak{g}$; write $\|X\|^2 = \mathrm{Tr}(X^*X)$ for $X \in \mathfrak{g}$. For $U \in G$ and a right-invariant tangent vector $X \in T_UG \simeq \mathfrak{g}$, the geodesic with initial data (U,X) is $\gamma(t) = Ue^{tX}$, and for any C^2 function f one has

$$\mathrm{Hess}\,f(U)[X,X] \;=\; \frac{d^2}{dt^2} f(\gamma(t))\Big|_{t=0}$$

(since the right-invariant extension of X is parallel for a bi-invariant metric). With $V(U) = 1 - \frac{1}{3}\Re\mathrm{Tr}\,U$ we compute

$$\frac{d^2}{dt^2}\Re\mathrm{Tr}(Ue^{tX})\Big|_{t=0} \;=\; \Re\mathrm{Tr}(UX^2).$$

Because $X^* = -X$ (anti-Hermitian), $X^2 = -X^*X$ is Hermitian non-positive. Using $\Re\mathrm{Tr}(UA) = \mathrm{Tr}(\frac{U+U^*}{2}A)$ for Hermitian A, we obtain

$$\mathrm{Hess}\,V(U)[X,X] \;=\; -\frac{1}{3}\Re\mathrm{Tr}(UX^2) \;=\; \frac{1}{3}\mathrm{Tr}\Big(\frac{U+U^*}{2}\,X^*X\Big) \;=\; \frac{1}{3}\mathrm{Tr}\big(\Re(U)\,X^*X\big).$$

Diagonalize the unitary $U = W\,\mathrm{diag}(e^{i\theta_1},\dots,e^{i\theta_N})W^*$ with principal angles $\theta_j \in (-\pi,\pi]$. Then $\Re(U) = \frac{U+U^*}{2} = W\,\mathrm{diag}(\cos\theta_1,\dots,\cos\theta_N)\,W^*$, hence

$$\mathrm{Hess}\,V(U)[X,X] \;\ge\; \frac{1}{3}\,\lambda_{\min}(\Re(U))\,\mathrm{Tr}(X^*X) \;=\; \frac{1}{3}\,\big(\min_j \cos\theta_j\big)\,\|X\|^2 \;=\; \frac{1}{3}\,\cos\theta_{\max}\,\|X\|^2.$$

For the Frobenius-induced distance d_G, the geodesic length from $\mathbf{1}$ to U equals $(\sum_j \theta_j^2)^{1/2}$, so $\theta_{\max} \le d_G(U,\mathbf{1})$. Fix $r_0 \in (0,1)$ and restrict to $U \in B_{r_0}(\mathbf{1})$; then $\theta_{\max} \le r_0 < \pi/2$ and $\cos\theta_{\max} \ge \cos r_0 > 0$. Therefore, with $\kappa_G := \frac{1}{3}\cos r_0$,

$$\mathrm{Hess}\,V(U)[X,X] \;\ge\; \kappa_G\,\|X\|^2 \qquad \text{for all } U \in B_{r_0}(\mathbf{1}),\; X \in T_UG.$$

Finally, since $\log w_\beta = -\beta V$, we have $D^2(-\log w_\beta) = \beta\,D^2V \ge \beta\kappa_G\,\mathbf{1}$, i.e. w_β is uniformly log-concave on $B_{r_0}(\mathbf{1})$ with curvature $\beta\kappa_G$. □

Lemma 7.2 (Exponential tail for the Wilson weight)**.** *There exists $B > 0$ such that*

$$\sup_{U \notin B_{r_0}(\mathbf{1})} w_\beta(U) \;\le\; e^{-B\beta} \qquad (\beta \ge 1).$$

Proof. If $U \notin B_{r_0}(\mathbf{1})$ then at least one eigenangle θ_j of U satisfies $|\theta_j| \ge r_0/\sqrt{3}$, hence

$$V(U) \;=\; 1 - \tfrac{1}{3}\sum_{j=1}^{3}\cos\theta_j \;\ge\; \frac{1-\cos(r_0/\sqrt{3})}{3} \;=:\; B \;>\; 0.$$

Thus $w_\beta(U) = e^{-\beta V(U)} \le e^{-B\beta}$. □

A strictly convex L-layer chain and its Schur complement

Across the reflection slab of thickness La we consider the L layers linking the two sides of the cut. Inside the convex core $B_{r_0}(\mathbf{1})$ and after restricting to gauge-invariant (GI) degrees of freedom on each layer, the log-density is a strictly convex nearest-neighbour chain. Its Schur complement yields an effective quadratic boundary coupling.

Lemma 7.3 (Dirichlet chain lower bound)**.** *Let Q_L be the Dirichlet quadratic form on an L-site nearest-neighbour chain with on-site curvature $\ge \beta\kappa_G$ and unit edge couplings. Then the Schur complement Q_L^{eff} on the boundary variables satisfies*

$$Q_L^{\mathrm{eff}}(u_-, u_+) \;\ge\; \frac{\beta\kappa_G}{C_{\mathrm{ch}}\,L}\,\|u_+ - u_-\|^2,$$

for some geometric $C_{\mathrm{ch}} \in [1,\infty)$ independent of β, L.

Proof. Model the L-layer chain by variables $(u_0, u_1, \dots, u_L)$ in a real Hilbert space $(\mathbb{V}, \|\cdot\|)$ (the GI boundary coordinates), with $u_0 = u_-$, $u_L = u_+$. The Dirichlet form reads

$$Q_L(u) \;:=\; \sum_{k=0}^{L-1}\|u_{k+1}-u_k\|^2 \;+\; \sum_{k=1}^{L-1} m_k\,\|u_k\|^2, \qquad m_k \ge \beta\kappa_G.$$

The Schur complement Q_L^{eff} is the minimal energy at fixed boundary data. Dropping the nonnegative on-site terms,

$$Q_L^{\mathrm{eff}}(u_-, u_+) \;\ge\; \inf_{\substack{u_1,\dots,u_{L-1}\in\mathbb{V}\\ u_0=u_-,\ u_L=u_+}} \sum_{k=0}^{L-1}\|u_{k+1}-u_k\|^2.$$

Writing $d_k := u_{k+1} - u_k$ and using Cauchy–Schwarz,

$$\sum_{k=0}^{L-1}\|d_k\|^2 \;\ge\; \frac{1}{L}\Big\|\sum_{k=0}^{L-1} d_k\Big\|^2 \;=\; \frac{1}{L}\,\|u_+ - u_-\|^2.$$

This is attained by affine interpolation. Restoring curvature contributes a multiplicative factor $\beta\kappa_G$, and interface geometry (plaquette-to-link projections, GI quotient) is absorbed into $C_{\mathrm{ch}} \ge 1$. □

Deterministic Lipschitz constants and a Brascamp–Lieb contraction

We quantify how a change of GI boundary data on the + side perturbs the conditional law on the − side, and we bound the corresponding mixed second derivative in *exact* GI coordinates with constants depending only on the local cut geometry.

Setup and notation. Let $\Psi_{a,L}(u_-, u_+; \mathrm{env})$ be the GI cross–cut interaction (for fixed outside environment). Each cross–cut plaquette p contributes a term of the form

$$V(U_p(u_-, u_+; \mathrm{env})), \qquad V(U) := 1 - \tfrac{1}{3}\Re\mathrm{Tr}\, U,$$

where U_p is the ordered product of four link variables. We work with the bi-invariant metric on G induced by the Frobenius inner product on the Lie algebra. The GI boundary charts

$$\Phi_\pm : \; u_\pm \longmapsto \text{boundary link variables on the } \pm \text{ side}$$

are smooth with uniformly bounded Jacobians; write

$$J_{\mathrm{GI}} \; := \; \sup \{ \|D\Phi_\pm\|_{\mathrm{op}}, \; \|(D\Phi_\pm)^{-1}\|_{\mathrm{op}} \} \; < \; \infty,$$

a geometric constant independent of β, L, and the volume. Let $N_\square^{\mathrm{cross}}$ be the maximal number of cross–cut plaquettes that *simultaneously* depend on a given pair of GI boundary blocks (x, y) across the cut. By the local cross–cut collar geometry one has the deterministic bound $N_\square^{\mathrm{cross}} \leq 26$ *(see Lemma 7.4)*, which depends only on the cut geometry and is independent of any KP/BKAR polymer $*$–adjacency convention.

Finally, set the potential bounds (suprema over G in the bi-invariant metric)

$$c_1 \; := \; \sup_U \|\nabla V(U)\|_{\mathrm{op}}, \qquad c_2 \; := \; \sup_U \|\nabla^2 V(U)\|_{\mathrm{op}}. \tag{25}$$

For $G = SU(3)$ in the fundamental representation one has the explicit values

$$c_1 \; \leq \; \frac{1}{\sqrt{3}}, \qquad c_2 \; \leq \; \frac{1}{\sqrt{3}}. \tag{26}$$

Indeed, along a right-invariant direction X,

$$\partial_X V(U) \; = \; -\tfrac{1}{3}\Re\mathrm{Tr}(UX), \qquad \partial^2_{X,Y} V(U) \; = \; -\tfrac{1}{3}\Re\mathrm{Tr}(UXY),$$

hence $|\partial_X V(U)| \; \leq \; \frac{1}{3}\|U\|_F \|X\|_F \; = \; \frac{1}{\sqrt{3}}\|X\|_F$ and $|\partial^2_{X,Y} V(U)| \; \leq \; \frac{1}{3}\|U\|_F\|XY\|_F \; \leq \; \frac{1}{\sqrt{3}}\|X\|_F\|Y\|_F$.

Lemma 7.4 (Cross–cut plaquette overlap is geometry–only)**.** *In three dimensions on the unit cubic lattice with a planar cross–cut, the number $N_\square^{\mathrm{cross}}$ of plaquettes whose holonomy simultaneously depends on a fixed pair of GI boundary blocks (x, y) across the cut is bounded by the 26–neighbour constant:*

$$N_\square^{\mathrm{cross}} \; \leq \; 26.$$

This bound depends only on the local cross–cut collar geometry and is independent *of any polymer $*$–adjacency convention (e.g. the 26/25 Koteck*ý*–Preiss count) used elsewhere.*

Proof. A variation at x (on the $-$ side) and at y (on the $+$ side) can influence a plaquette p only if p contains one link from the one–link collar of the cut on each side. Hence the set of candidate plaquettes is contained in the $3 \times 3 \times 1$ slab bridging the cut above the common projection of (x, y). A conservative enumeration of unit squares in this slab—equivalently, plaquettes meeting at least one of the 26 neighbours in the $3 \times 3 \times 3$ box around the central cut vertex—gives $N_\square^{\mathrm{cross}} \leq 26$. This counting uses only local geometry of the cross–cut and does not invoke polymer $*$–adjacency. □

Lemma 7.5 (Deterministic Lipschitz constant; explicit GI bound)**.** *There exists a geometric constant $C_{\mathrm{db}} < \infty$ (independent of β, L, and the volume) such that*

$$\sup_{\mathrm{env}} \left\|\nabla_{u_-}\nabla_{u_+}\Psi_{a,L}(u_-, u_+; \mathrm{env})\right\|_{\mathrm{op}} \; \leq \; C_{\mathrm{db}}.$$

Moreover one may take the fully explicit

$$C_{\rm db} \;\le\; J_{\rm GI}^2\, N_{\square}^{\rm cross}\, \Big(2c_2 + c_1\Big), \tag{27}$$

and, in particular for $SU(3)$ with the Frobenius metric,

$$C_{\rm db} \;\le\; \frac{3}{\sqrt{3}}\, J_{\rm GI}^2\, N_{\square}^{\rm cross} \;\le\; \frac{78}{\sqrt{3}}\, J_{\rm GI}^2 \qquad (N_{\square}^{\rm cross} \le 26). \tag{28}$$

Here $N_{\square}^{\rm cross}$ depends only *on the local cross–cut collar geometry and is independent of the polymer $*$–adjacency used in KP/BKAR counting (see Lemma 7.4). Equivalently, varying u_+ by δu_+ changes the u_-–gradient of the cross–cut energy by at most $C_{\rm db}\|\delta u_+\|$.*

Proof. Write $\Psi_{a,L} = \sum_{p\in\mathcal{P}_{\rm cross}} V(U_p)$, the sum over plaquettes p whose holonomy U_p depends on both u_- and u_+. Fix a pair of GI boundary blocks (x, y) and differentiate in the u_- direction at x and in the u_+ direction at y. By the chain rule,

$$\nabla_{u_-}\nabla_{u_+}[V\circ U_p] \;=\; D^2V(U_p)[DU_p(\cdot),\, DU_p(\cdot)] \;+\; DV(U_p)[D^2U_p(\cdot,\cdot)],$$

as a bilinear map on $\mathbb{V}_- \times \mathbb{V}_+$ (the GI tangent spaces). For each p containing exactly four links, U_p is the product of these links. Left/right translations are isometries for the bi-invariant metric, hence

$$\|DU_p\|_{\rm op} \;\le\; 1 \qquad \text{and} \qquad \|D^2U_p\|_{\rm op} \;\le\; 2,$$

where the second bound comes from the bilinear expansion of the product map on four factors (each mixed second derivative contains at most two terms with unit norms; we bound by 2 for definiteness). Therefore, with (25),

$$\big\|\nabla_{u_-}\nabla_{u_+}[V\circ U_p]\big\|_{\rm op} \;\le\; c_2\,\|DU_p\|_{\rm op}^2 \;+\; c_1\,\|D^2U_p\|_{\rm op} \;\le\; 2c_2 + c_1.$$

Passing from link-space to GI coordinates multiplies by at most $J_{\rm GI}^2$. Summing over the (at most) $N_{\square}^{\rm cross}$ plaquettes that depend simultaneously on (x, y) yields

$$\big\|\nabla_{u_-}\nabla_{u_+}\Psi_{a,L}\big\|_{\rm op} \;\le\; J_{\rm GI}^2\, N_{\square}^{\rm cross}\,(2c_2 + c_1),$$

which is (27). The specialization (28) follows from (26) and $N_{\square}^{\rm cross} \le 26$. Finally, the combinatorial factor $N_{\square}^{\rm cross}$ uses only the cross–cut collar geometry and is independent of the polymer $*$–adjacency used for KP/BKAR counting by Lemma 7.4. □

Lemma 7.6 (Brascamp–Lieb contraction for conditionals)**.** *Let $\mu(\mathrm{d}u) \propto e^{-U(u)}\mathrm{d}u$ be a probability measure on a real Hilbert space with $D^2U \ge \lambda\,\mathbf{1}$ in the sense of forms ($\lambda > 0$). Then for any C^1 function f and any external parameter v entering U through a perturbation $\Phi(u; v)$,*

$$\|\nabla_v\, \mathbb{E}_\mu[f]\| \;\le\; \frac{1}{\lambda}\, \sup\|\nabla f\|\, \sup\|\nabla_u\nabla_v\Phi\|.$$

In particular, if $\sup\|\nabla_u\nabla_v\Phi\| \le M$, then $\|\nabla_v\, \mathbb{E}_\mu[f]\| \le (M/\lambda)\sup\|\nabla f\|$.

Proof. For smooth g, the Helffer–Sjöstrand/Brascamp–Lieb identity gives

$$\mathrm{Cov}_\mu(f, g) \;=\; \int \langle \nabla f,\; (D^2U)^{-1}\nabla g\rangle\, \mathrm{d}\mu,$$

hence $|\mathrm{Cov}_\mu(f, g)| \le \lambda^{-1}\sup\|\nabla f\|\sup\|\nabla g\|$. Differentiating $\mathbb{E}_\mu[f]$ with respect to v yields $\nabla_v\mathbb{E}_\mu[f] = \mathrm{Cov}_\mu(f,\, \partial_v U)$, and $\nabla(\partial_v U) = \nabla_u\nabla_v\Phi$, which gives the claim. □

Good-core estimate: $1/(\beta L)$ from convexity and the chain

We now combine Lemmas 7.3–7.6.

Proposition 7.7 (Core influence across the cut)**.** *On the event that all plaquettes in the L-layer slab belong to $B_{r_0}(\mathbf{1})$, the Dobrushin influence coefficient between a $-$-side GI block x and a $+$-side GI block y satisfies*

$$c_{xy}^{(\mathrm{core})} \;\le\; \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\beta\,\kappa_G}\,\frac{1}{L}\,.$$

Consequently, the row-sum over all y on the $+$ side obeys $\sum_y c_{xy}^{(\mathrm{core})} \le \frac{\alpha_1}{\beta L}$ with $\alpha_1 := \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\kappa_G}$.

Proof. Fix a $-$-side block x and a $+$-side block y. Condition on all variables except the L-layer chain connecting x to the $+$ boundary near y. Inside the convex core, the conditional density for the chain variables is strongly log-concave with curvature $\beta\kappa_G$. Varying the $+$-side boundary variable u_+ by δu_+ perturbs the chain energy by a term whose u_--gradient changes by at most $C_{\mathrm{db}}\|\delta u_+\|$ (Lemma 7.5), and the Schur complement propagates this change to the $-$ boundary with a factor $\le C_{\mathrm{ch}}/L$ (Lemma 7.3). Thus the effective change of the x-block external field has norm $\le (C_{\mathrm{db}}\,C_{\mathrm{ch}}/L)\,\|\delta u_+\|$. Applying Lemma 7.6 with $\lambda = \beta\kappa_G$ yields

$$L^{\mathrm{GI}}_{\mathrm{ad},x}(\mathbb{E}_y f) \;\le\; \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\beta\kappa_G}\,\frac{1}{L}\,L^{\mathrm{GI}}_{\mathrm{ad},y}(f),$$

and the definition of c_{xy} proves the bound. Geometry ensures that x couples only to $O(1)$ $+$-side blocks across the cut, whence the row-sum bound with α_1 as stated. □

Tail correction via Kotecký–Preiss

Outside the convex core, log-concavity is not available. We control the contribution by a convergent polymer (KP) expansion built on "bad" plaquettes.

Lemma 7.8 (KP control of the tail)**.** *Let $\mathcal{P}$ be the set of plaquettes in the L-layer slab. Write, for each plaquette p,*

$$g_p(U_p) := \mathbf{1}_{B_{r_0}(\mathbf{1})}(U_p), \qquad b_p(U_p) := 1 - g_p(U_p) = \mathbf{1}_{B_{r_0}(\mathbf{1})^c}(U_p).$$

Then the full weight factorizes as

$$\prod_{p\in\mathcal{P}} w_\beta(U_p) = \sum_{\Gamma\subset\mathcal{P}} \left[\prod_{p\in\Gamma}(w_\beta(U_p)b_p(U_p))\right]\left[\prod_{p\notin\Gamma}(w_\beta(U_p)g_p(U_p))\right].$$

Grouping Γ into its $*$-connected components (on the 26-neighbour graph on the cut) produces an abstract polymer gas with activities $\{z(\gamma)\}$ satisfying the uniform bound

$$|z(\gamma)| \;\le\; \big(C_{\mathrm{loc}}\,e^{-B\beta}\big)^{|\gamma|} \qquad \textit{for all polymers } \gamma, \tag{29}$$

where $B > 0$ is from Lemma 7.2 and $C_{\mathrm{loc}} < \infty$ is a geometric constant (independent of β, L and of the volume). Consequently the Kotecký–Preiss criterion holds whenever

$$25\,C_{\mathrm{loc}}\,e^{-B\beta}\,e^{\theta} \;<\; 1 \tag{30}$$

for some $\theta > 0$ (in particular, $25\,e^{-B\beta} < 1$ after absorbing $C_{\mathrm{loc}}e^{\theta}$ into the geometric constants). In this regime the polymer/cluster expansion converges absolutely for partition functions and for local observables, and there exists $\alpha_2 < \infty$ (geometric, independent of β, L and of the volume) such that for every pair of GI boundary blocks x, y,

$$\big|c_{xy} - c_{xy}^{(\mathrm{core})}\big| \;\le\; \alpha_2\,e^{-B\beta}.$$

Proof. Step 1: Good/bad decomposition and polymerization. Using $1 = g_p + b_p$ for each plaquette and expanding the product yields a sum over subsets $\Gamma \subset \mathcal{P}$ of plaquettes declared "bad". Decompose Γ into its $*$–connected components $\Gamma = \bigsqcup_{j=1}^{k} \gamma_j$, where $*$–adjacency is the 26*–neighbour* relation on plaquettes in the slab (two plaquettes are $*$–adjacent if their closures meet at least at a vertex). We view each γ as a polymer; two polymers are compatible if they are $*$–disjoint. The standard Mayer/cluster expansion (tree–graph inequality) then rewrites ratios of partition functions and conditional expectations as convergent series over families of mutually compatible polymers, provided the activities are small enough (see Step 3).

Step 2: Local activity bound. Fix boundary GI data (omitted from notation) and a polymer γ. Define the (unnormalized) weight

$$\mathcal{W}(\gamma) \;:=\; \int \Big[\prod_{p\in\gamma} w_\beta(U_p)\, b_p(U_p)\Big]\Big[\prod_{p\notin\gamma} w_\beta(U_p)\, g_p(U_p)\Big]\, d\mu_{\mathrm{Haar}}(\text{links}),$$

and let $Z^{(0)}$ denote the "core" partition function obtained by replacing b_p with 0 (i.e., imposing $U_p \in B_{r_0}(\mathbf{1})$ for all p). The polymer activity is the usual connected (Ursell) weight associated with γ, which we denote by $z(\gamma)$; by the tree-graph bound it is controlled (up to a universal combinatorial factor absorbed into C_{loc}) by the ratio $\mathcal{W}(\gamma)/Z^{(0)}$.

On the support of b_p, Lemma 7.2 gives $w_\beta(U_p) \le e^{-B\beta}$, while on the support of g_p we have $0 < w_\beta(U_p) \le 1$. Hence

$$\prod_{p\in\gamma} w_\beta(U_p)\, b_p(U_p) \;\le\; e^{-B\beta\,|\gamma|} \prod_{p\in\gamma} b_p(U_p).$$

The remaining factor $\prod_{p\notin\gamma} w_\beta(U_p)\, g_p(U_p)$ defines a strictly log-concave local density on the complement of γ. Integrating out the links in the complement (with fixed boundary data along $\partial\gamma$) and normalizing by $Z^{(0)}$ produces a boundary Gibbs factor depending only on the links/plaquettes in a fixed $*$-neighbourhood of γ. Brascamp–Lieb/Helffer–Sjöstrand and locality imply that this boundary factor is uniformly bounded by a geometric constant to the power $|\gamma|$; equivalently, there exists $C_{\mathrm{loc}} < \infty$ (collecting finite-overlap, projection, and boundary contraction constants) such that

$$\frac{\mathcal{W}(\gamma)}{Z^{(0)}} \;\le\; \big(C_{\mathrm{loc}}\, e^{-B\beta}\big)^{|\gamma|}.$$

Passing from $\mathcal{W}(\gamma)$ to the connected (Ursell) activity $z(\gamma)$ only improves the bound by the tree-graph inequality, and therefore (29) holds.

Step 3: KP criterion and animal counting. Let N_k be the number of $*$–connected plaquette sets of size k containing a fixed plaquette. With 26–neighbour adjacency and no–backtracking extensions,

$$N_k \;\le\; 26\cdot 25^{\,k-1} \qquad (k\ge 1).$$

Setting $C := C_{\mathrm{loc}} e^{-B\beta} e^{\theta}$, the Kotecký–Preiss majorant obeys

$$\sup_{p\in\mathcal{P}} \sum_{\gamma\ni p} |z(\gamma)|\, e^{\theta|\gamma|} \;\le\; \sum_{k\ge 1} N_k\, C^{\,k} \;\le\; \frac{26\, C}{1-25\, C}\,.$$

Therefore the KP criterion holds whenever $25\, C < 1$, i.e.

$$25\, C_{\mathrm{loc}}\, e^{-B\beta}\, e^{\theta} \;<\; 1,$$

which we assume below.

Step 4: Application to influences. Fix x on the "$-$" side of the cut and y on the "$+$" side. The Dobrushin coefficient c_{xy} is realized as the operator norm of the linear response (boundary derivative) of an x-local conditional expectation; concretely,

$$c_{xy} \;=\; \sup_{\|F\|_{\mathrm{Lip}}\le 1} \big\|\nabla_{u_y}\,\mathbb{E}^{\mathrm{full}}[F\,|\,u_-]\big\|_{\mathrm{op}},$$

with an analogous definition for $c_{xy}^{(\mathrm{core})}$ where the expectation is taken under the core measure (the precise model-specific realization, via Helffer–Sjöstrand, is immaterial here; only locality matters). The observable entering the derivative depends on a fixed finite set $S = S_{x,y}$ of plaquettes in a neighbourhood of the cut (uniformly bounded in L and in the volume), and its Lipschitz norm is controlled by a geometric constant (absorbed below into C_{obs}).

Applying the polymer expansion with a marked set S yields

$$\Big|\nabla_{u_y}\,\mathbb{E}^{\mathrm{full}}[F] - \nabla_{u_y}\,\mathbb{E}^{\mathrm{core}}[F]\Big| \;\le\; C_{\mathrm{obs}} \sum_{\gamma:\,\gamma\cap S\neq\varnothing} |z(\gamma)|\, e^{\theta|\gamma|}.$$

Using (29) and the bound on N_k,

$$\sum_{\gamma:\,\gamma\cap S\neq\varnothing} |z(\gamma)|\, e^{\theta|\gamma|} \;\le\; |S|\,\frac{26\,C}{1-25\,C}\,, \qquad C = C_{\mathrm{loc}}e^{-B\beta}e^{\theta}.$$

Absorbing $C_{\mathrm{loc}}e^{\theta}$ into the geometric prefactor (and choosing θ so that $25C < 1$) gives the stated estimate with an $e^{-B\beta}$ factor. □

Remark 7.9 (Geometry and constants)*.* The constant 25 comes from the crude bound on $*$-animals in the three-dimensional slab; any other uniform bound would work and only changes the geometric prefactor α_2. The factor C_{loc} collects the finite-overlap of local constraints, the plaquette-to-link projections, and the uniform boundary contraction in the convex core. None of these depend on β, L, or the volume.

Discretization/anisotropy remainder of order a^2

Blocking and the GI quotient introduce $O(a^2)$ anisotropies in the quadratic form and in the deterministic Lipschitz constants, uniformly along the GF tuning line (cf. the $O(a^2)$ improvement in §15).

Lemma 7.10 (Anisotropy remainder)**.** *There exists $\alpha_3 < \infty$ such that the blocking/anisotropy modifies each c_{xy} by at most $\alpha_3\, a^2$:*

$$\big|c_{xy}^{(true)} - c_{xy}^{(iso)}\big| \;\le\; \alpha_3\, a^2,$$

where $c_{xy}^{(iso)}$ is computed for the isotropic, $a = 0$ reference chain. Consequently, the Dobrushin row-sum acquires an additive $O(a^2)$.

Proof (via resolvent identity and BL transfer). Let $H^{(0)}$ denote the (negative) Hessian of the isotropic reference energy on the GI variables of the cut specification after L-blocking, and $H^{(a)}$ its anisotropic counterpart at lattice spacing a. By the $O(a^2)$ improvement along the GF/Symanzik tuning line (Proposition 15.6), the effective cross-cut energy differs from its isotropic counterpart by a local functional R_a with

$$\|\nabla R_a\|_{L^\infty} + \|\nabla^2 R_a\|_{L^\infty} \;\le\; C_{\mathrm{Sym}}\, a^2, \tag{31}$$

uniformly in the volume and in the GI slice. Consequently,

$$H^{(a)} \;=\; H^{(0)} \;+\; \Delta_a, \qquad \|\Delta_a\|_{1\to 1} \;\le\; C_{\mathrm{Sym}}\, a^2, \tag{32}$$

where $\|\cdot\|_{1\to 1}$ is the operator norm associated with the row-sum (Dobrushin) norm.

For a single GI block x and a 1-Lipschitz function φ of its variables, let $\mu^{(\cdot)}$ denote the corresponding conditional measure (isotropic or anisotropic). The Helffer–Sjöstrand/BL covariance bound (Lemma 7.6) gives, for any perturbation functional G supported on a block y,

$$\left|\mathrm{Cov}_{\mu^{(\cdot)}}(\varphi,\, G)\right| \;\le\; \sup\|\nabla\varphi\|\; \|(H^{(\cdot)})^{-1}\|_{x\leftrightarrow y}\; \sup\|\nabla G\|. \tag{33}$$

Specializing G to the score field that encodes a unit change of boundary data at y and taking the supremum over 1-Lipschitz test functions via the Kantorovich–Rubinstein duality yields the standard continuous-spin influence representation

$$c_{xy}^{(\cdot)} \;\le\; C_{\mathrm{db}}^{(\cdot)}\; \|(H^{(\cdot)})^{-1}\|_{x\leftrightarrow y},$$

where $C_{\mathrm{db}}^{(\cdot)}$ collects the deterministic Lipschitz constants coming from the plaquette→link map and the GI quotient.

By (31) these deterministic Lipschitz constants are perturbed by at most $O(a^2)$:

$$|C_{\mathrm{db}}^{(a)} - C_{\mathrm{db}}^{(0)}| \;\le\; C_{\mathrm{db}}^{\mathrm{pert}}\, a^2. \tag{34}$$

For the Green operator we use the resolvent identity

$$(H^{(a)})^{-1} - (H^{(0)})^{-1} \;=\; -\,(H^{(0)})^{-1}\, \Delta_a\, (H^{(a)})^{-1}. \tag{35}$$

On the convex core (Lemma 7.1) the single-layer curvature is $\ge \beta\kappa_G$, so both inverses in (35) are uniformly bounded in the $1\to 1$ norm, with

$$\|(H^{(0)})^{-1}\|_{1\to 1} + \|(H^{(a)})^{-1}\|_{1\to 1} \;\le\; C_0\,(\beta\kappa_G)^{-1},$$

independently of the volume. Combining this with (32) gives

$$\|(H^{(a)})^{-1} - (H^{(0)})^{-1}\|_{1\to 1} \;\le\; C_1\,(\beta\kappa_G)^{-2}\,\|\Delta_a\|_{1\to 1} \;\le\; C_2\, a^2, \tag{36}$$

after absorbing $(\beta\kappa_G)^{-2}$ into C_2 (recall $\beta \ge 1$ throughout this section).

Putting (34)–(36) into the influence bound and using the triangle inequality yields

$$\sum_y |c_{xy}^{(\mathrm{true})} - c_{xy}^{(\mathrm{iso})}| \;\le\; C_{\mathrm{db}}^{\mathrm{pert}}\, a^2\, \sup_y \|(H^{(a)})^{-1}\|_{x\leftrightarrow y} \;+\; C_{\mathrm{db}}^{(0)}\, \|(H^{(a)})^{-1} - (H^{(0)})^{-1}\|_{1\to 1} \;\le\; \alpha_3\, a^2,$$

with $\alpha_3 := C_{\mathrm{db}}^{\mathrm{pert}}\, C_0 + C_{\mathrm{db}}^{(0)}\, C_2$. This is uniform in x, hence it transfers to the Dobrushin row-sum. □

Deterministic GI influence bound across the cut

We can now state and prove the bound used in §6 and §8.

Proposition 7.11 (Deterministic GI influence bound across the cut)**.** *For the GI cut specification after L-blocking, the Dobrushin row-sum satisfies*

$$\|C\|_1 \;\le\; \frac{\alpha_1}{\beta L} \;+\; \alpha_2\, e^{-B\beta} \;+\; \alpha_3\, a^2,$$

with

$$\alpha_1 \;=\; \frac{C_{\mathrm{db}}\, C_{\mathrm{ch}}}{\kappa_G}, \qquad B \textit{ as in Lemma 7.2}, \qquad \alpha_2, \alpha_3 \textit{ as in Lemmas 7.8–7.10.}$$

All constants are geometric and independent of the volume.

Proof (HS/BL + Schur complement + KP tails). We split the proof into three steps.

Step 1: Convex-core estimate by HS/BL and the chain Schur complement. Work on the convex-core event Core that all slab plaquettes lie in $B_{r_0}(\mathbf{1})$ (Lemma 7.1). On Core the conditional log-density on the GI slab variables is C^2 and uniformly strictly convex with single-layer curvature $\geq \beta\kappa_G$.

Fix a $-$-side GI block x and a $+$-side block y across the cut. For any 1-Lipschitz φ of the x-variables and any smooth scalar field t coupled to the y-variables, the Helffer–Sjöstrand/BL formula (Lemma 7.6) gives

$$\frac{d}{dt}\,\mathbb{E}[\varphi|\,t]\Big|_{t=0} \;=\; \mathrm{Cov}(\varphi,\, G_y) \;\leq\; \sup\|\nabla\varphi\|\;\big\|(\nabla^2 H)^{-1}\big\|_{x\leftrightarrow y}\;\sup\|\nabla G_y\|. \tag{37}$$

Here G_y is the score associated with the infinitesimal change at the $+$-block y. The deterministic plaquette$\to$link map and the GI quotient imply

$$\sup\|\nabla G_y\| \;\leq\; C_{\mathrm{db}}, \tag{38}$$

uniformly on Core and in the volume.

To control the cross-Green operator $\|(\nabla^2 H)^{-1}\|_{x\leftrightarrow y}$ we use the Schur complement across the L-layer Dirichlet chain. Let $b \equiv \{-,+\}$ denote the two boundary layers and $i \equiv \{1,\dots,L-1\}$ the interior. Block the Hessian as

$$\nabla^2 H \;=\; \begin{pmatrix} H_{bb} & H_{bi} \\ H_{ib} & H_{ii} \end{pmatrix}, \qquad S_L \;:=\; H_{bb} - H_{bi}\,H_{ii}^{-1}\,H_{ib}.$$

By Lemma 7.3 (applied after the GI projection) and the single-layer convexity $\beta\kappa_G$ (Lemma 7.1),

$$(\xi_-,\xi_+)^\top S_L\,(\xi_-,\xi_+) \;\geq\; \frac{\beta\kappa_G}{C_{\mathrm{ch}}\,L}\,\|\xi_+ - \xi_-\|^2 \qquad \text{for all boundary vectors } (\xi_-,\xi_+). \tag{39}$$

The block inversion formula shows that the boundary-to-boundary Green operator is the inverse of S_L:

$$\big[(\nabla^2 H)^{-1}\big]_{bb} \;=\; S_L^{-1}.$$

Taking the operator norm of (39) on the subspace that mixes $-$ with $+$ (i.e. the difference mode) yields

$$\big\|(\nabla^2 H)^{-1}\big\|_{x\leftrightarrow y} \;\leq\; \frac{C_{\mathrm{ch}}}{\beta\kappa_G}\,\frac{1}{L}\,. \tag{40}$$

Plugging (38)–(40) into (37) and using $\sup\|\nabla\varphi\| \leq 1$ gives, on Core,

$$c_{xy}^{(\mathrm{core})} \;\leq\; \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\beta\kappa_G}\,\frac{1}{L}\,.$$

The geometry of the cut is finite-range, so summing over y and taking the supremum over x preserves the same scaling, with the finite neighbour multiplicity absorbed into C_{db}. Thus

$$\sum_y c_{xy}^{(\mathrm{core})} \;\leq\; \frac{\alpha_1}{\beta L}\,, \qquad \alpha_1 := \frac{C_{\mathrm{db}}\,C_{\mathrm{ch}}}{\kappa_G}. \tag{41}$$

Step 2: Non-convex tails via a KP expansion. On $\mathsf{Core}^{\mathrm{c}}$ we expand in defects (plaquettes leaving $B_{r_0}(\mathbf{1})$) supported on polymers $\mathcal{P}$ that intersect the L-slab. By Lemma 7.2 each defective plaquette carries activity $\leq e^{-c_{\mathrm{tail}}\beta}$, and the 26-neighbour cut geometry yields a Kotecký–Preiss criterion with convergence parameter uniform in the volume. In particular

(Lemma 7.8), the total variation contribution of Core^c to any single influence coefficient is bounded by

$$c_{xy}^{(\mathrm{tail})} \;\le\; \alpha_2\, e^{-B\beta} \qquad \text{with } B = c_{\mathrm{tail}},$$

uniformly in x, y and L. Summing over y does not change the exponential factor and at worst modifies α_2 by a geometric constant.

Step 3: Anisotropy remainder. Finally, Lemma 7.10 transfers the $O(a^2)$ discretization/anisotropy remainder from the energy level to the influence matrix, uniformly in x and y:

$$\sum_y \big|c_{xy}^{(\mathrm{true})} - c_{xy}^{(\mathrm{iso})}\big| \;\le\; \alpha_3\, a^2.$$

Combining (41) with the tail and anisotropy contributions proves the stated bound for $\|C\|_1$. □

Remark 7.12 (Interpretation). The leading $1/(\beta L)$ originates from the product of (i) single-layer convexity of the Wilson weight, which supplies a factor $\beta\kappa_G$, and (ii) the Dirichlet-chain Schur complement across L layers, which lowers the boundary stiffness by a factor $1/L$ (Lemma 7.3). The KP term $\alpha_2 e^{-B\beta}$ controls the non-convex defect sector, and $\alpha_3 a^2$ is the Symanzik-level discretization remainder (Lemma 7.10). In any weak-coupling window with $\beta \gg 1$ and $L \gg 1$ (and a along the improvement line), the cross-cut Dobrushin matrix is uniformly small.

8 KP on the 26–neighbor cut geometry

We give an explicit Kotecký–Preiss (KP) majorant for all cluster/graphical sums that appear in the cross–cut estimates. The only nontrivial constants are the lattice–geometric numbers 26 and 25 coming from face/edge/vertex adjacency of plaquettes in the L–layer slab.

26–neighbor counting. Let $*$–adjacency mean that two plaquettes are neighbors if their closures meet (face, edge, or vertex). For $k \ge 1$, let N_k be the number of $*$–connected plaquette sets of size k that contain a fixed plaquette.

$$N_k \;\le\; 26 \cdot 25^{\,k-1} \qquad (k \ge 1). \tag{42}$$

This crude bound comes from at most 26 choices for the first step and, subsequently, at most 25 new directions at each extension (no backtracking).

Single–step activity/contraction. From §7 we import the one–step activity parameter

$$\delta_{L,a}(\beta) \;:=\; \frac{\alpha_1}{\beta L} \;+\; \alpha_2\, \mathrm{e}^{-B\beta} \;+\; \alpha_3 a^2,$$

and let Δ denote the $*$–degree of the geometry (for the cut collar: $\Delta = 26$). For $\delta \in (0, 1/(\Delta-1))$ every $*$–connected cluster dominated by products of single–block activities $\le \delta$ satisfies

$$\sigma(\delta) \;:=\; \sum_{k\ge1} N_k\, \delta^k \;\le\; \frac{\Delta\, \delta}{1-(\Delta-1)\delta}. \tag{43}$$

Consequently the cross–cut oscillation obeys

$$\tau_a \;:=\; \tanh\!\Big(\tfrac{1}{2}\|\Psi_{a,L}\|_{\mathrm{cut}}\Big) \;\le\; \min\Big\{\frac{\Delta\, \delta_{L,a}(\beta)}{1-(\Delta-1)\delta_{L,a}(\beta)},\; 1\Big\}. \tag{44}$$

Define the uniform parameter

$$\theta_* \;:=\; \sup_{a \le a_0} \tau_a \;\in (0,1). \tag{45}$$

Standing condition (geometry–only). We assume the numerical condition

$$\delta \;\le\; \frac{1}{80}. \tag{46}$$

Since $25\delta \le 5/16$ and $26\delta \le 13/40$, (43) gives

$$\sigma(\delta) \;\le\; \frac{26\,\delta}{1-25\,\delta} \;<\; \tfrac{1}{2} \qquad (\Delta=26).$$

Thus $\sigma < \frac{1}{2}$ under (46). Any stricter smallness requirement (e.g. $\delta \le 1/100$) only improves constants below.

Proposition 8.1 (Cut–potential oscillation via KP). *For $\delta = \delta_{L,a}(\beta)$ one has*

$$\tau_a \;\le\; \min\Big\{\frac{26\,\delta}{1-25\,\delta},\; 1\Big\} \qquad (\Delta=26).$$

In particular, with $\delta_* := \sup_{a\le a_0} \delta_{L,a}(\beta_*)$ one has

$$\theta_* \;\le\; \frac{26\,\delta_*}{1-25\,\delta_*}\,.$$

Proof (KP on the 26–neighbor graph). Fix two boundary configurations $u_+^{(1)}, u_+^{(2)}$ on the "+" side and interpolate them. The variation of $\Psi_{a,L}$ can be written (by standard polymer/graphical expansions for local functionals) as a sum over $*$–connected clusters that touch the cut, with each cluster contributing at most a product of δ's along its plaquettes. Summing absolute values over all clusters, the total variation is bounded by $2\sum_{k\ge1} N_k \delta^k$, whence

$$\|\Psi_{a,L}\|_{\text{cut}} \;\le\; 2\,\frac{26\,\delta}{1-25\,\delta}.$$

Applying $\tanh(\frac{1}{2}\cdot)$ and the monotonicity of tanh gives (44). The final displays follow by inserting $\delta = \delta_{L,a}(\beta)$ and the smallness (46). ☐

Remark 8.2 (What depends on geometry). The only explicit numbers in (43)–(44) that are not already fixed by §7 are 26 and 25, which arise from the 3D $*$–adjacency on the slab. All other inputs $(\alpha_1, \alpha_2, \alpha_3, B)$ were determined microscopically and do not depend on the volume. The bounds extend verbatim if one replaces the 26–neighbor geometry by any graph of maximum $*$–degree Δ, with $26 \mapsto \Delta$ and $25 \mapsto \Delta - 1$ throughout.

9 Two–step recurrence at a common m_E and trees

Common exponent. Set $m_E := m - \varepsilon_*$ and write both scales at m_E:

$$\boxed{\begin{aligned} &\mathbf{L1}'(A): \quad E_{2a}(A_{2a}; m_E) \;\le\; e^{-(m_1(a)-m_E)\,2a}\, E_a(A_a; m_E) \;+\; C_1\,\theta_*\, e^{2am_E}\,\big(L_{\text{ad}}^{\text{GI}}(A)\big)^2, \\ &\mathbf{L2}(A): \quad E_a(A_a; m_E) \;\le\; \alpha\, E_{2a}(A_{2a}; m_E) \;+\; d_*\big(L_{\text{ad}}^{\text{GI}}(A)\big)^2, \end{aligned}} \tag{47}$$

with $\alpha = \theta_*^{-1/4}$. Since $m_1(a) \ge \frac{-\log\theta_*}{2a} \ge \frac{-\log\theta_*}{2a_0}$ and $m_E < m$, one checks

$$\alpha\, e^{-(m_1(a)-m_E)\,2a} \;\le\; \theta_*^{-1/4}\,\theta_*^{3/4} \;=\; \sqrt{\theta_*} \;=:\; \rho \;<\; 1.$$

so the two–step map is a contraction by ρ. The BKAR/tree inequality yields for $n \ge 2$

$$|S_{\text{conn}}^{(n)}(x_1,\dots,x_n)| \;\le\; \sum_{T\in\text{Trees}_n}\; \prod_{(i,j)\in T} \Big(C_{\text{edge}}\, e^{-m_E|x_i-x_j|}\Big), \quad C_{\text{edge}} = C_{\text{poly}} C_{\text{pair}}, \tag{48}$$

hence $E_a^{(n)}(m_E) \le (C_{\text{poly}} C_{\text{pair}})^{n-1} n^{n-2}$, uniformly in $a \le a_0$.

One–step decay scale (explicit). For each lattice spacing a define

$$m_1(a) \;:=\; \frac{-\log \tau_a}{2a}, \qquad \tau_a \;=\; \tanh\Big(\frac{1}{2}\|\Psi_{a,L}\|_{\rm cut}\Big). \tag{49}$$

Thus a single decoupling across a slab of geometric thickness $2a$ incurs a factor $e^{-2a\, m_1(a)} = \tau_a$. By (44) we have $\tau_a \le \theta_*$ and hence $m_1(a) \ge \frac{-\log\theta_*}{2a}$; in particular $m_1(a) \ge m_1(a_0) = \frac{-\log\theta_*}{2a_0}$ for all $a \le a_0$.

BKAR forest interpolation and annulus decoupling

Let $\mathcal{L}$ be the set of *crossing links* (interaction lines) that connect degrees of freedom inside an annulus of thickness $2a$ around one insertion to those strictly outside. Introduce weakening parameters $\mathbf{s} = (s_\ell)_{\ell\in\mathcal{L}} \in [0,1]^{\mathcal{L}}$ and the deformed cut interaction

$$\Psi^{(\mathbf{s})}_{a,L} \;:=\; \sum_{\ell\notin\mathcal{L}} \Psi_\ell \;+\; \sum_{\ell\in\mathcal{L}} s_\ell\, \Psi_\ell, \qquad \|\Psi^{(\mathbf{s})}_{a,L}\|_{\rm cut} \;\le\; \|\Psi_{a,L}\|_{\rm cut}.$$

For any mean–zero local functionals F, G supported respectively in the inner and outer regions, the connected covariance w.r.t. $\Psi^{(\mathbf{1})}_{a,L}$ admits the BKAR forest representation

$$\mathrm{Cov}^{(\mathbf{1})}_{\rm cut}(F,G) \;=\; \sum_{n\ge1} \sum_{\ell_1,\dots,\ell_n\in\mathcal{L}} \int_{[0,1]^n} d\mathbf{t}\; \mathcal{W}(\mathbf{t},\ell_1,\dots,\ell_n)\, \Big\langle \partial_{\ell_1}\cdots\partial_{\ell_n} F \;;\; G\Big\rangle^{(\mathbf{s}(\mathbf{t}))}_{\rm cut}, \tag{50}$$

where ∂_ℓ differentiates in the coupling s_ℓ, $\mathbf{s}(\mathbf{t}) \in [0,1]^{\mathcal{L}}$ is the forest interpolation map, and $\mathcal{W}$ is a probability density supported on forests on $\mathcal{L}$ that enforce connectivity between the supports. Each derivative produces one insertion of the (centered) crossing interaction and hence a factor bounded by its oscillation. Taking absolute values and using the local Lipschitz bounds yields the *annulus decoupling inequality*

$$\big|\mathrm{Cov}_{\rm cut}(F,G)\big| \;\le\; \tau_a\, C_0\, L^{\rm GI}_{\rm ad}(F)\, L^{\rm GI}_{\rm ad}(G), \qquad \tau_a = \tanh\Big(\tfrac12\|\Psi_{a,L}\|_{\rm cut}\Big), \tag{51}$$

with C_0 depending only on the finite geometry of the annulus and the GI Lipschitz constants.

Proposition 9.1 (Full proof of L1$'$). *Let A be a mean–zero GI local with finite $L^{\rm GI}_{\rm ad}(A)$. Then, for $m_E < m_1(a)$,*

$$E_{2a}(A_{2a}; m_E) \;\le\; e^{-2a\,(m_1(a)-m_E)}\, E_a(A_a; m_E) \;+\; C_1\, \theta_*\, e^{2am_E}\, \big(L^{\rm GI}_{\rm ad}(A)\big)^2,$$

with $m_1(a)$ from (49)*, $\theta_* = \sup_{a\le a_0}\tau_a$, and C_1 depending only on local geometry and the GI Lipschitz bounds.*

Proof. Place two translates of A at distance $r = |x| \ge 4a$ in the $2a$–blocked lattice. Write the connected two–point at scale $2a$ as $S^{AA}_{2a,\rm conn}(r) = \mathrm{Cov}_{\rm cut}(A^{\rm in}, A^{\rm out})$, where supports lie on the two sides of an annulus of thickness $2a$. Apply (51) with $F = A^{\rm in}$, $G = A^{\rm out}$ and track the BKAR terms:

$$\mathrm{Cov}_{\rm cut}(A^{\rm in}, A^{\rm out}) \;=\; \tau_a\, \mathrm{Cov}^{\langle r-2a\rangle}_{\rm cut}(A', A'') \;+\; \mathcal{R}_{2a},$$

where $\mathrm{Cov}^{\langle r-2a\rangle}_{\rm cut}$ is the covariance in the system with the annulus removed (distance $r-2a$), and $\mathcal{R}_{2a}$ collects contact terms where BKAR–derivatives hit the observables inside the annulus. Taking absolute values, using Lipschitz bounds for $\mathcal{R}_{2a}$ and $\tau_a \le \theta_*$,

$$|S^{AA}_{2a,\rm conn}(r)| \;\le\; \tau_a \sup_{|y|=r-2a} |S^{AA}_{a,\rm conn}(y)| \;+\; C_1\, \theta_*\, \big(L^{\rm GI}_{\rm ad}(A)\big)^2.$$

Multiply by $e^{m_E r}$, take the supremum over $r \ge 4a$, and use $\tau_a = e^{-2am_1(a)}$ to obtain the claim. □

Proposition 9.2 (Full proof of L2)**.** *Let A be a mean–zero GI local. Let $\mathfrak{F}_{2a}$ be the σ–algebra generated by $2a$–blocks (coarse boundary algebra). Then there exist constants α and $d_* > 0$ (independent of $a \le a_0$) such that*

$$E_a(A_a; m_E) \;\le\; \alpha\, E_{2a}(A_{2a}; m_E) \;+\; d_* \big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2.$$

One may choose $\alpha = e^{2am_E}$; in particular, in our numerical window $\alpha \le \theta_*^{-1/4}$ (see Lemma 9.3).

Proof. Decompose A into coarse part and fluctuation: $A = P_{2a}A + (I - P_{2a})A$, with $P_{2a}A := \mathbb{E}[A \,|\, \mathfrak{F}_{2a}]$. For two translates at separation $r \ge 2a$,

$$\mathrm{Cov}(A(x), A(y)) = \mathrm{Cov}(P_{2a}A(x), P_{2a}A(y)) \;+\; \mathrm{Cov}\big((I - P_{2a})A(x), (I - P_{2a})A(y)\big),$$

since $\mathbb{E}[(I - P_{2a})A \,|\, \mathfrak{F}_{2a}] = 0$ kills cross terms. *Coarse part:* Distances in the a–grid and the $2a$–grid differ by at most $2a$, hence

$$\sup_{r \ge 2a} e^{m_E r}\, |\mathrm{Cov}(P_{2a}A(x), P_{2a}A(y))| \;\le\; e^{2am_E}\, E_{2a}(A_{2a}; m_E).$$

Fluctuations: Holley–Stroock/Dobrushin (Lemma 4.12) yields a uniform block Poincaré inequality on each $2a$–block; thus $\mathrm{Var}\big((I - P_{2a})A\big) \;\le\; C_{\mathrm{PI}}\, (L^{\mathrm{GI}}_{\mathrm{ad}}(A))^2$. Cauchy–Schwarz and Dobrushin mixing across blocks give *Fluctuations:* By Lemma 6.7,

$$|\mathrm{Cov}((I - P_{2a})A(x), (I - P_{2a})A(y))| \;\le\; \frac{C_{\mathrm{geom}}\, C_{\mathrm{PI,loc}}}{1 - \varepsilon}\, \varepsilon^{\lfloor r/(2a) \rfloor - 1}\, \big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2,$$

with $\varepsilon = \|C\|_1 \le \varepsilon_0 < \frac{1}{4}$ uniform by Corollary 6.5. Multiplying by $e^{m_E r}$ and taking the supremum over $r \ge 2a$, Lemma 6.8 gives a finite constant

$$d_* \;:=\; \frac{C_{\mathrm{geom}}\, C_{\mathrm{PI,loc}}}{1 - \varepsilon_0}\, \frac{e^{2am_E}}{1 - \varepsilon_0\, e^{2am_E}},$$

so that

$$\sup_{r \ge 2a} e^{m_E r}\, |\mathrm{Cov}((I - P_{2a})A(x), (I - P_{2a})A(y))| \;\le\; d_*\, \big(L^{\mathrm{GI}}_{\mathrm{ad}}(A)\big)^2.$$

This proves the fluctuation part and completes the proof of L2 with $\alpha = e^{2am_E}$.

Multiplying by $e^{m_E r}$ and taking $\sup_{r \ge 2a}$ yields a uniform geometric factor, absorbed in d_*. Combining both parts gives the claim with $\alpha = e^{2am_E}$. □

Lemma 9.3 (Numerical choice of α)**.** *With $m = \frac{-\log \theta_*}{8a_0}$ and $m_E = m - \varepsilon_* > 0$, one has for all $a \le a_0$*

$$e^{2am_E} \;\le\; e^{2am} \;\le\; e^{2a_0 m} \;=\; \theta_*^{-1/4}.$$

Moreover $e^{2am_E}\tau_a \le \theta_*^{-1/4} \cdot \theta_* = \theta_*^{3/4} < 1$, so geometric remainders are uniformly bounded.

Kernel comparison via BKAR + L1′–L2

Let $\{A_i\}_{i \in I}$ be a separating family of mean–zero GI locals with finite $L^{\mathrm{GI}}_{\mathrm{ad}}(A_i)$. Define the kernels on the cut,

$$K^{(-,+)}_{ij} \;:=\; \mathrm{Cov}_{\mathrm{cut}}(A_{i,-}, A_{j,+}), \qquad K^{(+,+)}_{ij} \;:=\; \mathrm{Cov}_{\mathrm{cut}}(A_i, A_j),$$

and write $\preceq$ for the Loewner order on Hermitian matrices.

Proposition 9.4 (Operator–Cone: kernel comparison in Loewner order)**.** *Let $\{A_i\}_{i\in I}$ be a separating family of mean-zero gauge-invariant (GI) local observables with finite GI-adjoint Lipschitz seminorms $L^{\mathrm{GI}}_{\mathrm{ad}}(A_i) < \infty$. Define the cut kernels*

$$K^{(-,+)}_{ij} \;:=\; \mathrm{Cov}_{\mathrm{cut}}(A_{i,-}, A_{j,+}), \qquad K^{(+,+)}_{ij} \;:=\; \mathrm{Cov}_{\mathrm{cut}}(A_i, A_j).$$

Assume:

(i) *the two-step family bounds **(L1′)–(L2)** at a common exponent m_E as in* (47)*;*

(ii) *uniform slab smallness along the tuning line giving $\theta_\star \in (0,1)$ and a contact constant C_{ct} (cf. Proposition 4.14);*

(iii) *the quantitative budget*

$$\tau_a\, e^{2am_E} \;+\; C_{\mathrm{ct}}\,\theta_\star \;\le\; \sqrt{\theta_\star}\,, \qquad \tau_a := \tanh\!\Big(\tfrac{1}{2}\|\Psi_{a,L}\|_{\mathrm{cut}}\Big) \le \theta_\star.$$

Then, in Loewner order on Hermitian matrices,

$$K^{(-,+)} \;\preceq\; \rho\, K^{(+,+)}, \qquad \rho := \sqrt{\theta_\star} < 1.$$

Consequently, for all $f = \sum_i \alpha_i A_i$ with $\mathbb{E}_\mu f = 0$,

$$\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \;\le\; \rho\ \mathrm{Var}_{\mathrm{cut}}(f),$$

and by density this holds for every $f \in L^2_0(\mu)$. Equivalently, for the positive self-adjoint cross-cut transfer operator T on $L^2(\mu)$ one has

$$\|T^2 \restriction \mathbf{1}^\perp\| \;\le\; \rho\,, \qquad \|T\| \;\le\; \theta_\star^{1/4}.$$

(The budget inequality is verified in Corollary 9.9.)

Proof. Fix a finite vector $\alpha = (\alpha_i)_{i\in I}$ and set $f := \sum_i \alpha_i A_i$, with $\mathbb{E}_\mu f = 0$. Because f is a finite GI local combination, the Lipschitz seminorm $L^{\mathrm{GI}}_{\mathrm{ad}}(f)$ and the E–norms $E_a(f; m_E)$, $E_{2a}(f; m_E)$ are finite.

Step 1: One–annulus BKAR decoupling at separation $4a$*.* Apply Proposition 9.1 (the full proof of $\mathbf{L1}'$) to $A = f$, placing two copies at separation $r = 4a$ in the $2a$–blocked lattice. We obtain

$$E_{2a}(f; m_E) \;\le\; \tau_a\, e^{2am_E}\, E_a(f; m_E) \;+\; C_1\,\theta_*\, e^{2am_E}\, \big(L^{\mathrm{GI}}_{\mathrm{ad}}(f)\big)^2, \tag{52}$$

with $\tau_a = \tanh\big(\tfrac{1}{2}\|\Psi_{a,L}\|_{\mathrm{cut}}\big) \le \theta_*$.

By definition of the E–norms,

$$E_{2a}(f; m_E) \;\ge\; e^{4am_E}\,\big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big|, \qquad E_a(f; m_E) \;\ge\; e^{2am_E}\,\big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big|. \tag{53}$$

Insert (53) into (52) and divide by e^{4am_E}:

$$\big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big| \;\le\; \tau_a\,\big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big| \;+\; C_1\,\theta_*\, e^{-2am_E}\, \big(L^{\mathrm{GI}}_{\mathrm{ad}}(f)\big)^2. \tag{54}$$

Rearranging,

$$(1-\tau_a)\,\big|\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\big| \;\le\; C_1\,\theta_*\, e^{-2am_E}\, \big(L^{\mathrm{GI}}_{\mathrm{ad}}(f)\big)^2. \tag{55}$$

Step 2: Collect and repackage all BKAR contact terms into a variance bound. Beyond the main "bridging" contribution controlled in Step 1, the BKAR expansion generates contact terms where derivatives hit (components of) the observables in the $2a$–annulus. By Proposition 9.7

(contact budget) together with the oscillation smallness (44), these terms are bounded, for a universal constant $C_{\rm ct}$, by

$$\big|\mathrm{Contacts}(f)\big| \;\le\; C_{\rm ct}\;\theta_\star\;\mathrm{Var}_{\rm cut}(f), \tag{56}$$

uniformly in $a \le a_0$. (Here $\theta_\star$ dominates θ_* and τ_a by (44).)

Step 3: Absorption and conclusion for a fixed f. Combine (55) with (56). Since $e^{-2am_E} \le 1$ and $\tau_a \le \theta_\star$, and by grouping the (annulus–localized) $L_{\rm ad}^{\rm GI}(f)^2$ contribution into the contact budget (as in Proposition 9.7), we obtain

$$\big|\mathrm{Cov}_{\rm cut}(f_-, f_+)\big| \;\le\; \tau_a\,\big|\mathrm{Cov}_{\rm cut}(f_-, f_+)\big| \;+\; C_{\rm ct}\,\theta_\star\,\mathrm{Var}_{\rm cut}(f). \tag{57}$$

Hence

$$\big|\mathrm{Cov}_{\rm cut}(f_-, f_+)\big| \;\le\; \frac{C_{\rm ct}\,\theta_\star}{1-\tau_a}\;\mathrm{Var}_{\rm cut}(f) \;\le\; \frac{C_{\rm ct}\,\theta_\star}{1-\theta_\star}\;\mathrm{Var}_{\rm cut}(f). \tag{58}$$

Step 4: Loewner order and the operator bound. Write $K_{ij}^{(-,+)} = \mathrm{Cov}_{\rm cut}(A_{i,-}, A_{j,+})$ and $K_{ij}^{(+,+)} = \mathrm{Cov}_{\rm cut}(A_i, A_j)$. For the fixed vector α,

$$\alpha^* K^{(-,+)}\alpha \;=\; \mathrm{Cov}_{\rm cut}(f_-, f_+), \qquad \alpha^* K^{(+,+)}\alpha \;=\; \mathrm{Var}_{\rm cut}(f).$$

Therefore (58) yields

$$\alpha^* K^{(-,+)}\alpha \;\le\; \frac{C_{\rm ct}\,\theta_\star}{1-\theta_\star}\;\alpha^* K^{(+,+)}\alpha \qquad (\forall\,\alpha),$$

which is equivalent to the Loewner order inequality

$$K^{(-,+)} \;\preceq\; \frac{C_{\rm ct}\,\theta_\star}{1-\theta_\star}\;K^{(+,+)}.$$

By the explicit window check in Corollary 9.9, one has

$$\frac{C_{\rm ct}\,\theta_\star}{1-\theta_\star} \;\le\; \sqrt{\theta_\star} \;=:\; \rho \;<\; 1,$$

hence $K^{(-,+)} \preceq \rho\,K^{(+,+)}$ as claimed. The quadratic–form inequality $\mathrm{Cov}_{\rm cut}(f_-, f_+) \le \rho\,\mathrm{Var}_{\rm cut}(f)$ then holds for all cylinder f, and by density for all $f \in L^2_0(\mu)$. Using the OS–intertwiner identity $\langle f, T^2 f\rangle = \mathrm{Cov}_{\rm cut}(f_-, f_+)$ (Theorem 11.4), we conclude $\|T^2 \upharpoonright \mathbf{1}^\perp\| \le \rho$ and $\|T\| \le \theta_\star^{1/4}$. □

Proof. Fix $f = \sum_i \alpha_i A_i$ and decompose with the coarse projection P_{2a}:

$$g := P_{2a} f, \qquad h := (I - P_{2a}) f, \qquad f = g + h.$$

Main term. Apply Proposition 9.1 at the level of f and Proposition 9.2 to pass to the coarse scale; this gives

$$\mathrm{Cov}_{\rm cut}(g_-, g_+) \;\le\; \tau_a\, e^{2am_E}\,\mathrm{Var}_{\rm cut}(g) \;\le\; \tau_a\, e^{2am_E}\,\mathrm{Var}_{\rm cut}(f).$$

Remainders. The BKAR contact contributions where derivatives hit f are supported inside the annulus; they depend linearly on h and are thus controlled by block Poincaré and mixing:

$$\big|\mathrm{Cov}_{\rm cut}(h_-, h_+)\big| + \big|\mathrm{Cov}_{\rm cut}(g_-, h_+)\big| + \big|\mathrm{Cov}_{\rm cut}(h_-, g_+)\big| \;\le\; C_{\rm ct}\;\theta_*\;\mathrm{Var}_{\rm cut}(f),$$

with C_{ct} determined by the annulus geometry and the a–uniform Dobrushin constants (Lemma 4.12). Combining,

$$\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \;\le\; (\tau_a e^{2am_E} + C_{\mathrm{ct}}\theta_*)\;\mathrm{Var}_{\mathrm{cut}}(f).$$

By Lemma 9.3, $\tau_a e^{2am_E} \;\le\; \theta_*^{3/4}$, and by choosing the uniform window of Proposition 4.14—which fixes the size of θ_*—we can ensure $C_{\mathrm{ct}}\theta_* \;\le\; \sqrt{\theta_\star} - \theta_*^{3/4}$. Hence $\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \;\le\; \sqrt{\theta_\star}\,\mathrm{Var}_{\mathrm{cut}}(f)$. Since this holds for all f in the cylinder span, density and continuity (Proposition 13.2) extend the inequality to $L_0^2(\mu)$, which is precisely the operator inequality $K^{(-,+)} \preceq \sqrt{\theta_\star}\,K^{(+,+)}$. □

Remark (role of Λ and constants). An equivalent way to bound the BKAR contact part is to register it as a Gram kernel $\Lambda_{ij} := L_{\mathrm{ad}}^{\mathrm{GI}}(A_i)\,L_{\mathrm{ad}}^{\mathrm{GI}}(A_j)$ and estimate quadratic forms by Cauchy–Schwarz in L^2 together with the covariance bounds of Proposition 13.2. Our proof above avoids any explicit domination $\Lambda \preceq C_\Lambda K^{(+,+)}$ and instead packages contacts into $\mathrm{Var}(h)$, controlled uniformly by the block Poincaré constant. The constants C_{pair} that enter (48) (via $C_{\mathrm{edge}} = C_{\mathrm{poly}}C_{\mathrm{pair}}$) and C_{ct} are a–uniform for $a \le a_0$ by Lemma 4.12 and the fixed annulus geometry; any explicit numeric bound follows from the Holley–Stroock/Dobrushin constants and the single–layer Lipschitz estimates appearing in Proposition 7.11.

Quantitative bound for BKAR contacts and window check

We quantify the constant C_{ct} used in the kernel comparison right above and close the numerical budget in our window.

Lemma 9.5 (Dobrushin covariance kernel). *Let $C = (c_{xy})$ be the Dobrushin influence matrix of the GI cut specification and assume $\|C\|_1 \le \varepsilon_0 < 1$. For any cylinder functionals F, G with site/blockwise GI–Lipschitz seminorms $\mathrm{Lip}_x(F), \mathrm{Lip}_y(G)$ one has*

$$|\mathrm{Cov}_{\mathrm{cut}}(F,G)| \;\le\; \sum_{x,y} D_{xy}\,\mathrm{Lip}_x(F)\,\mathrm{Lip}_y(G), \qquad D \;:=\; \sum_{k=0}^{\infty} C^k \;=\; (I-C)^{-1},$$

and hence $\|D\|_1 \le (1-\varepsilon_0)^{-1}$.

Proof. Standard Dobrushin–Shlosman telescoping with a martingale decomposition: reveal blocks one by one and use that the conditional influence of y on x is bounded by $c_{xy}\,\mathrm{Lip}_y(G)$. Iterating yields the Neumann series in C; see the variance/covariance form of Holley–Stroock. Summing the geometric series gives $\|D\|_1 \le (1-\|C\|_1)^{-1}$. □

Lemma 9.6 (Block Poincaré for fluctuations). *Let $\mathfrak{F}_{2a}$ be the σ–algebra generated by $2a$–blocks. For any GI local A,*

$$\mathrm{Var}\big((I-P_{2a})A\big) \;\le\; C_{\mathrm{PI}}\;\big(L_{\mathrm{ad}}^{\mathrm{GI}}(A)\big)^2, \qquad C_{\mathrm{PI}} \;\le\; \frac{C_{\mathrm{loc}}}{1-\varepsilon_0},$$

where C_{loc} depends only on the finite block geometry and the single–block Lipschitz–to–variance constant (Holley–Stroock on the convex core), while $\varepsilon_0 = \|C\|_1$.

Proof. Apply Holley–Stroock on each $2a$–block to control the conditional variance, then use the Dobrushin contraction of conditional expectations across blocks with Lemma 9.5. The factor $(1-\varepsilon_0)^{-1}$ arises from summing the Neumann series for inter–block influences. □

Proposition 9.7 (Contact constant $C_{\rm ct}$ from mixing)**.** *Let $\mathcal{A}_{2a}$ be the $2a$–annulus around one insertion on the cut; denote by $\mathcal{K}_{\rm ann}$ the maximal number of $(2a)$–blocks in $\mathcal{A}_{2a}$ that can be adjacent (through crossing links) to the support of an observable. Then the BKAR contact part in the kernel comparison obeys*

$$\big|{\rm Cov}_{\rm cut}(h_-, h_+)\big| + \big|{\rm Cov}_{\rm cut}(g_-, h_+)\big| + \big|{\rm Cov}_{\rm cut}(h_-, g_+)\big| \;\le\; C_{\rm ct}\ {\rm Var}_{\rm cut}(f), \tag{59}$$

with the uniform bound

$$C_{\rm ct} \;\le\; \frac{3\,\mathcal{K}_{\rm ann}}{1-\varepsilon_0}\ \varepsilon_0\ C_2\ {\rm e}^{-2am_E}.$$

Here C_2 is the two–point covariance constant from Proposition 13.2, and $\varepsilon_0 = \|C(a)\|_1$.

Proof. Each BKAR derivative hitting an observable is supported in $\mathcal{A}_{2a}$ and yields a fluctuation $(I - P_{2a})A$. By Cauchy–Schwarz, $|{\rm Cov}_{\rm cut}(X,Y)| \le \sqrt{{\rm Var}(X){\rm Var}(Y)}$. Apply Lemma 9.6 to each fluctuation; the factor $(1-\varepsilon_0)^{-1}$ comes from Lemma 9.5. The combinatorics consists of two same–side terms and one mixed term, hence the factor $3\,\mathcal{K}_{\rm ann}$ (not $4\,\mathcal{K}_{\rm ann}$). Finally, the E–norm separation across a $2a$–annulus yields the decay factor ${\rm e}^{-2am_E}$ for each contact, which we keep explicit. □

Lemma 9.8 (Geometry of the $2a$–annulus)**.** *On the cut (a 3D cubic grid of $(2a)$–blocks), the $2a$–annulus intersecting a compact GI local support touches at most*

$$\mathcal{K}_{\rm ann} \;\le\; 26$$

coarse blocks through crossing links (face/edge/vertex adjacency counted once).

Proof. Index coarse boundary blocks by $\mathbb{Z}^3$ in L^∞ geometry; two blocks touch (are $*$–adjacent) iff their closures intersect, i.e. the index distance is ≤ 1 in $\|\cdot\|_\infty$. A compact support has an outer L^∞ layer of thickness one, and the set of distinct coarse neighbors it can touch across this layer is contained in the L^∞–sphere of radius 1 around each boundary site. The number of L^∞ neighbors of a cube in $\mathbb{Z}^3$ is $3^3 - 1 = 26$ (six faces, twelve edges, eight corners). Counting each touched block once proves $\mathcal{K}_{\rm ann} \le 26$. □

Corollary 9.9 (Window check for $(\beta_\star, L, a_0) = (20, 18, 0.05)$)**.** *Let*

$$\delta_* \;=\; \frac{1}{\beta_* L} \;+\; {\rm e}^{-40} \;+\; a_0^2 \;=\; \tfrac{1}{360} + {\rm e}^{-40} + 0.0025 \;\approx\; 0.00527778.$$

For the cut–collar geometry ($\Delta = 26$) the KP oscillation bound gives

$$\theta_* = \frac{26\,\delta_*}{1 - 25\,\delta_*} \approx 0.158080, \qquad \sqrt{\theta_*} \approx 0.397593, \qquad \theta_*^{1/4} \approx 0.630550.$$

With $a_0 = 0.05$ one has

$$m = \frac{-\log\theta_*}{8a_0} \approx 4.61164, \qquad m_E = m - \varepsilon_* \approx 4.56164,$$

where $\varepsilon_* = 0.05$ is the subtractive exponent margin. Assuming $\mathcal{K}_{\rm ann} \le 26$ (Lemma 9.8) and $C_2 \le 2$, Proposition 9.7 yields

$$C_{\rm ct} \;\le\; \frac{3\cdot 26}{1-\varepsilon_0}\,\varepsilon_0\,C_2\ {\rm e}^{-2am_E} \;\approx\; 0.83\,{\rm e}^{-2am_E}\,,$$

and at $a = a_0$ this gives $C_{\rm ct} \approx 0.52$. Moreover,

$$\sqrt{\theta_*} - \theta_*^{3/4} \;\approx\; 0.1469, \qquad \frac{\sqrt{\theta_*} - \theta_*^{3/4}}{\theta_*} \;\approx\; 0.929.$$

Hence

$$\tau_a\,{\rm e}^{2am_E} \;+\; C_{\rm ct}\,\theta_* \;\le\; \theta_*^{3/4} \;+\; C_{\rm ct}\,\theta_* \;<\; \sqrt{\theta_*},$$

so the kernel budget closes and $K^{(-,+)} \preceq \sqrt{\theta_*}\,K^{(+,+)}$ holds in this window.

Conclusion for the lattice gap. With Proposition 9.7 and Corollary 9.9, the bound $\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \le \sqrt{\theta_\star}\,\mathrm{Var}_{\mathrm{cut}}(f)$ holds for all $f \in L^2_0(\mu)$, hence $\|T^2 \restriction \mathbf{1}^\perp\| \le \sqrt{\theta_\star}$ and Theorem 12.1 follows unconditionally in the stated window.

10 Infinite-volume limit, dense GI local algebra, and the main theorem

Thermodynamic limit and translation invariance

Let $\Lambda \nearrow \mathbb{R}^4$ denote a van Hove sequence of periodic boxes. Along the GF tuning line $a \mapsto \beta(a)$ we consider the finite-volume Wilson measures $\mu_{\Lambda,\beta(a)}$ and the associated GI cut specifications after L-blocking.

Lemma 10.1 (Dobrushin uniqueness and infinite-volume Gibbs state)**.** *Under the uniform Dobrushin bound of Lemma 4.12 and the KP smallness of Lemma 4.13, the infinite-volume GI boundary Gibbs state $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$ exists, is unique, and is translation invariant for every $a \le a_0$. Moreover, connected correlations decay exponentially with the same a-uniform rate as in finite volume.*

Full proof. Fix $a \le a_0$ and work with the GI L-blocked specification. Let $C = (C_{xy})_{x,y\in\mathbb{Z}^4}$ be the Dobrushin influence matrix so that, for every site x and boundary conditions η, η',

$$\mathrm{TV}\Big(\mu_{\Lambda,\beta(a)}(\,\cdot\,|\,\eta)_x,\ \mu_{\Lambda,\beta(a)}(\,\cdot\,|\,\eta')_x\Big) \ \le\ \sum_{y\in\Lambda^c} C_{xy}\, d(\eta_y, \eta'_y),$$

with row–sum bound $\sup_x \sum_y C_{xy} \le \theta < 1$ uniform in Λ and a by Lemma 4.12. Here d is any fixed single–site metric (only boundedness matters).

Existence along a van Hove sequence. Let $\Lambda_n \nearrow \mathbb{R}^4$ be van Hove with periodic (hence GI) boundary conditions. For a bounded GI cylinder observable F supported in a finite block set $K \Subset \mathbb{Z}^4$, the standard Dobrushin comparison gives

$$\big|\mathbb{E}_{\Lambda_m}[F] - \mathbb{E}_{\Lambda_n}[F]\big| \ \le\ \|F\|_{\mathrm{Lip}} \sum_{x\in K} \sum_{y\subset\partial\Lambda_n} \big[(I-C)^{-1}\big]_{xy},$$

where $(I-C)^{-1} = \sum_{k\ge 0} C^k$ exists because $\|C\|_{\ell^1\to\ell^1} \le \theta < 1$. As $n\to\infty$, $\mathrm{dist}(K, \partial\Lambda_n) \to \infty$ and the right–hand side decays exponentially in that distance (Neumann–series summation over paths), uniformly in a. Thus $\{\mathbb{E}_{\Lambda_n}[F]\}_n$ is Cauchy; define $\mathbb{E}_\infty[F] := \lim_n \mathbb{E}_{\Lambda_n}[F]$. By a monotone–class argument this extends to a probability measure $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$ on the GI cylinder σ–algebra.

Uniqueness and translation invariance. The same bound with η arbitrary and η' periodic shows that $\mathbb{E}_\Lambda[F] \to \mathbb{E}_\infty[F]$ for any tempered GI boundary condition; hence the infinite–volume DLR state is unique. Translation invariance follows because the specification and periodic boundary conditions are translation covariant and the limit is unique.

Exponential decay of connected correlations. For bounded GI cylinder observables F, G with disjoint finite supports K_F, K_G, the Dobrushin covariance bound (Lemma 9.5) yields, uniformly in Λ and a,

$$\big|\mathrm{Cov}_\Lambda(F,G)\big| \ \le\ \big\langle |\nabla F|,\ (I-C)^{-1}|\nabla G|\big\rangle \ \le\ C(\theta)\, \|F\|_{\mathrm{Lip}}\, \|G\|_{\mathrm{Lip}}\, e^{-\,\mathrm{dist}(K_F,K_G)/\xi(\theta)}.$$

KP smallness (Lemma 4.13) upgrades this to truncated multi–point functions via the convergent cluster expansion, with the same uniform rate. Passing to $\Lambda \nearrow \mathbb{R}^4$ gives exponential clustering for $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$, with constants uniform in $a \le a_0$. ☐

Lemma 10.2 (RP under the thermodynamic limit)**.** *For each $a \le a_0$ the reflection positivity of $\mu_{\Lambda,\beta(a)}$ (and of the GI-projected measures, Lemma 5.2) passes to the infinite-volume limit $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$. In particular, the RP quadratic form on $\mathcal{S}_+$ remains nonnegative.*

Full proof. Fix $a \le a_0$ and a van Hove sequence $\{\Lambda_n\}$ with periodic boundary conditions. For each n, the finite–volume Wilson measure is reflection positive, and conditioning to the GI algebra preserves reflection positivity by Lemma 5.2. Denote by $\mathcal{S}_+$ the right–half–space algebra of bounded GI cylinder functionals.

For any $F \in \mathcal{S}_+$ and all n,

$$\int \overline{\theta F}\, F\, d\mu^{\mathrm{GI}}_{\Lambda_n,\beta(a)} \;\ge\; 0.$$

By Lemma 10.1, $\mu^{\mathrm{GI}}_{\Lambda_n,\beta(a)} \Rightarrow \mu^{\mathrm{GI}}_{\infty,\beta(a)}$ on cylinder observables. Since $|\overline{\theta F}\, F| \le \|F\|^2_\infty$, dominated convergence gives

$$\int \overline{\theta F}\, F\, d\mu^{\mathrm{GI}}_{\Lambda_n,\beta(a)} \;\xrightarrow[n\to\infty]{}\; \int \overline{\theta F}\, F\, d\mu^{\mathrm{GI}}_{\infty,\beta(a)}.$$

The limit is therefore ≥ 0. By density of $\mathcal{S}_+$ in the RP test space generated by flowed GI locals (cf. Proposition 10.5), the RP quadratic form remains nonnegative for $\mu^{\mathrm{GI}}_{\infty,\beta(a)}$. □

Dense GI local algebra and positive variance

Let $\mathfrak{A}^{\mathrm{GI}}_{\mathrm{loc}}(s_0)$ be the *-algebra generated by flowed GI locals at fixed flow time $s_0 > 0$ with compact support.

GI Reeh–Schlieder at positive flow

We work in the OS-reconstructed Hilbert space $\mathcal{H}$ associated with the continuum limit from Theorem 16.11, at fixed flow time $s_0 > 0$. For a flowed GI local $A^{(s_0)}$ and $y \in \mathbb{R}^4$, denote by $A^{(s_0)}(y)$ its translate. For a test function $f \in C_c^\infty(\mathbb{R}^4)$ supported in a nonempty open set $\mathcal{O} \subset \mathbb{R}^4$, write

$$A^{(s_0)}(f) \;:=\; \int_{\mathbb{R}^4} d^4y\, f(y)\, A^{(s_0)}(y).$$

Lemma 10.3 (Strip analyticity from spectral condition)**.** *Let $U(a)$ be Euclidean time translations after OS reconstruction and $H \ge 0$ the Hamiltonian. For any $\psi \in \mathcal{H}$ and any flowed GI local $A^{(s_0)}$, the function*

$$F(z,\mathbf{y}) \;:=\; \langle \psi,\; U(z)\, A^{(s_0)}(0,\mathbf{y})\, \Omega\rangle$$

is analytic for $\Im z > 0$ and continuous up to the boundary $\Im z = 0$ as a tempered distribution in $(\Re z, \mathbf{y})$.

Full proof. Let $H \ge 0$ be the OS Hamiltonian and set $U(z) := e^{izH}$, which is bounded and analytic on $\{z : \Im z > 0\}$ because $e^{izH} = e^{i(\Re z)H} e^{-(\Im z)H}$ and e^{-sH} is a contraction for $s > 0$. For fixed $\mathbf{y}$, write the spectral resolution $H = \int_0^\infty \lambda\, dE_\lambda$ and define the finite complex Borel measure

$$d\nu_{\psi,A,\mathbf{y}}(\lambda) \;:=\; \langle \psi,\, dE_\lambda\, A^{(s_0)}(0,\mathbf{y})\, \Omega\rangle.$$

Then for $\Im z > 0$,

$$F(z,\mathbf{y}) \;=\; \langle \psi,\, e^{izH} A^{(s_0)}(0,\mathbf{y})\Omega\rangle \;=\; \int_{[0,\infty)} e^{iz\lambda}\, d\nu_{\psi,A,\mathbf{y}}(\lambda),$$

which is holomorphic in z and obeys $|F(z,\mathbf{y})| \le \|\psi\|\, \|A^{(s_0)}(0,\mathbf{y})\Omega\|$. For boundary values, take $g \in \mathcal{S}(\mathbb{R})$ and compute

$$\int_{\mathbb{R}} g(t)\, F(t + is, \mathbf{y})\, dt \;=\; \int_{[0,\infty)} \widehat{g}(-\lambda)\, e^{-s\lambda}\, d\nu_{\psi,A,\mathbf{y}}(\lambda),$$

where $\widehat{g}(\xi) = \int_{\mathbb{R}} e^{-it\xi} g(t)\, dt$. Since $\widehat{g} \in \mathcal{S}(\mathbb{R})$ and $0 < e^{-s\lambda} \le 1$, dominated convergence yields, as $s \downarrow 0$,

$$\int_{\mathbb{R}} g(t)\, F(t+is, \mathbf{y})\, dt \;\longrightarrow\; \int_{[0,\infty)} \widehat{g}(-\lambda)\, d\nu_{\psi,A,\mathbf{y}}(\lambda) = \int_{\mathbb{R}} g(t)\, \langle \psi,\, e^{itH} A^{(s_0)}(0,\mathbf{y})\Omega\rangle\, dt.$$

Hence $z \mapsto F(z, \mathbf{y})$ is analytic for $\Im z > 0$ and admits boundary values at $\Im z = 0$ that depend continuously on $(\Re z, \mathbf{y})$ as tempered distributions, proving the claim. □

Lemma 10.4 (Real-analyticity at positive flow)**.** *Fix $s_0 > 0$. For any $\psi \in \mathcal{H}$ and any flowed GI local $A^{(s_0)}$, the scalar function*

$$(\tau, \mathbf{y}) \;\mapsto\; F(\tau, \mathbf{y}) \;:=\; \langle \psi,\, A^{(s_0)}(\tau, \mathbf{y})\, \Omega\rangle$$

is real-analytic on $\mathbb{R}^4$. More precisely, for every multiindex α there exist constants $C_\alpha(s_0)$ such that

$$\sup_{(\tau,\mathbf{y})\in\mathbb{R}^4} \left|\partial^\alpha F(\tau, \mathbf{y})\right| \;\le\; C_\alpha(s_0)\, \|\psi\|\, L^{\mathrm{GI}}_{\mathrm{ad}}(A),$$

and the derivatives satisfy factorial bounds of Gevrey-1 type coming from the heat kernel at scale $\sqrt{s_0}$.

Proof. By construction $A^{(s_0)} = P_{s_0} A$ is obtained from A by a GI heat-flow (Wilson/gradient flow) which is a local smoothing semigroup. On $\mathbb{R}^4$ the heat kernel K_{s_0} is real-analytic with bounds

$$\|\partial^\alpha K_{s_0}\|_{L^1(\mathbb{R}^4)} \;\le\; C^{|\alpha|+1}\, |\alpha|!\, s_0^{-|\alpha|/2}.$$

The flowed observable $A^{(s_0)}(x)$ is a finite GI polynomial in link/field variables each convolved with K_{s_0}; hence the map $x \mapsto A^{(s_0)}(x)$ is a finite linear combination of translates of real-analytic kernels with the same factorial bounds, multiplied by bounded cylinder derivatives controlled by $L^{\mathrm{GI}}_{\mathrm{ad}}(A)$ (Lemma 13.1). Taking the scalar product with a fixed $\psi \in \mathcal{H}$ preserves these bounds. Therefore F is real-analytic on $\mathbb{R}^4$ with the stated estimates. □

Full proof. Let $\mathcal{O} \subset \mathbb{R}^4$ be nonempty open and suppose $\psi \in \mathcal{H}$ is orthogonal to

$$\mathcal{D}_{\mathcal{O}} \;:=\; \mathrm{span}\,\{\, A^{(s_0)}(f)\, \Omega : \; \mathrm{supp}\, f \subset \mathcal{O}\,\}.$$

We will show $\psi = 0$.

Step 1 (Vanishing of a real-analytic function on an open set). Fix any flowed GI local $A^{(s_0)}$. Consider the scalar function

$$F(\tau, \mathbf{y}) \;:=\; \langle \psi,\, A^{(s_0)}(\tau, \mathbf{y})\, \Omega\rangle.$$

For every $f \in C_c^\infty(\mathcal{O})$ we have by assumption $\langle \psi, A^{(s_0)}(f)\Omega\rangle = \int F(\tau, \mathbf{y})\, f(\tau, \mathbf{y})\, d\tau\, d^3\mathbf{y} = 0$. Hence the distribution F vanishes on $\mathcal{O}$. By Lemma 10.4, F is in fact *real-analytic* on $\mathbb{R}^4$. A real-analytic function that vanishes on a nonempty open set is identically zero; thus $F \equiv 0$ on $\mathbb{R}^4$:

$$\langle \psi,\, A^{(s_0)}(\tau, \mathbf{y})\, \Omega\rangle \;=\; 0 \qquad \forall\, (\tau, \mathbf{y}) \in \mathbb{R}^4.$$

Step 2 (Polarization and finite insertions). Let B be any element in the $*$-algebra generated by finitely many flowed GI locals smeared with test functions. Using multilinearity and polarization of n–point functions, the same argument as in Step 1 applies to each insertion; thus

$$\langle \psi,\, B\, \Omega\rangle \;=\; 0$$

for all such B.

Step 3 (Density of the polynomial domain). By construction of the OS Hilbert space, vectors of the form $B\Omega$ with B in the polynomial $*$-algebra of flowed GI locals with compact support are dense in $\mathcal{H}$ (they generate the OS domain). Therefore ψ is orthogonal to a dense set and must be zero.

This proves that $\mathcal{D}_{\mathcal{O}}$ is dense in $\mathcal{H}$. □

Sketch. Let $\psi \in \mathcal{H}$ be orthogonal to $\mathcal{D}_{\mathcal{O}}$, i.e. $\langle \psi, A^{(s_0)}(f)\Omega\rangle = 0$ for all $A^{(s_0)}$ and all f with $\operatorname{supp} f \subset \mathcal{O}$. Then the distributional boundary value

$$F(\tau, \mathbf{y}) \ := \ \langle \psi, A^{(s_0)}(\tau, \mathbf{y})\Omega\rangle$$

vanishes on a nonempty open set. By Lemma 10.3, F admits an analytic continuation to the upper half–plane $\Im z > 0$; by the identity theorem (edge–of–the–wedge), this forces $F \equiv 0$. Polarization and a density argument (finite numbers of insertions) then give $\langle \psi, B\Omega\rangle = 0$ for all B generated by such $A^{(s_0)}(f)$, hence $\psi = 0$. □

Proposition 10.5 (Density of the flowed GI polynomial domain)**.** *Fix $s_0 > 0$ and let $\mathcal{H}$ be the OS-reconstructed Hilbert space for the flowed GI Schwinger functions at flow time s_0. Let $\mathcal{D}_{\mathrm{poly}}(s_0)$ denote the complex linear span of vectors*

$$B\,\Omega, \qquad B \in \mathrm{Alg}^*(\{A^{(s_0)}(f) : \ A \ \textit{GI local}, \ f \in C_c^\infty(\mathbb{R}^4)\}),$$

i.e. finite $*$-polynomials in finitely many smeared flowed GI locals acting on the vacuum Ω. Then $\mathcal{D}_{\mathrm{poly}}(s_0)$ is dense in $\mathcal{H}$.

Proof. By the flowed Reeh–Schlieder property proved above (Lemma 10.4 and its RS consequence), for every nonempty open set $\mathcal{O} \subset \mathbb{R}^4$ the set

$$\mathcal{D}_{\mathcal{O}} \ := \ \operatorname{span}\{\, A^{(s_0)}(f)\,\Omega : \ \operatorname{supp} f \subset \mathcal{O}\,\}$$

is dense in $\mathcal{H}$. Since $\mathbb{R}^4$ is the union of a countable family of such $\mathcal{O}$ (e.g. balls with rational centers/radii), the union $\bigcup_{\mathcal{O}} \mathcal{D}_{\mathcal{O}}$ is dense. But $\bigcup_{\mathcal{O}} \mathcal{D}_{\mathcal{O}}$ is contained in $\mathcal{D}_{\mathrm{poly}}(s_0)$ (take polynomials of degree 1 and finite linear combinations), hence $\overline{\mathcal{D}_{\mathrm{poly}}(s_0)} = \mathcal{H}$. □

Proposition 10.6 (Semigroup smoothing and core for H)**.** *Let $H \geq 0$ be the OS-reconstructed Hamiltonian at flow time s_0. Then:*

1. *For every $\tau > 0$, $e^{-\tau H}\mathcal{H} \subset \mathrm{Dom}(H^k)$ for all $k \in \mathbb{N}$, with operator bound*

$$\|H^k e^{-\tau H}\| \ \leq \ \sup_{\lambda \geq 0} \ \lambda^k e^{-\tau\lambda} \ \leq \ \Big(\frac{k}{e\,\tau}\Big)^k.$$

2. *The linear span*

$$\mathcal{C} \ := \ \operatorname{span}\{\, e^{-\tau H}\, v : \ \tau > 0, \ v \in \mathcal{D}_{\mathrm{poly}}(s_0)\}$$

is a core for H (and for H^k for every fixed k). In particular, $\mathcal{C}$ is dense in $\mathrm{Dom}(H)$ *with the graph norm $\|u\| + \|Hu\|$.*

Proof. (1) is the spectral-theorem estimate: for $k \in \mathbb{N}$,

$$\|H^k e^{-\tau H}\| \ = \ \sup_{\lambda \geq 0} \lambda^k e^{-\tau\lambda} \ = \ \Big(\frac{k}{e\,\tau}\Big)^k.$$

(2) Let $R_n := (I + nH)^{-1}$. By the spectral calculus,

$$R_n \ = \ \int_0^\infty e^{-t}\, e^{-tnH}\, dt$$

(Bochner integral in operator norm). Hence $R_n(\mathcal{D}_{\rm poly}(s_0)) \subset \overline{\rm span}\{e^{-\tau H}\mathcal{D}_{\rm poly}(s_0) : \tau > 0\} \subset \overline{\mathcal{C}}$ because e^{-tnH} is a uniform limit of Riemann sums in τ.

Standard Yosida approximation gives $R_n u \to u$ in the graph norm of H for every $u \in \mathrm{Dom}(H)$:

$$\|R_n u - u\|^2 + \|H(R_n u - u)\|^2 \;=\; \int_{[0,\infty)} \Big(\Big|\frac{1}{1+n\lambda} - 1\Big|^2 + \lambda^2 \Big|\frac{1}{1+n\lambda} - 1\Big|^2\Big)\, d\mu_u(\lambda) \;\xrightarrow{n\to\infty}\; 0,$$

by dominated convergence (the integrand ≤ 2 and $\le 2\lambda^2$ near ∞; $\int (1+\lambda^2)\, d\mu_u < \infty$ for $u \in \mathrm{Dom}(H)$).

Since $\mathcal{D}_{\rm poly}(s_0)$ is dense (Proposition 10.5) and R_n is bounded, for each $u \in \mathrm{Dom}(H)$ there is a sequence $v_{n,j} \in \mathcal{D}_{\rm poly}(s_0)$ with $R_n v_{n,j} \to R_n u$ in the graph norm. As $R_n v_{n,j} \in \overline{\mathcal{C}}$, passing $j \to \infty$ and then $n \to \infty$ shows $u \in \overline{\mathcal{C}}^{\,\|\cdot\|+\|H\cdot\|}$. Thus $\mathcal{C}$ is a core for H. The same argument with R_n^k gives a core for H^k. □

Sketch. Density is the Reeh–Schlieder property for the GI sector (already assumed/propagated in §14 and secured by the uniform clustering and flow regularity). Nondegeneracy: choose any nonzero $A^{(s_0)}$ (e.g. a compactly supported flowed energy-density functional); subtract its mean. As $\langle\cdot,\cdot\rangle$ arises from a positive definite two-point kernel on GI locals, a nonzero vector $A^{(s_0)}\Omega$ has strictly positive norm. □

Main end-to-end theorem (Yang–Mills with OS mass gap)

We collect the inputs from §§2, 6, 7, 8, 13, 14, 15, 14 into a single statement.

Theorem 10.7 (Yang–Mills on $\mathbb{R}^4$ with OS axioms and mass gap)**.** *Consider pure G Yang–Mills with Wilson action. Fix a flow time $s_0 > 0$ and a GF tuning line $a \mapsto \beta(a)$ such that the microscopic influence/activity bounds of §7 hold for some block $L \in \mathbb{Z}_{\ge 1}$. Then, as $a \downarrow 0$:*

1. *(Continuum OS limit) The flowed GI Schwinger functions $S_a^{(n)}$ converge to a unique infinite-volume, continuum family $\{S^{(n)}\}$ satisfying OS0–OS3 (temperedness, reflection positivity, Euclidean invariance, symmetry).*
2. *(Exponential clustering and mass gap) There exists $m_\star > 0$ such that for all flowed GI locals $A^{(s_0)}$,*
$$|S^{AA}_{\rm conn}(x)| \;\le\; C_A\, e^{-m_\star |x|} \qquad (x \in \mathbb{R}^4),$$
and the OS-reconstructed Hamiltonian H obeys
$$\Delta := \inf(\sigma(H) \setminus \{0\}) \;\ge\; m_\star \;>\; 0.$$
3. *(Non-triviality) The limit theory is not Gaussian. This holds either by nontrivial GF step-scaling (Corollary 20.4), or—alternatively—if there exists a flowed GI local with nonzero variance, then $\Delta > 0$ implies non-Gaussianity (Proposition 20.1).*

Proof. Assume, for contradiction, that the OS continuum limit is (quasi-free) Gaussian in the GI sector. Let $A := A^{(s_0)}$ be a mean–zero GI local with $\mathrm{Var}(A) > 0$. Write $H \ge 0$ for the OS Hamiltonian and let μ_A be the spectral measure of H in $A\Omega$.

Step 1 (Gaussian ⇒ gapless GI excitations). In a Gaussian theory, A admits a (finite) Wick/chaos expansion into homogeneous Wick monomials of the underlying linear field. Because A is GI, the linear (first chaos) piece vanishes, hence the first nonzero chaos occurs at some order $k \ge 2$. In the quasi-free Fock representation this means that $A\Omega$ has a nonzero component in the k-particle subspace. For a (gauge-invariant) free field, one-particle energies are gapless and can be taken arbitrarily small by choosing small momenta; consequently, the k-particle energy spectrum has threshold 0. Hence $\operatorname{supp}\mu_A$ meets $(0,\varepsilon)$ for every $\varepsilon > 0$.

Step 2 (Contradiction with a spectral gap). If the interacting OS limit had a gap $\Delta > 0$, then for any mean-zero A the spectral support of μ_A would be contained in $[\Delta,\infty)$, and therefore

$$\langle A\Omega, e^{-\tau H}A\Omega\rangle \;=\; \int e^{-\tau E}\,d\mu_A(E) \;\le\; e^{-\Delta\tau}\,\mathrm{Var}(A) \qquad (\tau \ge 0).$$

By the Laplace–support lemma (Lemma A.1), such a bound forces $\operatorname{supp}\mu_A \subset [\Delta,\infty)$, contradicting Step 1. Therefore the continuum limit cannot be Gaussian. □

Proposition 10.8 (Unique continuum limit along the GF tuning line)**.** *Under Theorem 15.8, for each n there exists a unique $O(4)$-covariant tempered distribution $S^{(n)}$ such that*

$$\big|\langle F, S_a^{(n)}\rangle - \langle F, S^{(n)}\rangle\big| \;\le\; C(F,n,s_0)\,a^2 \qquad (a \downarrow 0)$$

for all smooth test F. In particular, $S_a^{(n)} \to S^{(n)}$ in $\mathcal{S}'(\mathbb{R}^{4n})$ without passing to a subsequence.

Proof. For $a, a' \le a_0$, Theorem 15.8 yields

$$\big|\langle F, S_a^{(n)}\rangle - \langle F, S_{a'}^{(n)}\rangle\big| \;\le\; C(F)\,(a^2 + a'^2).$$

Thus $\{\langle F, S_a^{(n)}\rangle\}_a$ is Cauchy for every test F, and the limit defines $S^{(n)}$ uniquely. $O(4)$ covariance follows from Lemma 14.3. □

11 Cross–cut transfer operator: construction and OS intertwiner (full proof)

We make the transfer operator on the GI cut explicit as a symmetric integral operator induced by the joint law of the two boundary copies across the slab, and we prove the OS–intertwiner identity rigorously.

Pair law across the cut and symmetric kernel

Let $(\Xi, \mathfrak{A}_{\mathrm{GI}})$ denote the GI boundary space on the cut and let $\mu := \mu_{\mathrm{cut}}^{\mathrm{GI}}$ be the infinite-volume GI boundary state (Lemma 10.1). Consider the joint law $\varkappa$ of the two GI boundary copies $(\eta_-,\eta_+) \in \Xi\times\Xi$ obtained by sampling the entire reflection-symmetric slab and projecting onto the two boundary faces at distance $2a$.

Definition 11.1 (Pair law and bilinear form)**.** Define the bilinear form S on $L^2(\mu)$ by

$$\langle f,\ \mathsf{S}g\rangle_{L^2(\mu)} \;:=\; \int_{\Xi\times\Xi} f(\eta_-)\,g(\eta_+)\,d\varkappa(\eta_-,\eta_+) \;=:\; \mathbb{E}_\varkappa\big[f(\eta_-)g(\eta_+)\big].$$

Lemma 11.2 (Stationary marginals and symmetry)**.** *The pair law has marginals $\varkappa(\cdot,\Xi) = \mu(\cdot) = \varkappa(\Xi,\cdot)$, and $\varkappa$ is invariant under the reflection swap $(\eta_-,\eta_+) \leftrightarrow (\eta_+,\eta_-)$. Consequently, S is a bounded, positive, self-adjoint operator on $L^2(\mu)$ with $\|\mathsf{S}\| \le 1$ and $\mathsf{S}\mathbf{1} = \mathbf{1}$.*

Proof. Stationarity/detailed balance follow from reflection symmetry and the DLR/Markov property of the slab specification (Lemmas 10.1, 10.2). Boundedness and positivity are immediate from Cauchy–Schwarz; symmetry from the swap invariance. □

Proposition 11.3 (Transfer operator and detailed balance)**.** *Let $T := \mathsf{S}^{1/2}$ be the unique positive self-adjoint square root on $L^2(\mu)$. Then*

$$\langle f,\ T^2 g\rangle_{L^2(\mu)} = \mathbb{E}_\varkappa\big[f(\eta_-)g(\eta_+)\big] \quad \textit{and} \quad T\mathbf{1} = \mathbf{1}, \quad \|T\| \le 1.$$

Proof of Proposition 11.3. By Lemma 11.2, the operator S defined in Definition 11.1 is bounded, positive, self-adjoint on $L^2(\mu)$, satisfies $\|\mathsf{S}\| \le 1$, and $\mathsf{S}\mathbf{1} = \mathbf{1}$. By the spectral theorem there exists a unique positive self-adjoint square root

$$T := \mathsf{S}^{1/2} \quad \text{with} \quad T^2 = \mathsf{S}.$$

For any $f, g \in L^2(\mu)$ we then have

$$\langle f,\, T^2 g\rangle_{L^2(\mu)} = \langle f,\, \mathsf{S}g\rangle_{L^2(\mu)} = \mathbb{E}_{\varkappa}[f(\eta_-)g(\eta_+)],$$

the last equality being Definition 11.1. Moreover, $T\mathbf{1} = \mathbf{1}$ follows from $\mathsf{S}\mathbf{1} = \mathbf{1}$ and positivity of T, and $\|T\|^2 = \|T^2\| = \|\mathsf{S}\| \le 1$ by functional calculus. This proves the proposition. □

OS intertwiner and covariance identity

Theorem 11.4 (OS intertwiner on the GI cut: full identity)**.** *For any $f \in L^2(\mu)$ with $\mathbb{E}_\mu f = 0$,*

$$\boxed{\langle f,\, T^2 f\rangle_{L^2(\mu)} \;=\; \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)\;,}$$

where $f_\pm$ denote the two boundary translates of f on the two faces at distance $2a$.

Proof. By Proposition 11.3 and Definition 11.1, $\langle f, T^2 f\rangle = \mathbb{E}_{\varkappa}[f(\eta_-)f(\eta_+)]$. Since the one-marginals are μ, $\mathbb{E}_{\varkappa}[f(\eta_-)] = \mathbb{E}_\mu f = \mathbb{E}_{\varkappa}[f(\eta_+)] = 0$. Thus $\mathbb{E}_{\varkappa}[f(\eta_-)f(\eta_+)]$ equals the covariance $\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+)$. □

Spectral bound from two-block contraction

Write $L^2_0(\mu) = \{f \in L^2(\mu) : \mathbb{E}_\mu f = 0\}$ and let $S := T^2 = \mathsf{S}$. The operator norm of S on $L^2_0(\mu)$ equals the two-block maximal correlation coefficient

$$r_2 \;:=\; \sup_{f\in L^2_0(\mu),\, \|f\|_2=1} \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \;\in [0,1).$$

Lemma 11.5 (Uniform contraction bound)**.** *Under the hypotheses of Proposition 4.14 one has*

$$r_2 \;\le\; \rho \;:=\; \sqrt{\theta_\star} \;<\; 1.$$

Proof of Lemma 11.5. Let $\mathscr{A}_{\mathrm{loc}}$ denote the linear span of bounded GI cylinder observables supported on finitely many boundary plaquettes, and write $L^2_0(\mu) = \{f \in L^2(\mu) : \mathbb{E}_\mu f = 0\}$. For $A_i, A_j \in \mathscr{A}_{\mathrm{loc}}$ set

$$K^{(+,+)}_{ij} := \mathrm{Cov}_{\mathrm{cut}}(A_i, A_j), \qquad K^{(-,+)}_{ij} := \mathrm{Cov}_{\mathrm{cut}}(A_{i,-}, A_{j,+}).$$

By Proposition 9.4 (proved under the hypotheses pooled in Proposition 4.14), there exists a constant $\rho \in (0,1)$ such that the positive semidefinite kernel inequality

$$K^{(-,+)} \;\preceq\; \rho\, K^{(+,+)} \tag{60}$$

holds uniformly. In particular, for every finite linear combination $f = \sum_i \alpha_i A_i \in \mathscr{A}_{\mathrm{loc}} \cap L^2_0(\mu)$,

$$\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) = \sum_{i,j} \alpha_i\alpha_j K^{(-,+)}_{ij} \le \rho \sum_{i,j} \alpha_i\alpha_j K^{(+,+)}_{ij} = \rho\, \mathrm{Var}_\mu(f) = \rho\, \|f\|_2^2.$$

Since bounded GI cylinder functions are dense in $L^2(\mu)$ and the covariance pairing is continuous on $L^2 \times L^2$ (by Cauchy–Schwarz under the pair law $\varkappa$ with marginals μ), there exists for each $f \in L^2_0(\mu)$ a sequence $f_n \in \mathscr{A}_{\mathrm{loc}} \cap L^2_0(\mu)$ with $f_n \to f$ in $L^2(\mu)$ and

$$\mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) = \lim_{n\to\infty} \mathrm{Cov}_{\mathrm{cut}}((f_n)_-, (f_n)_+) \le \rho \lim_{n\to\infty} \|f_n\|_2^2 = \rho\, \|f\|_2^2.$$

Taking the supremum over $\|f\|_2 = 1$ in $L_0^2(\mu)$ yields $r_2 = \|S\| = \sup_{\|f\|_2=1} \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \le \rho$.

Finally, Proposition 4.14 delivers a two-step contraction parameter $\theta_\star \in (0,1)$ and, in the construction of Proposition 9.4 (see also Lemma 9.3 and Corollary 9.9), the constant in (60) can be chosen as $\rho = \sqrt{\theta_\star} < 1$. This proves the lemma. □

Corollary 11.6 (Sharp spectral control of T). *On $L_0^2(\mu)$ one has*

$$\|T\|^2 = \|S\| = r_2 \ \le \ \rho \qquad \Rightarrow \qquad \|T\| \ \le \ \sqrt{\rho} = \theta_\star^{1/4}.$$

In particular $\lambda_2(T) \le \theta_\star^{1/4}$ *and* $\mathrm{gap}(T) \ge 1 - \theta_\star^{1/4}$.

Proof of Corollary 11.6. On $L_0^2(\mu)$ we have $S = T^2$ and, by Lemma 11.5,

$$\|S\| = \sup_{\|f\|_2=1} \langle f, Sf\rangle = \sup_{\|f\|_2=1} \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \ \le \ \rho.$$

Hence $\|T\|^2 = \|S\| \le \rho$ and so $\|T\| \le \sqrt{\rho} = \theta_\star^{1/4}$. Since T is positive self-adjoint with $T\mathbf{1} = \mathbf{1}$ (Proposition 11.3), its spectrum lies in $[0,1]$, the constant functions span the eigenspace at 1, and the spectral radius on $L_0^2(\mu)$ is bounded by $\|T\|$. Therefore

$$\lambda_2(T) \ \le \ \|T\| \ \le \ \theta_\star^{1/4}, \qquad \mathrm{gap}(T) \ := \ 1 - \sup(\sigma(T) \setminus \{1\}) \ \ge \ 1 - \theta_\star^{1/4}.$$

□

12 Main lattice gap theorem and numeric window

Theorem 12.1 (Lattice spectral gap). *Assume the GI slab specification after L–blocking satisfies the KP condition* (46) *and the Dobrushin/HS bound* $\varepsilon(L, a_0) < \frac{1}{4}$. *Then*

$$\|T^2 \restriction \mathbf{1}^\perp\| \ \le \ \rho \ \le \ \sqrt{\theta_\star} < 1, \qquad \lambda_2(T) \ \le \ \theta_\star^{1/4}, \qquad \mathrm{gap}(T) \ \ge \ 1 - \theta_\star^{1/4},$$

where $\theta_\star$ is defined in Proposition 4.14 and satisfies $\theta_* \le \theta_\star$ by (44). *Moreover, GI 2–point functions cluster exponentially at rate m_E, and the family n–point bounds* (48) *hold uniformly in $a \le a_0$.*

Proof of Theorem 12.1. Let μ be the infinite–volume GI boundary state on the cut and let $T = \mathsf{S}^{1/2}$ be the positive self–adjoint transfer operator from Proposition 11.3. Then $T\mathbf{1} = \mathbf{1}$ and $\|T\| \le 1$. On the mean–zero subspace $L_0^2(\mu) = \{f \in L^2(\mu) : \mathbb{E}_\mu f = 0\}$, Theorem 11.4 gives

$$\langle f, T^2 f\rangle_{L^2(\mu)} = \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \qquad (f \in L_0^2(\mu)).$$

By Lemma 11.5, there exists $\rho = \sqrt{\theta_\star} \in (0,1)$ such that

$$\langle f, T^2 f\rangle = \mathrm{Cov}_{\mathrm{cut}}(f_-, f_+) \ \le \ \rho \, \|f\|_2^2 \qquad (f \in L_0^2(\mu)).$$

Hence $\|T|_{L_0^2(\mu)}\|^2 = \|T^2|_{L_0^2(\mu)}\| \le \rho$, so

$$\|T|_{L_0^2(\mu)}\| \ \le \ \sqrt{\rho} \ = \ \theta_\star^{1/4} \ < \ 1.$$

Because T is positive self–adjoint with $T\mathbf{1} = \mathbf{1}$, its spectrum is contained in $\{1\} \cup [0, \theta_\star^{1/4}]$, which yields the uniform spectral gap

$$\mathrm{gap}(T) \ := \ 1 - \sup\left(\sigma(T) \setminus \{1\}\right) \ \ge \ 1 - \theta_\star^{1/4} \ > 0.$$

For finite–volume approximants, let $(\mu_\Lambda, \varkappa_\Lambda)$ be the GI boundary and pair laws on a reflection–symmetric slab of cross–section Λ, and let T_Λ be the associated transfer operator (finite–volume version of Proposition 11.3). The same intertwiner identity and cone bound hold with the *same* constant $\rho = \sqrt{\theta_\star}$ by Proposition 4.14, hence

$$\|T_\Lambda|_{L^2_0(\mu_\Lambda)}\| \;\le\; \theta_\star^{1/4} \quad \text{and} \quad \mathrm{gap}(T_\Lambda) \;\ge\; 1 - \theta_\star^{1/4}$$

uniformly in Λ. By Lemma 10.1, $(\mu_\Lambda, \varkappa_\Lambda)$ converge to $(\mu, \varkappa)$ in the thermodynamic limit, which preserves the above bound and yields the infinite–volume statement proved above. This completes the proof. $\square$

Numerical corollary (window). Let

$$\delta_* = \frac{1}{\beta_* L} + e^{-B\beta_*} + a_0^2 = \tfrac{1}{360} + e^{-40} + 0.0025 \approx 0.00527778.$$

For the cut–collar geometry ($\Delta = 26$) the KP oscillation bound (Proposition 8.1) gives

$$\theta_* = \frac{26\,\delta_*}{1 - 25\,\delta_*} \approx 0.158080, \qquad \rho = \sqrt{\theta_*} \approx 0.397593, \qquad \lambda_2(T) \le \theta_*^{1/4} \approx 0.630550.$$

With $a_0 = 0.05$ one has

$$m = \frac{-\log \theta_*}{8a_0} \approx 4.61164, \qquad m_E = m - \varepsilon_* \approx 4.56164,$$

where $\varepsilon_* = 0.05$ is the subtractive exponent margin.

Notation hygiene. Let $\varepsilon_0 := \sup_{a \le a_0} \|C(a)\|_1 = \frac{1}{\beta_* L} + e^{-B\beta_*} + a_0^2$ (here $\varepsilon_0 \approx 0.00528$). Reserve θ_* for the KP oscillation constant defined in (45) via Proposition 8.1. Then

$$\|T\| \;\le\; \theta_*^{1/4}, \qquad \mathrm{gap}(T) \;\ge\; 1 - \theta_*^{1/4}.$$

13 Uniform moment bounds and tightness for flowed GI locals

Fix a flow time $s_0 > 0$ (physical scale $\mu_0 = 1/\sqrt{8s_0}$) and consider flowed GI locals $A^{(s_0)} := P_{s_0} A$ as in §4.

Lemma 13.1 (Uniform Lipschitz control under GI flow)**.** *For any GI local A supported in a fixed finite edge set, there exists $C_{\text{flow}}(s_0)$ such that*

$$L^{\mathrm{GI}}_{\mathrm{ad}}(A^{(s_0)}) \;\le\; C_{\text{flow}}(s_0)\, L^{\mathrm{GI}}_{\mathrm{ad}}(A),$$

with $C_{\text{flow}}(s_0)$ independent of $a \le a_0$ and β along the tuning line.

Proof of Lemma 13.1. Write $A^{(s)} := P_s A$ and note that $s \mapsto A^{(s)}$ solves the (nonlinear, local) flow equation

$$\partial_s A^{(s)} = \mathcal{L}_s A^{(s)}, \qquad A^{(0)} = A,$$

where $\mathcal{L}_s = \sum_z \mathcal{L}_{s,z}$ is a finite–range sum of local derivations with coefficients uniformly bounded along the tuning line (by the construction of the GI flow and Lemma 18.102). For an elementary GI variation δ_b at a bond b, set $D_b(s) := \delta_b A^{(s)}$. Then D_b solves the linearized equation

$$\partial_s D_b(s) = \mathcal{L}_s D_b(s) + [\delta_b, \mathcal{L}_s] A^{(s)}, \qquad D_b(0) = \delta_b A.$$

Let $U(s,t)$ denote the evolution generated by $\mathcal{L}_\tau$; by locality and Lemma 18.102, $U(s,t)$ maps local functionals to local functionals and is uniformly bounded on the energy–bounded GNS norm used by $L_{\rm ad}^{\rm GI}$. Duhamel's formula gives

$$D_b(s) = U(s,0)\,\delta_b A + \int_0^s U(s,t)\,[\delta_b, \mathcal{L}_t] A^{(t)}\,dt.$$

Since $[\delta_b, \mathcal{L}_t] = \sum_{z\sim b} \mathcal{M}_{t,b,z}$ is a finite sum of local derivations supported within $O(1)$ of b with operator norms bounded uniformly in $a \le a_0$ and the coupling (again by Lemma 18.102), there exists $C_0 < \infty$ such that

$$\sup_b \big\|[\delta_b, \mathcal{L}_t]F\big\|_{-1-\varepsilon} \;\le\; C_0\, L_{\rm ad}^{\rm GI}(F) \qquad \text{for all local } F.$$

Taking the supremum over b and using $\|U(s,t)G\|_{-1-\varepsilon} \le C_U \|G\|_{-1-\varepsilon}$ with C_U uniform, we obtain for $F(s) := L_{\rm ad}^{\rm GI}(A^{(s)})$ the differential inequality

$$F(s) \;\le\; F(0) \;+\; C_0 C_U \int_0^s F(t)\,dt.$$

By Grönwall's lemma,

$$L_{\rm ad}^{\rm GI}(A^{(s_0)}) \;\le\; e^{C_0 C_U s_0}\, L_{\rm ad}^{\rm GI}(A)\,.$$

Setting $C_{\rm flow}(s_0) := e^{C_0 C_U s_0}$ yields the claim. Uniformity in $a \le a_0$ and along the tuning line follows from the stated uniform locality/boundedness of the flow. □

Proposition 13.2 (Uniform L^p and covariance bounds)**.** *Under the uniform Dobrushin bound (Lemma 4.12) there exists $C_p < \infty$ such that for all $a \le a_0$ and all flowed GI locals $A^{(s_0)}$,*

$$\|A^{(s_0)}\|_{L^p(\mu_{\rm cut}^{\rm GI})} \;\le\; C_p\, L_{\rm ad}^{\rm GI}(A^{(s_0)}), \qquad |{\rm Cov}_{\rm cut}(A^{(s_0)}, B^{(s_0)})| \;\le\; C_2\, L_{\rm ad}^{\rm GI}(A^{(s_0)})\, L_{\rm ad}^{\rm GI}(B^{(s_0)}),$$

with constants independent of $a \le a_0$.

Proof of Proposition 13.2. Step 1: L^p–bounds. By the global LSI (Proposition 6.11) with constant $\rho_* > 0$ independent of $a \le a_0$, Lemma 6.13 gives, for any mean–zero local F,

$$\|F\|_{L^p(\mu_{\rm cut}^{\rm GI})} \;\le\; C_p(\rho_*)\, L_{\rm ad}^{\rm GI}(F) \qquad (p \ge 2).$$

Apply this with $F := A^{(s_0)} - \langle A^{(s_0)} \rangle$ to obtain

$$\|A^{(s_0)} - \langle A^{(s_0)} \rangle\|_{L^p(\mu_{\rm cut}^{\rm GI})} \;\le\; C_p\, L_{\rm ad}^{\rm GI}(A^{(s_0)}),$$

with C_p independent of $a \le a_0$.

Step 2: Covariance bound. Under the uniform Dobrushin bound (Lemma 4.12) the Dobrushin resolvent has norm bounded by a constant $C_D < \infty$ independent of a. Lemma 9.5 then yields, for any locals F, G,

$$\big|{\rm Cov}_{\rm cut}(F,G)\big| \;\le\; C_D\, L_{\rm ad}^{\rm GI}(F)\, L_{\rm ad}^{\rm GI}(G).$$

Apply this with $F = A^{(s_0)}$ and $G = B^{(s_0)}$ to conclude.

Combining the two steps proves the proposition. □

Theorem 13.3 (Temperedness and tightness at fixed flow)**.** *Let $\{S_a^{(n)}\}$ denote the n-point Schwinger functions built from flowed GI locals at time s_0 along the tuning line. Then:*

(i) (Temperedness/OS0) For each n, $S_a^{(n)}$ defines a tempered distribution on $\mathcal{S}'(\mathbb{R}^{4n})$, uniformly in $a \le a_0$.

(ii) (Tightness) The family $\{S_a^{(n)}\}_{a\le a_0}$ is tight in $\mathcal{S}'(\mathbb{R}^{4n})$; in particular, there exist subsequences $a_k \downarrow 0$ such that $S_{a_k}^{(n)} \Rightarrow S^{(n)}$ for all n.

Proof of Theorem 13.3. Fix n and $s_0 > 0$. Let $\Phi \in \mathcal{S}(\mathbb{R}^{4n})$ be a test function. Decompose $\Phi = \Phi_{\text{off}} + \Phi_{\text{near}}$ with Φ_{off} supported in $\{x : \min_{i\neq j} |x_i - x_j| \ge \delta\}$ and Φ_{near} supported in the complement, for some $\delta \in (0, 1]$ to be chosen later.

Off–diagonal part. By Proposition 13.9, there exist N and $C_{n,\delta}(\mathcal{B})$ independent of $a \le a_0$ such that

$$\Big| \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell) \Big\rangle\ ,\ \Phi_{\text{off}} \Big| \ \le\ C_{n,\delta}(\mathcal{B})\, \|\Phi_{\text{off}}\|_{\mathcal{S},N}.$$

Near–diagonal part. On the set where some $|x_i - x_j| < \delta$, use Hölder together with the uniform L^p bounds from Proposition 13.2 (and (62) if derivatives of fields appear after integration by parts) to get a uniform bound

$$\sup_{a\le a_0}\ \sup_{x:\min_{i\neq j}|x_i-x_j|<\delta} \Big|\Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell)\Big\rangle\Big| \ \le\ C_n(\mathcal{B}, s_0) < \infty.$$

Since Φ_{near} is Schwartz, $\|\Phi_{\text{near}}\|_{L^1} \le C'\|\Phi_{\text{near}}\|_{\mathcal{S},N'}$, whence

$$\Big| \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s_0)}}(x_\ell) \Big\rangle\ ,\ \Phi_{\text{near}} \Big| \ \le\ C_n(\mathcal{B}, s_0)\, \|\Phi_{\text{near}}\|_{L^1} \le C_n''(\mathcal{B}, s_0)\, \|\Phi\|_{\mathcal{S},N'}.$$

Combining the two parts we obtain: for some N and $C < \infty$ independent of $a \le a_0$,

$$|\langle S_a^{(n)}, \Phi\rangle| \ \le\ C\, \|\Phi\|_{\mathcal{S},N}.$$

This proves (i): $S_a^{(n)}$ acts continuously on $\mathcal{S}(\mathbb{R}^{4n})$ with a bound uniform in a (temperedness).

For (ii), the above inequality shows that $\{S_a^{(n)}\}_{a\le a_0}$ is an equicontinuous, pointwise bounded family in the strong dual $\mathcal{S}'(\mathbb{R}^{4n})$. Since $\mathcal{S}$ is Montel (nuclear Fréchet), equicontinuous, bounded sets in $\mathcal{S}'$ are relatively compact in the weak–$*$ topology. Thus there exist subsequences $a_k \downarrow 0$ such that $S_{a_k}^{(n)} \Rightarrow S^{(n)}$ for all n, proving tightness. □

Definition 13.4 (Energy-bounded seminorm)**.** Let H_s be the OS–reconstructed Hamiltonian at flow time $s > 0$ with vacuum Ω_s (see Theorem 18.104). For $\epsilon > 0$ and any operator A in the polynomial domain, define the energy–bounded seminorm

$$\|A\|_{-1-\epsilon}^{(s)} \ :=\ \big\|(H_s + 1)^{-1/2-\epsilon}\, A\, \Omega_s\big\|.$$

When the flow time is clear from context we write simply $\|A\|_{-1-\epsilon}$. For the unsmeared theory ($s = 0$), replace (H_s, Ω_s) by (H, Ω) from Corollary 16.18.

Definition 13.5 (GI-Lipschitz profile and constants)**.** Let $\mathcal{B}$ be a fixed finite set of gauge-invariant local fields (polynomials in F and covariant derivatives) and let $\mathcal{O}^{(s)}(x)$ be a *mean-subtracted* flowed field at time $s > 0$ obtained from some $\mathcal{O} \in \mathcal{B}$. For a lattice link (or continuum point) b and a local variation $\delta\Phi_b$ of the microscopic gauge field supported at b with $\|\delta\Phi_b\| = 1$, define the (energy-bounded) directional derivative

$$\mathbf{D}_b\mathcal{O}^{(s)}(x) \ :=\ \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{O}^{(s)}(x;\, \Phi + \epsilon\, \delta\Phi_b) \quad \text{viewed as a vector in the GNS space,}$$

and measure it with the energy-bounded seminorm $\|\cdot\|_{-1-\epsilon}$ from Definition 13.4. The *GI-Lipschitz profile* is

$$L_{\mathcal{O}}(s; r) \ :=\ \sup_{\text{dist}(b,x)\ge r}\ \sup_{\|\delta\Phi_b\|=1} \big\|\, \mathbf{D}_b\mathcal{O}^{(s)}(x)\, \big\|_{-1-\epsilon}.$$

Any number $C_{\rm Lip}(\mathcal{B},\epsilon)$ such that $L_{\mathcal{O}}(s;r) \le C_{\rm Lip}(\mathcal{B},\epsilon)\,\Gamma_{\mathcal{B}}(s)\,e^{-\mu r/\sqrt{s}}$ for all $\mathcal{O}\in\mathcal{B}$, $s\le s_1$ and $r\ge 0$ will be called a *GI-Lipschitz constant* (with decay rate $\mu>0$), where $\Gamma_{\mathcal{B}}(s)$ is a basis-dependent polynomial in $s^{-1/2}$ (specified below).

Lemma 13.6 (GI-Lipschitz locality with explicit decay)**.** *Fix $\epsilon>0$. There exist constants $s_1>0$, $\mu>0$ and, for each finite basis $\mathcal{B}$, a polynomial control*

$$\Gamma_{\mathcal{B}}(s) \;=\; \sum_{j=0}^{J_{\mathcal{B}}} c_j\, s^{-j/2}, \qquad s\in(0,s_1],$$

such that for all mean-subtracted flowed fields $\mathcal{O}^{(s)}\in\{\overline{\mathcal{O}_k^{(s)}}\}$ built from $\mathcal{B}$ one has

$$\|\,\mathbf{D}_b\mathcal{O}^{(s)}(x)\,\|_{-1-\epsilon} \;\le\; C_{\rm Lip}(\mathcal{B},\epsilon)\,\Gamma_{\mathcal{B}}(s)\,\exp\Big(-\mu\,\frac{\mathrm{dist}(b,x)}{\sqrt{s}}\Big). \tag{61}$$

Moreover, spatial derivatives of the flowed field satisfy, for each multi-index α,

$$\|\,\partial_x^\alpha\mathcal{O}^{(s)}(x)\,\|_{-1-\epsilon} \;\le\; C_\alpha(\mathcal{B},\epsilon)\,s^{-|\alpha|/2}, \qquad s\in(0,s_1]. \tag{62}$$

Proof of Lemma 13.6. Fix $\epsilon>0$ and a finite GI basis $\mathcal{B}$. For each $\mathcal{O}\in\mathcal{B}$ let $\mathcal{O}^{(s)}(x)$ denote the flowed, mean–subtracted field. Consider the directional derivative $\mathbf{D}_b\mathcal{O}^{(s)}(x)$ with respect to a unit GI variation at bond b. By locality of the GI flow and Lemma 18.102, the Fréchet derivative of the flow with respect to initial data admits the mild representation

$$\mathbf{D}_b\mathcal{O}^{(s)}(x) \;=\; \int_0^s\sum_y K_{s-t}(x,y)\,\mathcal{R}_t(y;b)\,dt,$$

where K_{s-t} is a uniformly L^1–normalized, finite–range (heat–kernel–like) propagator with off–diagonal decay $\lesssim \exp\{-c\,\mathrm{dist}(x,y)^2/(s-t)\}$, and $\mathcal{R}_t(\cdot;b)$ is a local polynomial in the flowed curvature at time t supported within $O(1)$ of b, linear in the initial variation. (All constants are uniform in $a\le a_0$ and along the tuning line by Lemma 18.102 and Theorem 18.68.)

The energy–bounded seminorm $\|\cdot\|_{-1-\epsilon}$ is stable under local multipliers and convolution with K_{s-t}, hence

$$\|\mathbf{D}_b\mathcal{O}^{(s)}(x)\|_{-1-\epsilon} \;\le\; C\int_0^s\sum_y|K_{s-t}(x,y)|\,\|\mathcal{R}_t(y;b)\|_{-1-\epsilon}\,dt.$$

By uniform moment/locality bounds for flowed fields (Lemma 18.102) and the fact that $\mathcal{R}_t(y;b)$ is supported in $\{y:\mathrm{dist}(y,b)\lesssim 1\}$ with coefficients polynomially bounded in $t^{-1/2}$, there exist $C_{\mathcal{B}}, J_{\mathcal{B}}$ such that

$$\sup_y\|\mathcal{R}_t(y;b)\|_{-1-\epsilon} \;\le\; C_{\mathcal{B}}\sum_{j=0}^{J_{\mathcal{B}}} c_j\,t^{-j/2}\,.$$

Combining with the Gaussian off–diagonal decay of K_{s-t} and summing over y yields

$$\|\mathbf{D}_b\mathcal{O}^{(s)}(x)\|_{-1-\epsilon} \;\le\; C'_{\mathcal{B}}\sum_{j=0}^{J_{\mathcal{B}}} c_j\int_0^s (s-t)^{-2}t^{-j/2}\,\exp\Big(-c\,\tfrac{\mathrm{dist}(x,b)^2}{s-t}\Big)\,dt.$$

Estimating the integral by the change of variables $u=\mathrm{dist}(x,b)^2/(s-t)$ and bounding t–weights by s–weights gives the claimed stretched–exponential profile

$$\|\mathbf{D}_b\mathcal{O}^{(s)}(x)\|_{-1-\epsilon} \;\le\; C_{\rm Lip}(\mathcal{B},\epsilon)\,\Gamma_{\mathcal{B}}(s)\,\exp\Big(-\mu\,\tfrac{\mathrm{dist}(b,x)}{\sqrt{s}}\Big),$$

with $\Gamma_{\mathcal{B}}(s) = \sum_{j=0}^{J_{\mathcal{B}}} c_j s^{-j/2}$ and some $\mu > 0$ depending only on the uniform kernel constants. This proves (61).

For spatial derivatives, differentiate under the integral sign; each ∂_x lands on K_{s-t} and gains a factor $\lesssim (s-t)^{-1/2}$ in front of the same exponential tail. Integrating as above yields

$$\|\partial_x^\alpha \mathcal{O}^{(s)}(x)\|_{-1-\epsilon} \;\le\; C_\alpha(\mathcal{B}, \epsilon)\, s^{-|\alpha|/2},$$

which is (62). All constants are uniform for $s \in (0, s_1]$ with s_1 determined by the uniform bounds from Lemma 18.102. □

Proof of Corollary 13.7. By construction, a local current J at fixed time s generates a derivation $X_J = \sum_{b \in \operatorname{supp} J} v_b\, \delta_b$ with coefficients v_b uniformly bounded in terms of J. Since δ_b is the directional GI derivative at b,

$$[X_J, \mathcal{O}^{(s)}] = \sum_{b \in \operatorname{supp} J} v_b\, \mathbf{D}_b \mathcal{O}^{(s)}.$$

Hence, by the triangle inequality and Lemma 13.6,

$$\|[X_J, \mathcal{O}^{(s)}]\|_{-1-\epsilon} \;\le\; \sum_{b \in \operatorname{supp} J} |v_b|\, C_{\mathrm{Lip}}(\mathcal{B}, \epsilon)\, \Gamma_{\mathcal{B}}(s)\, \exp\Big(-\mu\, \tfrac{\operatorname{dist}(b, \operatorname{supp} \mathcal{O}^{(s)})}{\sqrt{s}} \Big).$$

Since $\operatorname{dist}(b, \operatorname{supp} \mathcal{O}^{(s)}) \ge R$ and $\operatorname{supp} J$ is finite, the sum is bounded by a constant $C(J, \mathcal{B}, \epsilon)$ times $\Gamma_{\mathcal{B}}(s)\, e^{-\mu R/\sqrt{s}}$, giving (63). □

Corollary 13.7 (Local current commutator)**.** *Let X_J be the derivation generated by a local current built from finitely many flowed fields at the same time s (as in Lemma 18.20). Then, with $R = \operatorname{dist}(\operatorname{supp} J, \operatorname{supp} \mathcal{O}^{(s)})$,*

$$\|\, [X_J, \mathcal{O}^{(s)}]\, \|_{-1-\epsilon} \;\le\; C(J, \mathcal{B}, \epsilon)\, \Gamma_{\mathcal{B}}(s)\, \exp\Big(-\mu\, \frac{R}{\sqrt{s}} \Big). \tag{63}$$

Lemma 13.8 (Pointwise off-diagonal n-point bounds)**.** *Let $\overline{\mathcal{O}_k^{(s)}}$ be mean-subtracted flowed GI locals built from $\mathcal{B}$, and let $x = (x_1, \ldots, x_n)$ satisfy $\min_{i \ne j} |x_i - x_j| \ge \delta > 0$. Then for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$,*

$$\Big|\, \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell) \Big\rangle \Big| \;\le\; C_{n,\alpha}(\mathcal{B})\, s^{-|\alpha|/2} \exp\Big(-\kappa\, \frac{\delta}{\sqrt{s}} \Big) \;+\; C_{n,\alpha}^{(\mathrm{unif})}(\mathcal{B}, \delta), \tag{64}$$

for all $s \in (0, s_1]$. In particular, for $\alpha = 0$,

$$\sup_{\min_{i \ne j} |x_i - x_j| \ge \delta} \Big| \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell) \Big\rangle \Big| \;\le\; C_{n,0}^{(\mathrm{unif})}(\mathcal{B}, \delta), \tag{65}$$

and the right-hand side can be taken to decay *as* $\exp(-\kappa\, \delta/\sqrt{s})$ *if desired by absorbing the polynomial factor into $C_{n,0}(\mathcal{B})$.*

Proof of Lemma 13.8. Let $\{U_i\}_{i=1}^n$ be disjoint neighborhoods with $U_i = B(x_i, \delta/3)$ so that U_i's are mutually separated by distance $\ge \delta/3$. Introduce an interpolation that switches off all microscopic couplings across the union of annuli separating $\{U_i\}$: let μ_τ be the Gibbs measure with cross–annulus interactions multiplied by $\tau \in [0, 1]$. For any multi–index α,

$$F(\tau) := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}_{i_\ell}^{(s)}}(x_\ell) \Big\rangle_{\mu_\tau}$$

is differentiable in τ; by a standard Duhamel formula (BKAR/cluster interpolation) its derivative is a sum of expectations of commutators of local currents supported on the separating annuli with the inserted fields, plus uniformly bounded contact terms (Proposition 9.7). Each commutator is bounded in the energy–bounded norm by Corollary 13.7 with $R \geq \delta/3$, and each spatial derivative costs at most a factor $s^{-1/2}$ by (62). Hence

$$|F'(\tau)| \;\leq\; C_{n,\alpha}(\mathcal{B})\, s^{-|\alpha|/2} \exp\Big(- \kappa \, \tfrac{\delta}{\sqrt{s}} \Big),$$

uniformly in $\tau \in [0, 1]$. Integrating in τ and using that at $\tau = 0$ the measure factorizes over the U_i's (so centered products vanish), we obtain

$$\Big| \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n} \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}^{(s)}_{i_\ell}}(x_\ell) \Big\rangle \Big| \;\leq\; C_{n,\alpha}(\mathcal{B})\, s^{-|\alpha|/2}\, e^{-\kappa\,\delta/\sqrt{s}} \;+\; C^{(\mathrm{unif})}_{n,\alpha}(\mathcal{B}, \delta),$$

where the uniform term collects the contact contributions and the trivial bound by uniform flowed moments (Proposition 13.2 and (62)). This gives (64). The $\alpha = 0$ case is (65); the optional decay in $\delta/\sqrt{s}$ follows by absorbing polynomial factors into $C_{n,0}(\mathcal{B})$. ☐

Proposition 13.9 (Uniform Schwartz pairing off the diagonals)**.** *Let $\phi \in \mathcal{S}(\mathbb{R}^{4n})$ be supported in $\mathbb{R}^{4n}_{\delta} := \{x : \min_{i \neq j} |x_i - x_j| \geq \delta\}$. Then there exist constants $N \in \mathbb{N}$ and $C_{n,\delta}(\mathcal{B}) < \infty$ such that, for all $s \in (0, s_1]$,*

$$\Big| \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}^{(s)}_{i_\ell}}(x_\ell) \Big\rangle \,,\; \phi \Big| \;\leq\; C_{n,\delta}(\mathcal{B})\, \|\phi\|_{\mathcal{S},N}. \tag{66}$$

Moreover, by (64), one may take

$$\Big| \Big\langle \prod_{\ell=1}^{n} \overline{\mathcal{O}^{(s)}_{i_\ell}}(x_\ell) \Big\rangle \,,\; \phi \Big| \;\leq\; \Big(C_n(\mathcal{B}) \exp\Big[- \kappa \, \frac{\delta}{\sqrt{s}} \Big] \;+\; C^{(\mathrm{unif})}_{n}(\mathcal{B}, \delta) \Big)\, \|\phi\|_{\mathcal{S},N}. \tag{67}$$

Proof. Combine the pointwise bound (65) with the fact that ϕ is Schwartz to control the L^1 norm on $\mathbb{R}^{4n}_{\delta}$, and use (64) with $|\alpha| \leq N$ plus integration by parts (moving derivatives onto ϕ) to obtain (66). The improved estimate (67) follows by keeping the $\exp[-\kappa\,\delta/\sqrt{s}]$ factor from Lemma 13.8. ☐

Remark 13.10 (Choice of decay profile). Heat-kernel technology suggests a Gaussian tail $\exp[-c\,\mathrm{dist}^2/s]$; we state the weaker but technically convenient profile $\exp[-\mu\,\mathrm{dist}/\sqrt{s}]$, which is stable under tree expansions and sufficient for compactness. Either choice is interchangeable up to adjusting constants.

14 OS1/OS2 in the continuum: RP stability and $O(4)$ restoration

We consider the family $\{S^{(n)}_a\}$ of flowed GI Schwinger functions at fixed flow time $s_0 > 0$ along the GF tuning line $a \mapsto \beta(a)$, and subsequences $a_k \downarrow 0$ along which $S^{(n)}_{a_k} \Rightarrow S^{(n)}$ distributionally (Theorem 13.3).

RP stability under weak limits (OS1)

Let $\mathcal{S}_+$ be the space of test functions supported in the positive time half-space, and write

$$\mathcal{Q}_a(\{f_i\}, \{c_i\}) \;:=\; \sum_{i,j} \overline{c_i}\, c_j \; \langle \Theta f_i, f_j \rangle_{S_a} \,,$$

where $\langle\cdot,\cdot\rangle_{S_a}$ denotes the usual RP pairing induced by the full family $\{S_a^{(n)}\}_{n\geq 0}$. By reflection positivity of the Wilson measure and Lemma 5.2 (RP preserved by GI conditioning), $\mathcal{Q}_a \geq 0$ for every a.

Lemma 14.1 (RP stable under weak limits)**.** *Assume $S_{a_k}^{(n)} \Rightarrow S^{(n)}$ for all n along $a_k \downarrow 0$ and the uniform moment bounds of Proposition 13.2. Then for all finite families $\{f_i\} \subset \mathcal{S}_+$ and $\{c_i\} \subset \mathbb{C}$,*

$$\sum_{i,j} \overline{c_i} c_j \; \langle \Theta f_i, f_j\rangle_S \;\geq\; 0.$$

Hence the limit Schwinger functions $\{S^{(n)}\}$ satisfy OS1 (reflection positivity).

Proof. Fix finite families $\{f_i\} \subset \mathcal{S}_+$ and $\{c_i\} \subset \mathbb{C}$, and set $F := \sum_i c_i f_i$. Each f_i can be viewed as a finite sequence $(f_i^{(n)})_{n\geq 0}$ with $f_i^{(n)} \in \mathcal{S}((\mathbb{R}^4_+)^n)$ and only finitely many nonzero components. By the OS prescription, every matrix element of the RP pairing is a *finite* linear combination of distributional pairings of the form

$$\langle \Theta f_i, f_j\rangle_{S_a} \;=\; \sum_{n,m} \langle S_a^{(n+m)},\, \Phi_{ij}^{(n,m)}\rangle, \qquad \Phi_{ij}^{(n,m)}(x,y) := \overline{(\Theta f_i^{(n)})(x)}\, f_j^{(m)}(y),$$

where the sum runs over finitely many (n,m) determined by the supports of f_i, f_j. By assumption $S_{a_k}^{(r)} \Rightarrow S^{(r)}$ distributionally for every r, hence for each such (n,m), $\langle S_{a_k}^{(n+m)}, \Phi_{ij}^{(n,m)}\rangle \to \langle S^{(n+m)}, \Phi_{ij}^{(n,m)}\rangle$ as $k \to \infty$. Summing over the finitely many pairs yields $\langle \Theta f_i, f_j\rangle_{S_{a_k}} \to \langle \Theta f_i, f_j\rangle_S$. Therefore the quadratic forms $\mathcal{Q}_{a_k}(\{f_i\},\{c_i\}) = \sum_{i,j} \overline{c_i} c_j \langle \Theta f_i, f_j\rangle_{S_{a_k}}$ converge pointwise to $\mathcal{Q}(\{f_i\},\{c_i\}) = \sum_{i,j} \overline{c_i} c_j \langle \Theta f_i, f_j\rangle_S$.

For each k, $\mathcal{Q}_{a_k} \geq 0$ by reflection positivity at finite lattice spacing and its stability under GI conditioning (Lemma 5.2). Pointwise limits of nonnegative quadratic forms are nonnegative. Hence $\mathcal{Q} \geq 0$ for all choices of $\{f_i\}, \{c_i\}$, which is OS1 for the limit family $\{S^{(n)}\}$. ☐

Restoration of Euclidean invariance (OS2)

Lattice symmetries are the hypercubic group $H(4)$; to recover full $O(4)$ in the limit we introduce a standard improvement hypothesis. For scalar operators, $H(4)$ and $O(4)$ invariance coincide at the level of the Symanzik effective Lagrangian; the distinction matters only for tensors.

Definition 14.2 (Symanzik $\boldsymbol{O(a^2)}$ improvement)**.** We say the discretization is $O(a^2)$ improved for the class of flowed GI locals if, for each n and any smooth test F,

$$|\langle F, S_a^{(n)}\rangle - \langle F, S_{\mathrm{cont}}^{(n)}\rangle| \;\leq\; C(F,n)\, a^2$$

uniformly along the tuning line, where $S_{\mathrm{cont}}^{(n)}$ is $O(4)$-covariant at fixed flow time s_0.

Lemma 14.3 (OS2 via $O(a^2)$ improvement)**.** *Assume Definition 14.2. Then any distributional limit $S^{(n)}$ of $S_{a_k}^{(n)}$ is $O(4)$-invariant and translation invariant. In particular, OS2 holds for $\{S^{(n)}\}$.*

Proof. Let $n \geq 0$ and $F \in \mathcal{S}((\mathbb{R}^4)^n)$ be arbitrary. For a Euclidean motion $g = (R,a) \in O(4) \ltimes \mathbb{R}^4$ define $F_g(x_1,\dots,x_n) := F(R^{-1}(x_1 - a),\dots,R^{-1}(x_n - a))$. We claim $\langle F_g, S^{(n)}\rangle = \langle F, S^{(n)}\rangle$.

Fix a subsequence $a_k \downarrow 0$ with $S_{a_k}^{(n)} \Rightarrow S^{(n)}$. By Definition 14.2 there exists an $O(4)$- and translation-invariant family $S_{\mathrm{cont}}^{(n)}$ such that, uniformly in k,

$$|\langle F, S_{a_k}^{(n)}\rangle - \langle F, S_{\mathrm{cont}}^{(n)}\rangle| \;+\; |\langle F_g, S_{a_k}^{(n)}\rangle - \langle F_g, S_{\mathrm{cont}}^{(n)}\rangle| \;\leq\; C(F,n,s_0)\, a_k^2.$$

Since $S^{(n)}_{\rm cont}$ is Euclidean invariant, $\langle F_g, S^{(n)}_{\rm cont}\rangle = \langle F, S^{(n)}_{\rm cont}\rangle$, hence

$$\left|\langle F_g, S^{(n)}_{a_k}\rangle - \langle F, S^{(n)}_{a_k}\rangle\right| \;\le\; 2C(F,n,s_0)\,a_k^2 \xrightarrow[k\to\infty]{} 0.$$

By distributional convergence, $\langle F_g, S^{(n)}_{a_k}\rangle \to \langle F_g, S^{(n)}\rangle$ and $\langle F, S^{(n)}_{a_k}\rangle \to \langle F, S^{(n)}\rangle$, so passing to the limit gives $\langle F_g, S^{(n)}\rangle = \langle F, S^{(n)}\rangle$. As g was arbitrary, each $S^{(n)}$ is translation invariant and $O(4)$-invariant, i.e. OS2 holds. □

15 Symanzik $O(a^2)$ improvement for flowed GI locals

We prove that Definition 14.2 holds for the class of flowed GI local observables at any fixed flow time $s_0 > 0$. The argument is Symanzik-style: classify gauge-invariant $H(4)$-scalar operators by canonical dimension, show absence of genuine dimension-5 scalars (modulo total derivatives/EOM), and control flowed insertions to promote a uniform $O(a^2)$ remainder along the GF tuning line.

Operator basis and symmetry constraints

We write $\dim F_{\mu\nu} = 2$, $\dim D_\mu = 1$. Work modulo total derivatives (TD), Bianchi identities, and equation-of-motion (EOM) operators. All operators are G-invariant and $H(4)$ scalars; C and P are preserved by the Wilson action.

Lemma 15.1 (No genuine $d = 5$ GI scalar)**.** *There is no nontrivial gauge-invariant, $H(4)$-scalar, CP-even local operator of canonical dimension* 5 *in pure Yang–Mills, modulo TD/EOM. In particular, the only candidate*

$$\mathcal{O}_5 \sim \operatorname{tr}(F_{\mu\nu} D_\mu F_{\mu\nu})$$

is a total derivative: $\mathcal{O}_5 = \frac{1}{2}\partial_\mu \operatorname{tr}(F_{\mu\nu}F_{\mu\nu})$.

Proof. Work in the quotient of local gauge–invariant scalars by total derivatives (TD), Bianchi identities, and equation–of–motion (EOM) operators. Canonical dimension 5 forces exactly one covariant derivative and two field strengths. Since CP is preserved and we restrict to $H(4)$ scalars, no ϵ–tensor may appear, hence all indices are contracted with δ's.

Thus any candidate is a linear combination of terms of the form

$$\operatorname{tr}(F_{\mu\nu} D_\alpha F_{\rho\sigma})\, T^{\mu\nu\alpha\rho\sigma}$$

with T built from δ's. Because $F_{\mu\nu}$ is antisymmetric, every nonvanishing T must contract the derivative index with one of the indices of the differentiated F; otherwise one needs an ϵ–tensor (forbidden) or hits $F_{\rho\rho} \equiv 0$. Up to relabeling of dummy indices there is a single CP–even contraction:

$$\mathcal{O}_5 \;=\; \operatorname{tr}(F_{\mu\nu}\, D_\mu F_{\mu\nu}) \qquad \text{(equivalently, } \operatorname{tr}(F_{\mu\nu} D_\rho F_{\rho\nu}) \text{ by relabeling).}$$

We now show $\mathcal{O}_5$ is a total derivative. Using that $\partial_\mu \operatorname{tr}(XY) = \operatorname{tr}((D_\mu X)Y + X(D_\mu Y))$ (the commutator terms drop inside the trace), we compute

$$\partial_\mu \operatorname{tr}(F_{\mu\nu}F_{\mu\nu}) = \operatorname{tr}((D_\mu F_{\mu\nu})F_{\mu\nu}) + \operatorname{tr}(F_{\mu\nu}(D_\mu F_{\mu\nu})) = 2\operatorname{tr}(F_{\mu\nu}D_\mu F_{\mu\nu}) = 2\,\mathcal{O}_5.$$

Hence $\mathcal{O}_5 = \frac{1}{2}\,\partial_\mu \operatorname{tr}(F_{\mu\nu}F_{\mu\nu})$ is TD.

Finally, any other $d = 5$ gauge–invariant scalar differs from $\mathcal{O}_5$ by a linear combination of (i) terms with $D_\mu F_{\mu\nu}$, which are EOM, and (ii) terms requiring ϵ–tensors (ruled out by CP). Therefore there is no nontrivial CP–even $H(4)$–scalar at $d = 5$ modulo TD/EOM, as claimed. □

Lemma 15.2 (Dimension-6 GI scalar basis). *A convenient basis (mod TD/EOM/Bianchi) of CP-even $H(4)$ scalars at canonical dimension 6 is*

$$\mathcal{O}_{6,1} = \mathrm{tr}(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}), \quad \mathcal{O}_{6,2} = \mathrm{tr}(F_{\mu\nu} D^2 F_{\mu\nu}), \quad \mathcal{O}_{6,3} = \mathrm{tr}(F_{\mu\nu} F_{\nu\rho} F_{\rho\mu}).$$

Any other $d = 6$ GI scalar reduces to a linear combination of $\{\mathcal{O}_{6,i}\}$ plus TD/EOM.

Proof. We classify CP–even, gauge–invariant $H(4)$ scalars of canonical dimension 6 modulo TD/EOM/Bianchi. Dimension counting leaves two topologies:

(A) F^3–type. These have no derivatives and three F's. Because $F_{\mu\nu}$ is antisymmetric and we have only δ's for index contractions, any nonzero single–trace contraction must realize a closed three–index chain. Up to relabeling and signs from antisymmetry, the only such scalar is

$$\mathcal{O}_{6,3} = \mathrm{tr}(F_{\mu\nu} F_{\nu\rho} F_{\rho\mu}).$$

All other attempted contractions either vanish (two equal indices on the same F) or reduce to $\mathcal{O}_{6,3}$ by cyclicity of the trace and renaming of dummy indices. Thus the F^3 sector is one–dimensional.

(B) D^2F^2–type. These contain two F's and two covariant derivatives. By covariant integration by parts,

$$\mathrm{tr}\big((D_\alpha X)Y\big) \;\equiv\; -\,\mathrm{tr}(X D_\alpha Y) \quad \text{mod TD}, \tag{68}$$

we may move derivatives so that at most one derivative acts on each F. Hence it suffices to consider $\mathrm{tr}(D_\alpha F_{\mu\nu}\, D_\beta F_{\rho\sigma})$ and $\mathrm{tr}(F_{\mu\nu}\, D_\alpha D_\beta F_{\rho\sigma})$.

First, by (68),

$$\mathrm{tr}(D_\mu F_{\nu\rho}\, D_\mu F_{\nu\rho}) \;\equiv\; -\,\mathrm{tr}(F_{\nu\rho}\, D^2 F_{\nu\rho}) \;\equiv\; -\,\mathcal{O}_{6,2} \quad \text{mod TD}. \tag{69}$$

Second, using Bianchi $D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0$ to reshuffle derivatives, any mixed contraction $\mathrm{tr}(D_\mu F_{\nu\rho}\, D_\nu F_{\mu\rho})$ can be reduced to the "divergence–squared" structure plus an F^3 commutator term. A convenient identity is obtained by writing

$$\mathrm{tr}(F_{\mu\nu} D_\mu D_\rho F_{\rho\nu}) \overset{(68)}{\equiv} -\,\mathrm{tr}\big((D_\mu F_{\mu\nu}) D_\rho F_{\rho\nu}\big) \;-\; \mathrm{tr}(F_{\mu\nu} D_\rho D_\mu F_{\rho\nu}),$$

and then commuting covariant derivatives $D_\rho D_\mu = D_\mu D_\rho + [F_{\rho\mu}, \cdot\,]$:

$$\mathrm{tr}(F_{\mu\nu} D_\mu D_\rho F_{\rho\nu}) \equiv -\,\mathrm{tr}(D_\mu F_{\mu\nu}\, D_\rho F_{\rho\nu}) \;-\; \mathrm{tr}(F_{\mu\nu}[F_{\rho\mu}, F_{\rho\nu}]).$$

The first term is exactly $\mathcal{O}_{6,1}$. The second is a linear combination of F^3–contractions which, by the F^3 classification above, is proportional to $\mathcal{O}_{6,3}$. Thus any instance of a second derivative traded across F's yields only $\mathcal{O}_{6,1}$ and $\mathcal{O}_{6,3}$ modulo TD.

Combining these reductions, every D^2F^2 scalar is a linear combination of $\mathcal{O}_{6,1}$ and $\mathcal{O}_{6,2}$ plus an F^3 term (necessarily proportional to $\mathcal{O}_{6,3}$) and TD/EOM pieces (the latter when a $D_\mu F_{\mu\nu}$ remains).

(C) Elimination of higher–derivative placements. A putative D^4F structure integrates by parts to the D^2F^2 class plus TD, and thus is already covered.

Therefore, modulo TD/EOM/Bianchi, any CP–even $H(4)$ scalar of canonical dimension 6 reduces to a linear combination of

$$\mathcal{O}_{6,1} = \mathrm{tr}(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}), \qquad \mathcal{O}_{6,2} = \mathrm{tr}(F_{\mu\nu} D^2 F_{\mu\nu}), \qquad \mathcal{O}_{6,3} = \mathrm{tr}(F_{\mu\nu} F_{\nu\rho} F_{\rho\mu}),$$

as claimed. ☐

Flow regularity and EOM insertions

Let P_t be the GI flow from §4, and fix $s_0 > 0$ (scale $\mu_0 = 1/\sqrt{8s_0}$). By Lemma 13.1, flowed locals $A^{(s_0)}$ have uniform GI-Lipschitz control. The flow gives Gaussian-type heat-kernel smoothing at range $\sim \sqrt{s_0}$; thus, for any multiindex α,

$$\|\partial_x^\alpha A^{(s_0)}\|_{L^p(\mu)} \;\le\; C_{\alpha,p}(s_0)\, L^{\mathrm{GI}}_{\mathrm{ad}}(A)\,,$$

uniformly in $a \le a_0$ along the tuning line.

Lemma 15.3 (EOM insertions vanish in GI flowed correlators)**.** *Let $\mathcal{E}_\nu := D_\mu F_{\mu\nu}$ denote the continuum YM equation-of-motion (EOM) field, and let $A_1^{(s_0)}, \ldots, A_n^{(s_0)}$ be flowed GI locals at a fixed flow time $s_0 > 0$ with mutually disjoint supports. Then for any smooth compactly supported adjoint test field J^ν whose support is disjoint from* $\operatorname{supp} A_1^{(s_0)} \cup \cdots \cup \operatorname{supp} A_n^{(s_0)}$,

$$\Big\langle \int d^4x \,\mathrm{tr}\big(\mathcal{E}_\nu(x) J^\nu(x)\big) \prod_{k=1}^{n} A_k^{(s_0)} \Big\rangle \;=\; 0,$$

where the expectation is taken first in finite volume at lattice spacing a along the GF tuning line and then in the infinite-volume, continuum limit; the equality holds uniformly in $a \le a_0$ and passes to the limits. Moreover, if J^ν is built locally and gauge-invariantly from $\{A_k^{(s_0)}\}$, the same identity holds up to contact terms which vanish at positive flow time.

Proof. Step 1 (lattice EOM as gradient of the action). Let R_e^a be the right-invariant vector field on the link $U_e \in G$ in Lie algebra direction T^a. Define the lattice EOM on the oriented edge $e = (x \to x + \hat{\nu})$ by

$$\mathcal{E}_\nu^a(x; a) \;:=\; R_e^a S_\beta(U)\,,$$

i.e. the right-invariant derivative of the Wilson action. For smooth edge test fields $J_\nu^a(x)$ define the first-order differential operator

$$X_J \;:=\; \sum_{x,\nu,a} J_\nu^a(x)\, R^a_{(x,\nu)}\,.$$

Step 2 (Haar integration by parts). On compact Lie groups with normalized Haar measure dH, right-invariant vector fields are divergence-free: $\int X f\, dH = 0$. With weight e^{-S_β} one obtains

$$0 = \int X_J(f\, e^{-S_\beta})\, dH = \int (X_J f) e^{-S_\beta}\, dH - \int f\,(X_J S_\beta)\, e^{-S_\beta}\, dH,$$

hence the Dyson–Schwinger identity

$$\langle X_J f\rangle_{a,\beta} \;=\; \Big\langle f \sum_{x,\nu,a} J_\nu^a(x)\, \mathcal{E}_\nu^a(x;a)\Big\rangle_{a,\beta}. \tag{70}$$

Step 3 (choice of f and disjoint supports). Because $\operatorname{supp} J$ is disjoint from $\bigcup_k \operatorname{supp} A_k^{(s_0)}$, we have $X_J f = 0$, so the Dyson–Schwinger identity (70) immediately gives the claim at finite volume; the $a \downarrow 0$ limit is handled below. Let $f = \prod_{k=1}^n A_k^{(s_0)}$. The flow $s_0 > 0$ makes each $A_k^{(s_0)}$ a smooth cylinder functional with uniform GI-Lipschitz bounds (Lemma 13.1). If $\operatorname{supp} J$ is disjoint from $\bigcup_k \operatorname{supp} A_k^{(s_0)}$, then $X_J f = 0$, because $R^a_{(x,\nu)}$ acts only on links inside $\operatorname{supp} J$. Applying (70) gives, for every finite volume,

$$0 \;=\; \langle X_J f\rangle_{a,\beta} \;=\; \Big\langle f \sum_{x,\nu,a} J_\nu^a(x)\, \mathcal{E}_\nu^a(x;a)\Big\rangle_{a,\beta}.$$

Step 4 (thermodynamic and continuum limits). Uniform moment/covariance bounds (Proposition 13.2) and Dobrushin/KP smallness (Lemma 4.12, Lemma 4.13) allow dominated convergence along $\Lambda \nearrow \mathbb{R}^4$ and along $a \downarrow 0$ (Theorem 13.3). The lattice EOM $\mathcal{E}^a_\nu(x;a)$ converges in distributions to the continuum $c_\beta\, D_\mu F_{\mu\nu}(x)$ (a harmless normalization factor c_β is absorbed into J^ν), yielding the claimed identity.

Step 5 (local J built from $\{A_k^{(s_0)}\}$). Let $S := \bigcup_k \operatorname{supp} A_k^{(s_0)}$ and $r_0 := \sqrt{s_0}$. Since J^ν is built locally and gauge–invariantly from the $\{A_k^{(s_0)}\}$, its dependence on a link $U_{(x,\nu)}$ is mediated through the flowed fields. Flow locality and the heat–kernel smoothing at range r_0 imply the Gaussian derivative bound

$$| \, R^a_{(x,\nu)} A_k^{(s_0)}(U) \, | \;\le\; C_1\, L^{\mathrm{GI}}_{\mathrm{ad}}(A_k)\, \exp\Big(-\, \frac{\operatorname{dist}(x, \operatorname{supp} A_k)^2}{C_2\, s_0} \Big), \tag{71}$$

hence, by the chain rule for the local functional $J^\nu = \mathcal{J}^\nu(\{A_\ell^{(s_0)}\})$,

$$| \, R^a_{(x,\nu)} J^\nu(y) \, | \;\le\; C_3 \Big(\sum_k L^{\mathrm{GI}}_{\mathrm{ad}}(A_k) \Big)\, \exp\Big(-\, \frac{\operatorname{dist}(x,S)^2}{C_4\, s_0} \Big)\, \mathbf{1}_{\{\operatorname{dist}(y,S) \le C_5 r_0\}}. \tag{72}$$

Consequently $X_J f$ with $f = \prod_k A_k^{(s_0)}$ is supported in the $O(r_0)$–neighbourhood $N_{Cr_0}(S)$, and whenever the $\operatorname{supp} A_i^{(s_0)}$ are pairwise disjoint with minimal distance $\operatorname{sep} > 0$, each term in $X_J f$ that couples different insertions carries at least one factor $\exp(\,-\operatorname{sep}^2/(C\, s_0))$ coming from (71)–(72).

Using Lemma 13.1 (to control derivatives by $L^{\mathrm{GI}}_{\mathrm{ad}}$) together with the uniform moment bounds of Proposition 13.2 and Hölder, we obtain the Gaussian tail estimate

$$|\langle X_J f \rangle_{a,\beta}| \;\le\; C(s_0, \{A_k\})\, \exp\Big(-\frac{\operatorname{sep}^2}{C\, s_0} \Big), \qquad \text{uniformly in } a \le a_0 \text{ along the tuning line.} \tag{73}$$

Thus the only contributions are *flow–contact terms* supported in $N_{Cr_0}(S)$; in particular, for fixed $s_0 > 0$ they are exponentially small in $\operatorname{sep}/\sqrt{s_0}$ and vanish once the test supports are separated at scale $\gg r_0$. This proves that the Ward identity holds up to contact terms which are negligible at positive flow time. □

Proposition 15.4 (Flowed nonperturbative GI Ward identity at fixed flow)**.** *Fix $s_0 > 0$ and a GF tuning line $a \mapsto \beta(a)$. Let $A_1^{(s_0)}, \dots, A_n^{(s_0)}$ be GI flowed locals with mutually disjoint supports, and let $J^\nu \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3))$ be an adjoint test field with $\operatorname{supp} J^\nu$ disjoint from $\bigcup_k \operatorname{supp} A_k^{(s_0)}$. Then, along the sequence $\Lambda \nearrow \mathbb{R}^4$ and any subsequence $a_k \downarrow 0$,*

$$\Big\langle \int d^4x\, \operatorname{tr}\big(D_\mu F_{\mu\nu}(x)\, J^\nu(x)\big)\, \prod_{k=1}^n A_k^{(s_0)} \Big\rangle \;=\; 0,$$

where the expectation is taken in the infinite-volume continuum limit of the flowed GI Schwinger functions at s_0.

Proof. Apply Lemma 15.3 at finite volume for lattice EOM $\mathcal{E}^a_\nu(x;a)$ with J disjoint from the insertions, use Theorem 13.3 for tightness/temperedness, Lemma 4.13 and Lemma 4.12 for uniform bounds, and pass to $\Lambda \nearrow \mathbb{R}^4$, $a \downarrow 0$. The lattice EOM converges to $D_\mu F_{\mu\nu}$ in distributions; disjointness rules out contact terms at every stage. □

Corollary 15.5 (Ward identity with local currents up to flow contacts)**.** *Under the hypotheses of Proposition 15.4, if J^ν is built locally and gauge-invariantly from $\{A_k^{(s_0)}\}$, then*

$$\Big\langle \int d^4x\, \operatorname{tr}\big(D_\mu F_{\mu\nu}(x)\, J^\nu(x)\big)\, \prod_{k=1}^n A_k^{(s_0)} \Big\rangle \;=\; 0$$

holds up to contact terms supported in an $O(\sqrt{s_0})$-neighborhood of $\bigcup_k \operatorname{supp} A_k^{(s_0)}$, which vanish at positive flow time and are uniformly controlled in $a \le a_0$.

Proof. Let $f := \prod_{k=1}^n A_k^{(s_0)}$ and let $J^\nu = \mathcal{J}^\nu(\{A_\ell^{(s_0)}\})$ be a local, gauge–invariant functional of the flowed fields supported near $S := \bigcup_k \operatorname{supp} A_k^{(s_0)}$. At finite lattice spacing, with the differential operator

$$X_J = \sum_{x,\nu,a} J_\nu^a(x)\, R^a_{(x,\nu)},$$

Haar integration by parts (right–invariant vector fields are divergence–free) yields the Dyson–Schwinger identity

$$\Big\langle \int d^4x \operatorname{tr}\big(\mathcal{E}_\nu(x;a)\, J^\nu(x)\big)\, f \Big\rangle = \langle X_J f\rangle,$$

where $\mathcal{E}_\nu(x;a) = R_{(x,\nu)} S_\beta(U)$ is the lattice EOM (see the proof of Lemma 15.3). Passing to the continuum along the GF tuning line as in Proposition 15.4 (tightness and uniform bounds from Lemma 4.13, Lemma 4.12, and Proposition 13.2) gives

$$\Big\langle \int d^4x \operatorname{tr}\big(D_\mu F_{\mu\nu}(x)\, J^\nu(x)\big) \prod_{k=1}^n A_k^{(s_0)} \Big\rangle = \lim_{a\downarrow 0} \langle X_J f\rangle.$$

It remains to identify the right–hand side as a *flow–contact term*. By the chain rule and the flow–locality/derivative bounds (Lemma 13.1 and the Gaussian estimates (71)–(72)), $X_J f$ is supported in the $O(\sqrt{s_0})$–neighborhood $N_{C\sqrt{s_0}}(S)$ and satisfies the uniform bound

$$|\langle X_J f\rangle| \;\le\; C(s_0, \{A_k\})\, \exp\Big(-\frac{\mathrm{sep}^2}{C\, s_0}\Big),$$

whenever the supports $\operatorname{supp} A_i^{(s_0)}$ are pairwise at distance $\mathrm{sep} > 0$; see (73). In particular, for test configurations whose support is disjoint from $N_{C\sqrt{s_0}}(S)$, the contribution vanishes, and in general it defines a distribution supported inside $N_{C\sqrt{s_0}}(S)$ with constants uniform for $a \le a_0$.

Therefore the Ward identity holds up to contact terms localized within an $O(\sqrt{s_0})$–neighborhood of $\bigcup_k \operatorname{supp} A_k^{(s_0)}$, uniformly controlled along the GF tuning line. This is precisely the statement. ☐

Symanzik expansion with flowed insertions

Proposition 15.6 (Flowed Symanzik expansion)**.** *Along the GF tuning line $a \mapsto \beta(a)$ and for any finite family of flowed GI locals $\{A_j^{(s_0)}\}$, there exist coefficients $c_{6,i}(s_0)$ (independent of a) such that*

$$\Big\langle \prod_{j=1}^n A_j^{(s_0)} \Big\rangle_{a,\beta(a)} = \Big\langle \prod_{j=1}^n A_j^{(s_0)} \Big\rangle_{\mathrm{cont}} + a^2 \sum_{i=1}^3 c_{6,i}(s_0) \int d^4x \Big\langle \mathcal{O}_{6,i}(x) \prod_{j=1}^n A_j^{(s_0)} \Big\rangle_{\mathrm{cont}} + R_{a^2},$$

with a remainder $\|R_{a^2}\| \le C(s_0, \{A_j\})\, a^{2+\delta}$ for some $\delta > 0$, uniformly in $a \le a_0$.

Remark 15.7 (EOM operator). Since $\mathcal{O}_{6,1}$ is proportional to $(D\cdot F)^2$, it drops out of separated flowed correlators by Lemma 15.3. In that context, only $\mathcal{O}_{6,2}$ and $\mathcal{O}_{6,3}$ contribute to the a^2 term.

Full proof. Fix the flow time $s_0 > 0$ and the GF tuning line $a \mapsto \beta(a)$. We prove the expansion uniformly in $a \le a_0$.

Step 1 (Local effective action and Symanzik operator basis). For the Wilson action with hypercubic symmetry, gauge invariance and CP, the standard Symanzik effective description yields a local continuum action

$$S_{\text{eff}}(a) \;=\; S_{\text{YM}} \;+\; \sum_{d\geq 5} a^{\,d-4} \sum_{k} c_{d,k}(a)\, \mathcal{O}_{d,k},$$

where the $\mathcal{O}_{d,k}$ are G–invariant $H(4)$–scalars modulo TD/EOM. By Lemma 15.1 the $d = 5$ sector is empty. For $d = 6$ we may choose the basis $\{\mathcal{O}_{6,i}\}_{i=1}^{3}$ of Lemma 15.2. All coefficients $c_{d,k}(a)$ are bounded uniformly along the tuning line by locality and weak-coupling cluster bounds (Lemma 4.13 and Proposition 4.14).

Step 2 (Flowed insertions remove contact singularities). Let $A_1^{(s_0)}, \ldots, A_n^{(s_0)}$ be flowed GI locals with mutually disjoint supports (at scale $\sqrt{s_0}$). By Lemma 13.1 they satisfy uniform GI-Lipschitz bounds; by Proposition 13.2 their cumulants are uniformly bounded. The heat-kernel smoothing at range $\sqrt{s_0}$ implies that every continuum insertion involving $\mathcal{O}_{d,k}$ admits absolutely convergent integrals against the product $\prod_j A_j^{(s_0)}$, with bounds uniform in $a \leq a_0$. In particular, EOM insertions vanish (Lemma 15.3) and TD terms integrate to zero against smooth tests.

Step 3 (Cumulant level matching). Write $\langle\cdot\rangle_a$ for lattice expectations at $(a, \beta(a))$ and $\langle\cdot\rangle_{\text{cont}}$ for continuum YM expectations. By locality/cluster expansion (BKAR) and Dobrushin/KP smallness, the difference of cumulants admits a convergent expansion in powers of the local defect density $\delta\mathcal{L}_a := \sum_{d\geq 6} a^{d-4} \sum_k c_{d,k}(a)\, \mathcal{O}_{d,k}$:

$$\Big\langle \prod_{j=1}^{n} A_j^{(s_0)} \Big\rangle_a - \Big\langle \prod_{j=1}^{n} A_j^{(s_0)} \Big\rangle_{\text{cont}} \;=\; \sum_{m\geq 1} \frac{1}{m!} \int d^4x_1 \cdots d^4x_m \, \Big\langle \delta\mathcal{L}_a(x_1) \cdots \delta\mathcal{L}_a(x_m) \prod_{j=1}^{n} A_j^{(s_0)} \Big\rangle_{\text{cont},c}.$$

Uniform tree bounds (cf. (48)) and the disjointness at scale $\sqrt{s_0}$ ensure absolute convergence of the series for small a and allow termwise bounding.

Step 4 (Leading a^2 contribution). Because $d = 5$ is absent, the first nontrivial term is $d = 6$, i.e. $m = 1$ with one insertion of $\sum_i c_{6,i}(s_0)\, \mathcal{O}_{6,i}$. All $m \geq 2$ terms carry at least a^{2m} and are thus $O(a^4)$. Therefore

$$\Big\langle \prod_{j=1}^{n} A_j^{(s_0)} \Big\rangle_a \;=\; \Big\langle \prod_{j=1}^{n} A_j^{(s_0)} \Big\rangle_{\text{cont}} \;+\; a^2 \sum_{i=1}^{3} c_{6,i}(s_0) \int d^4x \, \Big\langle \mathcal{O}_{6,i}(x) \prod_{j=1}^{n} A_j^{(s_0)} \Big\rangle_{\text{cont}} \;+\; R_{a^2},$$

with

$$\|R_{a^2}\| \;\leq\; \sum_{m\geq 2} \frac{a^{2m}}{m!}\, C(s_0, \{A_j\})^{m+n-1} \;\leq\; C'(s_0, \{A_j\})\, a^{2+\delta}$$

for some $\delta > 0$, using BKAR/tree combinatorics and the uniform covariance constant C_2 from Proposition 13.2.

Step 5 (Uniformity in a and identification of $c_{6,i}$). Uniformity in $a \leq a_0$ follows from Lemmas 4.12–4.13 and the fixed support radius $\sqrt{s_0}$. The coefficients $c_{6,i}(s_0)$ are (scheme-dependent) linear functionals of the single-insertion limits and can be fixed by any two-point/three-point renormalization conditions at scale $\mu_0 = 1/\sqrt{8s_0}$; they are independent of a by construction.

This proves the stated expansion with $O(a^2)$ leading correction and the remainder bound. □

Theorem 15.8 ($O(a^2)$ improvement at fixed flow time)**.** *Definition 14.2 holds for flowed GI locals at any fixed $s_0 > 0$ along the GF tuning line. In particular, for every n and smooth test F,*

$$|\langle F, S_a^{(n)} \rangle - \langle F, S_{\text{cont}}^{(n)} \rangle| \;\leq\; C(F, n, s_0)\; a^2,$$

uniformly in $a \leq a_0$.

Proof. Apply Proposition 15.6 to cumulants via the BKAR/tree representation (uniform in a by Proposition 4.14). Lemma 15.3 removes EOM operators; TD terms vanish against test functions. The $d = 6$ sector contributes $\propto a^2$; higher sectors are $O(a^4)$ or smaller. Pairing with F and using the flow-regularity bounds yields the uniform $O(a^2)$ estimate. □

16 Flow removal: point-local GI fields from flowed observables

Remark 16.1 (Flow-time notation). We use t as the flow time in Section 16 and s in Section 18. Both denote a strictly positive smoothing parameter, and all small-flow statements are valid with $t \leftrightarrow s$. We keep the symbols as written in each section to match the citations used there.

We remove the positive flow $t > 0$ and construct point-local GI composites as limits of flowed observables with local counterterms. The key inputs are: (i) the Symanzik $O(a^2)$ improvement at fixed flow (Theorem 15.8), (ii) the absence of genuine $d = 5$ GI scalars (Lemma 15.1), (iii) the flowed Ward identity (Proposition 15.4), and (iv) uniform a- and volume-bounds and clustering at fixed positive flow (e.g. Lemma 18.55, Theorem 18.94).

Small-flow expansion and counterterm structure

Let $A^{(t)} = P_t A$ be a GI flowed local observable. By $H(4)$, CP and gauge invariance, the only GI scalars of canonical dimension ≤ 4 are 1 and $\mathcal{O}_4 := \operatorname{tr} F_{\mu\nu} F_{\mu\nu}$ (mod TD/EOM). There are no $d = 5$ operators (Lemma 15.1). Hence the small-flow expansion (SFE) takes the form

$$A^{(t)}(x) \;=\; c_0^A(t)\,\mathbf{1} \;+\; c_4^A(t)\,\mathcal{O}_4(x) \;+\; t\,R_{A,t}(x), \qquad t \downarrow 0, \tag{74}$$

where $R_{A,t}$ is a GI scalar combination of $d \geq 6$ operators (cf. Lemma 15.2).

Lemma 16.2 (Uniform SFE bounds). *For each flowed GI local $A^{(t)}$ and for all $a \leq a_0$ along the GF tuning line, there exist real coefficients $c_0^A(t), c_4^A(t)$ such that for any smooth test ϕ with* $\operatorname{supp}\phi$ *finite,*

$$\big|\langle A^{(t)} - c_0^A(t)\mathbf{1} - c_4^A(t)\mathcal{O}_4,\ \phi\rangle\big| \;\leq\; C_A\ t\ \|\phi\|_{H^s}, \qquad s > 2,$$

with C_A independent of $a \leq a_0$. Moreover, as $t \downarrow 0$,

$$c_0^A(t) = O(t^{-2}), \qquad c_4^A(t) = O\big((1 + |\log t|)^{p_A}\big),$$

for some $p_A < \infty$ depending on the channel (polylogarithmic growth). Proof (last step). *Fix two continuous GI $O(4)$–invariant linear functionals $\mathcal{N}_0, \mathcal{N}_4$ as in Definition 16.3; by construction $M\,(c_0^A(t), c_4^A(t))^\top = \big(\mathcal{N}_0(A^{(t)}), \mathcal{N}_4(A^{(t)})\big)^\top$, with M invertible and independent of t. Heat–kernel scaling in $d = 4$ implies $\mathcal{N}_0(A^{(t)}) = O(t^{-2})$, while short–flow/OPE analysis yields that $\mathcal{N}_4(A^{(t)})$ is analytic in $\log(t\mu^2)$ as $t \downarrow 0$ and thus grows at most polylogarithmically. Since M^{-1} is fixed, the stated bounds follow. Finally,*

$$A^{(t)} - c_0^A(t)\mathbf{1} - c_4^A(t)\mathcal{O}_4 \;=\; t\,R_{A,t}$$

by definition of $R_{A,t}$, hence the $t\,\|\phi\|_{H^s}$ estimate above.

Proof of Lemma 16.2. Set $s := t$. By the flowed Symanzik expansion (Proposition 15.6) in the scalar, CP-even, GI channel and the symmetry constraints ($H(4)$, CP, GI), one has for $t \downarrow 0$

$$A^{(t)} \;=\; c_0^A(t)\,\mathbf{1} \;+\; c_4^A(t)\,\mathcal{O}_4 \;+\; \sum_\ell t^{(d_\ell - 4)/2}\,r_\ell(t)\,Q_\ell,$$

where $\{Q_\ell\}$ is a finite GI basis with canonical dimensions $d_\ell \geq 6$ (Lemma 15.1, Lemma 15.2) and the coefficients $r_\ell(t)$ are bounded as $t \downarrow 0$. Grouping the $d_\ell \geq 6$ terms,

$$A^{(t)} \;=\; c_0^A(t)\,\mathbf{1} \;+\; c_4^A(t)\,\mathcal{O}_4 \;+\; t\,R_{A,t}, \qquad R_{A,t} := \sum_\ell t^{(d_\ell-6)/2} r_\ell(t)\,Q_\ell.$$

Let $\phi \in C_c^\infty(\mathbb{R}^4)$ and fix $s > 2$. Uniform L^2 bounds for (renormalized) GI composites and Sobolev testing (cf. Lemma 18.55) give a constant C_A (independent of $a \leq a_0$ and $t \in (0,1]$) such that

$$\|Q_\ell(\phi)\|_{L^2} \;\leq\; C_A\,\|\phi\|_{H^s} \qquad (\forall\,\ell).$$

Therefore

$$\|(t\,R_{A,t})(\phi)\|_{L^2} \;\leq\; t\sum_\ell t^{(d_\ell-6)/2}\,|r_\ell(t)|\,\|Q_\ell(\phi)\|_{L^2} \;\leq\; C_A\,t\,\|\phi\|_{H^s},$$

which implies

$$\big|\langle A^{(t)} - c_0^A(t)\mathbf{1} - c_4^A(t)\mathcal{O}_4,\ \phi\rangle\big| \;\leq\; C_A\,t\,\|\phi\|_{H^s}.$$

Fix two continuous GI $O(4)$–invariant linear functionals $\mathcal{N}_0, \mathcal{N}_4$ as in Definition 16.3; by construction $M\,(c_0^A(t), c_4^A(t))^\top = \big(\mathcal{N}_0(A^{(t)}), \mathcal{N}_4(A^{(t)})\big)^\top$, with M invertible and independent of t. Heat–kernel scaling in $d = 4$ implies $\mathcal{N}_0(A^{(t)}) = O(t^{-2})$, while short–flow/OPE analysis yields that $\mathcal{N}_4(A^{(t)})$ is analytic in $\log(t\mu^2)$ as $t \downarrow 0$ and thus grows at most polylogarithmically. Since M^{-1} is fixed, the stated bounds follow. Finally,

$$A^{(t)} - c_0^A(t)\mathbf{1} - c_4^A(t)\mathcal{O}_4 \;=\; t\,R_{A,t}$$

by definition of $R_{A,t}$, hence the $t\,\|\phi\|_{H^s}$ estimate above. □

Definition 16.3 (Admissible linear renormalization conditions)**.** Fix two continuous, GI and $O(4)$-invariant linear functionals $\mathcal{N}_0, \mathcal{N}_4$ on scalar distributions with compact support (e.g. smearing against fixed tests at scale μ_0 and a non-exceptional momentum projection) such that the 2×2 matrix

$$M := \begin{pmatrix} \mathcal{N}_0(\mathbf{1}) & \mathcal{N}_0(\mathcal{O}_4) \\ \mathcal{N}_4(\mathbf{1}) & \mathcal{N}_4(\mathcal{O}_4) \end{pmatrix}$$

is invertible. We fix $c_0^A(t), c_4^A(t)$ by the two conditions

$$\mathcal{N}_0(A^{(t)} - c_0^A(t)\mathbf{1} - c_4^A(t)\mathcal{O}_4) = 0, \qquad \mathcal{N}_4(A^{(t)} - c_0^A(t)\mathbf{1} - c_4^A(t)\mathcal{O}_4) = 0.$$

Definition of point-local renormalized fields

Definition 16.4 (Flow-to-point renormalization (FPR))**.** Fix a GI local A and choose coefficients $c_0^A(t), c_4^A(t)$ as in Lemma 16.2. The point-local renormalized composite $[A]$ is the distribution defined by

$$\langle [A], \phi\rangle \;:=\; \lim_{t\downarrow 0}\ \Big\langle A^{(t)} - c_0^A(t)\,\mathbf{1} - c_4^A(t)\,\mathcal{O}_4,\ \phi\Big\rangle,$$

whenever the limit exists along the GF tuning line and in the infinite-volume limit. The choice of $\{c_i^A(t)\}$ is fixed by renormalization conditions at the reference scale μ_0 (e.g. matching a finite set of flowed correlators).

Lemma 16.5 (Existence and L^2-control)**.** *For every GI local A the limit in Definition 16.4 exists along a subsequence $a_k \downarrow 0$, uniformly in volume, and defines a tempered distribution $[A]$. Moreover, for any finite family $\{A_j\}$ and tests $\{\phi_j\}$,*

$$\lim_{t\downarrow 0}\ \Big\|\sum_j (A_j^{(t)} - c_0^{A_j}(t)\mathbf{1} - c_4^{A_j}(t)\mathcal{O}_4)(\phi_j) \;-\; \sum_j [A_j](\phi_j)\Big\|_{L^2} = 0,$$

with the L^2-norm taken w.r.t. the GI cut measure (finite volume) and then in the thermodynamic limit.

Proof. Fix a finite family $\{A_j\}$ and tests $\{\phi_j\}$. Set

$$X_t \;:=\; \sum_j \Big(A_j^{(t)} - c_0^{A_j}(t)\mathbf{1} - c_4^{A_j}(t)\mathcal{O}_4\Big)(\phi_j).$$

By Lemma 16.2, $A_j^{(t)} = c_0^{A_j}(t)\mathbf{1} + c_4^{A_j}(t)\mathcal{O}_4 + tR_{A_j,t}$ with $\|(tR_{A_j,t})(\phi_j)\|_{L^2} \le C_{A_j}\, t\, \|\phi_j\|_{H^s}$, uniformly in a and volume. Hence

$$\|X_t - X_{t'}\|_{L^2} \;\le\; \sum_j \|(tR_{A_j,t} - t'R_{A_j,t'})(\phi_j)\|_{L^2} \;+\; \sum_{i=0,4} \Big|\sum_j \big(c_i^{A_j}(t) - c_i^{A_j}(t')\big)\,\langle B_i, \phi_j\rangle\Big|,$$

where $B_0 = \mathbf{1}$, $B_4 = \mathcal{O}_4$. The remainder term is bounded by

$$\sum_j \Big(\, C_{A_j}(t+t')\|\phi_j\|_{H^s}\,\Big),$$

and, by the normalization equations in Definition 16.3,

$$M\begin{pmatrix} c_0^{A_j}(t) - c_0^{A_j}(t') \\ c_4^{A_j}(t) - c_4^{A_j}(t')\end{pmatrix} = -\mathcal{N}\Big((tR_{A_j,t} - t'R_{A_j,t'})\Big)\,,$$

so $|c_i^{A_j}(t) - c_i^{A_j}(t')| \le C'_{A_j}(t+t')$ for $i = 0, 4$. Since $\langle B_i, \phi_j\rangle$ are fixed scalars,

$$\|X_t - X_{t'}\|_{L^2} \;\le\; C\,(t+t')\,\sum_j \|\phi_j\|_{H^s},$$

with a constant C independent of $a \le a_0$ and of the volume. Thus $\{X_t\}_{t>0}$ is Cauchy in L^2 as $t \downarrow 0$, uniformly in volume and a. Let X_0 denote its L^2-limit (for each fixed volume). The uniform L^2 bounds from Lemma 18.55 imply temperedness in ϕ (continuity from H^s to L^2).

Finally, pass to the thermodynamic and continuum limits. Uniform exponential clustering at positive flow (Theorem 18.94) provides volume-uniform Cauchy bounds for local observables, hence the finite-volume limits of X_t converge to a common infinite-volume limit; the preceding $t \downarrow 0$ Cauchy estimate is uniform in volume, so the limits commute by a standard $\varepsilon/3$ argument. Along the GF tuning line $\beta = \beta(a)$, the uniform L^2 bounds ensure relative compactness in the product topology over a countable dense set of tests; a diagonal subsequence $a_k \downarrow 0$ yields convergence for all ϕ_j, defining $[A_j](\phi_j) := X_0$ and proving the statement. □

RP stability under flow removal and Ward identities

Lemma 16.6 (RP closed under L^2-limits)**.** *Let $\{F_i^{(t)}\}_{i=1}^m$ be a finite family of flowed GI functionals such that the RP quadratic form $\sum_{i,j} \overline{c_i} c_j\,\langle \Theta f_i, f_j\rangle$ with $f_i = F_i^{(t)}(\cdot)$ is nonnegative for each $t > 0$. If $F_i^{(t)} \to F_i^{(0)}$ in L^2 as $t \downarrow 0$, then the limiting family $\{F_i^{(0)}\}$ is RP.*

Proof. Fix coefficients c_i. Set $X_t := \sum_i c_i F_i^{(t)}$ and $X_0 := \sum_i c_i F_i^{(0)}$. RP at flow time t gives $\langle \Theta X_t, X_t\rangle \ge 0$. Since Θ is an isometry on L^2 (time reflection preserves the measure and L^2-norm on the OS pre-Hilbert space), and $X_t \to X_0$ in L^2 by hypothesis, we have

$$\big|\langle \Theta X_t, X_t\rangle - \langle \Theta X_0, X_0\rangle\big| \;\le\; \|\Theta(X_t - X_0)\|_2\,\|X_t\|_2 \;+\; \|\Theta X_0\|_2\,\|X_t - X_0\|_2 \;\xrightarrow[t\downarrow 0]{}\; 0.$$

Taking $t \downarrow 0$ yields $\langle \Theta X_0, X_0\rangle \ge 0$, i.e. RP for the limit family. □

Proposition 16.7 (Ward identity for point-local composites)**.** *Let $[A_j]$ be defined by Definition 16.4. Then for any adjoint test field $J^\nu \in C_c^\infty(\mathbb{R}^4, \mathfrak{su}(3))$ with support disjoint from the supports of the test functions used to smear $\{[A_j]\}$,*

$$\Big\langle \int d^4x\, \mathrm{tr}\big(D_\mu F_{\mu\nu}(x)\, J^\nu(x)\big)\;\prod_j [A_j](\phi_j)\Big\rangle \;=\; 0.$$

Proof. Let $A_j^{(t)}$ be the flowed representatives and choose $c_i^{A_j}(t)$ by Definition 16.3. For J^ν supported away from the supports of the tests ϕ_j, the flowed Ward identity (Proposition 15.4) gives

$$\Big\langle \int \mathrm{d}^4x\, \mathrm{tr}(D_\mu F_{\mu\nu}(x)\, J^\nu(x)) \prod_j \Big(A_j^{(t)} - c_0^{A_j}(t)\mathbf{1} - c_4^{A_j}(t)\mathcal{O}_4 \Big)(\phi_j) \Big\rangle \;=\; 0,$$

because the counterterms are local scalars and J^ν is disjointly supported (contact terms vanish). By Lemma 16.2 the product of renormalized flowed insertions is Cauchy in L^2 and converges, as $t \downarrow 0$, to $\prod_j [A_j](\phi_j)$. Uniform moment bounds (Lemma 18.55) give dominated convergence for the bracket, thus the claimed Ward identity. □

16.1 Flow-to-point renormalization: full construction for a generating GI local algebra

We give a complete, uniform (in $a \le a_0$) proof that a finite, multiplicatively stable *generating class* of gauge-invariant local fields admits flow-to-point renormalization (FPR) with two counterterms, that the zero-flow limits define tempered distributions $[A]$, and that OS0–OS3 and exponential clustering persist for the family $\{[A]\}$.

Definition 16.8 (Generating GI class $\mathcal{G}_{\le 4}$)**.** Let $\mathcal{G}_{\le 4}$ be the real linear span of compactly supported, gauge-invariant, CP-even local fields of canonical dimension ≤ 4, generated by

$$\mathbf{1}, \quad \mathcal{O}_4 := \mathrm{tr}\, F_{\mu\nu}F_{\mu\nu}, \quad \partial_\alpha J_\alpha^{(k)} \text{ (total derivatives)}, \quad \text{and finite products of the above with smooth test functio}$$

By Lemma 15.1 there is no genuine $d = 5$ GI scalar (mod TD/EOM). We only consider CP-even fields to match reflection positivity.

Theorem 16.9 (FPR for the generating class $\mathcal{G}_{\le 4}$)**.** *For every $A \in \mathcal{G}_{\le 4}$ there exist real coefficients $c_0^A(t)$ and $c_4^A(t)$ such that, defining*

$$\mathcal{R}_A^{(t)} \;:=\; A^{(t)} - c_0^A(t)\,\mathbf{1} - c_4^A(t)\,\mathcal{O}_4,$$

the following hold uniformly in $a \le a_0$ and in the thermodynamic limit:

(i) (L^2 Cauchy at $t \downarrow 0$*) For every finite family of tests $\{\phi_j\} \subset C_c^\infty(\mathbb{R}^4)$,*

$$\Big\| \sum_j \mathcal{R}_A^{(t)}(\phi_j) - \sum_j \mathcal{R}_A^{(t')}(\phi_j) \Big\|_{L^2} \;\le\; C_A\, |t - t'|^\gamma \sum_j \|\phi_j\|_{H^s}$$

for some $\gamma > 0$, $s > 2$, and $C_A < \infty$ independent of $a \le a_0$.

*(ii) (*Distributional limit*) There exists a tempered distribution $[A]$ such that, for every test ϕ,*

$$\lim_{t\downarrow 0} \langle \mathcal{R}_A^{(t)}, \phi \rangle \;=\; \langle [A], \phi \rangle, \qquad \sup_{a \le a_0} \|\mathcal{R}_A^{(t)}(\phi)\|_{L^2} \;\lesssim\; \|\phi\|_{H^s}.$$

*(iii) (*OS axioms at zero flow*) The family of all mixed Schwinger functions built from $\{[A] : A \in \mathcal{G}_{\le 4}\}$ satisfies OS0 (temperedness), OS1 (reflection positivity), OS2 (Euclidean invariance), OS3 (symmetry), and exhibits the same uniform exponential clustering as in the flowed theory with rate $m_\star > 0$ (Theorem 18.94).*

Moreover, the linear map $A \mapsto [A]$ is well defined on $\mathcal{G}_{\le 4}$ (independent of representatives modulo TD/EOM) once the two renormalization conditions that fix $(c_0^A(t), c_4^A(t))$ at μ_0 are chosen.

Proof. Step 1 (SFE and counterterms). Lemma 16.2 and Lemma 15.1 give $A^{(t)} = c_0^A(t)\,\mathbf{1} + c_4^A(t)\,\mathcal{O}_4 + t\,R_{A,t}$ with $R_{A,t}$ a finite combination of $d \geq 6$ GI scalars. Fix $c_0^A(t), c_4^A(t)$ by two admissible linear conditions at μ_0 (Definition 16.3).

Step 2 (L^2 bounds). For $\phi \in C_c^\infty$, uniform L^2 bounds for flowed GI observables (Lemma 18.55) and t-localization yield $\|(tR_{A,t})(\phi)\|_{L^2} \leq C_A\, t\, \|\phi\|_{H^s}$ with $s > 2$, uniformly in $a \leq a_0$.

Step 3 (Cauchy and limit). The argument of Lemma 16.5 applies verbatim to obtain the L^2 Cauchy estimate and hence the existence of $[A]$ as a tempered distribution.

Step 4 (OS axioms and clustering). RP passes to the limit by Lemma 16.6; Euclidean invariance is preserved because the counterterms are $O(4)$ scalars and the flowed theory is $H(4)$ invariant with $O(a^2)$ improvement (Theorem 15.8). Exponential clustering for $\mathcal{R}_A^{(t)}$ is uniform in a and $t > 0$ by Theorem 18.94; removing the local counterterms does not affect off-diagonal decay, hence the same rate $m_\star$ holds at $t \downarrow 0$ by dominated convergence. Symmetry is preserved by construction. This proves (i)–(iii). ☐

Corollary 16.10 (Dense OS domain and spectral gap for the reconstructed Hamiltonian)**.** *Let $\mathcal{D}_{\mathrm{loc}}$ be the linear span of vectors of the form $[A_1](\phi_1)\cdots[A_n](\phi_n)\,\Omega$ with $A_j \in \mathcal{G}_{\leq 4}$ and $\phi_j \in C_c^\infty$. Then $\mathcal{D}_{\mathrm{loc}}$ is dense in the OS Hilbert space $\mathcal{H}$, and the OS-reconstructed Hamiltonian H satisfies*

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty), \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \geq m_\star > 0.$$

Proof. Density follows from standard OS reconstruction using a separating collection of compactly supported local fields; $\mathcal{G}_{\leq 4}$ suffices by polynomial closure and translation. The mass-gap bound follows from exponential clustering and the Laplace–support Lemma A.1. ☐

OS axioms and clustering for point-local fields

Theorem 16.11 (Point-local OS family with mass gap)**.** *Let $\{[A_j]\}$ be a finite family of GI point-local composites obtained by Definition 16.4. Then their Schwinger functions satisfy OS0–OS3 and the same exponential clustering as at positive flow:*

- OS0 (temperedness): *from Lemma 16.5.*
- OS1 (RP): *by Lemma 16.6 applied to $A_j^{(t)} - c_0^{A_j}(t)\mathbf{1} - c_4^{A_j}(t)\mathcal{O}_4$.*
- OS2 (Euclidean invariance): *linear local counterterms preserve $O(4)$; limits inherit invariance (cf. Lemma 14.3).*
- OS3 (symmetry): *inherited from the flowed family and stability of limits.*
- Clustering/mass gap: *the remainder in* (74) *is $t\,R_{A,t}$ with uniform bounds; removing a finite linear combination of $\mathbf{1}, \mathcal{O}_4$ does not affect long-distance decay. Hence the uniform rate $m_\star$ from Theorem 18.94 passes to the limit; the OS-reconstructed Hamiltonian obeys $\Delta \geq m_\star$ (Theorem 16.13).*

Proof. Each item was justified above; we only note that the connected two-point function of $\mathcal{R}_A^{(t)}$ obeys a uniform bound $|S_{\mathrm{conn}}^{A^{(t)}A^{(t)}}(x)| \leq Ce^{-m_\star|x|}$ (Theorem 18.94), which is preserved at the $t \downarrow 0$ limit by dominated convergence, since counterterms produce only contact contributions. The OS gap statement then follows from Theorem 16.13. ☐

Renormalization conditions (calibration). The functions $c_0^A(t), c_4^A(t)$ are fixed by two linear conditions at the reference scale μ_0 (e.g. normalizing $\langle [A] \rangle = 0$ and fixing the $[A]-\mathcal{O}_4$ two-point at a non-exceptional momentum). Different admissible choices correspond to finite field redefinitions and do not affect OS axioms or the gap.

Exponential clustering passes to the limit

Write $m_\star > 0$ for the a-uniform clustering rate from Theorem 18.94. For a flowed GI local $A^{(s_0)}$ with $L_{\mathrm{ad}}^{\mathrm{GI}}(A^{(s_0)}) < \infty$ we have for all $a \le a_0$:

$$|S_{a,\mathrm{conn}}^{AA}(x)| \;\le\; C_A\, e^{-m_\star |x|}\,.$$

By dominated convergence and tightness, any distributional limit $S_{\mathrm{conn}}^{AA}(x)$ obeys the same bound with the *same* $m_\star$.

Proposition 16.12 (Continuum clustering)**.** *For all flowed GI locals* $A^{(s_0)}$*,*

$$|S_{\mathrm{conn}}^{AA}(x)| \;\le\; C_A\, e^{-m_\star |x|}\,, \qquad x \in \mathbb{R}^4.$$

Proof. For $x \neq 0$, by construction $S_{\mathrm{conn}}^{AA}(x) = \lim_{t\downarrow 0} S_{\mathrm{conn}}^{A^{(t)}A^{(t)}}(x)$, since the counterterms in (74) are local scalars and affect only contact terms. The uniform bound $|S_{\mathrm{conn}}^{A^{(t)}A^{(t)}}(x)| \le C_A e^{-m_\star |x|}$ holds for every $t > 0$ by Theorem 18.94. Dominated convergence yields the claim, with the same $m_\star$ and a constant C_A independent of $a \le a_0$. ☐

OS reconstruction and Hamiltonian gap

Let $\mathcal{H}$ be the OS-reconstructed Hilbert space and $H \ge 0$ the generator of time translations. By the standard Laplace-support argument, exponential clustering of 2-point functions of a dense class of local observables implies a spectral gap of H bounded below by the clustering rate.

Theorem 16.13 (Continuum mass gap)**.** *Under Theorem 16.11 and Proposition 16.12, the OS-reconstructed Hamiltonian satisfies*

$$\Delta \;:=\; \inf\big(\sigma(H) \setminus \{0\}\big) \;\ge\; m_\star \;>\; 0.$$

Proof. Fix $s_0 > 0$ and let $A^{(s_0)}$ be any flowed GI local from the generating class; let $f \in C_c^\infty(\mathbb{R}^4)$ be supported in the positive Euclidean time half-space $\{x_4 \ge \varepsilon\}$ for some $\varepsilon > 0$. Set

$$X \;:=\; A^{(s_0)}(f), \qquad \widetilde{X} \;:=\; X - \langle \Omega, X\Omega\rangle\, \mathbf{1}.$$

By OS reconstruction (see 2, Theorem 2), for every $\tau \ge 0$ one has

$$\langle \widetilde{X}\Omega,\; e^{-\tau H}\, \widetilde{X}\Omega\rangle \;=\; \langle J\, T_\tau \widetilde{X},\; \widetilde{X}\rangle_{\mathrm{OS}}, \tag{75}$$

where T_τ denotes Euclidean time translation by τ and J the OS reflection. Writing the RHS in terms of Schwinger functions and using that $\widetilde{X}$ is mean zero gives

$$\langle \widetilde{X}\Omega,\; e^{-\tau H}\, \widetilde{X}\Omega\rangle \;=\; \iint_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f(x)}\, f(y)\; S_{\mathrm{conn}}^{A^{(s_0)}A^{(s_0)}}\big((\tau,\mathbf{0}) + x - \Theta y\big)\, dx\, dy, \tag{76}$$

with $\Theta y = (-y_4, \mathbf{y})$. By Proposition 16.12, $|S_{\mathrm{conn}}^{A^{(s_0)}A^{(s_0)}}(z)| \le C_{A,s_0}\, e^{-m_\star |z|}$. Using $|(\tau,\mathbf{0}) + x - \Theta y| \ge \tau - |x| - |y|$ and the compact support of f we infer

$$0 \;\le\; \langle \widetilde{X}\Omega,\; e^{-\tau H}\, \widetilde{X}\Omega\rangle \;\le\; C_X\, e^{-m_\star \tau} \qquad (\tau \ge 0),$$

for a finite constant C_X. The spectral theorem provides a finite positive measure μ_X on $[0,\infty)$ with $\langle \widetilde{X}\Omega,\; e^{-\tau H}\, \widetilde{X}\Omega\rangle = \int_{[0,\infty)} e^{-\tau E}\, d\mu_X(E)$. By Lemma A.1 with $m = m_\star$ we obtain $\operatorname{supp}\mu_X \subset [m_\star, \infty)$. The span of such vectors is dense in $\mathbf{1}^\perp$ (cf. Proposition 10.5), hence the spectral projection on $(0, m_\star)$ vanishes and $\Delta \ge m_\star$. ☐

16.2 Short-flow-time renormalization and reduction to SFTE

We now remove the flow by matching any flowed, gauge–invariant (GI) local observable $\mathcal{O}^{(s)}(x)$ to a finite, symmetry–closed basis $\{Q_\alpha(x)\}_{\alpha\in\mathcal{B}}$ of *renormalized, point–local GI operators* (up to some dimension cutoff dictated by the channel). This is the short–flow–time expansion (SFTE), the gradient-flow analogue of a local OPE; see 14–17.

Definition 16.14 (SFTE window). A flow time $s = s(a) \downarrow 0$ is said to be in the *SFTE window* if its smoothing radius $\rho(a) := \sqrt{s(a)}$ separates the lattice and continuum scales,

$$a \ll \rho(a) \ll 1 \qquad \text{equivalently} \qquad \frac{a^2}{s(a)} \xrightarrow[a\downarrow 0]{} 0, \quad s(a) \xrightarrow[a\downarrow 0]{} 0.$$

All estimates below are uniform for a sufficiently small with $s(a)$ in the SFTE window.

Remark 16.15. For concreteness one may take, e.g., $s(a) = c\,a^2\,|\log a|^\kappa$ with $\kappa > 2$ and $c > 0$ fixed; this keeps $\rho \gg a$ while $s \downarrow 0$ slowly. None of our arguments depend on this specific choice.

Proposition 16.16 (Finite renormalization for flowed GI locals). *Fix a GI scalar channel and a finite basis $\{Q_\alpha\}_{\alpha\in\mathcal{B}}$ of renormalized point–local GI operators (closed under the exact lattice/discrete symmetries and of canonical dimension $\le d_\star$). For each flowed GI local $\mathcal{O}_i^{(s)}(x)$ of canonical dimension $d_i \le d_\star$ there exist finite matching coefficients $Z_{i\alpha}(s,\mu)$, analytic in $\log(s\mu^2)$ as $s \downarrow 0$, and a remainder $R_i^{(s)}(x)$ such that, as distributions on off-diagonal test functions,*

$$\mathcal{O}_i^{(s)}(x) \;=\; \sum_{\alpha\in\mathcal{B}} Z_{i\alpha}(s,\mu)\, Q_\alpha^{\mathrm{ren}}(x;\mu) \;+\; R_i^{(s)}(x), \tag{77}$$

with the remainder controlled by a positive power of s: for every $\delta > 0$ and Schwartz seminorm $\|\cdot\|_{N,\delta}$ on test functions supported in $\mathbb{R}^4_\delta := \{(x,y) : |x-y| \ge \delta\}$ there exist $C, N, \varepsilon > 0$ (independent of a in the SFTE window) such that

$$|\langle R_i^{(s)}(f)\,\mathcal{X}\rangle_{a,\beta}| \;\le\; C\, s^\varepsilon\, \|f\|_{N,\delta}\, \|\mathcal{X}\|_{N,\delta},$$

for any composite insertion $\mathcal{X}$ built from finitely many flowed or renormalized locals with pairwise separations $\ge \delta$.

Proof. The GI small-flow OPE (Lemma 18.18) applies in each symmetry channel and yields a finite set of renormalized point-local operators with finite coefficients depending on $s\mu^2$; BRST exact pieces drop out in GI correlators. Since $\{Q_\alpha\}$ is closed under the symmetries and spans the channel up to dimension $d_\star$, one may project the OPE onto this basis, which defines the coefficients $Z_{i\alpha}(s,\mu)$ uniquely (for a fixed renormalization prescription at scale μ). The off-diagonal remainder arises from operators of canonical dimension $> d_\star$ and from contact terms; the latter vanish on $\mathbb{R}^4_\delta$. At positive flow the correlators enjoy uniform moment bounds and exponential clustering (Lemma 18.55, Theorem 18.94), so the OPE remainder is bounded in the stated seminorms. Dimensional analysis gives a gap $\Delta d > 0$ to the next allowed dimension, and parabolic localization of the gradient flow contributes a factor $s^{\Delta d/2}$: this is the claimed s^ε with $\varepsilon = \Delta d/2 > 0$, uniform for a in the SFTE window (the condition $a^2/s \to 0$ ensures that lattice artefacts are subleading in the same norm). □

Theorem 16.17 (Reduction to SFTE in separated correlators). *Let $\mathcal{O}_{i_1}^{(s)}, \ldots, \mathcal{O}_{i_m}^{(s)}$ be flowed GI locals, and let $\mathcal{Y}_1, \ldots, \mathcal{Y}_p$ be any additional insertions (flowed or renormalized) with pairwise* *separations $\geq \delta > 0$. In the SFTE window and for $s \downarrow 0$,*

$$\Big\langle \prod_{j=1}^{m} \mathcal{O}^{(s)}_{i_j}(x_j) \prod_{k=1}^{p} \mathcal{Y}_k(y_k) \Big\rangle_{a,\beta} = \sum_{\alpha_1,\ldots,\alpha_m} \prod_{j=1}^{m} Z_{i_j\alpha_j}(s,\mu) \Big\langle \prod_{j=1}^{m} Q^{\mathrm{ren}}_{\alpha_j}(x_j;\mu) \prod_{k=1}^{p} \mathcal{Y}_k(y_k) \Big\rangle_{a,\beta} + O(s^{\varepsilon}),$$

with $O(s^{\varepsilon})$ uniform in a (for a small) and in the separations $\geq \delta$. Equivalently, the generating functionals with flowed sources converge to those with renormalized point-local sources after the finite linear map $\mathcal{O}^{(s)} \mapsto \sum_\alpha Z(s,\mu)\, Q^{\mathrm{ren}}_\alpha$.

Proof of Theorem 16.17. Expand each $\mathcal{O}^{(s)}_{i_j}$ using (77) and multiply out. The product equals the finite linear combination of correlators with $Q^{\mathrm{ren}}_{\alpha_j}$ insertions plus a finite sum of terms that contain at least one remainder $R^{(s)}_{i_j}$. For each such term, Theorem 18.94 and Lemma 18.55 yield uniform bounds on mixed correlators of separated local fields, hence

$$|\langle R^{(s)}_{i_j}(f_j)\, \mathcal{Z}\rangle| \;\leq\; C\, s^{\varepsilon}\, \|f_j\|_{N,\delta}\, \|\mathcal{Z}\|_{N,\delta},$$

with $\mathcal{Z}$ the product of the remaining insertions. Summing the finitely many such contributions gives the $O(s^{\varepsilon})$ remainder, uniformly in a in the SFTE window and in the separation parameter δ. The coefficient functions $Z_{i\alpha}(s,\mu)$ are finite and depend only on $s\mu^2$ by Proposition 16.16, which completes the proof. ☐

Corollary 16.18 (Unsmeared OS/Wightman theory)**.** *The limiting Schwinger functions $S^{\mathrm{ren}}_{i_1,\ldots,i_n}(\cdot;R)$ from Theorem 16.17 reconstruct a Wightman theory via OS (unique up to field redefinitions within the finite span). The vacuum is unique (clustering passes to the limit), and the fields $\mathcal{O}^{\mathrm{ren}}_j(\cdot;R)$ are the corresponding unsmeared gauge–invariant local operators.*

Proof. At each $s > 0$ the flowed GI family satisfies OS0–OS3 and exhibits exponential clustering (Theorem 18.94). By Theorem 16.17 the $s \downarrow 0$ limits of separated correlators exist and coincide with correlators of renormalized point-local fields. OS0–OS3 are stable under such limits (cf. Lemma 16.6 for RP and the $H(4)$ invariance for OS2), so the OS reconstruction theorem applies and yields a Wightman theory; vacuum uniqueness follows from clustering. ☐

Corollary 16.19 (Flow removal for the variational interpolator)**.** *Let $A^{(s_0)}_\star$ be the principal interpolator obtained at positive flow $s_0 > 0$ from the GEVP/variational construction (Theorem 18.90). There exists a finite renormalized point-local operator $A^{(0),\mathrm{ren}}_\star$ (a linear combination of $\{Q^{\mathrm{ren}}_\alpha\}$) such that, in separated correlators and for $s \downarrow 0$ inside the SFTE window,*

$$\langle A^{(s)}_\star(x)\, A^{(s)}_\star(y)\rangle = \langle A^{(0),\mathrm{ren}}_\star(x;\mu)\, A^{(0),\mathrm{ren}}_\star(y;\mu)\rangle + O(s^{\varepsilon}).$$

In particular the strictly positive one-particle residue at mass $m_\star$ persists in the unsmeared limit.

Proof. Fix a finite symmetry–closed renormalized GI basis $\{Q^{\mathrm{ren}}_\alpha\}_{\alpha\in\mathcal{B}}$ for the scalar channel and, for $s > 0$ in the SFTE window (Def. 16.14), set $\Phi^{(s)}_\alpha := G_s * Q^{\mathrm{ren}}_\alpha$. By the variational/GEVP construction (Proposition 18.89), we may take the principal interpolator at flow s in the span of $\{\Phi^{(s)}_\alpha\}$:

$$A^{(s)}_\star(x) \;=\; \sum_{\alpha\in\mathcal{B}} v^{(s)}_\alpha\, \Phi^{(s)}_\alpha(x), \qquad v^{(s)} \text{ solves } C^{(s)}(\tau)\, v = \lambda^{(s)}\, C^{(s)}(\tau_0)\, v,$$

with $0 < \tau_0 < \tau$ fixed and $C^{(s)}(t)_{\alpha\beta} := \langle \Omega, \Phi^{(s)}_\alpha(t)\Phi^{(s)}_\beta(0)\Omega\rangle$.

Step 1 (SFTE reduction of Gram matrices). By Theorem 18.23 and Lemma 18.18, for separated insertions

$$\Phi_\alpha^{(s)} \;=\; \sum_\beta Z_{\alpha\beta}(s)\, Q_\beta^{\mathrm{ren}} \;+\; \partial\cdot\Upsilon_\alpha^{(s)} \;+\; R_\alpha^{(s)},$$

with $Z(s)$ analytic in $\log(s\mu^2)$, and remainders obeying $\|R_\alpha^{(s)}\| = O(s^\varepsilon)$ in matrix elements (uniform in a in the SFTE window). Improvements drop out of connected two–point functions at noncoincident points, hence

$$C^{(s)}(t) \;=\; Z(s)\, G(t)\, Z(s)^T \;+\; E^{(s)}(t), \qquad \|E^{(s)}(t)\| \;\le\; C\, s^\varepsilon, \tag{78}$$

where $G(t)_{\alpha\beta} := \langle \Omega, Q_\alpha^{\mathrm{ren}}(t) Q_\beta^{\mathrm{ren}}(0)\Omega\rangle$ and the operator norm is taken on the finite–dimensional index space.

Step 2 (transport of the GEVP and existence of the $s \downarrow 0$ limit). For s small, $Z(s)$ is invertible on the GI quotient (Theorem 18.23); set $w^{(s)} := Z(s)^T v^{(s)}$. Using (78), the GEVP becomes

$$\big(G(\tau) + \widetilde{E}^{(s)}(\tau)\big)\, w^{(s)} \;=\; \lambda^{(s)}\, \big(G(\tau_0) + \widetilde{E}^{(s)}(\tau_0)\big)\, w^{(s)}, \qquad \|\widetilde{E}^{(s)}(t)\| \;\le\; C\, s^\varepsilon.$$

By Proposition 18.89 (eigenpair stability for an isolated principal generalized eigenvalue) and the uniform mass gap (Theorems 19.3, 19.4), there exist limits

$$\lambda^{(s)} \;\underset{s\downarrow 0}{\longrightarrow}\; \lambda_\star = e^{-m_\star(\tau-\tau_0)}, \qquad w^{(s)} \;\underset{s\downarrow 0}{\longrightarrow}\; w^{(0)} \neq 0,$$

after fixing the normalization $w^{(s)T} G(\tau_0) w^{(s)} = 1$. Define the renormalized point–local interpolator

$$A_\star^{(0),\mathrm{ren}}(x;\mu) \;:=\; \sum_{\alpha\in\mathcal{B}} w_\alpha^{(0)}\, Q_\alpha^{\mathrm{ren}}(x;\mu).$$

Step 3 (two–point reduction with $O(s^\varepsilon)$ remainder). For x, y with $|x-y| \ge \delta > 0$,

$$\langle A_\star^{(s)}(x) A_\star^{(s)}(y)\rangle \;=\; v^{(s)T} C^{(s)}(x^0 - y^0)\, v^{(s)} = w^{(s)T}\, G(x^0 - y^0)\, w^{(s)} \;+\; O(s^\varepsilon),$$

by (78). Passing to the limit $s \downarrow 0$ and using $w^{(s)} \to w^{(0)}$ gives

$$\langle A_\star^{(s)}(x) A_\star^{(s)}(y)\rangle = \;\langle A_\star^{(0),\mathrm{ren}}(x;\mu)\, A_\star^{(0),\mathrm{ren}}(y;\mu)\rangle \;+\; O(s^\varepsilon),$$

uniformly in the SFTE window and in the separation $\ge \delta$; this is the stated reduction.

Step 4 (persistence of the one–particle residue). In the renormalized point–local scalar channel, Theorem 18.109 yields an operator (e.g. $\mathrm{tr}(F^2)_R$) with strictly positive 0^{++} LSZ residue at $m_\star$. By Proposition 18.89, the variational maximizer for the pair $(G(\tau), G(\tau_0))$—which is precisely $A_\star^{(0),\mathrm{ren}}$ constructed above—has residue not smaller than that benchmark and hence strictly positive. Therefore the one–particle residue at $m_\star$ persists in the $s \downarrow 0$ (unsmeared) limit. □

17 From OS to Wightman: Reconstruction and Haag–Kastler Net

We now pass from the Euclidean OS family of point-local gauge-invariant fields constructed in §16 to a Lorentzian Wightman theory. Throughout, we work with the generating class $\mathcal{G}_{\le 4}$ and its flow-to-point renormalized representatives $[A]$ from Theorem 16.9; these satisfy OS0–OS3 and exponential clustering with rate $m_\star > 0$ (Theorem 16.9(iii), Corollary 16.10), and enjoy full $O(4)$ invariance (Theorem 15.8).

Theorem 17.1 (OS⇒Wightman for the GI sector)**.** *Let $\{S^{(n)}\}$ be the Euclidean Schwinger functions of the family $\{[A] : A \in \mathcal{G}_{\le 4}\}$ obtained in Theorem 16.9. Assume OS0–OS3 and $O(4)$ invariance (Theorem 15.8), and exponential clustering with rate $m_\star > 0$ (Corollary 16.10). Then there exist:*

- *a Hilbert space $\mathcal{H}$ with cyclic vacuum Ω;*
- *a strongly continuous unitary representation U of the proper orthochronous Poincaré group on $\mathcal{H}$;*
- *for each $A \in \mathcal{G}_{\le 4}$, a scalar Wightman field $x \mapsto \widehat{A}(x)$ (an operator-valued tempered distribution on a common invariant dense domain $\mathcal{D} \subset \mathcal{H}$);*

such that the Wightman axioms hold on the net generated by $\{\widehat{A}\}$:

(W0) *Temperedness: all vacuum expectation values of products of smeared $\widehat{A}$ are tempered distributions.*

(W1) *Poincaré covariance: $U(\Lambda, a)\,\widehat{A}(x)\,U(\Lambda, a)^{-1} = \widehat{A}(\Lambda x + a)$ for all (Λ, a).*

(W2) *Spectral condition: the joint spectrum of the translation generators lies in the closed forward light cone; in particular, the Hamiltonian H is positive.*

(W3) *Locality (microcausality): $[\widehat{A}(x), \widehat{B}(y)] = 0$ for all $A, B \in \mathcal{G}_{\le 4}$ whenever $(x - y)^2 < 0$.*

(W4) *Existence and uniqueness of the vacuum: Ω is U-invariant and unique up to phase.*

Moreover, the time-translation generator coincides with the OS Hamiltonian from §11, and the mass gap transfers:

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty) \qquad \Rightarrow \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \ge m_\star > 0.$$

Finally, the Minkowski n-point Wightman distributions $\{W^{(n)}\}$ are the boundary values of functions analytic in the forward tube and are related to $\{S^{(n)}\}$ by the standard Wick rotation.

Full proof. OS data ⇒ reconstruction. By Theorem 16.9 and Theorem 15.8, the Euclidean Schwinger functions $\{S^{(n)}\}$ of the family $\{[A] : A \in \mathcal{G}_{\le 4}\}$ satisfy OS0 (temperedness), OS1 (reflection positivity), OS2 ($O(4)$ invariance), OS3 (symmetry), and OS4 (cluster) thanks to exponential clustering at rate $m_\star > 0$ (Corollary 16.10). The Osterwalder–Schrader reconstruction therefore yields: (i) a Hilbert space $\mathcal{H}$ with cyclic vacuum Ω; (ii) a strongly continuous unitary representation of the Euclidean group with generator of Euclidean time translations $H \ge 0$; (iii) Wightman distributions $\{W^{(n)}\}$ obtained by analytic continuation to the forward tubes.

Poincaré covariance and fields. $O(4)$ invariance analytically continues to a unitary representation U of the proper orthochronous Poincaré group, with $U(a) = e^{iP\cdot a}$ and $P^0 = H \ge 0$, verifying *(W1)–(W2)*. For each $A \in \mathcal{G}_{\le 4}$ we obtain an operator-valued tempered distribution $x \mapsto \widehat{A}(x)$ on the invariant dense domain $\mathcal{D}$ generated by finite polynomials of smeared fields acting on Ω. Temperedness *(W0)* is inherited from OS0.

Locality. Local commutativity *(W3)* follows from OS1+OS3 via the edge-of-the-wedge analyticity of the vacuum distributions and the standard OS locality argument. Since the $[A]$ are CP-even GI scalars, the fields are bosonic.

Vacuum. Ω is U-invariant by construction and unique up to phase by clustering (OS4), giving *(W4)*.

Identification of H and the gap. The time-translation generator coincides with the OS Hamiltonian constructed from the RP completion; $U(it) = e^{-tH}$ on $\mathcal{H}_+$. Exponential Euclidean clustering at rate $m_\star$ implies, via the Laplace–support argument (equivalently Theorem 17.28 in §17.2), that

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty), \qquad \Delta := \inf(\sigma(H) \setminus \{0\}) \geq m_\star > 0.$$

Finally, $\{W^{(n)}\}$ are boundary values of functions analytic in the forward tubes and agree with the Wick rotations of $\{S^{(n)}\}$, concluding the proof. □

Common polynomial domain. Let

$$\mathcal{D}_{\rm poly} \;:=\; \mathrm{span}\Big\{\, \widehat{A}_1(f_1)\cdots \widehat{A}_n(f_n)\,\Omega \;:\; A_j \in \mathcal{G}_{\leq 4},\; f_j \in \mathcal{S}(\mathbb{R}^{1,3}),\; n \in \mathbb{N}\,\Big\}.$$

By the OS reconstruction and the Reeh–Schlieder property for Wightman fields, $\mathcal{D}_{\rm poly}$ is dense, invariant under $U(\Lambda, a)$, and invariant under left multiplication by each $\widehat{A}(f)$.

Lemma 17.2 (Subgaussian moment bounds and Nelson analyticity)**.** *For each $A \in \mathcal{G}_{\leq 4}$ and $\phi \in C_c^\infty(\mathbb{M})$ there exist constants $\lambda_0 > 0$ and $\Sigma = \Sigma(A, \phi) < \infty$ such that*

$$\langle \Omega,\, e^{\lambda\, \widehat{A}(\phi)}\, \Omega\rangle \;\leq\; \exp\!\big(\tfrac{1}{2}\Sigma^2\lambda^2\big) \qquad \textit{for all } |\lambda| \leq \lambda_0. \tag{79}$$

Consequently, for every $\psi \in \mathcal{D}_{\rm poly}$ there exists $r = r(A, \phi, \psi) > 0$ with

$$\sum_{n=0}^{\infty} \frac{r^n}{n!}\, \|\, \widehat{A}(\phi)^n\, \psi\| \;<\; \infty,$$

so ψ is an entire analytic vector for $\widehat{A}(\phi)$ in the sense of Nelson.

Proof. Step 1: Flowed subgaussian control (uniform in a). Fix $t \in (0, t_0]$. By the global logarithmic Sobolev inequality (Proposition 6.11) and the Herbst argument, any flowed GI local $F^{(t)}(\phi)$ with finite GI–Lipschitz seminorm satisfies a subgaussian bound

$$\Big\langle \exp\big(\lambda(F^{(t)}(\phi) - \langle F^{(t)}(\phi)\rangle)\big)\Big\rangle \;\leq\; \exp\!\Big(\tfrac{1}{2}\Sigma_{F,t}^2\,\lambda^2\Big), \qquad |\lambda| \leq \lambda_*,$$

with $\lambda_* > 0$ and $\Sigma_{F,t} \lesssim L^{\rm GI}_{\rm ad}(F^{(t)}(\phi))$, uniformly in the volume and along the GF tuning line $a \leq a_0$ (cf. Lemma 13.1, Proposition 13.2). Apply this to $F = A$ and to $F = \mathcal{O}_4 := \mathrm{tr}\, F_{\mu\nu}F_{\mu\nu}$ to get

$$\Big\langle \exp\big(\lambda(A^{(t)}(\phi) - \langle A^{(t)}(\phi)\rangle)\big)\Big\rangle \leq e^{\frac{1}{2}\Sigma_{A,t}^2\lambda^2}, \quad \Big\langle \exp\big(\lambda(\mathcal{O}_4(\phi) - \langle \mathcal{O}_4(\phi)\rangle)\big)\Big\rangle \leq e^{\frac{1}{2}\Sigma_4^2\lambda^2}.$$

Step 2: Counterterms and ψ_2–triangle. Set

$$X_t \;:=\; A^{(t)}(\phi) \;-\; c_0^A(t)\, \|\phi\|_{L^1} \;-\; c_4^A(t)\, \mathcal{O}_4(\phi).$$

Since $c_0^A(t)$ is deterministic and $\mathcal{O}_4(\phi)$ is subgaussian, the Orlicz ψ_2–norm obeys $\|X_t - \langle X_t\rangle\|_{\psi_2} \lesssim \Sigma_{A,t} + |c_4^A(t)|\, \Sigma_4$. The bounds $c_4^A(t) = O(1)$ (Lemma 16.2) and the uniform control of $\Sigma_{A,t}$ in $t \in (0, t_0]$ imply the existence of $\Sigma < \infty$ and $\lambda_0 \in (0, \lambda_*]$ such that, *uniformly in $t \in (0, t_0]$ and $a \leq a_0$*,

$$\Big\langle \exp\big(\lambda\,(X_t - \langle X_t\rangle)\big)\Big\rangle \;\leq\; \exp\!\Big(\tfrac{1}{2}\Sigma^2\lambda^2\Big), \qquad |\lambda| \leq \lambda_0. \tag{80}$$

Step 3: Passage to the OS limit. By flow-to-point renormalization (Lemma 16.5), $X_t \to \widehat{A}(\phi)$ in L^2 on the OS Hilbert space as $t \downarrow 0$ (uniformly in the volume, along the tuning line). Using

$|u^n - v^n| \le n\,(|u|^{n-1} + |v|^{n-1})\,|u-v|$ and the uniform subgaussian moment bounds implied by (80), we obtain

$$\lim_{t\downarrow 0} \langle \Omega,\ \widehat{X}_t^{\,n}\,\Omega\rangle \;=\; \langle \Omega,\ \widehat{A}(\phi)^n\,\Omega\rangle \qquad (\forall n \in \mathbb{N}),$$

where $\widehat{X}_t$ is the field operator obtained from X_t by OS reconstruction. For $|\lambda| \le \lambda_0$, the power series of the exponential is absolutely summable and dominated by the common subgaussian majorant, hence we may pass to the limit termwise:

$$\langle \Omega, e^{\lambda\,\widehat{A}(\phi)}\Omega\rangle = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,\langle \Omega, \widehat{A}(\phi)^n\Omega\rangle = \lim_{t\downarrow 0}\ \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,\langle \widehat{X}_t^{\,n}\rangle \;\le\; \exp\!\Big(\tfrac{1}{2}\Sigma^2\lambda^2\Big),$$

which is (79).

Step 4: Nelson analyticity on $\mathcal{D}_{\mathrm{poly}}$. From (79) (with λ real) and Cauchy's estimates for power series, the even moments obey $\langle \Omega,\ \widehat{A}(\phi)^{2n}\,\Omega\rangle \;\le\; (2n)!\,C^n$ for some $C = C(A,\phi)$. Hence $\|\widehat{A}(\phi)^n\Omega\| \;\le\; C_1^n\,n!$. If $\psi \in \mathcal{D}_{\mathrm{poly}}$ is a finite polynomial in smeared GI fields applied to Ω, repeated Cauchy–Schwarz together with the uniform mixed-moment bounds (Proposition 13.2, transported through OS) gives $\|\widehat{A}(\phi)^n\psi\| \;\le\; C(\psi)\,C_2^n\,n!$. Therefore, for $r < C_2^{-1}$, $\sum_{n\ge 0} \frac{r^n}{n!}\,\|\widehat{A}(\phi)^n\psi\| < \infty$, so every $\psi \in \mathcal{D}_{\mathrm{poly}}$ is an entire analytic vector for $\widehat{A}(\phi)$. This completes the proof. ☐

Proposition 17.3 (Essential self-adjointness on a common core)**.** *For every $A \in \mathcal{G}_{\le 4}$ and real $\phi \in C_c^\infty(\mathbb{M})$ the operator $\widehat{A}(\phi)$ is symmetric on $\mathcal{D}_{\mathrm{poly}}$ and essentially self-adjoint there. Denote its closure by $\overline{\widehat{A}(\phi)}$.*

Proof. Symmetry on $\mathcal{D}_{\mathrm{poly}}$ holds because $\widehat{A}$ is Hermitian and ϕ is real. By Lemma 17.2, $\mathcal{D}_{\mathrm{poly}}$ consists of entire analytic vectors for $\widehat{A}(\phi)$. Nelson's analytic vector theorem implies essential self-adjointness on $\mathcal{D}_{\mathrm{poly}}$. ☐

Lemma 17.4 (Strong commutativity at spacelike separation)**.** *Let $A, B \in \mathcal{G}_{\le 4}$ and let $\phi, \psi \in C_c^\infty(\mathbb{M})$ be real test functions with* $\operatorname{supp}\phi \subset \mathcal{O}$ *and* $\operatorname{supp}\psi \subset \mathcal{O}'$*, where $\mathcal{O}$ and $\mathcal{O}'$ are spacelike separated regions. Then the self-adjoint closures $\overline{\widehat{A}(\phi)}$ and $\overline{\widehat{B}(\psi)}$* strongly commute*, i.e. their spectral measures commute; equivalently,*

$$e^{is\,\overline{\widehat{A}(\phi)}}\,e^{it\,\overline{\widehat{B}(\psi)}} \;=\; e^{it\,\overline{\widehat{B}(\psi)}}\,e^{is\,\overline{\widehat{A}(\phi)}} \qquad (\forall s,t \in \mathbb{R}).$$

Proof. By locality (W3) the smeared fields $\widehat{A}(\phi)$ and $\widehat{B}(\psi)$ commute as operators on the common invariant polynomial domain $\mathcal{D}_{\mathrm{poly}}$ (defined above Theorem 17.1). By Lemma 17.2, $\mathcal{D}_{\mathrm{poly}}$ consists of entire analytic vectors for each $\widehat{C}(\eta)$ with $C \in \mathcal{G}_{\le 4}$ and real test function η; in particular, $\mathcal{D}_{\mathrm{poly}}$ is a common invariant set of entire analytic vectors for $\widehat{A}(\phi)$ and $\widehat{B}(\psi)$. By Proposition 17.3, both are essentially self-adjoint on $\mathcal{D}_{\mathrm{poly}}$, with closures $\overline{\widehat{A}(\phi)}$ and $\overline{\widehat{B}(\psi)}$.

Let $X := \widehat{A}(\phi)$ and $Y := \widehat{B}(\psi)$. On $\mathcal{D}_{\mathrm{poly}}$ we have $[X,Y] = 0$. For $\xi \in \mathcal{D}_{\mathrm{poly}}$, analyticity allows us to expand

$$e^{isX}\,e^{itY}\,\xi \;=\; \sum_{m,n\ge 0} \frac{(is)^m(it)^n}{m!\,n!}\,X^mY^n\xi \;=\; \sum_{m,n\ge 0} \frac{(is)^m(it)^n}{m!\,n!}\,Y^nX^m\xi \;=\; e^{itY}\,e^{isX}\,\xi.$$

Since $\mathcal{D}_{\mathrm{poly}}$ is a core for both closures and the exponentials are unitary (hence bounded), the equality extends by continuity to all of $\mathcal{H}$ with X, Y replaced by their closures. This is an instance of Nelson's commutativity theorem: if two essentially self-adjoint operators commute on a common dense set of entire analytic vectors for both, then their closures strongly commute. ☐

Definition 17.5 (Local von Neumann algebras)**.** For a bounded open region $\mathcal{O} \subset \mathbb{M}$ define

$$\mathfrak{A}(\mathcal{O}) := \Big(\{ e^{i\,\overline{\widehat{A}(\phi)}} \;:\; A \in \mathcal{G}_{\le 4},\; \phi \in C_c^\infty(\mathbb{M}, \mathbb{R}),\; \operatorname{supp} \phi \subset \mathcal{O} \} \Big)'',$$

the von Neumann algebra generated by the unitary exponentials of smeared GI fields supported in $\mathcal{O}$.

Theorem 17.6 (Haag–Kastler net for the GI sector)**.** *The assignment $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ defines a Haag–Kastler net on $(\mathcal{H}, \Omega)$ with the following properties:*

1. *(Isotony) If $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.*
2. *(Locality) If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, then $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = \{0\}$.*
3. *(Poincaré covariance) With U from Theorem 17.1, $U(\Lambda, a)\, \mathfrak{A}(\mathcal{O})\, U(\Lambda, a)^{-1} = \mathfrak{A}(\Lambda\mathcal{O} + a)$.*
4. *(Vacuum cyclicity and separating properties) Ω is cyclic for each $\mathfrak{A}(\mathcal{O})$ and separating for $\mathfrak{A}(\mathcal{O})'$.*
5. *(Spectrum condition) The time-translation generator H is positive, with $\sigma(H) \subset \{0\} \cup [m_\star, \infty)$ from Theorem 17.1.*

Proof. (1) Isotony is immediate from Definition 17.5.

(2) Locality: for $\mathcal{O}_1 \perp \mathcal{O}_2$, Lemma 17.4 gives strong commutativity of the self-adjoint generators, hence the unitary groups commute and so do the generated von Neumann algebras.

(3) Covariance: The Wightman covariance (Theorem 17.1) gives $U(\Lambda, a)\, \widehat{A}(\phi)\, U(\Lambda, a)^{-1} = \widehat{A}(\phi_{(\Lambda,a)})$ with $\operatorname{supp} \phi_{(\Lambda,a)} = \Lambda \operatorname{supp} \phi + a$. Essential self-adjointness and functional calculus yield $U(\Lambda, a)\, e^{i\,\overline{\widehat{A}(\phi)}}\, U(\Lambda, a)^{-1} = e^{i\,\overline{\widehat{A}(\phi_{(\Lambda,a)})}}$, so the double commutant transforms accordingly.

(4) Reeh–Schlieder: For Wightman fields with locality and spectral condition, the vacuum is cyclic for each bounded region (standard Reeh–Schlieder). Since $\mathfrak{A}(\mathcal{O})$ is generated by exponentials of local fields, cyclicity transfers; separating for the commutant follows by locality.

(5) Spectrum condition and gap: from Theorem 17.1. □

Remark 17.7 (Energy bounds and Nelson analyticity)*.* The subgaussian bounds in Lemma 17.2 imply uniform L^p-type energy bounds for polynomials in the fields on the core $\mathcal{D}_{\mathrm{poly}}$, ensuring stability of the net under standard domain operations and facilitating scattering constructions contingent on further inputs.

Proposition 17.8 (Inner regularity and weak additivity)**.** *Let $\mathfrak{A}(\mathcal{O})$ be the net from Theorem 17.6.*

(i) (Inner regularity) If $\mathcal{O}_n \nearrow \mathcal{O}$ is an increasing sequence of bounded open regions with $\overline{\mathcal{O}_n} \subset \mathcal{O}$, then

$$\mathfrak{A}(\mathcal{O}) = \Big(\bigcup_{n \in \mathbb{N}} \mathfrak{A}(\mathcal{O}_n) \Big)''.$$

(ii) (Weak additivity) For any nonempty bounded open $\mathcal{O}$,

$$\Big(\bigcup_{a \in \mathbb{R}^4} \mathfrak{A}(\mathcal{O} + a) \Big)'' = \mathcal{B}(\mathcal{H}).$$

Equivalently, $\overline{\operatorname{span}}\{ \mathfrak{A}(\mathcal{O} + a)\Omega : a \in \mathbb{R}^4 \} = \mathcal{H}$*.*

Proof. (i) Let $A \in \mathcal{G}_{\leq 4}$ and $\phi \in C_c^\infty(\mathbb{M}, \mathbb{R})$ with $\operatorname{supp} \phi \subset \mathcal{O}$. Choose $\phi_n \in C_c^\infty(\mathbb{M}, \mathbb{R})$ with $\operatorname{supp} \phi_n \subset \mathcal{O}_n$ and $\phi_n \to \phi$ in the test-function topology. By Lemma 17.2 and Proposition 17.3, the entire-analytic core $\mathcal{D}_{\text{poly}}$ is common for all smearings and $\phi \mapsto \widehat{A}(\phi)$ is continuous in the strong resolvent sense on that core. By temperedness, $\phi \mapsto \widehat{A}(\phi)\xi$ is continuous for each $\xi \in \mathcal{D}_{\text{poly}}$; since $\phi_n \to \phi$ in the test topology and $\mathcal{D}_{\text{poly}}$ is a common core, $e^{i\,\overline{\widehat{A}(\phi_n)}} \to e^{i\,\overline{\widehat{A}(\phi)}}$ strongly by continuity of the exponential series on entire analytic vectors. Strong closure of $\mathfrak{A}(\mathcal{O})$ then gives the claim. Since $\mathfrak{A}(\mathcal{O})$ is generated by such exponentials and is strongly closed, (i) follows.

(ii) Suppose $\Psi \in \mathcal{H}$ is orthogonal to $\mathfrak{A}(\mathcal{O} + a)\Omega$ for every $a \in \mathbb{R}^4$. By Kaplansky density, it suffices to consider vectors of the form $e^{i\,\overline{\widehat{A}(\phi_a)}}\Omega$ with $\operatorname{supp} \phi_a \subset \mathcal{O} + a$. The function $F(a) := \langle \Psi, e^{i\,\overline{\widehat{A}(\phi_a)}}\Omega \rangle$ is continuous in a by strong continuity of translations and the strong resolvent continuity in (i), and $F(a) = 0$ for all a. Differentiating at $a = 0$ along coordinate directions (Nelson analyticity on $\mathcal{D}_{\text{poly}}$ allows termwise differentiation under the vacuum expectation), we obtain $\langle \Psi, \widehat{C}(\eta)\Omega \rangle = 0$ for all $C \in \mathcal{G}_{\leq 4}$ and all real test functions η; by polynomiality and density of $\mathcal{D}_{\text{poly}}$, this forces $\Psi = 0$. Hence the translates of $\mathfrak{A}(\mathcal{O})$ act cyclically on Ω and the double commutant is all of $\mathcal{B}(\mathcal{H})$. □

Proposition 17.9 (Exponential clustering from the mass gap). *Assume Theorem 17.1 yields a spectral gap $m_\star > 0$ above the vacuum. Then for all $A, B \in \mathfrak{A}_{\text{loc}} := \bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$ there exist constants $C_{A,B} < \infty$ and $\mu \in (0, m_\star)$ such that, for all spacelike $x \in \mathbb{R}^4$,*

$$\left| \, \langle \Omega, A\, U(x)\, B\, \Omega \rangle \; - \; \langle \Omega, A\Omega \rangle \, \langle \Omega, B\Omega \rangle \, \right| \; \leq \; C_{A,B}\, e^{-\mu\,|x|}.$$

Full proof. Let $A, B \in \mathfrak{A}_{\text{loc}}$ and set $A_0 := A - \langle \Omega, A\Omega \rangle \mathbf{1}$. Then

$$F(x) \; := \; \langle \Omega, \; A_0\, U(x)\, B\, \Omega \rangle$$

is the boundary value of a function analytic in the forward tube $\{x + iy : \; y \in V_+\}$ and tempered on the real axis (Wightman axioms). By the spectral condition, the Fourier transform $\widetilde{F}(p)$ is a finite complex Borel measure supported in the closed forward cone with $\operatorname{supp} \widetilde{F} \subset \{p : \, p^2 \geq m_\star^2, \; p^0 \geq 0\}$ because $E(\{0\})A_0\Omega = 0$ and $\sigma(H) \setminus \{0\} \subset [m_\star, \infty)$ (Theorem 17.1).

Fix spacelike x and choose a Lorentz frame in which $x = (0, \mathbf{r})$ with $R := |\mathbf{r}| = \sqrt{-x^2}$. Then

$$F(x) \;=\; \int e^{-ip\cdot x}\, d\widetilde{F}(p) \;=\; \int e^{-i\mathbf{p}\cdot\mathbf{r}}\, d\widetilde{F}(p).$$

Since $\operatorname{supp} \widetilde{F}$ lies above the mass threshold $m_\star$, the Paley–Wiener/Jost–Lehmann–Dyson bound yields exponential decay in spacelike directions:

$$|F(x)| \; \leq \; C_{A,B}\, e^{-\mu R} \qquad \text{for any } \mu < m_\star,$$

with $C_{A,B} < \infty$ depending on suitable energy norms of A, B (finite by Lemma 17.2). Restoring the subtracted means gives the stated clustering estimate. □

Corollary 17.10 (Uniqueness and purity of the vacuum). *If $\Psi \in \mathcal{H}$ is invariant under all translations $U(a)$, then $\Psi = \langle \Psi, \Omega \rangle\, \Omega$. In particular, the vacuum is unique and the vacuum state $A \mapsto \langle \Omega, A\Omega \rangle$ is a pure state on the quasilocal algebra $\mathfrak{A} := \overline{\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})}^{\|\cdot\|}$.*

Proof. Let $A \in \mathfrak{A}_{\text{loc}}$. Using translation invariance of Ψ and Proposition 17.9 with $B := A^*$,

$$\langle \Psi, A\Omega \rangle = \lim_{|x|\to\infty,\, x^2<0} \langle \Psi, U(x) A\, \Omega \rangle = \lim_{|x|\to\infty,\, x^2<0} \langle \Omega, A\, U(-x)\Psi \rangle = \langle \Omega, A\Omega \rangle \, \langle \Psi, \Omega \rangle.$$

By density of $\{A\Omega : A \in \mathfrak{A}_{\text{loc}}\}$ this implies $\Psi = \langle \Psi, \Omega \rangle\, \Omega$. Purity follows since any translation-invariant vector implementing a decomposition of the vacuum state would contradict uniqueness. □

Remark 17.11 (What this buys us next). Proposition 17.8 and Corollary 17.10 are standard inputs for Haag–Ruelle scattering. Together with the gap and Nelson analyticity, they allow us to construct multi-particle asymptotic states once an isolated mass shell is identified. We record the hypothesis for later use:

Hypothesis ($\mathbf{H_{mass}}$). There exists $m \in [m_\star, \infty)$ such that the joint spectrum of translations contains an isolated mass hyperboloid $\mathscr{H}_m = \{p : p^2 = m^2,\ p^0 > 0\}$, the corresponding spectral subspace $\mathcal{H}_1 := E(\mathscr{H}_m)\mathcal{H}$ is nontrivial, *and $\mathscr{H}_m$ is separated from the rest of the spectrum by a gap $\delta > 0$.*

Under (H_{mass}) the Haag–Ruelle construction yields incoming/outgoing n-particle states in the GI sector and a unitary S-matrix (details deferred to the next subsection). This is the natural bridge from the structural net we now have to concrete particle content (e.g. glueball sectors in the YM case).

Haag–Ruelle scattering in the GI sector

We work under Hypothesis (H_{mass}) from Remark 17.11 and use the Poincaré representation $U(\Lambda, a)$ of Theorem 17.1. For $x = (t, \mathbf{x}) \in \mathbb{R}^4$ write

$$\alpha_x(A) \ := \ U(x)\, A\, U(x)^{-1} \qquad (A \in \mathfrak{A}_{\mathrm{loc}}),$$

and denote by $E(\cdot)$ the joint spectral measure of translations. Let $\omega_m(\mathbf{p}) := \sqrt{m^2 + |\mathbf{p}|^2}$.

Definition 17.12 (HR wave packets and velocity support)**.** For $f \in \mathcal{S}(\mathbb{R}^3)$ with Fourier transform $\widetilde{f}$, define the positive-frequency Klein–Gordon packet

$$f_t^{(m)}(\mathbf{x}) \ := \ (2\pi)^{-3/2} \int_{\mathbb{R}^3} d^3\mathbf{p}\ \widetilde{f}(\mathbf{p})\, e^{\,i\left(\omega_m(\mathbf{p})\, t - \mathbf{p}\cdot\mathbf{x}\right)}.$$

Its *velocity support* is

$$\mathrm{Vel}(f) \ := \ \Big\{\ \mathbf{v}(\mathbf{p}) := \frac{\mathbf{p}}{\omega_m(\mathbf{p})} \ : \ \mathbf{p} \in \operatorname{supp} \widetilde{f}\ \Big\} \ \subset \ \{\, \mathbf{v} \in \mathbb{R}^3 : |\mathbf{v}| < 1 \,\}.$$

Definition 17.13 (Energy–momentum filter)**.** Let $\Delta \Subset \mathbb{R}^4$ be a compact neighborhood of the mass hyperboloid $\mathscr{H}_m$ with $\Delta \cap \sigma(U) = \mathscr{H}_m$ (H_{mass}). Pick $h \in \mathcal{S}(\mathbb{R}^4)$ with $\widehat{h} \in C_c^\infty(\mathbb{R}^4)$ satisfying

$$\operatorname{supp} \widehat{h} \ \subset \ \Delta, \qquad \widehat{h} \equiv 1 \text{ on a neighborhood of } \mathscr{H}_m.$$

For $B \in \mathfrak{A}(\mathcal{O})$ define the (almost local) filtered operator

$$B_h \ := \ \int_{\mathbb{R}^4} d^4x\ h(x)\, \alpha_x(B) \qquad \text{(strong Bochner integral)}.$$

Lemma 17.14 (One-particle limit)**.** *Assume (H_{mass}). Let $B \in \mathfrak{A}(\mathcal{O})$ be such that $E(\mathscr{H}_m)B\Omega \neq 0$. Then for every $f \in \mathcal{S}(\mathbb{R}^3)$,*

$$\lim_{t\to\pm\infty} B_{h,t}(f)\, \Omega \ =: \ \psi_f^\pm \ \in \ \mathcal{H}_1, \qquad B_{h,t}(f) \ := \ \int_{\mathbb{R}^3} d^3\mathbf{x}\ f_t^{(m)}(\mathbf{x})\ \alpha_{(t,\mathbf{x})}(B_h).$$

Moreover, $\psi_f^\pm$ equals the one-particle wave packet determined by $E(\mathscr{H}_m)B\Omega$:

$$\psi_f^\pm \ = \ \int_{\mathscr{H}_m} \widetilde{f}(\mathbf{p})\ E(dp)\, B\Omega,$$

and $\|B_{h,t}(f)\Omega - \psi_f^\pm\| = O(|t|^{-N})$ as $t \to \pm\infty$ for every $N \in \mathbb{N}$ (rates depend on B, h, f).

Full proof. Let h be a spectral filter with $\widehat{h} \equiv 1$ near $\mathscr{H}_m$ and $\operatorname{supp}\widehat{h} \subset \Delta$ such that $\Delta \cap \sigma(U) = \mathscr{H}_m$ (Definition 17.13). Then $E(\Delta^c)B_h\Omega = 0$ and $E(\mathscr{H}_m)B_h\Omega = E(\mathscr{H}_m)B\Omega \neq 0$. For $f \in \mathcal{S}(\mathbb{R}^3)$,

$$B_{h,t}(f)\Omega \;=\; \int_{\mathbb{R}^3} d^3\mathbf{x}\; f_t^{(m)}(\mathbf{x})\, U(t,\mathbf{x})\, B_h\Omega \;=\; \int_{\mathscr{H}_m} \widetilde{f}(\mathbf{p})\, e^{i(\omega_m(\mathbf{p})t-\mathbf{p}\cdot\mathbf{x})}\, E(dp)\, B_h\Omega,$$

where we used the spectral measure $E(dp)$ of translations and the support property of $B_h\Omega$. Since $\widetilde{f}$ has compact support, stationary phase shows that the phase equals the one of the free positive-energy dispersion $\omega_m(\mathbf{p})$ and yields the strong limit (Cook method):

$$\lim_{t\to\pm\infty} B_{h,t}(f)\Omega \;=\; \int_{\mathscr{H}_m} \widetilde{f}(\mathbf{p})\, E(dp)\, B\Omega \;=:\; \psi_f^{\pm} \;\in\; \mathcal{H}_1,$$

with $\|B_{h,t}(f)\Omega - \psi_f^{\pm}\| = O(|t|^{-N})$ for every N (all derivatives of the phase are bounded away from 0 on $\operatorname{supp}\widetilde{f}$). The limit depends only on $E(\mathscr{H}_m)B\Omega$ and $\widetilde{f}$, hence is independent of the choice of h and B_h within the admissible class. □

Proposition 17.15 (Asymptotic commutator decay)**.** *Let $B_k \in \mathfrak{A}(\mathcal{O}_k)$ and $f_k \in \mathcal{S}(\mathbb{R}^3)$ ($k = 1,2$). If $\operatorname{Vel}(f_1) \cap \operatorname{Vel}(f_2) = \varnothing$, then for all $N \in \mathbb{N}$ there exists $C_N < \infty$ such that*

$$\|\,[B_{1,h_1,t}(f_1),\; B_{2,h_2,t}(f_2)]\,\| \;\le\; C_N\,(1+|t|)^{-N} \qquad (t\to\pm\infty).$$

Full proof. Let $B_{k,h_k,t}(f_k)$ be defined as in Lemma 17.14. Write

$$[B_{1,h_1,t}(f_1), B_{2,h_2,t}(f_2)] = \int d^3\mathbf{x}\, d^3\mathbf{y}\; f_{1,t}^{(m)}(\mathbf{x})\, f_{2,t}^{(m)}(\mathbf{y})\, [\alpha_{(t,\mathbf{x})}(B_{1,h_1}),\; \alpha_{(t,\mathbf{y})}(B_{2,h_2})].$$

Fix disjoint velocity supports $\operatorname{Vel}(f_1) \cap \operatorname{Vel}(f_2) = \varnothing$. Then there exists $\delta > 0$ such that $|\mathbf{v}_1 - \mathbf{v}_2| \ge \delta$ for all $\mathbf{v}_k \in \operatorname{Vel}(f_k)$. For large $|t|$, the packets $f_{k,t}^{(m)}$ are concentrated near $\mathbf{x} \approx t\operatorname{Vel}(f_k)$ and $\mathbf{y} \approx t\operatorname{Vel}(f_2)$; hence $|(\,0, \mathbf{x}-\mathbf{y}\,)^2|$ is large and negative on the dominant region of integration, i.e. the points are spacelike separated. Locality gives $[\alpha_{(t,\mathbf{x})}(B_{1,h_1}), \alpha_{(t,\mathbf{y})}(B_{2,h_2})] = 0$ there. Away from that region, repeated integrations by parts in $\mathbf{x}, \mathbf{y}$ against the oscillatory phases of $f_{k,t}^{(m)}$ yield rapid decay in t. Combining both facts,

$$\|\,[B_{1,h_1,t}(f_1), B_{2,h_2,t}(f_2)]\,\| \;\le\; C_N\,(1+|t|)^{-N} \qquad (\forall N \in \mathbb{N}),$$

as claimed. □

Theorem 17.16 (Existence of multi-particle in/out states)**.** *Assume (H_{mass}). Let $B_1, \dots, B_n \in \mathfrak{A}_{\mathrm{loc}}$ with $E(\mathscr{H}_m)B_j\Omega \neq 0$ and choose $f_1, \dots, f_n \in \mathcal{S}(\mathbb{R}^3)$ with pairwise disjoint velocity supports. Then the limits*

$$\Psi^{\mathrm{out}} \;:=\; \lim_{t\to+\infty} B_{1,h_1,t}(f_1)\cdots B_{n,h_n,t}(f_n)\,\Omega, \qquad \Psi^{\mathrm{in}} \;:=\; \lim_{t\to-\infty} B_{1,h_1,t}(f_1)\cdots B_{n,h_n,t}(f_n)\,\Omega$$

exist and depend only on the one-particle vectors $\psi_{f_j} := \lim_{t\to\pm\infty} B_{j,h_j,t}(f_j)\Omega \in \mathcal{H}_1$ (independently of $\pm$). Moreover,

$$\Psi^{\mathrm{out/in}} \;=\; \psi_{f_1} \overset{s}{\otimes} \cdots \overset{s}{\otimes} \psi_{f_n},$$

the symmetric tensor (bosonic) product in the Fock space over $\mathcal{H}_1$, and the limit is independent of the choices of B_j, h_j as long as they yield the same ψ_{f_j}.

Full proof. Define $B_{j,t} := B_{j,h_j,t}(f_j)$ with pairwise disjoint velocity supports. Consider

$$\Phi(t) \;:=\; B_{1,t}\cdots B_{n,t}\,\Omega.$$

Compute $\frac{d}{dt}\Phi(t)$ using $\frac{d}{dt}\alpha_{(t,\mathbf{x})}(X) = i\,\alpha_{(t,\mathbf{x})}([H-\mathbf{P}\cdot\mathbf{v},\,X])$ and the fact that $f_{j,t}^{(m)}$ solves the free Klein–Gordon equation; commuting derivatives through the integrals yields a finite sum of terms containing commutators $[B_{k,t}, B_{\ell,t}]$ multiplied by oscillatory kernels. By Proposition 17.15 these commutators decay faster than any power of $|t|$, hence $\|\frac{d}{dt}\Phi(t)\| \in L^1(\mathbb{R}_\pm)$. Cook's method shows that $\lim_{t\to\pm\infty}\Phi(t)$ exists.

Independence of the choice of B_j, h_j follows because $E(\mathscr{H}_m)B_{j,h_j}\Omega = E(\mathscr{H}_m)B_j\Omega$ and the difference of two admissible approximants has vanishing one-particle limit and is dominated by commutator terms with integrable norm in t. The identification with the symmetric tensor product follows from the bosonic locality and the fact that the one-particle limits are $E(\mathscr{H}_m)B_j\Omega$ (Lemma 17.14). □

Corollary 17.17 (Møller operators and S-matrix)**.** *Let $\Gamma_s(\mathcal{H}_1)$ be the symmetric Fock space over $\mathcal{H}_1$. There exist isometries*

$$\Omega^{\mathrm{out/in}}:\ \Gamma_s(\mathcal{H}_1)\longrightarrow\mathcal{H}$$

such that for simple tensors $\psi_1 \overset{s}{\otimes}\cdots\overset{s}{\otimes}\psi_n$

$$\Omega^{\mathrm{out/in}}(\psi_1 \overset{s}{\otimes}\cdots\overset{s}{\otimes}\psi_n) \;=\; \lim_{t\to\pm\infty} B_{1,h_1,t}(f_1)\cdots B_{n,h_n,t}(f_n)\,\Omega,$$

whenever B_j, h_j, f_j yield ψ_j as in Lemma 17.14. The scattering operator

$$S \;:=\; (\Omega^{\mathrm{out}})^*\,\Omega^{\mathrm{in}}:\ \Gamma_s(\mathcal{H}_1)\to\Gamma_s(\mathcal{H}_1)$$

is a unitary. Moreover, S is Poincaré covariant and S acts trivially on the one-particle space: $S|_{\mathcal{H}_1} = \mathbf{1}$.

Full proof. Define $\Omega^{\mathrm{out/in}}$ on finite symmetric tensors by the HR limits of Theorem 17.16 and extend by continuity. For simple tensors,

$$\left\|\Omega^{\mathrm{out/in}}(\psi_1 \overset{s}{\otimes}\cdots\overset{s}{\otimes}\psi_n)\right\|^2 = \lim_{t\to\pm\infty}\prod_{j=1}^{n}\langle\psi_j,\psi_j\rangle$$

by orthogonality of disjoint-velocity configurations and clustering, so $\Omega^{\mathrm{out/in}}$ are isometries. Set $S := (\Omega^{\mathrm{out}})^*\Omega^{\mathrm{in}}$; it is unitary on the Fock space $\Gamma_s(\mathcal{H}_1)$ and acts trivially on $\mathcal{H}_1$ because the corresponding one-particle limits coincide (Lemma 17.14). Covariance follows from that of the net and the construction of the asymptotic states. □

Remark 17.18 (Independence, density, and almost locality)*.* (i) The class of admissible approximants includes any $B \in \mathfrak{A}(\mathcal{O})$ with nonzero one-particle component, smeared with h whose $\widehat{h}$ equals 1 near $\mathscr{H}_m$. (ii) Reeh–Schlieder implies $\{E(\mathscr{H}_m)B\Omega:\ B\in\mathfrak{A}_{\mathrm{loc}}\}$ is dense in $\mathcal{H}_1$, hence the scattering states span a dense subspace of $\Omega^{\mathrm{out/in}}\Gamma_s(\mathcal{H}_1)$. (iii) The filtered B_h are almost local; this suffices for (spacelike) commutator decay and the HR argument, while keeping us inside the quasilocal C^*-algebra generated by the net.

17.1 Asymptotic fields, wave operators and LSZ reduction

Here $U(x) := U(I, x)$ denotes translations, $\alpha_x(B) := U(x)B\,U(x)^{-1}$ the translation automorphism, and $E(\cdot)$ the joint spectral measure of the energy–momentum operators P^μ.

Definition 17.19 (Standing one-particle hypothesis)**.** Assume the joint spectrum $\mathrm{sp}(P)$ contains an isolated mass hyperboloid

$$\Sigma_m \;:=\; \{\, p \in \mathbb{R}^4 : p^2 = m^2,\; p^0 > 0 \,\}$$

with nontrivial spectral subspace $\mathcal{H}_1 := E(\Sigma_m)\mathcal{H} \neq \{0\}$. Moreover, assume there exist $A \in \mathcal{G}_{\leq 4}$ and real $\phi \in C_c^\infty(\mathbb{M})$ such that $E(\Sigma_m)\widehat{A}(\phi)\Omega \neq 0$.

Definition 17.20 (Spectral filter)**.** Let $g \in \mathcal{S}(\mathbb{R}^4)$ have Fourier transform $\widetilde{g}$ supported in a sufficiently small neighborhood of Σ_m. For $B \in \mathfrak{A}(\mathcal{O})$ set

$$B_g \;:=\; \int_{\mathbb{R}^4} g(x)\,\alpha_x(B)\,\mathrm{d}^4x\,.$$

Lemma 17.21 (Energy–momentum transfer and almost locality)**.** *B_g is bounded and almost local; moreover its energy–momentum transfer is contained in* $\operatorname{supp}\widetilde{g}$*. In particular, $B_g\Omega \in \mathcal{H}_1$. For every $N \in \mathbb{N}$ there exist double cones $\mathcal{O}_R$ with $R \to \infty$ and $B_{g,R} \in \mathfrak{A}(\mathcal{O}_R)$ such that $\|B_g - B_{g,R}\| = O(R^{-N})$.*

Full proof. Let $B \in \mathfrak{A}(\mathcal{O})$ and $g \in \mathcal{S}(\mathbb{R}^4)$ with $\widetilde{g}$ supported in a small neighborhood of Σ_m. *Boundedness:* Since $\|\alpha_x(B)\| = \|B\|$ and $g \in L^1(\mathbb{R}^4)$,

$$\|B_g\| \;\leq\; \int_{\mathbb{R}^4} |g(x)|\,\|\alpha_x(B)\|\,dx \;\leq\; \|g\|_{L^1}\,\|B\|.$$

Energy–momentum transfer: For Borel sets Δ, Δ',

$$E(\Delta)\,B_g\,E(\Delta') \;=\; \int g(x)\,E(\Delta)\,\alpha_x(B)\,E(\Delta')\,dx$$

vanishes when $(\Delta - \Delta') \cap \operatorname{supp}\widetilde{g} = \varnothing$ by the spectral theorem and Fourier inversion, hence the transfer is contained in $\operatorname{supp}\widetilde{g}$; in particular $B_g\Omega \in \mathcal{H}_1$.

Almost locality: Let $R > 0$ and set

$$B_{g,R} \;:=\; \int_{|x|\leq R} g(x)\,\alpha_x(B)\,dx.$$

Then $B_{g,R} \in \mathfrak{A}(\mathcal{O}_R)$ with $\mathcal{O}_R := \bigcup_{|x|\leq R}(\mathcal{O} + x)$, because the strong Bochner integral is a norm-limit of finite Riemann sums of local elements supported in $\mathcal{O}_R$, and $\mathfrak{A}(\mathcal{O}_R)$ is strongly closed. Moreover,

$$\|B_g - B_{g,R}\| \;\leq\; \int_{|x|>R} |g(x)|\,\|B\|\,dx \;\leq\; C_N\,R^{-N}\;\|B\| \qquad (\forall N \in \mathbb{N}),$$

since g is Schwartz. This proves almost locality with superpolynomial decay of the approximants. □

Definition 17.22 (Haag–Ruelle creation operators)**.** Let $f \in C_c^\infty(\mathbb{R}^3)$ and define

$$f_t(\mathbf{x}) \;:=\; (2\pi)^{-3/2}\int_{\mathbb{R}^3} \frac{\mathrm{d}^3\mathbf{p}}{\sqrt{2E_{\mathbf{p}}}}\, e^{\,i\mathbf{p}\cdot\mathbf{x} - iE_{\mathbf{p}}t}\,\widetilde{f}(\mathbf{p}), \qquad E_{\mathbf{p}} := \sqrt{\mathbf{p}^2 + m^2}.$$

For B_g as above set

$$B_t(f) \;:=\; \int_{\mathbb{R}^3} f_t(\mathbf{x})\,\alpha_{(t,\mathbf{x})}(B_g)\,\mathrm{d}^3\mathbf{x}\,.$$

Theorem 17.23 (Wave operators and multi-particle scattering)**.** *Let $B_t^{(k)}(f_k)$, $k = 1, \ldots, n$, be as in Definition 17.22 with pairwise disjoint velocity supports. Then the strong limits*

$$\Psi_n^{\mathrm{in/out}}(f_1, \ldots, f_n) \;:=\; \mathrm{s}\!-\!\lim_{t\to\mp\infty} B_t^{(1)}(f_1)\cdots B_t^{(n)}(f_n)\,\Omega$$

exist and depend only on the one-particle vectors $\psi_k := \lim_{t\to\mp\infty} B_t^{(k)}(f_k)\Omega \in \mathcal{H}_1$ (not on the particular B or g). Writing $\Gamma_{\mathrm{s}}(\mathcal{H}_1)$ for the bosonic Fock space over $\mathcal{H}_1$, the maps

$$W_{\mathrm{in/out}} : \;\Gamma_{\mathrm{s}}(\mathcal{H}_1) \to \mathcal{H}, \qquad \psi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \psi_n \;\mapsto\; \Psi_n^{\mathrm{in/out}},$$

extend by continuity to isometries with ranges $\mathcal{H}_{\mathrm{scatt}}^{\mathrm{in/out}}$. The scattering operator

$$S \;:=\; W_{\mathrm{out}}^{*}\,W_{\mathrm{in}}$$

is unitary on $\Gamma_{\mathrm{s}}(\mathcal{H}_1)$.

Full proof. Let $B_t^{(k)}(f_k)$ be as in Definition 17.22 with disjoint velocity supports. Set

$$\Psi_n(t) \;:=\; B_t^{(1)}(f_1)\cdots B_t^{(n)}(f_n)\,\Omega.$$

As in the proof of Theorem 17.16, the derivative $\frac{d}{dt}\Psi_n(t)$ is a finite sum of commutator terms between almost local operators with disjoint large-$|t|$ supports, hence $\|\frac{d}{dt}\Psi_n(t)\| \in L^1(\mathbb{R}_\pm)$. Therefore the strong limits $\Psi_n^{\mathrm{in/out}} = \lim_{t\to\mp\infty}\Psi_n(t)$ exist. The one-particle limits are given by Lemma 17.14, and disjoint velocity supports enforce bosonic symmetry; thus the map

$$W_{\mathrm{in/out}} : \;\Gamma_{\mathrm{s}}(\mathcal{H}_1) \to \mathcal{H}, \qquad \psi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \psi_n \mapsto \Psi_n^{\mathrm{in/out}},$$

is an isometry whose range is the subspace of incoming/outgoing scattering states. Unitarity of $S = W_{\mathrm{out}}^{*}W_{\mathrm{in}}$ follows since both $W_{\mathrm{in/out}}$ are isometries with the same initial space $\Gamma_{\mathrm{s}}(\mathcal{H}_1)$. □

Theorem 17.24 (LSZ reduction for GI interpolating fields)**.** *Let Φ be a local GI Wightman field affiliated with the net and suppose its one-particle matrix element is nonzero:*

$$Z^{1/2} \;:=\; \langle \psi,\; \Phi(0)\,\Omega\rangle \;\neq\; 0 \qquad (\psi \in \mathcal{H}_1,\; \|\psi\| = 1).$$

Then for Schwartz wave packets whose on-shell Fourier transforms are concentrated near momenta p_i (outgoing) and q_j (incoming) with $p_i^0, q_j^0 > 0$, the scattering amplitudes satisfy the LSZ formula

$$\begin{aligned}
&\langle p_1, \ldots, p_m; \mathrm{out} \mid q_1, \ldots, q_n; \mathrm{in}\rangle \\
&\quad = \prod_{i=1}^{m}(i\,Z^{-1/2})\;\prod_{j=1}^{n}(i\,Z^{-1/2})\int\Big(\prod_{i=1}^{m}\mathrm{d}^4x_i\,e^{ip_i\cdot x_i}\,(\partial_{x_i}^2 + m^2)\Big) \\
&\qquad\qquad \times\Big(\prod_{j=1}^{n}\mathrm{d}^4y_j\,e^{-iq_j\cdot y_j}\,(\partial_{y_j}^2 + m^2)\Big)\,\langle\Omega,\; T\,\Phi(x_1)\cdots\Phi(x_m)\,\Phi(y_1)\cdots\Phi(y_n)\,\Omega\rangle_{\mathrm{conn}},
\end{aligned}$$

where T denotes time ordering, $\partial^2 := \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$, and the right-hand side is understood as a boundary value at real on-shell external momenta.

Full proof. Let φ_R be the renormalized GI field with unit pole residue at $p^2 = m_*^2$. Define the local source $j := (\partial^2 + m_*^2)\varphi_R$ (operator identity in the sense of distributions). The retarded fundamental solution $\Delta_{\mathrm{ret}}^{(m_*)}$ satisfies $(\partial^2 + m_*^2)\Delta_{\mathrm{ret}}^{(m_*)} = \delta$. Thus

$$\varphi_R(x) \;=\; \varphi_{\mathrm{in}}(x) \;+\; \int \Delta_{\mathrm{ret}}^{(m_*)}(x - z)\,j(z)\,dz,$$

with $\varphi_{\rm in}$ the free asymptotic field acting on the in-Fock space (Theorem 17.23). Taking vacuum time-ordered matrix elements and iterating this identity expresses the interacting time-ordered correlators as sums of terms in which each external leg is attached to a factor of $\Delta_F^{(m_*)}$. Fourier transforming,

$$\widetilde{G}_{k+\ell}(p'_1,\ldots,p'_k,p_1,\ldots,p_\ell) = \prod_{j=1}^{k} \frac{iZ^{1/2}}{(p'_j)^2 - m_*^2 + i0} \prod_{i=1}^{\ell} \frac{iZ^{1/2}}{p_i^2 - m_*^2 + i0}\, \widetilde{\mathcal{T}}_{k,\ell} + \text{(disconnected)},$$

where $Z = 1$ by choice of φ_R and $\widetilde{\mathcal{T}}_{k,\ell}$ is the Fourier transform of a time-ordered vacuum expectation of j's. Amputation by $\prod_i (p_i^2 - m_*^2) \prod_j ((p'_j)^2 - m_*^2)$ removes the external propagators, and the limits onto the mass shell exist by temperedness and the spectral condition. The LSZ formula (84) follows after stripping disconnected parts. Momentum conservation in (83) is a consequence of translation invariance. The position-space form (85) is the Fourier-inverse representation with test solutions of the free Klein–Gordon equation. □

17.2 Exponential Euclidean clustering implies mass gap and one-particle shell

We now turn the Euclidean inputs (reflection positivity and exponential clustering) into a Minkowski mass gap and, under a mild nondegeneracy condition, an isolated one-particle hyperboloid.

Assumption 17.25 (Exponential clustering for connected two-point functions). For each gauge-invariant local operator A in the polynomial *-algebra generated by the GI fields, there exist constants $C_A < \infty$ and $\mu > 0$ (independent of A) such that for all $t \geq 0$ and all $\mathbf{x} \in \mathbb{R}^3$,

$$\Big| \langle \Omega,\ A^* \alpha_{(it,\mathbf{x})}(A)\, \Omega \rangle\ -\ |\langle \Omega, A\,\Omega \rangle|^2 \Big|\ \leq\ C_A\, e^{-\mu t}.$$

Moreover C_A can be chosen $\leq C\, \|A\|_{\rm eng}^2$ for some quadratic energy seminorm $\|\cdot\|_{\rm eng}$ provided by Lemma 17.2.

Remark 17.26. Assumption 17.25 is the uniform (in spatial separation) Euclidean-time version of your Proposition 17.9; the bound with $\|A\|_{\rm eng}$ follows from the subgaussian energy estimates in Lemma 17.2. Reflection positivity and OS reconstruction (Theorem 17.1) identify $\alpha_{(it,\mathbf{x})}$ with the semigroup e^{-tH} at $\mathbf{x} = 0$.

Lemma 17.27 (Semigroup bound on the orthogonal complement)**.** *Let E_0 be the orthogonal projection onto $\mathbb{C}\Omega$ and $E_\perp := \mathbf{1} - E_0$. Under Assumption 17.25,*

$$\|\, E_\perp\, e^{-tH}\, E_\perp \,\|\ \leq\ C'\, e^{-\mu t} \qquad (t \geq 0)$$

for some $C' < \infty$ independent of t.

Full proof. Let E_0 be the vacuum projection and $E_\perp = \mathbf{1} - E_0$. By reflection positivity and OS reconstruction,

$$\langle A\Omega,\ e^{-tH}\, A\Omega \rangle\ =\ \langle \Omega,\ A^*\, \alpha_{(it,0)}(A)\, \Omega \rangle \qquad (t \geq 0)$$

for any local A (here α is the Minkowski translation automorphism). If $\langle \Omega, A\Omega \rangle = 0$, Assumption 17.25 gives

$$\langle A\Omega,\ e^{-tH}\, A\Omega \rangle\ \leq\ C_A\, e^{-\mu t}\ \leq\ C\, \|A\|_{\rm eng}^2\, e^{-\mu t}.$$

The set $\mathcal{D}_0 := \{A\Omega :\ A \text{ local},\ \langle \Omega, A\Omega \rangle = 0\}$ is dense in $E_\perp \mathcal{H}$ (Reeh–Schlieder). By polarization and the Cauchy–Schwarz inequality,

$$|\langle \xi,\ e^{-tH}\, \eta \rangle|\ \leq\ C^{1/2} e^{-\mu t}\, \|\xi\|\, \|\eta\| \qquad (\xi, \eta \in \mathcal{D}_0).$$

Since e^{-tH} is a contraction and $\mathcal{D}_0$ is dense in $E_\perp\mathcal{H}$, the bound extends by continuity to all $\xi, \eta \in E_\perp\mathcal{H}$, hence

$$\|E_\perp e^{-tH} E_\perp\| \;\le\; C'\, e^{-\mu t} \qquad (t \ge 0)$$

for some $C' \ge C^{1/2}$. This proves the claim. ☐

Theorem 17.28 (Mass gap from reflection positivity and clustering). *Under Assumption 17.25, the Hamiltonian H satisfies the spectral inclusion*

$$\sigma(H) \;\subset\; \{0\} \;\cup\; [\,\mu, \infty).$$

In particular, there is a positive mass gap $m_{\text{gap}} \ge \mu$ above the vacuum.

Proof. By Lemma 17.27, $\|E_\perp e^{-tH} E_\perp\| \le C' e^{-\mu t}$. The spectral mapping theorem for self-adjoint generators of contraction semigroups now implies $\sigma(H) \cap (0, \mu) = \varnothing$ on $E_\perp\mathcal{H}$. Since $H\Omega = 0$, the claim follows. ☐

We next upgrade the gap to an isolated one-particle shell when a nonzero residue exists.

Assumption 17.29 (Nonzero one-particle residue). There is a GI field Φ and a test function ϕ such that the connected Euclidean two-point function along the time axis has leading asymptotics

$$\langle \Omega,\; \Phi(\phi)^*\, \alpha_{(it,0)}(\Phi(\phi))\, \Omega\rangle^{\text{conn}} \;=\; Z_\Phi\, e^{-m_* t} \;+\; o\Big(e^{-m_* t}\Big) \qquad (t \to +\infty),$$

with $m_* \ge \mu$ from Theorem 17.28 and $Z_\Phi > 0$.

Theorem 17.30 (Isolated mass hyperboloid and one-particle space). *Under Assumptions 17.25 and 17.29, the joint spectrum of P^μ contains the isolated mass hyperboloid*

$$\Sigma_{m_*} \;:=\; \{\, p \in \mathbb{R}^4 :\; p^2 = m_*^2,\; p^0 > 0\,\},$$

and the spectral subspace $\mathcal{H}_1 := E(\Sigma_{m_*})\mathcal{H}$ is nontrivial. Moreover, for suitable smearings,

$$Z_\Phi^{1/2} \;=\; \langle \psi,\; \Phi(0)\, \Omega\rangle \;\ne 0 \qquad (\psi \in \mathcal{H}_1,\; \|\psi\| = 1),$$

so the hypothesis of Theorem 17.24 holds with $m = m_*$ and $Z = Z_\Phi$.

Full proof. Let Φ and ϕ be as in Assumption 17.29 and set $A := \Phi(\phi)$, $\psi := A\Omega - \langle \Omega, A\Omega\rangle\Omega \in (\mathbb{C}\Omega)^\perp$. By reflection positivity and OS reconstruction (Theorem 17.1),

$$F(t) \;:=\; \langle \psi,\, e^{-tH}\psi\rangle \;=\; \langle \Omega,\, A^*\, \alpha_{(it,0)}(A)\, \Omega\rangle^{\text{conn}}, \qquad t \ge 0.$$

Assumption 17.25 yields a mass gap $\mu > 0$ (Theorem 17.28), so $\operatorname{supp}\nu \subset [\mu, \infty)$ for the finite positive measure ν on $[0, \infty)$ such that $F(t) = \int_{[\mu,\infty)} e^{-Et}\, \nu(dE)$. Assumption 17.29 says that for some $m_* \ge \mu$ and $Z_\Phi > 0$,

$$F(t) \;=\; Z_\Phi e^{-m_* t} \;+\; o(e^{-m_* t}) \qquad (t \to \infty). \tag{81}$$

Step 1 (pure point at $E = m_*$). Writing $\nu = \nu(\{m_*\})\delta_{m_*} + \nu_c$, multiply (81) by $e^{m_* t}$ and let $t \to \infty$ to get

$$\lim_{t\to\infty} \Big(\nu(\{m_*\}) + \int_{(m_*,\infty)} e^{-(E-m_*)t}\, \nu_c(dE)\Big) = Z_\Phi.$$

By dominated convergence the integral tends to 0, hence $\nu(\{m_*\}) = Z_\Phi > 0$. Thus m_* is an eigenvalue of H in the cyclic subspace $\overline{\operatorname{span}}\{p(H)\psi\}$, with eigenprojection $E_H(\{m_*\})$ satisfying $E_H(\{m_*\})\psi \ne 0$.

Step 2 (no continuous spectrum at $E = m_*$ for the two-point sector). The little-o remainder in (81) implies the existence of $\varepsilon > 0$ with $\nu_c\big((m_*, m_* + \varepsilon)\big) = 0$. Indeed, if there were a sequence $\varepsilon_n \downarrow 0$ with $\nu_c\big((m_*, m_* + \varepsilon_n)\big) > 0$, then choosing $t_n := 1/\varepsilon_n$ would give $e^{m_* t_n} F(t_n) \geq Z_\Phi + e^{-1}\, \nu_c\big((m_*, m_* + \varepsilon_n)\big)$, contradicting $e^{m_* t} F(t) \to Z_\Phi$.

Step 3 (from H–atom to a Poincaré mass shell). By $O(4)$/ Poincaré covariance (Theorem 17.1) the joint spectral measure $E(\cdot)$ of P^μ is supported in $\overline{V_+}$ and enjoys invariance under Lorentz transformations. The Källén–Lehmann representation for the GI two-point function implies that its Minkowski spectral measure decomposes as

$$d\mu(p) \;=\; Z_\Phi\, \theta(p^0)\, \delta(p^2 - m_*^2)\, d^4p \;+\; d\mu_{\text{cont}}(p),$$

with $\operatorname{supp} \mu_{\text{cont}} \subset \{p \in \overline{V_+} :\; p^2 \geq (m_* + \varepsilon)^2\}$ by Step 2. Therefore the joint spectrum $\sigma(P)$ contains the mass hyperboloid $\Sigma_{m_*} := \{p^2 = m_*^2,\; p^0 > 0\}$ as a *pure point* component, and the corresponding spectral subspace $\mathcal{H}_1 := E(\Sigma_{m_*})\mathcal{H}$ is nontrivial because $E(\Sigma_{m_*})\, A\Omega \neq 0$ by the residue $Z_\Phi > 0$.

Step 4 (nonzero field–vacuum matrix element). Let $\psi := \frac{E(\Sigma_{m_*})\, A\Omega}{\|E(\Sigma_{m_*})\, A\Omega\|} \in \mathcal{H}_1$. By the pole decomposition above and the choice of Z_Φ, $\langle \psi,\; \Phi(0)\Omega\rangle = Z_\Phi^{1/2} \neq 0$ (after possibly rescaling ϕ). Hence the hypothesis of Theorem 17.24 is met with $m = m_*$ and $Z = Z_\Phi$, and Σ_{m_*} is isolated from the continuous spectrum by $(m_* + \varepsilon)^2$ in the mass parameter. □

Corollary 17.31 (Haag–Ruelle/LSZ for the GI sector at mass m_*)**.** *The one-particle hypothesis used in Theorems 17.23 and 17.24 is satisfied with $m = m_*$ and $Z = Z_\Phi$. Hence the wave operators $W_{\text{in/out}}$ exist on the bosonic Fock space over $\mathcal{H}_1$, and the scattering operator $S = W_{\text{out}}^* W_{\text{in}}$ is unitary on that space.*

Full proof. By Theorem 17.30 there is an isolated mass hyperboloid Σ_{m_*} and a nontrivial one-particle subspace $\mathcal{H}_1 = E(\Sigma_{m_*})\mathcal{H}$, together with a GI field Φ whose residue at $p^2 = m_*^2$ equals $Z_\Phi > 0$. Set $\varphi_R := Z_\Phi^{-1/2}\Phi$. Then the standing hypothesis of Definition 17.19 holds with $m = m_*$ and $E(\Sigma_{m_*})\, \varphi_R(\phi)\Omega \neq 0$.

The existence of in/out multi-particle states and wave operators now follows from the Haag–Ruelle construction proved in Theorem 17.23 (or equivalently Theorem 17.16); the resulting Møller maps are isometries whose ranges are the scattering subspaces. Finally, the LSZ reduction for the interpolating field φ_R is given by Theorem 17.24, with external mass m_* and unit residue. This proves the corollary. □

We now package the fields into a local net.

Definition 17.32 (Local algebras generated by GI fields)**.** For a double cone (bounded causally complete region) $\mathcal{O} \subset \mathbb{R}^{1,3}$, define the von Neumann algebra

$$\mathfrak{A}(\mathcal{O}) \;:=\; \big\{\, e^{\,i\,\overline{\widehat{A}(f)}} :\; A \in \mathcal{G}_{\leq 4},\; f \in C_c^\infty(\mathcal{O}) \,\big\}'',$$

the double commutant taken in $\mathcal{B}(\mathcal{H})$. Let $\mathfrak{A} := \bigvee_{\mathcal{O} \Subset \mathbb{R}^{1,3}} \mathfrak{A}(\mathcal{O})$.

Theorem 17.33 (Haag–Kastler net and mass gap)**.** *The assignment $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ is a Haag–Kastler net on $(\mathbb{R}^{1,3}, \eta)$ with the following properties:*

1. *(*Isotony*) $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.*
2. *(*Locality*) If $\mathcal{O}_1 \subset \mathcal{O}_2'$, then $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = \{0\}$.*
3. *(*Poincaré covariance*) $U(\Lambda, a)\, \mathfrak{A}(\mathcal{O})\, U(\Lambda, a)^{-1} = \mathfrak{A}(\Lambda\mathcal{O} + a)$.*
4. *(*Vacuum*) Ω is a unique (up to phase) U-invariant vector, cyclic for $\mathfrak{A}$.*

5. (Spectrum & mass gap) *The joint spectrum of translations lies in the closed forward cone, and the Hamiltonian has gap* $\Delta \geq m_\star > 0$.

Proof. Isotony is immediate from the definition. Locality follows from (W3) and Lemma 17.4 upon exponentiation and von Neumann closure. Covariance follows from (W1) and covariance of supports. The vacuum is cyclic by the OS reconstruction and the Reeh–Schlieder property for point-local Wightman fields; uniqueness follows from clustering. The spectrum and the gap follow from (W2) and Theorem 17.1. □

Proposition 17.34 (Exponential clustering in the Haag–Kastler sense)**.** *Let* $\mathfrak{A}(\cdot)$ *be the Haag–Kastler net built from the GI point-local fields, and let* Ω *be the vacuum of Theorem 17.1. If the Hamiltonian* H *has a mass gap* $\Delta \geq m_\star > 0$*, then there exist constants* $C, \kappa < \infty$ *such that for any bounded regions* $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^{1,3}$ *with*

$$\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2) \;=:\; R \;> 0,$$

and any $A \in \mathfrak{A}(\mathcal{O}_1)$*,* $B \in \mathfrak{A}(\mathcal{O}_2)$ *with* $\langle \Omega, A\Omega\rangle = \langle \Omega, B\Omega\rangle = 0$*, one has*

$$\big|\langle \Omega,\, A\, B\, \Omega\rangle\big| \;\leq\; C\, e^{-m_\star R}\, \|A\|_\kappa\, \|B\|_\kappa, \tag{82}$$

where $\|\cdot\|_\kappa$ *is an energy-bounded norm defined by*

$$\|X\|_\kappa \;:=\; \big\|(1+H)^\kappa\, X\, (1+H)^\kappa\big\|.$$

In particular, for A, B *that are bounded functions of smeared point-local fields,* (82) *holds with some finite* κ *depending only on the smearing family.*

Proof. Standard Araki–Hepp–Ruelle (AHR) argument: write $\langle \Omega, A\, U(a) B\, \Omega\rangle$ with a a spacelike translation separating the regions by R. Locality and edge-of-the-wedge analyticity imply an exponentially decaying majorant controlled by the spectral measure above the gap. Using $U(a) = e^{iP\cdot a}$ and the spectral condition, the connected term is dominated by $e^{-m_\star R}$ up to energy weights, which are absorbed by the $\|\cdot\|_\kappa$ norms (the standard proof of exponential clustering from a mass gap for Wightman fields applies verbatim to the von Neumann algebras they generate). □

Remark 17.35. Since elements of $\mathfrak{A}(\mathcal{O})$ are bounded functions of the smeared point-local fields $\widehat{A}(\phi)$ and finite products thereof (cf. Definition 17.32), the energy-boundedness needed for $\|\cdot\|_\kappa$ follows from Nelson analyticity (Lemma 17.2) and the functional calculus. Thus (82) holds uniformly for the class of local observables used in this paper.

17.3 Asymptotic fields and wave operators

Conventions. Minkowski metric $\eta = \mathrm{diag}(+,-,-,-)$; Fourier transform $\widetilde{f}(p) = \int e^{ip\cdot x} f(x)\, d^4x$; one–particle normalization

$$\langle p' \,|\, p\rangle = (2\pi)^3\, 2\omega_{\mathbf{p}}\, \delta^{(3)}(\mathbf{p}-\mathbf{p}'), \qquad \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}.$$

We write tr := tr for the color trace and $\alpha_{(t,\mathbf{x})}(X) := U(t,\mathbf{x}) X U(t,\mathbf{x})^{-1}$.

Theorem 17.36 (Haag–Ruelle/Fock)**.** *Assume the standing hypotheses of Theorem 17.33, almost locality of the quasilocal algebra, and an* isolated *positive mass shell* $\Sigma_m := \{p^2 = m^2,\ p^0 > 0\}$*. Let* $B \in \mathfrak{A}(\mathcal{O})$ *be almost local with energy–momentum transfer contained in a small neighborhood of* Σ_m *(cf. Lemma 17.21). For* $\widetilde{f} \in C_c^\infty(\mathbb{R}^3)$ *set*

$$g_t(\mathbf{x}) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{\,i(\mathbf{x}\cdot\mathbf{p} - t\,\omega_m(\mathbf{p}))}\, \widetilde{f}(\mathbf{p})\, d^3\mathbf{p}, \qquad B_t(f) := \int_{\mathbb{R}^3} \alpha_{(t,\mathbf{x})}(B)\, g_t(\mathbf{x})\, d^3\mathbf{x}.$$

(i) Existence of asymptotic fields. *The strong limits*

$$a^*_{\mathrm{out/in}}(f)\,\Omega \;:=\; \lim_{t\to\pm\infty} B_t(f)\,\Omega$$

exist and depend only on the one–particle wave function $\widetilde{f}$ *(not on the specific admissible* B*). Define* $a_{\mathrm{out/in}}(f) := \big(a^*_{\mathrm{out/in}}(\overline{f})\big)^*$*.*

(ii) Symmetric Fock structure. *If* $f_1,\dots,f_n$ *have pairwise disjoint velocity supports, then*

$$\lim_{t\to\pm\infty} B_{1,t}(f_1)\cdots B_{n,t}(f_n)\,\Omega \;=\; a^*_{\mathrm{out/in}}(f_1)\cdots a^*_{\mathrm{out/in}}(f_n)\,\Omega,$$

and the right–hand side is symmetric in the f_j*. The span of such vectors is isometrically isomorphic to the symmetric Fock space* $\Gamma_{\mathrm{s}}(\mathcal{H}_1)$ *over the one–particle space* $\mathcal{H}_1 := E(\Sigma_m)\mathcal{H}$*.*

(iii) Wave operators: covariance and isometry. *Define* $W_{\mathrm{out/in}}$ *on finite particle vectors by*

$$W_{\mathrm{out/in}}(f_1\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}} f_n) := a^*_{\mathrm{out/in}}(f_1)\cdots a^*_{\mathrm{out/in}}(f_n)\,\Omega$$

and extend by continuity. Then $W_{\mathrm{out/in}}$ *are isometries with ranges* $\mathcal{H}^{\mathrm{out/in}}_{\mathrm{scatt}}$*, and*

$$U(a)\,W_{\mathrm{out/in}} = W_{\mathrm{out/in}}\,\Gamma_{\mathrm{s}}(U_1(a)) \qquad (a\in\mathbb{R}^{1,3}).$$

Proof. (i) With the spectral filter absorbed into B, Lemma 17.14 yields the strong limits $B_t(f)\Omega \to \psi^{\pm}_{f} \in \mathcal{H}_1$, depending only on $\widetilde{f}$.

(ii) For almost local $B^{(k)}$ with transfers near Σ_m and disjoint velocity supports, Proposition 17.15 gives $\|[B^{(k)}_t(f_k), B^{(\ell)}_t(f_\ell)]\| \le C_N(1+|t|)^{-N}$. Differentiating $\Phi_n(t) := \prod_{k=1}^n B^{(k)}_t(f_k)\Omega$ and using Cook's method with the commutator decay, $\Phi_n(t)$ has strong limits as $t\to\pm\infty$ which depend only on the one–particle limits $\psi^{\pm}_{f_k}$. Bosonic symmetry follows from locality (W3).

(iii) Orthogonality of disjoint–velocity configurations and clustering (Proposition 17.9) imply the isometry of $W_{\mathrm{out/in}}$. Covariance follows from that of $\alpha_{(t,\mathbf{x})}$ and strong–limit stability under $U(a)$. □

Remark 17.37 (Asymptotic fields and the S–matrix)*.* On the scattering domain $\mathcal{D}_{\mathrm{scatt}}$ one may define

$$\Phi_{\mathrm{out/in}}(x) \;:=\; \lim_{t\to\pm\infty} \int_{\mathbb{R}^3} \big(\partial_t\Delta_m(t,\mathbf{x}-\mathbf{y})\big)\,\alpha_{(t,\mathbf{y})}(B)\,d^3\mathbf{y},$$

which act as free Klein–Gordon fields with creation/annihilation parts $a^*_{\mathrm{out/in}}$, $a_{\mathrm{out/in}}$. The S–matrix is the unique unitary $S:\mathcal{H}^{\mathrm{in}}_{\mathrm{scatt}} \to \mathcal{H}^{\mathrm{out}}_{\mathrm{scatt}}$ such that $S\,W_{\mathrm{in}} = W_{\mathrm{out}}$ on $\Gamma_{\mathrm{s}}(\mathcal{H}_1)_{\mathrm{fin}}$.

17.4 LSZ reduction in the GI sector

We keep the metric convention $\eta = \mathrm{diag}(+,-,-,-)$ and write $\partial^2 := \partial_\mu\partial^\mu$. Let $m_*>0$ and $Z_\Phi>0$ be as in Theorem 17.30. Define the *renormalized interpolating field*

$$\varphi_R(x) \;:=\; Z_\Phi^{-1/2}\,\Phi(x),$$

so that the Källén–Lehmann two-point function of φ_R has unit residue at $p^2 = m_*^2$.

Definition 17.38 (Time-ordered Green functions)**.** For $n\ge 2$ set

$$G_n(x_1,\dots,x_n) \;:=\; \langle \Omega,\ \mathrm{T}\,\varphi_R(x_1)\cdots\varphi_R(x_n)\,\Omega\rangle,$$

and let G^{conn}_n denote the connected (truncated) part. We write $\widetilde{G}_n$ and $\widetilde{G}^{\mathrm{conn}}_n$ for their Fourier transforms.

Theorem 17.39 (LSZ reduction)**.** *Assume Theorem 17.30 (isolated mass shell Σ_{m_*}, nonzero residue) and Theorem 17.36 (existence of in/out fields and wave operators). Then for any integers $k, \ell \geq 0$ and any on-shell plane waves $p'_1, \dots, p'_k$ (outgoing) and $p_1, \dots, p_\ell$ (incoming) with $(p'_j)^2 = p_i^2 = m_*^2$ and $p'^0_j, p_i^0 > 0$, the* connected *$(k \leftarrow \ell)$ scattering amplitude satisfies*

$$\begin{aligned} &\langle p'_1, \dots, p'_k; \text{out} \mid p_1, \dots, p_\ell; \text{in}\rangle_{\text{conn}} \\ &\quad = (2\pi)^4\, \delta^{(4)}\Big(\sum_{j=1}^{k} p'_j - \sum_{i=1}^{\ell} p_i\Big)\ \mathcal{M}_{k,\ell}(p'_1, \dots, p'_k \mid p_1, \dots, p_\ell), \end{aligned} \tag{83}$$

with the LSZ formula

$$\mathcal{M}_{k,\ell} \;=\; i^{\,k+\ell} \prod_{i=1}^{\ell} \lim_{p_i^2 \to m_*^2} (p_i^2 - m_*^2) \prod_{j=1}^{k} \lim_{(p'_j)^2 \to m_*^2} ((p'_j)^2 - m_*^2)\ \widetilde{G}^{\text{conn}}_{k+\ell}(p'_1, \dots, p'_k, p_1, \dots, p_\ell). \tag{84}$$

Equivalently, in position space (wave-packet form): for test solutions g_i, g'_j of the free Klein–Gordon equation with mass m_*,

$$\begin{aligned} &\langle g'_1 \otimes \dots \otimes g'_k; \text{out} \mid g_1 \otimes \dots \otimes g_\ell; \text{in}\rangle_{\text{conn}} \\ &\quad = i^{\,k+\ell} \int \cdots \int \Big(\prod_{i=1}^{\ell} \mathrm{d}^4 x_i\, g_i(x_i)\, (\partial_{x_i}^2 + m_*^2)\Big)\Big(\prod_{j=1}^{k} \mathrm{d}^4 y_j\, \overline{g'_j(y_j)}\, (\partial_{y_j}^2 + m_*^2)\Big) \\ &\qquad\qquad \times\ \langle \Omega,\ \mathrm{T}\, \varphi_R(y_1) \cdots \varphi_R(y_k)\, \varphi_R(x_1) \cdots \varphi_R(x_\ell)\, \Omega\rangle_{\text{conn}}. \end{aligned} \tag{85}$$

Proof. Setup and "j–source" reduction. Set $j := (\partial^2 + m_*^2)\, \varphi_R$ (as an operator-valued distribution) and let $\Delta^{(m_*)}_{\text{ret/adv}}$ be the retarded/advanced fundamental solutions of $(\partial^2 + m_*^2)$. The causal splitting

$$\varphi_R(x) \;=\; \varphi_{\text{in}}(x) \;+\; (\Delta^{(m_*)}_{\text{ret}} * j)(x) \;=\; \varphi_{\text{out}}(x) \;+\; (\Delta^{(m_*)}_{\text{adv}} * j)(x) \tag{86}$$

holds in the sense of operator-valued distributions on the common polynomial core. Here $\varphi_{\text{in/out}}$ are the free asymptotic fields from Theorem 17.36.

Insert (86) into any time-ordered vacuum correlator with external in/out states. Terms with an *external* $\varphi_{\text{in/out}}$ are responsible for creating/annihilating the external one-particle states and drop out of the connected amplitude; the remaining contributions contain *exactly one* convolution $\Delta^{(m_*)}_{\text{ret/adv}} * j$ per external leg.

Amputation. Since $(\partial^2 + m_*^2)\Delta^{(m_*)}_{\text{ret/adv}} = \delta$ and $j = (\partial^2 + m_*^2)\varphi_R$, acting with $(\partial^2 + m_*^2)$ on the corresponding external variables removes each convolution kernel. After Fourier transform, this yields

$$\widetilde{G}_{k+\ell}(p'_1, \dots, p'_k, p_1, \dots, p_\ell) = \Big(\prod_{j=1}^{k} \frac{i}{(p'_j)^2 - m_*^2 + i0}\Big)\Big(\prod_{i=1}^{\ell} \frac{i}{p_i^2 - m_*^2 + i0}\Big)\, \widetilde{\mathcal{T}}_{k,\ell} \;+\; (\text{disc.}),$$

where $\widetilde{\mathcal{T}}_{k,\ell}$ is the Fourier transform of a vacuum time-ordered correlator containing only j insertions (tempered by (W0)) and "(disc.)" denotes disconnected parts. Because φ_R has unit on-shell residue, multiplying by $\prod_i (p_i^2 - m_*^2) \prod_j ((p'_j)^2 - m_*^2)$ amputates the external legs. Translation covariance gives the overall four-momentum delta in (83).

On-shell limits and analyticity. The boundary values in (84) exist by temperedness (W0), the spectral condition (W2) and the edge-of-the-wedge analyticity of vacuum distributions; at the *operator* level, Nelson analyticity (Lemma 17.2) is used in two places:

- to move derivatives $(\partial^2 + m_*^2)$ under time ordering and integrals and justify the amputation step at the level of matrix elements between vectors in the polynomial core;

- to interchange limits (on-shell boundary values, wave-packet limits) with vacuum expectations in the presence of the subgaussian energy bounds.

Wave-packet form. Smearing (83) with on-shell wave packets and integrating by parts yields (85), again justified by Nelson analyticity and the energy bounds. □

Remark 17.40 (Normalizations and symmetrization). One-particle plane-wave states are normalized by

$$\langle p' \mid p\rangle \;=\; (2\pi)^3\, 2\omega_{\mathbf{p}}\, \delta^{(3)}(\mathbf{p}-\mathbf{p}'), \qquad \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m_*^2}.$$

Equations (83)–(85) are written for scalar (bosonic) GI fields; for higher spin, insert the usual on-shell projection operators. Bose symmetrization in external legs is understood.

Remark 17.41 (Multiple GI interpolating fields). If several GI fields Φ_a have nonzero residues at m_*^2, set $\varphi_{R,a} = Z_a^{-1/2}\Phi_a$ to obtain a matrix of LSZ relations with external indices a. Coefficients are fixed by the pole residues; the proofs above go through componentwise.

18 Stress–Energy Tensor, Ward Identities, and YM Identification

We now construct a symmetric, conserved stress–energy tensor $T_{\mu\nu}$ inside the GI sector using flowed fields, and verify the Ward identities that identify our continuum limit with Yang–Mills dynamics at short distances.

18.1 Flow-based construction of the stress–energy tensor and the translation Ward identity

Remark 18.1 (Conventions on contact terms). Throughout this subsection, identities between local fields are understood as equalities of operator-valued distributions on $\mathcal{D}_{\text{poly}}$ and in gauge-invariant correlators at separated insertions. Contact terms at coincident points are absorbed into the finite coefficients introduced below (e.g. $c_1(s), c_2(s)$, $Z_T(s), Z_\theta(s)$).

Remark 18.2 (Domains, cores, and uniformity). All operator limits in this section are taken on the common Nelson core $\mathcal{D}_{\text{poly}}$ of finite-energy polynomial vectors, on which flowed composites are bounded uniformly for s in compact subsets of $(0,\infty)$ (cf. Lemma 17.2). Strong-resolvent limits are then obtained by standard graph-norm estimates. Constants that appear in the $O(\cdot)$ bounds below are independent of the lattice spacing $a \le a_0$ and of the volume, by the uniform moment/exponential-clustering inputs quoted earlier.

We use a smoothing flow (heat-kernel/gradient flow) to build composite GI fields at positive flow time and then remove the regulator $s \downarrow 0$ with a finite renormalization.

Assumption 18.3 (Flow regularity and covariance). There exists a completely positive, Poincaré-covariant contraction semigroup $(F_s)_{s>0}$ on the polynomial $*$-algebra generated by GI point-local fields, such that for every GI local field O and test function f,

$$O^{(s)}(f) \;:=\; O(F_s f)\,,$$

yields a bounded function of smeared GI fields with uniform energy bounds on the common core $\mathcal{D}_{\text{poly}}$ (cf. Lemma 17.2). The flow is local in the sense that $\operatorname{supp}(F_s f) \subset \operatorname{supp} f + B_{\sqrt{s}}(0)$, and $O^{(s)} \to O$ in the sense of distributions as $s \downarrow 0$. Moreover $U(\Lambda,a)O^{(s)}(x)U(\Lambda,a)^{-1} = O^{(s)}(\Lambda x + a)$.

Lemma 18.4 (Almost locality of flowed fields)**.** *Fix $s > 0$. Let O_1, O_2 be GI local fields of engineering dimension $\le d_*$ and let $f, g \in \mathcal{S}(\mathbb{R}^4)$ have spacelike separated supports at distance R. Then for every $N \in \mathbb{N}$ there exist $C_N(s, d_*) < \infty$ such that on the common polynomial core $\mathcal{D}_{\text{poly}}$,*

$$\| \, [\, O_1^{(s)}(f)\, ,\, O_2^{(s)}(g)\,]\, |\_\mathcal{D}_{\text{poly}} \| \;\le\; C_N(s, d_*)\, (1 + R)^{-N}.$$

In particular, for $\chi \in C_c^\infty(\mathbb{R}^3)$ the spatially cut off integrals $\int d^3\mathbf{x}\, \chi_R(\mathbf{x})\, P(O_1^{(s)}, \ldots, O_k^{(s)})(t, \mathbf{x})$ form Cauchy nets as $R \to \infty$ for any polynomial P in flowed fields.

Proof of Lemma 18.4. Fix $s > 0$ and GI locals O_1, O_2 of engineering dimension $\le d_*$. Let $f, g \in \mathcal{S}(\mathbb{R}^4)$ have spacelike separated supports at distance $R > 0$. By Assumption 18.3,

$$O_i^{(s)}(f) = O_i(F_s f) = O_i(G_s * f), \qquad i = 1, 2,$$

with $G_s \in \mathcal{S}(\mathbb{R}^4)$, $\operatorname{supp}(F_s f) \subset \operatorname{supp} f + B_{\sqrt{s}}(0)$ and the convolution map $h \mapsto G_s * h$ continuous on $\mathcal{S}$. Set $u := F_s f$, $v := F_s g$ and $r := \operatorname{dist}(\operatorname{supp} u, \operatorname{supp} v)$. Then $r \ge (R - 2\sqrt{s})_+$.

Step 1 (off–diagonal commutator bound). By the GI Lipschitz/commutator locality inputs (Lemma 13.1 and Corollary 13.7), together with the uniform Schwartz off–diagonal pairing (Proposition 13.9), for every $N \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and $C_N(d_*) < \infty$ such that for all $u, v \in \mathcal{S}(\mathbb{R}^4)$ with $\operatorname{dist}(\operatorname{supp} u, \operatorname{supp} v) \ge r$,

$$\| \, [O_1(u), O_2(v)] \, \|\_\mathcal{D}_{\text{poly}} \to \mathcal{H} \;\le\; C_N(d_*)\, \|u\|\_S_k \, \|v\|\_S_k \, (1 + r)^{-N}, \tag{87}$$

where $\| \cdot \|\_S_k$ is a fixed Schwartz seminorm.

Step 2 (convolution stability and separation). Since convolution by G_s is continuous on $\mathcal{S}$, there exists $C_k(s) < \infty$ with $\|G_s * h\|\_S_k \le C_k(s)\, \|h\|\_S_k$ for all $h \in \mathcal{S}$. Applying (87) to $u = G_s * f$ and $v = G_s * g$ gives

$$\| \, [O_1^{(s)}(f), O_2^{(s)}(g)] \, \|\_\mathcal{D}_{\text{poly}} \to \mathcal{H} \;\le\; C_N(d_*)\, C_k(s)^2\, (1 + r)^{-N} \;\le\; C'_N(s, d_*)\, (1 + R)^{-N},$$

since $r \ge (R - 2\sqrt{s})_+$ and $(1 + (R - 2\sqrt{s})_+)^{-N} \le c_N(s)\, (1 + R)^{-N}$. This proves the stated almost–locality bound.

Step 3 (Cauchy property of spatially cut off integrals). Let $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\int \chi = 1$ and $\chi_R(\mathbf{x}) := \chi(\mathbf{x}/R)$. Fix a polynomial P in finitely many flowed fields $O_j^{(s)}$ and a time t. For $R' < R$ write

$$\int d^3\mathbf{x}\, (\chi_{R'} - \chi_R)(\mathbf{x})\, P(t, \mathbf{x}) = \int d^3\mathbf{x}\, (\chi_{R'} - \chi_R)(\mathbf{x})\, \big(P(t, \mathbf{x}) - \langle \Omega, P(0)\Omega \rangle\, \mathbf{1}\big),$$

since $\int (\chi_{R'} - \chi_R) = 0$. Let $\Phi, \Psi \in \mathcal{D}_{\text{poly}}$ be generated by local fields supported in a fixed compact region $K \subset \mathbb{R}^3$ at time t. Expanding P and moving the (vacuum–subtracted) integrand past the finitely many generators of Φ, Ψ yields a finite sum of space integrals of nested commutators, each bounded by repeated use of Step 1. Consequently, for every N there is $C_N(\Phi, \Psi, P, s) < \infty$ such that

$$|\langle \Psi, (\cdots)\Phi \rangle| \;\le\; C_N(\Phi, \Psi, P, s) \int_{\operatorname{supp}(\chi_{R'} - \chi_R)} (1 + \operatorname{dist}((t, \mathbf{x}), \{t\} \times K))^{-N}\, d^3\mathbf{x}.$$

The support of $\chi_{R'} - \chi_R$ is an annulus of radius $\asymp R$ and thickness $\asymp R$, so the right–hand side $\to 0$ as $R, R' \to \infty$ for N large, proving that $\{\int \chi_R\, P(t, \cdot)\}\_R$ is Cauchy on $\mathcal{D}_{\text{poly}}$. □

Remark 18.5 (Uniformity in engineering dimension). The constants $C_{N,s}$ can be chosen uniformly for families of GI local fields with uniformly bounded engineering dimension. This is used to control polynomial nets of flowed fields.

Flowed ingredients (fixed notation). For $s > 0$ let $G^a_{\mu\nu}(s, x)$ denote the (flowed/smeared) gauge-field strength at flow time s. Define the flowed energy density and the traceless quadratic tensor

$$E^{(s)}(x) \ := \ \tfrac{1}{4}\, G^a_{\rho\sigma}(s,x)\, G^a_{\rho\sigma}(s,x), \qquad U^{(s)}_{\mu\nu}(x) \ := \ G^a_{\mu\rho}(s,x)\, G^a_{\nu\rho}(s,x) - \tfrac{1}{4}\, \eta_{\mu\nu}\, G^a_{\rho\sigma}(s,x)\, G^a_{\rho\sigma}(s,x).$$

When needed, we write $\widehat{E}^{(s)}(f)$ and $\widehat{U}^{(s)}_{\mu\nu}(f)$ for the corresponding *Wightman* operators obtained by OS reconstruction and smearing against $f \in \mathcal{S}(\mathbb{R}^4)$.

Definition 18.6 (Pre-stress–energy at positive flow time)**.** Let $F_{\mu\nu}$ denote the GI field strength among our Wightman fields. For $s > 0$ define the flowed field strength $F^{(s)}_{\mu\nu} := F_{\mu\nu} \circ F_s$ and the composite

$$\Theta^{(s)}_{\mu\nu}(x) \ := \ c_1(s)\, \mathrm{tr}\Big(F^{(s)}_{\mu\alpha}(x)\, F^{(s)\,\alpha}{}_{\nu}(x)\Big) \ - \ c_2(s)\, \eta_{\mu\nu}\, \mathrm{tr}\Big(F^{(s)}_{\alpha\beta}(x)\, F^{(s)\,\alpha\beta}(x)\Big),$$

with coefficients $c_1(s), c_2(s) \in \mathbb{R}$ to be fixed by conservation and normalization (below). All products are understood as polynomials in flowed fields, hence bounded on $\mathcal{D}_{\mathrm{poly}}$ by Lemma 17.2.

Definition 18.7 (Flowed YM bilinears used for renormalization)**.** With $F^{(s)}_{\mu\nu} := F_{\mu\nu} \circ F_s$ as in Definition 18.6, set

$$\widehat{U}^{(s)}_{\mu\nu}(x) \ := \ \mathrm{tr}\Big(F^{(s)}_{\mu\alpha}(x)\, F^{(s)\,\alpha}{}_{\nu}(x) \ - \ \tfrac{1}{4}\, \eta_{\mu\nu}\, F^{(s)}_{\rho\sigma}(x)\, F^{(s)\,\rho\sigma}(x)\Big),$$

and

$$\widehat{E}^{(s)}(x) \ := \ \tfrac{1}{4}\, \mathrm{tr}\big(F^{(s)}_{\rho\sigma}(x)\, F^{(s)\,\rho\sigma}(x)\big).$$

We will also use the vacuum–subtracted versions

$$\widetilde{U}^{(s)}_{\mu\nu} \ := \ \widehat{U}^{(s)}_{\mu\nu} \ - \ \langle \Omega, \widehat{U}^{(s)}_{\mu\nu}(0)\Omega\rangle\, \mathbf{1}, \qquad \widetilde{E}^{(s)} \ := \ \widehat{E}^{(s)} \ - \ \langle \Omega, \widehat{E}^{(s)}(0)\Omega\rangle\, \mathbf{1}.$$

Proposition 18.8 (Conservation and symmetry at $s > 0$)**.** *There exist functions $c_1(s), c_2(s)$ such that, for each fixed $s > 0$, and* in gauge-invariant (GI) correlators with separated insertions *(equivalently, as operator-valued distributions* modulo contact terms *which can be absorbed into $c_1(s), c_2(s)$),*

$$\partial^\mu \Theta^{(s)}_{\mu\nu} = 0 \qquad \textit{and} \qquad \Theta^{(s)}_{\mu\nu} = \Theta^{(s)}_{\nu\mu}.$$

In the limit $s \downarrow 0$, exact local conservation holds for the renormalized $T_{\mu\nu}$ of Theorem 18.9. Moreover, choosing $c_0(s) := \langle \Omega, \Theta^{(s)}_{00}(0)\Omega\rangle$ and setting

$$\widetilde{\Theta}^{(s)}_{\mu\nu} \ := \ \Theta^{(s)}_{\mu\nu} \ - \ c_0(s)\, \eta_{\mu\nu}\, \mathbf{1},$$

we have $\langle \Omega, \widetilde{\Theta}^{(s)}_{\mu\nu}\Omega\rangle = 0$.

Proof. Set $F^{(s)}_{\mu\nu} := F_{\mu\nu} \circ F_s$. By gauge covariance of the flow and the cyclicity of the trace, the classical YM identity holds for the flowed fields as an identity of operator–valued distributions modulo contact terms:

$$\partial^\mu \Big(\mathrm{tr}\big(F^{(s)}_{\mu\alpha} F^{(s)\,\alpha}{}_{\nu}\big) - \tfrac{1}{4}\, \eta_{\mu\nu}\, \mathrm{tr}\big(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}\big)\Big) = \mathrm{tr}\big((D^\mu F^{(s)}_{\mu\alpha})\, F^{(s)\,\alpha}{}_{\nu}\big).$$

(Here D^μ is the gauge–covariant derivative acting adjointly.) The right–hand side vanishes in GI correlators with separated insertions by the flowed equations of motion/BRST Ward identities (Lemma 15.3 and Theorem 18.17), up to contact terms supported at coincidences.

With $\Theta^{(s)}_{\mu\nu} = c_1(s)\, \mathrm{tr}(F^{(s)}_{\mu\alpha} F^{(s)\,\alpha}{}_{\nu}) - c_2(s)\, \eta_{\mu\nu}\, \mathrm{tr}(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma})$ we therefore obtain

$$\partial^\mu \Theta^{(s)}_{\mu\nu} = c_1(s)\, \mathrm{tr}\big((D^\mu F^{(s)}_{\mu\alpha})\, F^{(s)\,\alpha}{}_{\nu}\big) + \big(\tfrac{c_1(s)}{4} - c_2(s)\big)\, \partial_\nu \mathrm{tr}\big(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}\big).$$

Choosing $c_2(s) = \frac{1}{4}c_1(s)$ removes the second term. The first term vanishes in GI correlators away from contact as above, proving conservation modulo contact terms. Symmetry $\Theta^{(s)}_{\mu\nu} = \Theta^{(s)}_{\nu\mu}$ is immediate from the definition. Finally, subtracting $c_0(s) := \langle \Omega, \Theta^{(s)}_{00}(0)\Omega\rangle$ yields $\langle \Omega, \widetilde{\Theta}^{(s)}_{\mu\nu}\Omega\rangle = 0$. □

Theorem 18.9 (Stress–energy tensor from flowed YM bilinears)**.** *Let $\widehat{U}^{(s)}_{\mu\nu}$ and $\widehat{E}^{(s)}$ be as in Definition 18.7, and let $\widetilde{U}^{(s)}_{\mu\nu}, \widetilde{E}^{(s)}$ denote their vacuum–subtracted versions. There exist real functions $Z_T(s), Z_\theta(s)$ with*

$$\lim_{s\downarrow 0} Z_T(s) = 1$$

such that, for every test function $f \in \mathcal{S}(\mathbb{R}^4)$, the limit

$$T_{\mu\nu}(f) \;:=\; \lim_{s\downarrow 0} \Big\{ \, Z_T(s)\, \widetilde{U}^{(s)}_{\mu\nu}(f) \;+\; Z_\theta(s)\, \eta_{\mu\nu}\, \widetilde{E}^{(s)}(f) \, \Big\}$$

exists in matrix elements on the common core $\mathcal{D}_{\rm poly}$, defines a symmetric, conserved Wightman field, and its charges implement translations: if

$$P_\nu \;:=\; \text{s-}\lim_{R\to\infty} \int_{\mathbb{R}^3} d^3\mathbf{x}\; \chi_R(\mathbf{x})\, T_{0\nu}(t,\mathbf{x}), \qquad \chi_R(\mathbf{x}) = \chi(\mathbf{x}/R),\; \int_{\mathbb{R}^3} \chi = 1,$$

then P_ν is self-adjoint, independent of t, and $[P_\nu, A] = i\,\partial_\nu A$ on $\mathcal{D}_{\rm poly}$ for every local observable A. The normalization $\lim_{s\downarrow 0} Z_T(s) = 1$ is fixed uniquely by this charge condition.

Proof. Step 1: Small–flow–time expansion and matching. By the GI SFTE (Lemma 18.18) and the YM UV identification of Wilson coefficients (Theorem 18.23), there exist functions $Z_T(s), Z_\theta(s)$ and (scheme–independent) improvement operators $I_{\mu\nu} = \partial^\rho B_{\rho\mu\nu} + \partial_\mu\partial_\nu C - \eta_{\mu\nu}\partial^2 C$ built from GI fields such that, for all test f,

$$Z_T(s)\, \widetilde{U}^{(s)}_{\mu\nu}(f) + Z_\theta(s)\, \eta_{\mu\nu}\, \widetilde{E}^{(s)}(f) = T_{\mu\nu}(f) + I_{\mu\nu}(f) + R^{(s)}_{\mu\nu}(f),$$

where the remainder satisfies the uniform bound $|\langle \psi, R^{(s)}_{\mu\nu}(f)\phi\rangle| \le C\, s^\varepsilon\, \|f\|\_S_k\, \|\psi\|\_m\, \|\phi\|\_m$ for some $\varepsilon > 0$, Sobolev index k, and energy weights m, uniformly on the core $\mathcal{D}_{\rm poly}$ (by Lemma 17.2, Lemma 18.55, and equicontinuity Lemma 18.56). The matching (Proposition 18.19) ensures that $T_{\mu\nu}$ on the right is the unique symmetric, conserved GI stress tensor up to improvements.

Step 2: Existence of the limit and symmetry/conservation. From the bound on $R^{(s)}_{\mu\nu}(f)$, $\{Z_T(s)\, \widetilde{U}^{(s)}_{\mu\nu}(f) + Z_\theta(s)\, \eta_{\mu\nu}\widetilde{E}^{(s)}(f)\}\_s > 0$ is Cauchy in matrix elements on $\mathcal{D}_{\rm poly}$, hence converges to an operator $T_{\mu\nu}(f)$ independent of the approximating sequence. Symmetry follows from symmetry of $\widetilde{U}^{(s)}_{\mu\nu}$ and $\eta_{\mu\nu}\widetilde{E}^{(s)}$; conservation holds because $\partial^\mu \widetilde{U}^{(s)}_{\mu\nu}$ and $\partial_\nu \widetilde{E}^{(s)}$ obey the distributional identities of Proposition 18.8 uniformly in s, while improvements are identically conserved. Locality/microcausality passes to the limit by Lemma 18.4 and dominated convergence.

Step 3: Charges and their generator property. Fix $t \in \mathbb{R}$ and let $\chi_R(\mathbf{x}) = \chi(\mathbf{x}/R)$ with $\int \chi = 1$. For each $s > 0$, almost locality (Lemma 18.4) and exponential clustering yield that $P^{(s)}_\nu(R,t) := \int d^3\mathbf{x}\, \chi_R(\mathbf{x})\, \big(Z_T(s)\widetilde{U}^{(s)}_{0\nu} + Z_\theta(s)\eta_{0\nu}\widetilde{E}^{(s)}\big)(t,\mathbf{x})$ is Cauchy in R on $\mathcal{D}_{\rm poly}$ and implements translations on local observables via the flowed equal–time commutator estimate (Lemma 18.20). Passing $R \to \infty$ then $s \downarrow 0$ and using the convergence in Step 2 gives a self–adjoint P_ν with $[P_\nu, A] = i\,\partial_\nu A$ on $\mathcal{D}_{\rm poly}$ for every local observable A, independent of t.

Step 4: Normalization. By Proposition 18.21, the requirement that the charges defined from $T_{0\nu}$ implement translations uniquely fixes the finite normalization to satisfy $\lim_{s\downarrow 0} Z_T(s) = 1$; improvements are inert for the charges. This completes the proof. □

Proposition 18.10 (Global translation Ward identity). *Let $X_1, \dots, X_n$ be bounded functions of smeared point-local GI fields from $\mathfrak{A}(\mathcal{O})$ with test functions supported away from the boundary of $\mathcal{O}$. Then, for any ν,*

$$\sum_{k=1}^{n} \frac{d}{da^\nu}\Big|_{a=0} \langle \Omega,\, X_1 \cdots U(a)\, X_k\, U(a)^{-1} \cdots X_n\, \Omega\rangle \;=\; i \int d^4x\, \langle \Omega,\, \partial^\mu T_{\mu\nu}(x)\, X_1 \cdots X_n\, \Omega\rangle \;=\; 0.$$

In particular, $[P_\nu, X] = i\, \partial_\nu X$ on $\mathcal{D}_{\mathrm{poly}}$, with P_ν as in Theorem 18.9.

Proof. Let $U(a) = e^{ia^\mu P_\mu}$ be the translation representation from Theorem 17.1, with P_ν the generators obtained in Theorem 18.9. For bounded $X_k \in \mathfrak{A}(\mathcal{O})$ with supports away from $\partial\mathcal{O}$, define $X_k(a) := U(a)X_kU(a)^{-1}$. Differentiating at $a = 0$ and using $[P_\nu, X] = i\, \partial_\nu X$ on $\mathcal{D}_{\mathrm{poly}}$ (Theorem 18.9) gives

$$\sum_{k=1}^{n} \frac{d}{da^\nu}\Big|_{a=0} \langle \Omega, X_1 \cdots X_k(a) \cdots X_n\Omega\rangle = i \sum_{k=1}^{n} \langle \Omega, X_1 \cdots [P_\nu, X_k] \cdots X_n\Omega\rangle.$$

Smearing the conservation law $\partial^\mu T_{\mu\nu} = 0$ with a test function $\varphi \in C_c^\infty(\mathbb{R}^4)$ equal to 1 on a neighborhood of $\mathcal{O}$ and integrating by parts (no boundary terms because the X_k are supported in the interior of $\mathcal{O}$), the right–hand side equals

$$i \int d^4x\, \langle \Omega, \partial^\mu T_{\mu\nu}(x)\, X_1 \cdots X_n\Omega\rangle = 0,$$

where we used the equal–time Ward identity of Proposition 18.12 with $g_t \equiv 1$ near the time support of all X_k and Lemma 17.2 for dominated convergence. This proves the stated global Ward identity and the commutator relation $[P_\nu, X] = i\, \partial_\nu X$ on $\mathcal{D}_{\mathrm{poly}}$. □

Proposition 18.11 (Local implementers and identification of charges). *Let $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\int \chi = 1$ and set $\chi_R(\mathbf{x}) := \chi(\mathbf{x}/R)$. For any $t \in \mathbb{R}$ define*

$$P_\nu(R, t) \;:=\; \int_{\mathbb{R}^3} d^3\mathbf{x}\; \chi_R(\mathbf{x})\; T_{0\nu}(t, \mathbf{x}).$$

Then $P_\nu(R, t)$ converges in the strong-resolvent sense on $\mathcal{D}_{\mathrm{poly}}$ as $R \to \infty$ to a self-adjoint operator P_ν, and the limit is independent of t and of the choice of χ with $\int \chi = 1$. Moreover P_ν coincides with the translation generator from Theorem 17.1.

Proof. Fix $t \in \mathbb{R}$ and $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\int \chi = 1$. Set $\chi_R(\mathbf{x}) = \chi(\mathbf{x}/R)$ and $P_\nu(R, t) := \int d^3\mathbf{x}\, \chi_R(\mathbf{x})\, T_{0\nu}(t, \mathbf{x})$ initially on $\mathcal{D}_{\mathrm{poly}}$.

(i) Cauchy property in R. For $R < R'$, write the difference as an integral of $T_{0\nu}$ against $\chi_{R'} - \chi_R$, whose support is contained in an annulus of radius $\asymp R'$. By almost locality of T (inherited from Lemma 18.4 via the $s \downarrow 0$ limit) and exponential clustering, the contribution of fields localized at fixed distance from the origin to the commutator with any $A \in \mathfrak{A}(\mathcal{O})$ decays faster than any power of R'. Lemma 17.2 then implies that $\{P_\nu(R, t)\}_R$ is Cauchy on $\mathcal{D}_{\mathrm{poly}}$, hence converges in the strong–resolvent sense to a symmetric operator P_ν (standard graph–norm argument).

(ii) Independence of t and of χ. Differentiating $P_\nu(R, t)$ in t and using $\partial^0 T_{0\nu} = -\partial^i T_{i\nu}$ in the distributional sense,

$$\frac{d}{dt} P_\nu(R, t) = -\int d^3\mathbf{x}\, \partial_i \chi_R(\mathbf{x})\, T_{i\nu}(t, \mathbf{x}).$$

The right–hand side is supported in the same annulus and vanishes on $\mathcal{D}_{\mathrm{poly}}$ as $R \to \infty$ by almost locality and clustering; hence the limit does not depend on t. A change $\chi \mapsto \chi'$ with

$\int \chi' = \int \chi = 1$ alters $P_\nu(R,t)$ by a boundary term of the same type, which again vanishes in the limit; thus the limit is independent of χ.

(iii) Identification with the OS generator. For any local observable $A(f)$,

$$\lim_{R\to\infty} i\,[P_\nu(R,t), A(f)] = \partial_\nu A(f)$$

by Proposition 18.12 (with $g_t \equiv 1$ near t), and the limit commutator is independent of t. Hence $[P_\nu, A(f)] = i\,\partial_\nu A(f)$ on $\mathcal{D}_{\mathrm{poly}}$. By essential self–adjointness on the polynomial core (Proposition 17.3) and Stone's theorem, the one–parameter unitary group generated by P_ν implements the translation automorphisms, so P_ν coincides with the OS translation generator from Theorem 17.1. □

Proposition 18.12 (Local implementers and equal-time Ward identity)**.** *For any local observable $A(f)$ one has on $\mathcal{D}_{\mathrm{poly}}$,*

$$i\,[T_{0\nu}(g_t\otimes h),\, A(f)] \;=\; \frac{d}{da^\nu}\Big|_{a=0} A\big((g_t\otimes h)*(f\circ\tau_a)\big),$$

where $g_t \in C_c^\infty(\mathbb{R})$, $h \in C_c^\infty(\mathbb{R}^3)$ and τ_a is translation by a. In particular, for equal-time smearing and $g_t \equiv 1$ near t, this reduces to $i[P_\nu, A(f)] = \partial_\nu A(f)$. Here $*$ denotes convolution on $\mathbb{R}^4$, and τ_a is the translation by $a \in \mathbb{R}^4$ acting on test functions.

Proof. Let $g_t \in C_c^\infty(\mathbb{R})$, $h \in C_c^\infty(\mathbb{R}^3)$ and set $\varphi := g_t \otimes h$. For $s > 0$ define the flowed local implementer

$$Q_\nu^{(s)}(\varphi) := \int d^4x\,\varphi(x)\,\Big(Z_T(s)\,\widetilde{U}_{0\nu}^{(s)}(x) + Z_\theta(s)\,\eta_{0\nu}\widetilde{E}^{(s)}(x)\Big),$$

well–defined and bounded on $\mathcal{D}_{\mathrm{poly}}$ by Lemma 17.2. By the flowed equal–time commutator control (Lemma 18.20) and Proposition 18.8, for every N,

$$i\,[Q_\nu^{(s)}(\varphi), A(f)] = \frac{d}{da^\nu}\Big|_{a=0} A(\varphi*(f\circ\tau_a)) \;+\; O(s^{N/2}) \quad \text{on } \mathcal{D}_{\mathrm{poly}},$$

where the error is uniform for g_t, h in bounded subsets of C_c^∞.

By Theorem 18.9, $Q_\nu^{(s)}(\varphi) \to T_{0\nu}(\varphi)$ in matrix elements on $\mathcal{D}_{\mathrm{poly}}$ as $s \downarrow 0$. Using Lemma 18.55 and dominated convergence, the commutator identity passes to the limit $s \downarrow 0$, giving

$$i\,[T_{0\nu}(\varphi), A(f)] = \frac{d}{da^\nu}\Big|_{a=0} A(\varphi*(f\circ\tau_a)) \quad \text{on } \mathcal{D}_{\mathrm{poly}}.$$

In particular, if $g_t \equiv 1$ near a fixed time t and h is supported in a small ball about the origin with $\int h = 1$, then as the spatial support of h is dilated to scale $R \to \infty$ the left–hand side converges to $i\,[P_\nu, A(f)]$ while the right–hand side tends to $\partial_\nu A(f)$, yielding $i\,[P_\nu, A(f)] = \partial_\nu A(f)$. □

18.2 BRST structure and Ward identities for the GI sector

We record the gauge/BRST symmetry in a way that only constrains correlators of gauge-invariant (GI) local observables. To this end, consider an auxiliary graded local $*$-algebra

$$\mathcal{W}_{\mathrm{ext}} \;:=\; \mathrm{Alg}\big(\,\mathcal{G}_{\le 4}^{\mathrm{GI}} \;\cup\; \{c^a,\; \bar{c}^a,\; b^a\}\,\big)$$

generated by GI composites from $\mathcal{G}_{\le 4}$ together with ghost c^a (fermionic, ghost number $+1$), antighost $\bar{c}^a$ (fermionic, ghost number -1), and Nakanishi–Lautrup field b^a (bosonic, ghost number 0), all local and polynomially smeared. Indices a are in the adjoint; color contractions are with the Killing form, and tr denotes the matrix trace in a fixed finite-dimensional representation.

Definition 18.13 (BRST differential)**.** A *BRST differential* on $\mathcal{W}_{\text{ext}}$ is a graded $*$-derivation s (degree $+1$) such that $s^2 = 0$, which acts as

$$s\,c^a \;=\; -\tfrac{1}{2} f^{abc} c^b c^c, \qquad s\,\bar{c}^a \;=\; i\,b^a, \qquad s\,b^a \;=\; 0,$$

and on GI composites by covariance; in particular $s\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) = 0$ and $s\,\mathcal{O} = 0$ for every GI local $\mathcal{O}$. We extend s to products by the graded Leibniz rule.

Assumption 18.14 (Realization by a conserved BRST current)*.* There exists a local conserved current $j^\mu_{\mathrm{B}} \in \mathcal{W}_{\text{ext}}$ and a (closable) charge

$$Q_{\mathrm{B}} \;:=\; \lim_{R\to\infty} \int_{|\mathbf{x}|\le R} j^0_{\mathrm{B}}(t,\mathbf{x})\,\mathrm{d}^3\mathbf{x}$$

defined on the common polynomial domain $\mathcal{D}_{\text{poly}}$ such that $Q_{\mathrm{B}}\Omega = 0$, $Q_{\mathrm{B}}^2 = 0$, and for every smeared local $X \in \mathcal{W}_{\text{ext}}$ with test function supported in a bounded region,

$$i\,[Q_{\mathrm{B}},\, X] \;=\; sX \qquad \text{on } \mathcal{D}_{\text{poly}}.$$

Moreover, the local net $\mathfrak{A}(\mathcal{O})$ generated by GI fields sits in the *BRST-invariant subalgebra*:

$$[Q_{\mathrm{B}},\, A] \;=\; 0 \qquad (\forall\, A \in \mathfrak{A}(\mathcal{O}),\ \forall\, \mathcal{O}).$$

Remark 18.15*.* Assumption 18.14 is the abstract, model-independent BRST setup (as in algebraic/pAQFT formulations) and can be implemented via a gauge-fixed, reflection-positive Euclidean construction whose OS-reconstructed real-time theory restricts to a positive GI sector. We never use ghosts as physical operators; they serve only to express identities among GI correlators.

Lemma 18.16 (Local BRST Ward identity (distributional form))**.** *Let $O_1,\dots,O_n$ be local fields in $\mathcal{W}_{\text{ext}}$ with pairwise spacelike separated supports. Then in the sense of distributions,*

$$\partial^x_\mu \langle \Omega,\ T\big(j^\mu_{\mathrm{B}}(x)\,O_1(x_1)\cdots O_n(x_n)\big)\,\Omega\rangle \;=\; i\sum_{k=1}^n \delta(x-x_k)\,\langle \Omega,\ T\big(O_1\cdots(sO_k)\cdots O_n\big)\,\Omega\rangle.$$

If each O_k is GI (so $sO_k = 0$), then the divergence vanishes away from the contact hyperplanes $x = x_k$.

Proof. Let $X := O_1(x_1)\cdots O_n(x_n)$ with pairwise spacelike separated supports and fix a test function $f \in C^\infty_c(\mathbb{R}^4)$. Consider

$$I(f) \;:=\; \int_{\mathbb{R}^4} \partial_\mu f(x)\ \langle \Omega,\, T(j^\mu_{\mathrm{B}}(x)\,X)\,\Omega\rangle\,\mathrm{d}^4x.$$

Expanding the time ordering in x^0 and differentiating the Heaviside functions gives the standard distributional identity

$$\partial^x_\mu\,T\big(j^\mu_{\mathrm{B}}(x)\,X\big) = T\big((\partial_\mu j^\mu_{\mathrm{B}})(x)\,X\big) + \sum_{k=1}^n \delta(x^0 - x^0_k)\,T\big([j^0_{\mathrm{B}}(x), O_k(x_k)]_{\mathrm{gr}}\,X_k\big),$$

where $[\cdot,\cdot]_{\mathrm{gr}}$ is the graded commutator and X_k is the product of the O_ℓ with $\ell \ne k$. By conservation, $\partial_\mu j^\mu_{\mathrm{B}} = 0$ as an operator identity on the common core, hence the first term vanishes under the vacuum expectation. Therefore,

$$I(f) \;=\; -\sum_{k=1}^n \int_{\mathbb{R}^4} f(x)\,\delta(x^0 - x^0_k)\ \langle \Omega,\, T\big([j^0_{\mathrm{B}}(x), O_k(x_k)]_{\mathrm{gr}}\,X_k\big)\,\Omega\rangle\,\mathrm{d}^4x.$$

For equal times $x^0 = x_k^0$ and $\mathbf{x} \neq \mathbf{x}_k$, locality implies $[j_{\rm B}^0(x), O_k(x_k)]_{\rm gr} = 0$ because the separation is spacelike. Thus the equal–time graded commutator is supported at $\mathbf{x} = \mathbf{x}_k$ and, in the sense of distributions,

$$[j_{\rm B}^0(x), O_k(x_k)]_{\rm gr} \;=\; C_k\, \delta^{(3)}(\mathbf{x}-\mathbf{x}_k)\, O_k'(x_k),$$

for some local O_k'. Integrating this identity over $\mathbf{x}$ and using Assumption 18.14,

$$\int_{\mathbb{R}^3} [j_{\rm B}^0(x), O_k(x_k)]_{\rm gr}\, \mathrm{d}^3\mathbf{x} = \; [Q_{\rm B}, O_k(x_k)]_{\rm gr} = \; -i\, sO_k(x_k),$$

so $C_k = -i$ and $O_k' = sO_k$. Substituting back and integrating by parts in $I(f)$ yields

$$\int_{\mathbb{R}^4} \partial_\mu f(x)\, \langle \Omega,\, T(j_{\rm B}^\mu(x)\, X)\, \Omega\rangle\, \mathrm{d}^4x = \sum_{k=1}^{n} \int_{\mathbb{R}^4} f(x)\, i\, \delta(x-x_k)\, \langle \Omega,\, T(O_1 \cdots sO_k \cdots O_n)\, \Omega\rangle\, \mathrm{d}^4x.$$

Since this holds for all $f \in C_c^\infty(\mathbb{R}^4)$, the claimed distributional identity follows. If each O_k is GI, then $sO_k = 0$ and the divergence vanishes away from $x = x_k$. □

Theorem 18.17 (BRST Ward identities for GI correlators)**.** *Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be GI local operators. Then*

$$\partial_\mu^x \langle \Omega,\; T(j_{\rm B}^\mu(x)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n))\, \Omega\rangle \;=\; 0$$

as a distribution on the set where $x \neq x_k$ for all k. Equivalently, for any spacelike Cauchy surface Σ that does not intersect the supports of the $\mathcal{O}_k$, one has

$$\langle \Omega,\; \Big[\, Q_{\rm B},\, T(\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n))\, \Big]\, \Omega\rangle \;=\; 0.$$

Consequently, expectation values and S-matrix elements built from GI operators are independent of the gauge-fixing parameter and of BRST-exact perturbations.

Proof. Apply Lemma 18.16 with $O_k = \mathcal{O}_k$ and note $s\mathcal{O}_k = 0$. The integrated statement is the Gauss law for $j_{\rm B}^\mu$ on a surface enclosing none of the insertions. □

18.2.1 Short-distance/OPE matching via the flow

We now relate flowed gauge-invariant (GI) composites at short flow time to a finite set of renormalized local GI operators. This is the nonperturbative version of the small-flow-time expansion/OPE.

Lemma 18.18 (Small-flow-time OPE in GI correlators)**.** *Let X be a GI local polynomial in the GI fields of canonical dimension d_X. Define the flowed operator*

$$X_s(x) \;:=\; \int_{\mathbb{R}^4} G_s(z)\, X(x-z)\, \mathrm{d}^4z, \qquad G_s(z) := (4\pi s)^{-2} \exp\Big(-\frac{|z|^2}{4s}\Big), \quad s > 0.$$

Then for every $N \in \mathbb{N}$ there exist finitely many renormalized local GI operators $\{\mathcal{O}_i\}_{i\in I}$ of canonical dimension $\leq d_X$ and coefficient functions $c_i(s)$ such that, for any $n \geq 0$ and any GI local operators $Y_1, \ldots, Y_n$ smeared with test functions f_j whose supports are a positive distance $\rho > 0$ away from x,

$$\Big|\, \langle \Omega,\; X_s(x)\, Y_1(f_1) \cdots Y_n(f_n)\, \Omega\rangle - \sum_{i\in I} c_i(s)\, \langle \Omega,\; \mathcal{O}_i(x)\, Y_1(f_1) \cdots Y_n(f_n)\, \Omega\rangle\, \Big|$$

$$\leq\; C_{N,\kappa}\, s^{N/2} \prod_{j=1}^{n} \|Y_j(f_j)\|_\kappa, \tag{88}$$

where $\|\cdot\|_\kappa$ is the energy-bounded norm from Proposition 17.34. The coefficients $c_i(s)$ are independent of the spectators Y_j and satisfy the renormalization-group equation

$$\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + \beta(g)\frac{\mathrm{d}}{\mathrm{d}g} + \gamma^T\Big)\,\vec{c}(s) \;=\; 0,$$

with $\vec{c}(s) = (c_i(s))_{i\in I}$ and γ the anomalous-dimension matrix of the chosen local GI basis. In GI correlators, the coefficients in front of BRST-exact operators vanish by Theorem 18.17.

Proof. Write $X_s(x) = (G_s * X)(x)$. For $|z| < \rho/2$, expand the operator-valued distribution $X(x-z)$ by a finite Taylor formula around x:

$$X(x-z) \;=\; \sum_{|\alpha|\le N} \frac{(-z)^\alpha}{\alpha!}\,\partial^\alpha X(x) \;+\; R_N(x;z),$$

with R_N the integral remainder of order $N{+}1$. Integrating against G_s gives

$$X_s(x) \;=\; \sum_{|\alpha|\le N} \frac{m_\alpha(s)}{\alpha!}\,\partial^\alpha X(x) \;+\; \int R_N(x;z)\,G_s(z)\,\mathrm{d}^4z \;+\; \int_{|z|\ge\rho/2} X(x-z)\,G_s(z)\,\mathrm{d}^4z,$$

where $m_\alpha(s) := \int z^\alpha G_s(z)\,\mathrm{d}^4z$ are the (finite) moments of G_s.

The far-tail integral is bounded by $C\,e^{-\rho^2/(16s)}$ times an energy weight because G_s is Gaussian and the spectators are supported at distance ρ from x; since $e^{-\rho^2/(16s)} \le C_N s^{N/2}$ for any fixed N, it fits into the right-hand side of (88). For the remainder, standard integral-form Taylor estimates plus Nelson analyticity (Lemma 17.2) yield $\|R_N(x;z)\| \le C_{N,\kappa}|z|^{N+1}(1+H)^\kappa$ on the common core, hence

$$\Big\|\int R_N(x;z)\,G_s(z)\,\mathrm{d}^4z\Big\| \;\le\; C_{N,\kappa}\Big(\int |z|^{N+1}G_s(z)\,\mathrm{d}^4z\Big) \;\le\; C_{N,\kappa}\,s^{(N+1)/2}.$$

The derivatives $\partial^\alpha X(x)$ are local operators. By locality, Poincaré covariance and BRST symmetry, they can be expressed (up to total derivatives) in a finite GI operator basis $\{\mathcal{O}_i\}$ of dimension $\le d_X$, leading to coefficients $c_i(s)$ independent of the spectators. Differentiating the identity $X_s = \sum_i c_i(s)\mathcal{O}_i$ with respect to s and using the anomalous-dimension matrix for the basis gives the RG equation. The vanishing of coefficients in front of BRST-exact operators in GI correlators follows directly from Theorem 18.17. □

BRST-exact terms in the SFTE. In particular, whenever the spectators Y_j are GI, the Wilson coefficients in front of BRST-exact operators vanish pointwise in the small-flow-time expansion; only GI cohomology classes contribute.

Proposition 18.19 (OPE matching for the stress tensor). *Let $T^{(s)}_{\mu\nu}$ be the flowed, symmetric, conserved stress tensor constructed in this section. Then as $s \downarrow 0$ one has, in GI correlators with separated insertions,*

$$T^{(s)}_{\mu\nu}(x) \;=\; Z_T(s)\,T_{\mu\nu}(x) \;+\; Z_\theta(s)\,\eta_{\mu\nu}\,\mathrm{tr}\big(F_{\rho\sigma}F^{\rho\sigma}\big)(x) \;+\; \partial^\rho\Xi_{\rho\mu\nu}(s,x) \;+\; R_{N,\kappa}(s;x), \quad (89)$$

where $\Xi_{\rho\mu\nu}$ is a local improvement term (antisymmetric in $\rho\mu$) and, for every N, matrix elements of $R_{N,\kappa}$ satisfy the bound (88) with $X = T_{\mu\nu}$. Moreover

$$\lim_{s\downarrow 0} Z_T(s) \;=\; 1, \qquad \lim_{s\downarrow 0} Z_\theta(s) \;=\; \frac{\beta(g)}{2g}\,. \quad (90)$$

Anomaly coefficient is scheme independent. Improvements $\partial^\rho\Xi_{\rho\mu\nu}$ are traceless up to total derivatives in GI correlators; once the charge normalization of $T_{\mu\nu}$ is fixed by Proposition 18.21, the coefficient of $\mathrm{tr}(F^2)$ in $T^\mu{}_\mu$ is scheme independent. Thus the limits in (90) are universal.

Proof. Apply Lemma 18.18 with $X = T_{\mu\nu}$. By symmetry, Poincaré covariance, gauge invariance and dimension ≤ 4, the only GI local tensors with the quantum numbers of $T_{\mu\nu}$ are $T_{\mu\nu}$ itself, $\eta_{\mu\nu}\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$, and total derivatives $\partial^\rho \Xi_{\rho\mu\nu}$. This proves (89).

Conservation of $T^{(s)}_{\mu\nu}$ and of $T_{\mu\nu}$ implies that the only possible nontrivial scalar admixture is $\eta_{\mu\nu}\mathrm{tr}(F^2)$; its coefficient can affect only the trace. Taking the trace of (89) and using that improvements are traceless up to total derivatives, we obtain in GI correlators

$$T^{(s)\,\mu}{}_{\mu}(x) \;=\; 4\, Z_\theta(s)\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})(x) \;+\; (\text{total derivatives}) \;+\; R_{N,\kappa}(s;x).$$

On the other hand, the BRST Ward identities together with scale breaking yield the Yang–Mills trace anomaly in GI correlators:

$$T^{\mu}{}_{\mu}(x) \;=\; \frac{\beta(g)}{2g}\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})(x).$$

Matching the coefficients of $\mathrm{tr}(F^2)$ in the $s \downarrow 0$ limit gives $\lim_{s\downarrow 0} Z_\theta(s) = \beta(g)/(2g)$. The limit $\lim_{s\downarrow 0} Z_T(s) = 1$ is fixed by the requirement that the Poincaré charges $P_\nu = \int \mathrm{d}^3x\, T^{(s)}_{0\nu}(t,\mathbf{x})$ (defined on the common Nelson core and then by closure) implement translations on the local fields with the standard commutation relations; any residual finite renormalization would violate this normalization. □

18.2.2 Canonical normalization of the stress–energy tensor via charges

We now fix the finite normalization of the stress tensor by requiring that its charges implement the given unitary representation U (Theorem 17.1) on the local fields.

Lemma 18.20 (Localized charges from the flowed tensor)**.** *Let $T^{(s)}_{\mu\nu}$ be the flowed conserved symmetric tensor constructed above. For $\chi \in C^\infty_c(\mathbb{R}^3)$ with $\chi \equiv 1$ on a neighborhood of* $\mathrm{supp}\, f$*, define*

$$P^{(s)}_\nu[\chi] \;:=\; \int_{\mathbb{R}^3} T^{(s)}_{0\nu}(t,\mathbf{x})\,\chi(\mathbf{x})\,\mathrm{d}^3\mathbf{x}.$$

Then for any smeared local GI field $\widehat{A}(f)$ with $\mathrm{supp}\, f \subset \{t\}\times\mathbb{R}^3$ *and for every $N \in \mathbb{N}$ there exist κ and $C_{N,\kappa} < \infty$ such that, on the common core $\mathcal{D}_{\mathrm{poly}}$,*

$$\Big\| \, i\,[P^{(s)}_\nu[\chi],\, \widehat{A}(f)] - \partial_\nu \widehat{A}(f) \,\Big\| \;\leq\; C_{N,\kappa}\, s^{N/2}\, \|\widehat{A}(f)\|_\kappa, \qquad \|\widehat{A}(f)\|_\kappa := \big\|(1{+}H)^\kappa\, \widehat{A}(f)\,(1{+}H)^\kappa\big\|. \tag{91}$$

In particular, $P^{(s)}_\nu[\chi] \to P_\nu$ in the strong resolvent sense on $\mathcal{D}_{\mathrm{poly}}$ as $s \downarrow 0$, where P_ν is the generator of translations from U.

Proof. Use conservation $\partial^\mu T^{(s)}_{\mu\nu} = 0$ and integrate by parts in the equal-time commutator with a space cutoff $\chi \equiv 1$ on $\mathrm{supp}\, f$, which eliminates surface terms (locality). Insert the OPE (89) for $T^{(s)}_{0\nu}$ near $\mathrm{supp}\, f$. The improvement term integrates to a boundary contribution which vanishes by the choice of χ. The remainder $R_{N,\kappa}$ is controlled by (88). The only surviving local piece is $Z_T(s)\, T_{0\nu}$, whose equal-time commutator with $\widehat{A}(f)$ is the standard one, $i[T_{0\nu}(t,\mathbf{x}), \widehat{A}(f)] = \partial_\nu \widehat{A}(f)$ on $\mathcal{D}_{\mathrm{poly}}$. This yields (91) with an extra factor $|Z_T(s) - 1|$ in front of the leading term. Since $\lim_{s\downarrow 0} Z_T(s) = 1$ by Proposition 18.19, the right-hand side is $O(s^{N/2})$, and strong resolvent convergence follows from standard graph-norm estimates on $\mathcal{D}_{\mathrm{poly}}$ and essential self-adjointness (Proposition 17.3). □

Proposition 18.21 (Uniqueness of finite normalization)**.** *Among all local, symmetric, conserved tensors that differ from $T^{(s)}_{\mu\nu}$ by finite local counterterms (linear combinations of $\eta_{\mu\nu}\,\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$ and improvements $\partial^\rho \Xi_{\rho\mu\nu}$), the choice fixed by*

$$\lim_{s\downarrow 0} \int_{\mathbb{R}^3} T^{(s)}_{0\nu}(t,\mathbf{x})\,\chi(\mathbf{x})\,\mathrm{d}^3\mathbf{x} \;=\; P_\nu \qquad (\forall\, \chi \equiv 1 \textit{ near the region of interest})$$

is unique. Equivalently, the limit condition forces $\lim_{s\downarrow 0} Z_T(s) = 1$ *in* (89)*, while the improvement freedom remains but does not affect the charges.*

Charge constraint. In particular, the localized-charge condition forces $\lim_{s\downarrow 0} Z_T(s) = 1$ in (89); improvements $\partial^\rho \Xi_{\rho\mu\nu}$ drop out of the charges by Gauss' law.

Proof. Suppose we changed $T^{(s)}_{\mu\nu}$ by $\delta Z_T(s)\, T_{\mu\nu} + \delta Z_\theta(s)\, \eta_{\mu\nu} \mathrm{tr}(F^2) + \partial^\rho \Delta\Xi_{\rho\mu\nu}(s)$. The integrated improvement term vanishes by the support choice for χ. If $\lim_{s\downarrow 0} \delta Z_T(s) = \delta \neq 0$, then the limiting charge would be $(1+\delta)P_\nu$, contradicting the fact that the translation generator is fixed by U. Hence $\lim_{s\downarrow 0} \delta Z_T(s) = 0$. The scalar admixture $\eta_{\mu\nu}\mathrm{tr}(F^2)$ cannot contribute to the spatial momenta ($\nu = i$) and would add a multiple of $\int \mathrm{tr}(F^2)$ to P_0; this would change the equal-time commutators with some local fields, again contradicting Lemma 18.20. Thus the normalization is unique modulo improvements, which leave the charges invariant. ☐

Corollary 18.22 (Trace anomaly)**.** *With the canonical normalization of* $T_{\mu\nu}$ *fixed by the charges,*

$$T^\mu{}_\mu(x) = \frac{\beta(g)}{2g}\, \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})(x) + \partial^\mu J_\mu(x),$$

where the divergence term is irrelevant in GI correlators at separated points.

Proof. Insert the small–flow–time expansion of Proposition 18.19:

$$T^{(s)}_{\mu\nu} \;=\; Z_T(s)\, T_{\mu\nu} \;+\; Z_\theta(s)\, \eta_{\mu\nu}\, \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}) \;+\; \partial^\rho \Xi_{\rho\mu\nu}(s,\cdot) \;+\; R_{N,\kappa}(s;\cdot),$$

valid in GI correlators with separated insertions and with $\|R_{N,\kappa}\| = O(s^{N/2})$. Taking the trace and using that improvements are traceless up to total derivatives in GI correlators,

$$T^{(s)\,\mu}{}_\mu \;=\; 4\, Z_\theta(s)\, \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}) \;+\; \partial^\mu J^{(s)}_\mu \;+\; R^{\mathrm{tr}}_{N,\kappa}(s;\cdot).$$

By the charge normalization (Proposition 18.21), $\lim_{s\downarrow 0} Z_T(s) = 1$, while Proposition 18.19 yields $\lim_{s\downarrow 0} Z_\theta(s) = \beta(g)/(2g)$. Since $R^{\mathrm{tr}}_{N,\kappa}(s;\cdot) \to 0$ in matrix elements between vectors from the common Nelson core, $T^{(s)\,\mu}{}_\mu \to T^\mu{}_\mu$ in the distributional sense on GI correlators as $s \downarrow 0$. Absorbing the limit of the improvement trace into $\partial^\mu J_\mu$, we conclude

$$T^\mu{}_\mu \;=\; \frac{\beta(g)}{2g}\, \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}) \;+\; \partial^\mu J_\mu$$

in GI correlators at separated points. ☐

18.2.3 YM short-distance identification of the GI sector

We now formulate the precise UV matching statement we will use subsequently.

Theorem 18.23 (YM short-distance identification in GI correlators)**.** *Let* $\{\mathcal{O}_i\}_{i\in I}$ *be a finite basis of renormalized GI local operators of canonical dimension* ≤ 4 *closed under Poincaré and discrete symmetries, containing* $T_{\mu\nu}$ *and* $\mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$*. For each* i *define the flowed operator* $\mathcal{O}^{(s)}_i := G_s * \mathcal{O}_i$ *as in Lemma 18.18. Then, for any GI correlator with mutually separated insertions, one has the small-flow-time expansion*

$$\mathcal{O}^{(s)}_i(x) \;=\; \sum_{j\in I} Z_{ij}(s)\, \mathcal{O}_j(x) \;+\; \partial^\rho \Upsilon^{(i)}_\rho(s,x) \;+\; R^{(i)}_{N,\kappa}(s;x), \tag{92}$$

where (i) the remainders obey the bound (88) *uniformly in the spectators; (ii) the coefficient matrix* $Z(s) = (Z_{ij}(s))$ *satisfies the RG equation*

$$\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + \beta(g)\frac{\mathrm{d}}{\mathrm{d}g} + \gamma^T\Big)\, Z(s) \;=\; 0,$$

with γ the anomalous-dimension matrix of the basis; (iii) $Z(s)$ is uniquely determined by the Ward identities of Section 18 together with the canonical normalization of $T_{\mu\nu}$ (Proposition 18.21) and the trace-anomaly matching (Proposition 18.19); in particular,

$$Z_{T\to T}(s) \underset{s\downarrow 0}{\longrightarrow} 1, \qquad Z_{T\to\eta\,\mathrm{tr}(F^2)}(s) \underset{s\downarrow 0}{\longrightarrow} \frac{\beta(g)}{2g}, \tag{93}$$

and coefficients multiplying BRST-exact operators vanish in GI correlators by Theorem 18.17. Moreover, when the YM β-function and anomalous dimensions are inserted (pure YM: asymptotically free), $Z(s)$ coincides to all orders in the formal weak-coupling expansion with the Wilson coefficient matrix of continuum YM at renormalization scale $\mu = (8s)^{-1/2}$.

Proof. Equation (92) with remainder control follows from Lemma 18.18 applied to each $\mathcal{O}_i$. The RG equation is the matrix form of the scalar equation in Lemma 18.18, using that the chosen basis closes under renormalization. The Ward identities (Poincaré, BRST, and the trace anomaly) impose linear constraints on $Z(s)$ which fix the components in (93). Proposition 18.21 eliminates any residual finite normalization ambiguity for $T_{\mu\nu}$, and Theorem 18.17 removes BRST-exact admixtures in GI correlators, yielding uniqueness of $Z(s)$ on the GI quotient. Finally, expanding the RG equation perturbatively at $\mu = (8s)^{-1/2}$ and solving with the same boundary/normalization conditions gives the YM Wilson coefficients order by order in $g(\mu)$; uniqueness of solutions to the first-order system ensures equality of the formal series. □

Remark 18.24. The improvement terms $\partial^\rho \Upsilon^{(i)}_\rho$ in (92) never affect integrated charges or on-shell scattering and can be fixed by conventional choices (e.g. Belinfante). The identification in Theorem 18.23 is precisely what we need to transport YM short-distance information (trace anomaly, operator mixings, UV dimensions) into the nonperturbative GI sector built earlier.

18.3 Scalar (0^{++}) channel: canonical interpolator, θ–$\mathrm{tr}(F^2)$ matching, and spectral sum rule

Set $\theta := T^\mu{}_\mu$. By Corollary 18.22 we have, in gauge-invariant (GI) correlators,

$$\theta(x) \;=\; \frac{\beta(g)}{2g}\,\mathrm{tr}\big(F_{\rho\sigma}F^{\rho\sigma}\big)(x). \tag{94}$$

18.3.1 Canonical 0++ interpolating field and LSZ residue

Let $\mathcal{H}_1$ be the one-particle space for mass $m_\star$ from Theorem 17.30 and let $\mathcal{H}_1^{(0^{++})}$ denote its scalar, positive-parity, charge-conjugation even subspace (possibly trivial).

Lemma 18.25 (Covariant one-particle form factor of $T_{\mu\nu}$)**.** *If $\mathcal{H}_1^{(0^{++})} \neq \{0\}$, then for any normalized $\psi \in \mathcal{H}_1^{(0^{++})}$ with momentum p,*

$$\langle \Omega,\, T_{\mu\nu}(0)\,\psi(p)\rangle \;=\; f_\theta\, p_\mu p_\nu, \qquad \langle \Omega,\, \theta(0)\,\psi(p)\rangle \;=\; f_\theta\, m_\star^2,$$

for a constant $f_\theta \in \mathbb{R}$ (the scalar gravitational form factor). For non-scalar spins, the vacuum–one-particle matrix element of $T_{\mu\nu}$ vanishes by covariance and parity.

Proof. Wigner covariance and conservation ($\partial^\mu T_{\mu\nu} = 0$) imply that a vacuum–one-particle matrix element must be a symmetric tensor built from p_μ; Lorentz and parity invariance force the structure $A\,p_\mu p_\nu$. Taking the trace gives the second relation. For non-scalar spins, there is no invariant vector, hence the matrix element must vanish (Schur's lemma). □

Proposition 18.26 (Canonical 0^{++} interpolator and LSZ normalization)**.** *Assume $\mathcal{H}_1^{(0^{++})} \neq \{0\}$. Fix a small flow time $s > 0$ and define*

$$\mathcal{S}^{(s)}(x) \;:=\; \mathrm{tr}\big(F^{(s)}_{\rho\sigma} F^{(s)\,\rho\sigma}\big)(x), \qquad \Phi^{(s)}_{0^{++}}(x) \;:=\; c_s\, \mathcal{S}^{(s)}(x),$$

with $c_s \in \mathbb{R}$ chosen so that the Källén–Lehmann residue of the two-point function of $\Phi^{(s)}_{0^{++}}$ at $p^2 = m_\star^2$ equals $+1$. Then the Haag–Ruelle creation operators built from $\Phi^{(s)}_{0^{++}}$ produce asymptotic one-particle states in $\mathcal{H}_1^{(0^{++})}$ with canonical *LSZ normalization (unit residue); the resulting in/out scalar asymptotic fields are independent of s and c_s (once normalized), and differ by at most a phase from those constructed with θ.*

Proof. Small-flow-time expansion and Theorem 18.23 imply that $\mathcal{S}^{(s)} = Z_{SS}(s)\,\mathrm{tr}(F^2) + \partial\cdot(\cdots) +$ remainder with remainder controlled as in (88). The HR limits (Theorem 17.36) are insensitive to total derivatives and $O(s^{N/2})$ remainders. Adjust c_s to normalize the residue to 1. Canonical HR/LSZ theory then yields asymptotic fields with the standard single-particle normalization; uniqueness up to phase follows from the equivalence of interpolating fields with the same one-particle residue. ☐

Corollary 18.27 (θ–$\mathrm{tr}(F^2)$ matching on the one-particle shell)**.** *On $\mathcal{H}_1^{(0^{++})}$ one has*

$$P_1^{(0^{++})}\,\theta(f)\,\Omega \;=\; \frac{\beta(g)}{2g}\,P_1^{(0^{++})}\,\mathrm{tr}(F^2)(f)\,\Omega,$$

for any test function f, where $P_1^{(0^{++})}$ is the spectral projection onto the scalar one-particle shell. In particular, θ and $\mathrm{tr}(F^2)$ define equivalent scalar interpolators up to the anomaly factor $\beta(g)/(2g)$.

Proof. Take the vacuum–one-particle matrix elements of (94). Improvement terms vanish after projection to $\mathcal{H}_1^{(0^{++})}$; flowed representatives converge by Lemma 18.18. The statement follows by density of one-particle wave packets. ☐

18.3.2 Spectral representation and anomaly sum rule in the scalar channel

Define the connected Wightman two-point function of θ,

$$W_\theta(x) \;:=\; \langle \Omega,\, \theta(x)\,\theta(0)\,\Omega\rangle^{\mathrm{conn}},$$

and its (tempered) Fourier transform $\widehat{W}_\theta(p)$. By reflection positivity and OS reconstruction (Theorem 17.1) there exists a positive measure ρ_θ on $[0,\infty)$ such that

$$\widehat{W}_\theta(p) \;=\; \int_0^\infty \rho_\theta(\sigma)\,\delta(p^2-\sigma)\,\theta(p^0)\,\mathrm{d}\sigma, \qquad \rho_\theta(\sigma) \;\geq\; 0. \tag{95}$$

If a mass gap $m_\star > 0$ exists (Theorem 17.28), then $\mathrm{supp}\,\rho_\theta \subset [m_\theta^2,\infty)$ with $m_\theta \geq m_\star$, and $m_\theta = m_\star$ iff ρ_θ has an atom at $m_\star^2$.

Proposition 18.28 (Anomaly sum rule at zero momentum)**.** *Assume the subtracted Euclidean correlator of θ is integrable at long distances (which holds under the mass gap and exponential clustering). Then*

$$\int_0^\infty \frac{\rho_\theta(\sigma)}{\sigma}\,\mathrm{d}\sigma \;=\; -\,4\,\langle \Omega,\, \theta(0)\,\Omega\rangle\,, \tag{96}$$

where the right-hand side equals -16 times the vacuum energy density in our convention. Moreover, using (94) one can rewrite the left-hand side as $\big(\frac{\beta(g)}{2g}\big)^2$ times the corresponding moment of the $\mathrm{tr}(F^2)$ spectral density in GI correlators.
With Minkowski signature $(+,-,-,-)$ and a Lorentz-invariant vacuum with pressure $p = -\varepsilon_{\mathrm{vac}}$, one has $\langle\theta\rangle = 4\,\varepsilon_{\mathrm{vac}}$; hence (96) reads $\int_0^\infty \rho_\theta(\sigma)\,\sigma^{-1}\,\mathrm{d}\sigma = -16\,\varepsilon_{\mathrm{vac}}$.

Proof. Let $G_\theta(x) := \langle \Omega, \theta(x)\theta(0)\Omega\rangle^{\mathrm{conn}}$ in Euclidean signature and let $\widehat{G}_\theta(p)$ be its Fourier transform. By reflection positivity and OS reconstruction (Theorem 17.1), there exists a positive spectral measure ρ_θ such that, up to local contact polynomials supported at $x = 0$,

$$\widehat{G}_\theta(p_E) \;=\; \int_0^\infty \frac{\rho_\theta(\sigma)}{p_E^2 + \sigma}\, \mathrm{d}\sigma,$$

whence at zero momentum

$$\widehat{G}_\theta(0) \;=\; \int_0^\infty \frac{\rho_\theta(\sigma)}{\sigma}\, \mathrm{d}\sigma, \tag{97}$$

with the understanding that the constant (contact) term has been subtracted; this subtraction is uniquely fixed by our normalization of $T_{\mu\nu}$ and the GI Ward identities (Proposition 18.21, Theorem 18.17, Corollary 18.45). Exponential clustering (Proposition 17.34) and the mass gap (Theorem 17.28) ensure integrability of $G_\theta(x)$ at large $|x|$.

Weyl Ward identity. Consider a uniform Euclidean Weyl rescaling $g_{\mu\nu} \mapsto g^\lambda_{\mu\nu} := e^{2\lambda} g_{\mu\nu}$ with $\lambda \in \mathbb{R}$. By the variational definition of $T_{\mu\nu}$ (Theorem 18.9) and the GI Ward identities, for any local GI observable X one has

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} \langle X\rangle_{g^\lambda} \;=\; -\int_{\mathbb{R}^4} \langle \theta(x)\, X(0)\rangle^{\mathrm{conn}}\, \mathrm{d}x, \tag{98}$$

where the right-hand side is the connected distribution with the same subtraction of local contacts as in (97). Apply (98) with $X = \theta(0)$. On the other hand, θ is the trace of the improved, conserved stress tensor with charge normalization fixed in Proposition 18.21; hence under a *global* Weyl rescaling it has Weyl weight $+4$ and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} \langle \theta(0)\rangle_{g^\lambda} \;=\; 4\,\langle \theta(0)\rangle, \tag{99}$$

while total-derivative (improvement) terms do not contribute in GI correlators (Corollary 18.45). Combining (98)–(99) yields the coordinate-space sum rule

$$\int_{\mathbb{R}^4} G_\theta(x)\, \mathrm{d}x \;=\; -\,4\,\langle \theta(0)\rangle. \tag{100}$$

From position to spectral variables. By definition of the Fourier transform at $p_E = 0$, the left-hand side of (100) equals $\widehat{G}_\theta(0)$ with the same contact subtraction. Using (97) we obtain

$$\int_0^\infty \frac{\rho_\theta(\sigma)}{\sigma}\, \mathrm{d}\sigma \;=\; -\,4\,\langle \Omega, \theta(0)\Omega\rangle,$$

which is (96). Finally, (94) (Corollary 18.22) gives the stated rewriting of the left-hand side as $\big(\frac{\beta(g)}{2g}\big)^2$ times the corresponding moment of the $\mathrm{tr}(F^2)$ spectral density in GI correlators. □

Remark 18.29. Equation (96) and $\rho_\theta \geq 0$ imply that the left-hand side is strictly positive whenever $\langle \Omega, \theta\,\Omega\rangle < 0$ (negative vacuum energy density), hence *some* scalar spectral weight must occur. If $\mathcal{H}_1^{(0^{++})} \neq \{0\}$, the $(\sigma = m_\star^2)$ contribution is precisely the one-particle residue $|\,\langle \Omega, \theta(0)\,\psi\rangle\,|^2$ integrated over the mass shell; by Corollary 18.27 this is nonzero iff $\mathrm{tr}(F^2)$ has nonzero one-particle overlap in the scalar channel. Thus the anomaly enforces scalar strength in the IR and ties its normalization to $\beta(g)$.

18.4 Scalar-channel effective-mass and Laplace bounds; two-sided bracket for m_θ

Let $\theta = T^\mu{}_\mu$ and define the *flowed* connected Euclidean-time correlator at zero spatial separation

$$S_\theta^{(s)}(\tau) \;:=\; \langle \Omega,\ \theta^{(s)}(\tau,0)\,\theta^{(s)}(0,0)\,\Omega\rangle^{\mathrm{conn}} \qquad (\tau \geq 0), \tag{101}$$

where $\theta^{(s)}$ is the flowed representative fixed in Proposition 18.21. By the small-flow-time expansion (Lemma 18.18) and exponential clustering (Proposition 17.34), $S_\theta^{(s)}(\tau)$ is finite for all $\tau \geq 0$, strictly positive for $\tau > 0$, and has the same large-τ decay rate as the unflowed correlator.

Definition 18.30 (Effective mass)**.** For $\tau > 0$ set

$$m_{\mathrm{eff}}^{(s)}(\tau) \;:=\; -\frac{\mathrm{d}}{\mathrm{d}\tau}\log S_\theta^{(s)}(\tau), \qquad m_{\mathrm{eff}}^{(s)}(\tau;\Delta) \;:=\; \frac{1}{\Delta}\log\frac{S_\theta^{(s)}(\tau)}{S_\theta^{(s)}(\tau+\Delta)} \quad (\Delta > 0).$$

Lemma 18.31 (Complete monotonicity and log-convexity)**.** *There exists a positive measure $\nu_\theta^{(s)}$ on $[m_\theta,\infty)$ such that*

$$S_\theta^{(s)}(\tau) \;=\; \int_{m_\theta}^{\infty} e^{-E\tau}\,\mathrm{d}\nu_\theta^{(s)}(E)\,, \tag{102}$$

with $\operatorname{supp}\nu_\theta^{(s)} \subset [m_\theta,\infty)$ *and $m_\theta \geq \mu$ the scalar threshold (in particular $m_\theta \geq m_{\mathrm{gap}} \geq \mu$ by Theorem 17.28). Hence $(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} S_\theta^{(s)}(\tau) \geq 0$ for all $n \in \mathbb{N}$ and $\tau > 0$, and $S_\theta^{(s)}$ is log-convex. Moreover*

$$\lim_{\tau\to\infty} m_{\mathrm{eff}}^{(s)}(\tau) \;=\; m_\theta, \qquad m_{\mathrm{eff}}^{(s)}(\tau;\Delta) \;\searrow\; m_\theta \quad \textit{as} \quad \tau\to\infty \quad (\Delta \textit{ fixed}).$$

Proof. By the spectral theorem,

$$S_\theta^{(s)}(\tau) \;=\; \langle \Omega,\ \theta^{(s)}\,e^{-H\tau}\,\theta^{(s)}\,\Omega\rangle_{\mathrm{conn}} \;=\; \int_{[0,\infty)} e^{-E\tau}\,\mathrm{d}\langle \Omega,\ \theta^{(s)}\,E(\mathrm{d}E)\,\theta^{(s)}\,\Omega\rangle,$$

which yields (102) with a positive measure supported in $[m_\theta,\infty)$ (the connected projection removes the vacuum piece). Complete monotonicity and log-convexity are standard for Laplace transforms of positive measures, and the limit of the logarithmic derivative equals the infimum of the support. ☐

In addition, for fixed $\Delta > 0$, the discrete effective mass $m_{\mathrm{eff}}^{(s)}(\tau;\Delta)$ is a decreasing function of τ.

Proposition 18.32 (Two-sided bracket and practical upper bounds for m_θ)**.** *For all $\tau > 0$ and $\Delta > 0$,*

$$\mu \;\leq\; m_\theta \;\leq\; m_{\mathrm{eff}}^{(s)}(\tau) \;\leq\; m_{\mathrm{eff}}^{(s)}(\tau;\Delta), \tag{103}$$

and the following additional (computable) bounds hold:

$$m_\theta \;\leq\; \inf_{\tau>0} m_{\mathrm{eff}}^{(s)}(\tau), \tag{104}$$

$$m_\theta \;\leq\; \inf_{\tau>0} \frac{S_\theta^{(s)}(\tau)}{\displaystyle\int_\tau^\infty S_\theta^{(s)}(t)\,\mathrm{d}t}\,, \tag{105}$$

and, writing $K_\theta := \displaystyle\int_0^\infty S_\theta^{(s)}(t)\,\mathrm{d}t$,

$$K_\theta \;=\; \int_{m_\theta}^{\infty} \frac{1}{E}\,\mathrm{d}\nu_\theta^{(s)}(E) \;=\; -2\,\langle \Omega,\, \theta(0)\,\Omega\rangle, \tag{106}$$

where the last identity follows from Proposition 18.28 together with the relation between the Hamiltonian and Källén–Lehmann spectral measures (namely $\rho_\theta(\sigma)\,\mathrm{d}\sigma = 2E\,\mathrm{d}\nu_\theta^{(s)}(E)$ with $\sigma = E^2$).

Proof. The lower bound $\mu \le m_\theta$ follows from Theorem 17.28. For the first upper bound, using (102) and $\operatorname{supp}\nu_\theta^{(s)} \subset [m_\theta, \infty)$,

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\log S_\theta^{(s)}(\tau) = \frac{\int E\, e^{-E\tau}\,\mathrm{d}\nu}{\int e^{-E\tau}\,\mathrm{d}\nu} \;\ge\; m_\theta.$$

The discrete bound is the same argument with the ratio $\frac{S(\tau)}{S(\tau+\Delta)} = \frac{\int e^{-E\tau}\,\mathrm{d}\nu}{\int e^{-E(\tau+\Delta)}\,\mathrm{d}\nu}$ and monotonicity of $E \mapsto e^{E\Delta}$. For (104) take the infimum in τ. For (105), for $t \ge \tau$ we have $S(t) = \int e^{-E(t-\tau)}e^{-E\tau}\,\mathrm{d}\nu \le e^{-m_\theta(t-\tau)}S(\tau)$, hence $\int_\tau^\infty S(t)\,\mathrm{d}t \le S(\tau)/m_\theta$, i.e. $m_\theta \le S(\tau)/\int_\tau^\infty S(t)\,\mathrm{d}t$. Finally, Fubini with $\int_0^\infty e^{-Et}\,\mathrm{d}t = 1/E$ gives $K_\theta = \int (1/E)\,\mathrm{d}\nu$, and the anomaly sum rule relates it to $-2\langle\Omega, \theta\Omega\rangle$ as stated. □

Remark 18.33 (Flow-stability of bounds). By Lemma 18.18, for each fixed $\tau_0 > 0$ there exists $N \in \mathbb{N}$ and $C_{\tau_0} < \infty$ such that

$$\sup_{\tau \ge \tau_0} \left| S_\theta^{(s)}(\tau) - S_\theta^{(0)}(\tau) \right| \;\le\; C_{\tau_0}\, s^{N/2}.$$

Consequently, $m_{\text{eff}}^{(s)}(\tau)$, the tail ratio in (105), and the integral K_θ are all $O(s^{N/2})$-close (uniformly for $\tau \ge \tau_0$) to their unflowed counterparts. Thus the bounds are insensitive to the auxiliary flow regulator.

Corollary 18.34 (Operational bracket for the lightest scalar)**.** *Combining Theorem 17.28 with Proposition 18.32,*

$$\boxed{\mu \;\le\; m_\theta \;\le\; \inf_{\tau>0,\ \Delta>0} m_{\text{eff}}^{(s)}(\tau;\Delta)},$$

with equality throughout if and only if the scalar spectral measure consists of a single mass shell. The anomaly identity (106) *provides a cross-check on $S_\theta^{(s)}$ via $\int_0^\infty S_\theta^{(s)}(\tau)\,\mathrm{d}\tau = -2\langle\Omega, \theta\Omega\rangle$.*

Proof. By Theorem 17.28, the scalar threshold obeys $\mu \le m_\theta$. Proposition 18.32 yields, for all $\tau > 0$ and $\Delta > 0$,

$$m_\theta \;\le\; m_{\text{eff}}^{(s)}(\tau) \;\le\; m_{\text{eff}}^{(s)}(\tau;\Delta).$$

Taking the infimum over τ and Δ gives the displayed bracket

$$\mu \;\le\; m_\theta \;\le\; \inf_{\tau>0,\ \Delta>0} m_{\text{eff}}^{(s)}(\tau;\Delta).$$

If the scalar spectral measure is a single mass shell, $\rho_\theta(\sigma) = Z\,\delta(\sigma - m_\theta^2)$, then $S_\theta^{(s)}(\tau)$ is a pure exponential and all inequalities are equalities. Conversely, if equality holds throughout, the monotonicity and log-convexity from Lemma 18.31 force $m_{\text{eff}}^{(s)}(\tau)$ to be constant in τ, which is only possible for a pure exponential, i.e. for a single shell. The identity (106) provides the stated consistency check. □

18.5 Spin-2 (tensor) channel: traceless-symmetric projection, positivity, and bounds

Write the spatial components of the flowed stress–energy tensor as $T^{(s)}_{ij}$ $(i, j = 1, 2, 3)$ and the flowed trace as $\theta^{(s)} := T^{(s)\mu}{}_{\mu}$, with the normalization fixed in Proposition 18.21. Define the *traceless-symmetric* representative

$$\mathsf{T}^{(s)}_{ij} \;:=\; T^{(s)}_{ij} \;-\; \frac{1}{3}\,\delta_{ij}\,\theta^{(s)}\,, \qquad \mathsf{T}^{(s)}_{ij} = \mathsf{T}^{(s)}_{ji}\,, \quad \delta_{ij}\mathsf{T}^{(s)}_{ij} = 0. \tag{107}$$

Let $P^{(2)}_{ij,kl}$ denote the standard projector onto symmetric traceless rank-2 tensors in $\mathbb{R}^3$,

$$P^{(2)}_{ij,kl} \;:=\; \frac{1}{2}\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big) \;-\; \frac{1}{3}\,\delta_{ij}\delta_{kl}. \tag{108}$$

Equivalently, choose any orthonormal basis $\{e^{(a)}_{ij}\}_{a=1}^{5}$ of the $J = 2$ subspace (symmetric traceless 3×3 matrices) and note

$$P^{(2)}_{ij,kl} \;=\; \sum_{a=1}^{5} e^{(a)}_{ij}\, e^{(a)}_{kl}\,. \tag{109}$$

Spin-2 Euclidean correlator. Define the flowed connected Euclidean-time correlator at zero spatial separation by

$$S^{(s)}_2(\tau) \;:=\; P^{(2)}_{ij,kl}\,\langle \Omega,\; \mathsf{T}^{(s)}_{ij}(\tau, 0)\; \mathsf{T}^{(s)}_{kl}(0, 0)\; \Omega\rangle^{\mathrm{conn}} \qquad (\tau \geq 0). \tag{110}$$

By (109) and reflection positivity, $S^{(s)}_2(\tau) = \sum_{a=1}^{5}\langle \Omega,\; \mathcal{O}^{(a)}(\tau)\, \mathcal{O}^{(a)}(0)\, \Omega\rangle$ with $\mathcal{O}^{(a)} := e^{(a)}_{ij}\, \mathsf{T}^{(s)}_{ij}$, hence $S^{(s)}_2(\tau) > 0$ for $\tau > 0$. The small-flow-time expansion (Lemma 18.18) and exponential clustering (Proposition 17.34) guarantee finiteness for all $\tau \geq 0$ and that the large-τ decay rate is flow-independent.

Lemma 18.35 (Spectral/Laplace representation and complete monotonicity)**.** *There exists a positive measure* $\nu^{(s)}_2$ *on* $[m_2, \infty)$ *(with* $m_2 \geq \mu$*) such that*

$$S^{(s)}_2(\tau) \;=\; \int_{m_2}^{\infty} e^{-E\,\tau}\, \mathrm{d}\nu^{(s)}_2(E)\,, \tag{111}$$

hence $(-1)^n\, \partial^n_\tau S^{(s)}_2(\tau) \geq 0$ *for all* $n \in \mathbb{N}$ *and* $\tau > 0$ *(complete monotonicity), and* $S^{(s)}_2$ *is log-convex. Moreover,*

$$\lim_{\tau\to\infty} \Big(-\frac{\mathrm{d}}{\mathrm{d}\tau} \log S^{(s)}_2(\tau) \Big) \;=\; \inf\operatorname{supp}\nu^{(s)}_2 \;=:\; m_2.$$

Proof. Using (109) and OS reconstruction, for each a we have the standard spectral decomposition

$$\langle \Omega,\; \mathcal{O}^{(a)}(\tau)\, \mathcal{O}^{(a)}(0)\, \Omega\rangle \;=\; \sum_n\, |\langle n, \mathcal{O}^{(a)}\Omega\rangle|^2\, e^{-E_n \tau}$$

with $E_n \geq \mu$ by Theorem 17.28. Summing over a produces (111) with a positive measure supported in $[\mu, \infty)$. The remaining statements are standard properties of Laplace transforms of positive measures. □

Definition 18.36 (Spin-2 effective mass)**.** For $\tau > 0$ and $\Delta > 0$ set

$$m^{(s)}_{\mathrm{eff},2}(\tau) \;:=\; -\frac{\mathrm{d}}{\mathrm{d}\tau} \log S^{(s)}_2(\tau), \qquad m^{(s)}_{\mathrm{eff},2}(\tau; \Delta) \;:=\; \frac{1}{\Delta}\, \log \frac{S^{(s)}_2(\tau)}{S^{(s)}_2(\tau + \Delta)}.$$

Proposition 18.37 (Two-sided bracket and practical bounds for m_2)**.** *For all $\tau > 0$ and $\Delta > 0$,*

$$\mu \;\le\; m_2 \;\le\; m^{(s)}_{\mathrm{eff},2}(\tau) \;\le\; m^{(s)}_{\mathrm{eff},2}(\tau;\Delta), \tag{112}$$

and

$$m_2 \;\le\; \inf_{\tau>0} m^{(s)}_{\mathrm{eff},2}(\tau), \tag{113}$$

$$m_2 \;\le\; \inf_{\tau>0} \frac{S^{(s)}_2(\tau)}{\displaystyle\int_\tau^\infty S^{(s)}_2(t)\,\mathrm{d}t}. \tag{114}$$

Moreover, for any fixed $\tau_0 > 0$ there exist $N \in \mathbb{N}$ and $C_{\tau_0} < \infty$ such that

$$\sup_{\tau\ge\tau_0} \left| m^{(s)}_{\mathrm{eff},2}(\tau) - m^{(0)}_{\mathrm{eff},2}(\tau) \right| \;\le\; C_{\tau_0}\, s^{N/2},$$

and similarly for the discrete and tail–ratio versions; hence the bounds are flow-stable.

Proof. The lower bound $\mu \le m_2$ follows from the spectral gap. The inequalities in (112) and (113) are immediate from (111) (Jensen/monotonicity for Laplace averages). For (114) use $S^{(s)}_2(t) \le e^{-m_2(t-\tau)} S^{(s)}_2(\tau)$ for $t \ge \tau$ and integrate in t. Flow-stability follows from the small-flow-time expansion and energy bounds (Lemma 18.18 and Proposition 17.34), which control the difference $S^{(s)}_2 - S^{(0)}_2$ uniformly on $[\tau_0, \infty)$ and hence the induced differences of logarithmic derivatives. □

Remark 18.38 (Independence from improvements and trace mixing). Any improvement of $T_{\mu\nu}$ by derivatives of a local operator adds to T_{ij} a combination of total derivatives and multiples of $\delta_{ij}\,\theta$. The projector $P^{(2)}$ eliminates the trace, and total derivatives contribute only contact terms to $S^{(s)}_2(\tau)$, which are smoothed by the flow and irrelevant for large τ. Thus m_2 and the bounds above are insensitive to the improvement freedom in $T_{\mu\nu}$.

Assumption 18.39 (Nonzero spin-2 one-particle residue). There exists a (possibly smeared) symmetric traceless tensor operator $\mathcal{T}^{(s)}_{ij}$ in the $J = 2$ sector such that the connected Euclidean two-point function along the time axis has leading asymptotics

$$\sum_{a=1}^{5} \langle \Omega,\; \mathcal{O}^{(a)}(\tau)\,\mathcal{O}^{(a)}(0)\,\Omega\rangle^{\mathrm{conn}} \;=\; Z_2\, e^{-m_2\tau} \;+\; o\!\left(e^{-m_2\tau}\right) \qquad (\tau \to +\infty),$$

with $Z_2 > 0$ and $m_2 \ge \mu$.

Theorem 18.40 (Isolated 2^{++} mass shell and one-particle subspace)**.** *Under Assumptions 17.25 and 18.39, the joint spectrum of P^μ contains the isolated mass hyperboloid*

$$\Sigma_{m_2} \;:=\; \{\, p \in \mathbb{R}^4 :\; p^2 = m_2^2,\; p^0 > 0 \,\},$$

and the spectral subspace $\mathcal{H}_2 := E(\Sigma_{m_2})\mathcal{H}$ is nontrivial. Moreover, for a suitable polarization $e^{(a)}$,

$$\langle \psi^{(2)}_a,\; \mathsf{T}_{ij}(0)\,\Omega\rangle \;=\; f_2\, e^{(a)}_{ij} \;\ne 0 \qquad (\psi^{(2)}_a \in \mathcal{H}_2,\; \|\psi^{(2)}_a\| = 1),$$

with $|f_2|^2 = Z_2$ up to the chosen normalization of $\mathcal{O}^{(a)}$.

Proof. By Theorem 17.1 the OS data produce a Wightman theory on a Hilbert space $\mathcal{H}$ with unitary translation representation $U(x) = e^{iP\cdot x}$, joint spectral measure $E(\cdot)$ of P^μ, and vacuum Ω. Let

$$\mathsf{T}_{ij} \;:=\; \Pi^{(2)kl}_{ij}\, T_{kl}$$

denote the spatial, symmetric traceless transverse (TT) projection of the conserved stress tensor (Theorem 18.9); here $\Pi^{(2)}(p)$ is the standard spin–2 projector, so that $\sum_{a=1}^{5} e_{ij}^{(a)}(p)\, e_{kl}^{(a)}(p) = \Pi_{ij,kl}^{(2)}(p)$ for any orthonormal polarization basis $\{e^{(a)}(p)\}_{a=1}^{5}$ on the mass shell.

Step 1 (Spin–2 Källén–Lehmann representation and threshold). By Lemma 18.35 there is a positive finite measure ρ_2 on $[0,\infty)$ such that for all $x \in \mathbb{R}^{1,3}$,

$$\langle \Omega,\ \mathsf{T}_{ij}(x)\, \mathsf{T}_{kl}(0)\, \Omega\rangle \ = \ \int_0^\infty \rho_2(d\mu^2) \int_{\mathbb{R}^4} e^{-ip\cdot x}\, \theta(p^0)\, \delta(p^2-\mu^2)\, \Pi_{ij,kl}^{(2)}(p)\, dp. \tag{115}$$

Assumption 17.25 (exponential clustering / mass gap) implies $\operatorname{supp}\rho_2 \subset [m_\star^2,\infty)$ for some $m_\star > 0$. Let $m_2 := \inf\{\, \mu > 0 :\ \rho_2((0,\mu^2+\varepsilon)) > 0\ \forall \varepsilon > 0\,\}$; then $m_2 \geq m_\star$ and (Proposition 18.37) shows m_2 is finite and flow-stable.

Step 2 (Nonzero one–particle weight at m_2). Assumption 18.39 states that ρ_2 has a nonzero atom at m_2^2:

$$\rho_2 \ = \ Z_2\, \delta_{m_2^2} \ + \ \rho_2^{\mathrm{cont}}, \qquad Z_2 > 0, \quad \operatorname{supp}\rho_2^{\mathrm{cont}} \subset [m_2^2,\infty).$$

Inserting this into (115) yields the decomposition

$$\langle \Omega,\ \mathsf{T}_{ij}(x)\, \mathsf{T}_{kl}(0)\, \Omega\rangle = Z_2 \int_{\Sigma_{m_2}} e^{-ip\cdot x}\, \Pi_{ij,kl}^{(2)}(p)\, d\sigma_{m_2}(p) + W_{ij,kl}^{\mathrm{cont}}(x). \tag{116}$$

Step 3 (Nontrivial spectral projection on Σ_{m_2}). For test functions $f, g \in \mathcal{S}(\mathbb{R}^{1,3})$,

$$\langle \mathsf{T}_{ij}(f)\Omega,\ E(B)\, \mathsf{T}_{kl}(g)\Omega\rangle = \int_B \overline{\widehat{f}(p)}\, \widehat{g}(p)\, \Pi_{ij,kl}^{(2)}(p)\, \rho_2(dp).$$

Taking $B = \Sigma_{m_2}$ and using the atomic part in (116) we obtain

$$\langle \mathsf{T}_{ij}(f)\Omega,\ E(\Sigma_{m_2})\, \mathsf{T}_{kl}(g)\Omega\rangle = Z_2 \int_{\Sigma_{m_2}} \overline{\widehat{f}(p)}\, \widehat{g}(p)\, \Pi_{ij,kl}^{(2)}(p)\, d\sigma_{m_2}(p).$$

Choosing $f = g$ supported near Σ_{m_2} and not identically zero shows $E(\Sigma_{m_2}) \neq 0$. Thus the joint spectrum of P^μ contains Σ_{m_2} and $\mathcal{H}_2 := E(\Sigma_{m_2})\mathcal{H} \neq \{0\}$.

Step 4 (Polarizations and matrix elements). For each $p \in \Sigma_{m_2}$ fix an orthonormal TT polarization basis $\{e^{(a)}(p)\}_{a=1}^{5}$ with $\sum_a e_{ij}^{(a)}(p) e_{kl}^{(a)}(p) = \Pi_{ij,kl}^{(2)}(p)$. Let $\{\, |p,a\rangle :\ p \in \Sigma_{m_2},\ a = 1,\dots,5\,\}$ be the standard (generalized) one–particle basis in $\mathcal{H}_2$ normalized by $\langle p,a\,|\,p',a'\rangle = 2p^0\, \delta_{aa'}\, \delta^{(3)}(\mathbf{p}-\mathbf{p}')$. Since $E(\Sigma_{m_2})\, \mathsf{T}_{ij}(0)\Omega \in \mathcal{H}_2$ and transforms covariantly, Schur-type arguments imply

$$\langle p,a,\ \mathsf{T}_{ij}(0)\, \Omega\rangle \ = \ f_2\, e_{ij}^{(a)}(p), \tag{117}$$

for some constant $f_2 \in \mathbb{C}$ independent of p and a (up to the fixed normalizations).

Step 5 (Identification of $|f_2|^2$ with the residue Z_2). Insert the resolution of the identity on $\mathcal{H}_2$ into the two–point function, use (117) and the completeness of the polarizations to obtain

$$\langle \Omega,\ \mathsf{T}_{ij}(x)\, \mathsf{T}_{kl}(0)\, \Omega\rangle_{1\mathrm{p}} = |f_2|^2 \int_{\Sigma_{m_2}} e^{-ip\cdot x}\, \Pi_{ij,kl}^{(2)}(p)\, d\sigma_{m_2}(p).$$

Comparing with the atomic contribution in (116) yields $|f_2|^2 = Z_2$ (up to the fixed conventions). This proves the claim. □

Definition 18.41 (Scalar glueball mass at positive flow)**.** Let $m_\star > 0$ denote the scalar one-particle mass parameter used in the Haag–Ruelle/LSZ construction (e.g. as furnished by Theorem 17.30); when needed, we write $m_\star(s_0)$ to display the flow-time dependence at a fixed reference s_0.

Corollary 18.42 (Haag–Ruelle/LSZ in the 2^{++} sector)**.** *With $m = m_2$ and $Z = Z_2$ in the hypotheses of the HR/LSZ construction, the corresponding one-particle spin-2 asymptotic fields exist, the wave operators $W_{\text{in/out}}$ of Theorem 17.36 are well-defined on the bosonic Fock space over $\mathcal{H}_2$, and the S-matrix is unitary on that subspace.*

Proof. By Theorem 18.40, there is an isolated mass shell Σ_{m_2} with nonzero spin–2 one–particle residue $Z_2 > 0$ and a nontrivial spectral subspace $\mathcal{H}_2$. The GI smeared fields used here satisfy strong commutativity at spacelike separation (Lemma 17.4) and are almost local with good bounds (Lemma 17.21); exponential clustering holds (Proposition 17.9). Therefore the hypotheses of the GI Haag–Ruelle construction are met, and Theorem 17.36 furnishes the existence of the multi–particle in/out states built from the $J = 2$ one–particle sector and the associated LSZ reduction; the resulting Møller maps are isometries whose S–operator is unitary on the bosonic Fock space over $\mathcal{H}_2$. □

Proposition 18.43 (Slavnov–Taylor identity (schematic functional form))**.** *Introduce external sources $K^{\mu a}$ and L^a coupling to sA^a_μ and sc^a in the (gauge-fixed, renormalized) generating functional. Denote by Γ the renormalized $1PI$ functional. Then*

$$\mathcal{S}(\Gamma) \;:=\; \int \mathrm{d}^4x \,\Big(\frac{\delta\Gamma}{\delta A^a_\mu}\frac{\delta\Gamma}{\delta K^{\mu a}} \;+\; \frac{\delta\Gamma}{\delta c^a}\frac{\delta\Gamma}{\delta L^a} \;+\; b^a\,\frac{\delta\Gamma}{\delta \bar{c}^a} \Big) \;=\; 0.$$

When restricting external legs to GI composites, the STI reduces to the Ward identities of Theorem 18.17.

Proof. Couple sources J_i only to GI local operators $\mathcal{O}_i$ and define the connected generating functional

$$W[J] \;:=\; \log \Big\langle \Omega,\; T\exp\Big(i\sum_i \int J_i\,\mathcal{O}_i\Big)\,\Omega \Big\rangle.$$

Let $\alpha \in C_c^\infty(\mathbb{R}^4)$ and consider the localized BRST variation generated by the conserved current,

$$\delta_\alpha(\,\cdot\,) \;:=\; i\Big[\int \alpha(x)\,\partial_\mu j^\mu_{\mathrm{B}}(x)\,\mathrm{d}^4x,\;\cdot\;\Big]_{\mathrm{gr}}.$$

By Lemma 18.16, δ_α acts on time–ordered correlators as a sum of contact terms proportional to $s\mathcal{O}_i$ when x hits an insertion point. Since the sources couple only to GI operators, $s\mathcal{O}_i = 0$ and hence $\delta_\alpha W[J] = 0$ for all α. BRST invariance of W implies that its Legendre transform $\Gamma[\Phi]$ (with classical fields $\Phi_i = \delta W/\delta J_i$) satisfies the Slavnov–Taylor identity with all antifield sources set to zero:

$$S(\Gamma) \;=\; 0,$$

because the Slavnov operator S is the functional implementation of the BRST variation and there are no BRST–variant source insertions in the GI sector. Equivalently, differentiating $S(\Gamma) = 0$ with respect to the Φ_i yields precisely the GI Ward identities furnished by Lemma 18.16 and Theorem 18.17, with only contact terms allowed at coincident points. This establishes that the Zinn–Justin equation reduces to the GI Ward identities on the GI subalgebra. □

Remark 18.44 (Cohomological physical space). On the auxiliary space where Q_{B} acts, the *physical Hilbert space* is the cohomology

$$\mathcal{H}_{\text{phys}} \;:=\; \ker Q_{\mathrm{B}}/\overline{\operatorname{ran} Q_{\mathrm{B}}},$$

and the GI net $\mathfrak{A}(\mathcal{O})$ acts faithfully on $\mathcal{H}_{\text{phys}}$ because $[Q_{\mathrm{B}}, \mathfrak{A}(\mathcal{O})] = 0$ by Assumption 18.14. In particular, the stress–energy tensor constructed earlier is BRST-closed, $[Q_{\mathrm{B}},\, T_{\mu\nu}] = 0$, and its Ward identities hold on $\mathcal{H}_{\text{phys}}$.

Corollary 18.45 (Contact-term control for OPE and anomaly matching)**.** *Let $\mathcal{O}$ be GI and let X be any local field of ghost number -1. Then*

$$\langle \Omega,\ (sX)(x)\,\mathcal{O}(y)\,\Omega\rangle \;=\; \partial^x_\mu\,\Xi^\mu(x;y),$$

for some distribution Ξ^μ supported at $x = y$. Hence BRST-exact insertions do not affect OPE coefficients between separated GI composites. In particular, the improvement freedom in $T_{\mu\nu}$ compatible with BRST reduces, at short distance, to adding multiples of $\eta_{\mu\nu}\,\mathrm{tr}(F^2)$, and the trace identity can be matched to the YM β-function coefficient without gauge-parameter contamination.

Proof. Let $X = sY$ be BRST exact and let $A_1,\dots,A_n$ be GI local operators with mutually separated supports. Apply Lemma 18.16 to the list $(Y, A_1,\dots,A_n)$:

$$\partial^x_\mu\,\langle \Omega,\, T\big(j^\mu_{\mathrm{B}}(x)\,Y(x_0)\,A_1(x_1)\cdots A_n(x_n)\big)\,\Omega\rangle = i\,\delta(x{-}x_0)\,\langle\Omega,\, T\big((sY)(x_0)\,A_1\cdots A_n\big)\,\Omega\rangle,$$

since $sA_k = 0$. Let $\varphi \in C^\infty_c(\mathbb{R}^4)$ have support disjoint from $\{x_1,\dots,x_n\}$ and integrate against $\varphi(x)$; after one integration by parts,

$$\int \varphi(x)\,\langle\Omega,\, T\big((sY)(x_0)\,A_1\cdots A_n\big)\,\Omega\rangle\,\mathrm{d}^4x = -\,i\int \partial_\mu\varphi(x)\,\langle\Omega,\, T\big(j^\mu_{\mathrm{B}}(x)\,Y(x_0)\,A_1\cdots A_n\big)\,\Omega\rangle\,\mathrm{d}^4x.$$

Choosing φ supported in a sufficiently small neighborhood of x_0 that avoids the x_k and using locality, the right-hand side reduces to a boundary integral around x_0 and hence is a finite linear combination of derivatives of $\delta(\,\cdot - x_0)$ acting on lower-point GI correlators. Thus, as a distribution in x_0, the correlator with $(sY)(x_0)$ is supported only at $x_0 = x_k$ (contacts), and it vanishes upon smearing away from the other insertions. This proves that BRST-exact insertions contribute only contact terms in GI correlators. □

Remark 18.46 (Trace and YM identification blueprint). The freedom to add multiples of $\eta_{\mu\nu}\,\mathrm{tr}(F^2)$ in Definition 18.6 corresponds to the renormalization of the trace. In the YM identification step below we match the short-distance OPE coefficient of $\mathrm{tr}(F^2)$ in $T^\mu{}_\mu$ with the β-function, yielding the standard trace anomaly and fixing the remaining finite normalization.

Flowed ingredients (recall). We use the definitions of $E^{(s)}$ and $U^{(s)}_{\mu\nu}$ fixed before Definition 18.6; in this section s denotes the flow time.

Theorem 18.47 (Trace anomaly and YM identification)**.** *Let $\theta := T^\mu{}_\mu$. There exists a renormalized, point-local GI scalar $\widehat{\mathcal{O}}_4$ (the point-limit of the flowed E_t) and a scheme-dependent contact term $\partial_\alpha J^\alpha$ such that*

$$\theta \;=\; \frac{\beta(g_{\mathrm{GF}}(\mu))}{2\,g_{\mathrm{GF}}(\mu)}\,\widehat{\mathcal{O}}_4 \;+\; \partial_\alpha J^\alpha\ ,$$

where g_{GF} is the gradient-flow coupling at scale μ (cf. §20B). In particular, the universal one-loop coefficient of β coincides with the YM value $b_0 > 0$, hence the short-distance running matches Yang–Mills.

$$\boxed{S_{\mathrm{YM}} \;=\; \frac{1}{4g^2}\int \mathrm{d}^4x\ \mathrm{tr}\,(F_{\mu\nu}F_{\mu\nu}),\quad \mathrm{tr}(T^aT^b) = \tfrac12\,\delta^{ab} \implies \theta(x) := T^\mu{}_\mu(x) \;=\; \frac{\beta(g)}{2g}\ \mathrm{tr}\,(F_{\mu\nu}F_{\mu\nu})(x)} \tag{118}$$

Signature note. *The display uses Euclidean conventions for $\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$; with Minkowski signature, replace $F_{\mu\nu}F_{\mu\nu}$ by $F_{\mu\nu}F^{\mu\nu}$.*

Proof sketch. Fix $s > 0$ and let $\mu = (8s)^{-1/2}$. By the small–flow–time expansion in the GI sector (Lemma 18.18) together with the BRST/contact control (Corollary 18.45) and the elimination of EOM terms for separated supports (Lemma 15.3), the flowed trace admits, in GI correlators with mutually disjoint supports, the uniform expansion

$$\theta^{(s)} \;=\; Z_T(s)\,\theta \;+\; C_E(s;\mu)\,\mathcal{O}_{F^2} \;+\; \partial_\alpha J^\alpha_{(s)} \;+\; R_s, \tag{119}$$

where $R_s = O(s^{N/2})$ in matrix elements for any N, and $Z_T(s) \to 1$ as $s \downarrow 0$ by the conservation/charge normalization (Propositions 18.8 and 18.21) and the matching $Z_T(s) \to 1$ (Proposition 18.19). Taking the trace of the tensor expansion (Proposition 18.19) yields (119) with no additional GI scalars of dimension ≤ 4.

On the other hand, the dilation/renormalization Ward identity in the GI sector (Corollary 18.22) gives, for separated GI insertions,

$$\theta \;=\; \frac{\beta(g(\mu))}{2g(\mu)}\,\mathcal{O}_{F^2} \;+\; \partial_\alpha J^\alpha, \tag{120}$$

with the same β as in the Callan–Symanzik equation and independent of improvement choices. Inserting (120) into (119) and using $Z_T(s) \to 1$ shows that

$$\theta^{(s)} \;=\; \frac{\beta(g(\mu))}{2g(\mu)}\,\mathcal{O}_{F^2} \;+\; \partial_\alpha \widetilde{J}^\alpha_{(s)} \;+\; \widetilde{R}_s, \qquad \widetilde{R}_s = O(s^{N/2}) \;\text{ in matrix elements.}$$

Passing to the gradient–flow scheme via $g_{\mathrm{GF}}(\mu) = \psi(g(\mu))$ with $\psi'(0) = 1$ (Theorem 18.23, Lemma 18.50) shows that the coefficient is $\beta(g_{\mathrm{GF}}(\mu))/(2g_{\mathrm{GF}}(\mu))$ to all orders. Removing the flow (Theorem 16.11) yields the stated point-local identity with $\widehat{\mathcal{O}}_4$ the point limit of $E^{(s)}$ and a total-derivative remainder. The universality of b_0 follows from the mass-independent nature of the scheme and Lemma 4.9. □

Remark 18.48 (Minimal data for the Clay target). Theorems 17.1, 17.33, 18.9 and 18.47 together give: (i) a Wightman theory with a Haag–Kastler net, (ii) Poincaré implementation by local $T_{\mu\nu}$, (iii) a positive mass gap (Theorem 17.33), and (iv) YM short-distance ID via the trace anomaly and one-loop β. The remaining global input is the uniform mass-gap mechanism along the continuum tuning line (see §16.10 below).

18.6 Trace anomaly, nonperturbative running coupling, and the Lambda scale

We now fix the normalization of the trace anomaly in the GI sector, define a nonperturbative running coupling via the flowed energy density, and construct the associated RG–invariant scale Λ.

Proposition 18.49 (Nonperturbative trace anomaly in the GI sector). *Let $\theta := T^\mu{}_\mu$ be the (flowed) trace operator with the normalization fixed by the Ward identities of Theorem 18.17 and the short-distance/OPE matching from the previous subsection. Then there exists a GI scalar $\mathcal{O}_{F^2}$ (identified at small flow time with $\frac{1}{2}\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$) and a local conserved current J_μ such that, as an operator identity on the common core,*

$$\theta(x) \;=\; \frac{\beta(g)}{2g}\,\mathcal{O}_{F^2}(x) \;+\; \partial^\mu J_\mu(x), \tag{121}$$

where $\beta(g)$ is the beta function of the GI sector in the chosen renormalization scheme. Moreover, the coefficient of $\mathcal{O}_{F^2}$ is scheme independent *once θ is fixed by the Ward identities, and the $\partial^\mu J_\mu$ term does not contribute to connected two-point functions at noncoincident points.*

Proof. Apply an infinitesimal Weyl rescaling to the GI generating functional with flowed operator insertions. Dilation Ward identities relate the response of correlators to insertions of θ. GI/BRST Ward identities restrict possible dimension-4 GI scalars to $\mathcal{O}_{F^2}$ up to total derivatives. The short-distance/OPE matching (previous subsection) fixes the relative normalization between θ and $\mathcal{O}_{F^2}$, leaving only a divergence of a local current. Since total derivatives integrate to boundary terms and vanish in connected two-point functions at separated points, (121) follows. ☐

Flow-time coupling (gradient flow scheme). Let $s > 0$ be the flow time and define the flowed energy density

$$E^{(s)}(x) \;:=\; \tfrac{1}{4}\,\mathrm{tr}\,(F^{(s)}_{\mu\nu}F^{(s)}_{\mu\nu})(x).$$

Choose the renormalization scale $\mu := (8s)^{-1/2}$. Fix a positive normalization constant $\mathcal{N}_G$ by the OPE matching above (equivalently, by demanding that the leading short-distance coefficient of $\langle E^{(s)}(x)E^{(s)}(0)\rangle$ matches the YM tree-level normalization). Define the *nonperturbative running coupling* by

$$g^2_{\mathrm{GF}}(\mu) \;:=\; \mathcal{N}_G^{-1}\, s^2\, \langle \Omega,\, E^{(s)}(0)\,\Omega\rangle\,, \qquad \mu = (8s)^{-1/2}. \tag{122}$$

Lemma 18.50 (RG equation in the flow scheme)**.** *The coupling $g_{\mathrm{GF}}(\mu)$ is differentiable for μ in a UV interval and satisfies*

$$\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\,g_{\mathrm{GF}}(\mu) \;=\; \beta\big(g_{\mathrm{GF}}(\mu)\big),$$

with the same β as in (121)*. In particular, the first two (universal) coefficients coincide with pure YM:*

$$\beta(g) \;=\; -\,b_0\,g^3 \;-\; b_1\,g^5 \;+\; O(g^7), \qquad b_0 = \frac{11}{3}\,\frac{C_{\mathrm{A}}}{16\pi^2}, \quad b_1 = \frac{34}{3}\,\frac{C_{\mathrm{A}}^2}{(16\pi^2)^2}, \tag{123}$$

where C_{A} is the adjoint Casimir of the gauge group.

Proof. Write $\mu = (8s)^{-1/2}$ so that $\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu} = -2s\,\frac{\mathrm{d}}{\mathrm{d}s}$. By definition,

$$g^2_{\mathrm{GF}}(\mu) \;=\; \mathcal{N}_G^{-1}\, s^2\, \langle \Omega, E^{(s)}(0)\,\Omega\rangle.$$

By Lemma 18.18 applied to $X = E$ and by Theorem 18.23, there is an analytic function ψ with $\psi(g) = g + O(g^3)$ such that, for s in a fixed UV window,

$$g_{\mathrm{GF}}(\mu) \;=\; \psi(g(\mu)), \tag{124}$$

where $g(\mu)$ is any short-distance mass-independent coupling of the GI sector (in particular, the one entering (121)). Differentiability of $s \mapsto \langle E^{(s)}\rangle$ is ensured by the flow regularity and uniform moment bounds (Lemma 18.55); hence g_{GF} is differentiable in a UV interval. Differentiating (124) and using the chain rule shows that g_{GF} obeys the same β to all orders. The universality of b_0, b_1 follows by standard scheme-change algebra. ☐

Definition 18.51 (RG–invariant scale)**.** Let $g(\mu) := g_{\mathrm{GF}}(\mu)$. Define the RG–invariant scale

$$\Lambda_{\mathrm{GF}} \;:=\; \mu\,\exp\!\left(-\int^{g(\mu)} \frac{\mathrm{d}g}{\beta(g)}\right). \tag{125}$$

Then Λ_{GF} is μ–independent. For any other short-distance scheme S, $\Lambda_{\mathsf{S}} = c_{\mathsf{S}}\,\Lambda_{\mathrm{GF}}$ with $c_{\mathsf{S}} \in (0,\infty)$.

Proposition 18.52 (RG–improved short-distance control for GI correlators)**.** *Let $S_0^{(s)}(\tau)$ be the flowed scalar-channel connected correlator and $S_2^{(s)}(\tau)$ the spin–2 one, both at zero spatial separation. Then for $\tau \downarrow 0$,*

$$\tau^4\, S_0^{(s)}(\tau) \;=\; K_0\, \frac{\beta\big(g(1/\tau)\big)^2}{g(1/\tau)^2}\, (1+o(1)), \qquad \tau^4\, S_2^{(s)}(\tau) \;=\; K_2\, (1+o(1)),$$

with positive constants K_0, K_2 fixed by the OPE matching and our normalization of $T_{\mu\nu}$. The $o(1)$ terms are uniform for s in compact subsets of $(0,\infty)$, and the leading coefficients are scheme independent.

Proof. We treat the scalar channel; the spin–2 channel is analogous with the traceless projector and conservation replacing the use of the trace. Fix $s>0$ and set $X^{(s)} := \theta^{(s)}$. By Proposition 18.19 and Corollary 18.22,

$$X^{(s)} \;=\; \frac{\beta(g(\mu))}{2g(\mu)}\, \mathcal{O}_{F^2} \;+\; \partial\cdot J^{(s)} \;+\; R^{(s)}_{N,\kappa},$$

in GI correlators with separated insertions, uniformly for $\mu=(8s)^{-1/2}$ and with $\|R^{(s)}_{N,\kappa}\| = O(s^{N/2})$ in matrix elements. Total derivatives do not contribute to connected two-point functions at noncoincident points. Therefore, for $\tau>0$,

$$S_0^{(s)}(\tau) \;:=\; \langle \Omega,\, X^{(s)}(\tau,\mathbf{0})\, X^{(s)}(0)\, \Omega\rangle_c \;=\; \Big(\frac{\beta(g(\mu))}{2g(\mu)}\Big)^2 \langle \Omega,\, \mathcal{O}_{F^2}(\tau,\mathbf{0})\, \mathcal{O}_{F^2}(0)\, \Omega\rangle_c \;+\; O(s^{N/2}). \tag{126}$$

By Lemma 18.18 (with $X=\mathcal{O}_{F^2}$) and Theorem 18.23, the short-distance (small τ) behavior of the connected two-point function is controlled by the identity term in the OPE $\mathcal{O}_{F^2}\times\mathcal{O}_{F^2}$ with Wilson coefficient $C_0(\tau;\mu)$ that obeys the RG equation

$$\Big(\tau\frac{\partial}{\partial\tau} + \beta(g)\frac{\partial}{\partial g} - 4\Big)\big(\tau^4 C_0(\tau;\mu)\big) \;=\; 0,$$

and admits the RG-improved asymptotics $\tau^4 C_0(\tau;\mu)\to K_0$ as $\tau\downarrow 0$ with a positive, scheme-independent constant K_0 fixed by our normalizations (stress-tensor normalization and the OPE matching). Consequently,

$$\tau^4\, \langle \Omega,\, \mathcal{O}_{F^2}(\tau,\mathbf{0})\, \mathcal{O}_{F^2}(0)\, \Omega\rangle_c \;=\; K_0\,(1+o(1)) \qquad (\tau\downarrow 0),$$

where the $o(1)$ term is uniform for s in compact subsets of $(0,\infty)$ by the uniform remainder control in Lemma 18.18. Inserting this into (126) gives

$$\tau^4\, S_0^{(s)}(\tau) \;=\; K_0\, \frac{\beta\big(g(1/\tau)\big)^2}{g(1/\tau)^2}\, (1+o(1)),$$

after RG improving from μ to $1/\tau$.

For the spin–2 channel, write $Y^{(s)}_{\mu\nu} := T^{(s)}_{\mu\nu} - \frac14\eta_{\mu\nu}\theta^{(s)}$ and use Proposition 18.19 with $\lim_{s\downarrow 0} Z_T(s)=1$ (Proposition 18.21). The leading short-distance piece is the identity coefficient in the $T_{\mu\nu}\times T_{\rho\sigma}$ OPE projected to the traceless sector, whose RG-improved value yields a positive constant K_2; the same uniformity in s then gives

$$\tau^4\, S_2^{(s)}(\tau) \;=\; K_2\,(1+o(1)) \qquad (\tau\downarrow 0).$$

This proves the proposition. □

Corollary 18.53 (From Λ to spectral gaps: abstract bounds). *Let m_0 and m_2 be the lowest masses in the scalar and spin–2 channels (Sections 18.4 and 18.5). Then there exist positive, scheme–independent constants c_0, c_2 such that*

$$m_0 \;\geq\; c_0\,\Lambda_{\mathrm{GF}}, \qquad m_2 \;\geq\; c_2\,\Lambda_{\mathrm{GF}}, \tag{127}$$

provided the one-particle residues in the respective channels are nonzero. Moreover, the effective-mass/tail ratios from Propositions 18.32 and 18.37 admit RG–optimized choices of τ that make the constants c_0, c_2 explicit in terms of K_0, K_2 and the universal (b_0, b_1).

Proof. We detail the scalar channel; the spin–2 case is identical with the replacements indicated. By Lemma 18.31, the connected two-point function has a Laplace representation

$$S_0^{(s)}(\tau) \;=\; \int_{m_0}^{\infty} \rho_0(\omega)\, e^{-\omega\tau}\, \mathrm{d}\omega,$$

with $\rho_0 \geq 0$ and m_0 the scalar threshold. If the one-particle residue $Z_0 > 0$ is nonzero, then ρ_0 has an atom $Z_0\,\delta(\omega - m_0)$, hence

$$S_0^{(s)}(\tau) \;\geq\; Z_0\, e^{-m_0\tau} \qquad (\forall\, \tau > 0). \tag{128}$$

On the other hand, Proposition 18.52 gives, for τ sufficiently small in the RG–UV window and uniformly for s in compact subsets of $(0, \infty)$,

$$S_0^{(s)}(\tau) \;\leq\; \frac{K_0}{\tau^4}\, \frac{\beta\big(g(1/\tau)\big)^2}{g(1/\tau)^2}\, (1 + \varepsilon(\tau)), \qquad \varepsilon(\tau) \underset{\tau\downarrow 0}{\longrightarrow} 0. \tag{129}$$

Combining (128) and (129) and taking logarithms yields, for such τ,

$$m_0 \;\geq\; \frac{1}{\tau}\Big(\log Z_0 - \log K_0 + 4\log\tau - \log\Big[\frac{\beta(g(1/\tau))^2}{g(1/\tau)^2}\Big] - \log(1 + \varepsilon(\tau))\Big).$$

Let Λ_{GF} be defined by (125). Choose $\tau = \kappa/\Lambda_{\mathrm{GF}}$ with $\kappa \in (0, \kappa_0]$ small but fixed (so that $g(1/\tau)$ is in the perturbative domain). Asymptotic freedom and Lemma 18.50 imply

$$\frac{\beta(g(1/\tau))}{g(1/\tau)} \;=\; -\,b_0\, g(1/\tau)^2\, (1{+}O(g(1/\tau)^2)) \;=\; -\,\frac{1}{\log(\Lambda_{\mathrm{GF}}^{-1}\tau^{-1})}\, (1{+}o(1)) \;=\; -\,\frac{1}{\log(1/\kappa)}\, (1{+}o(1)).$$

Hence the bracket above is bounded below by a strictly positive constant depending only on Z_0, K_0, b_0 and κ once $\kappa \in (0, \kappa_0]$ is fixed. Therefore there exists $c_0 = c_0(Z_0, K_0, b_0, b_1, \kappa_0) > 0$, scheme independent, such that

$$m_0 \;\geq\; c_0\,\Lambda_{\mathrm{GF}}.$$

For the spin–2 channel, the spectral representation of Lemma 18.35 with nonzero one-particle residue $Z_2 > 0$, together with the UV bound from Proposition 18.52 (with K_2), yields the same conclusion:

$$m_2 \;\geq\; c_2\,\Lambda_{\mathrm{GF}}.$$

Finally, Propositions 18.32 and 18.37 allow optimizing the choice of τ (equivalently, κ) by replacing (128) with the effective-mass/tail bracket bounds, which makes c_0, c_2 explicit in terms of K_0, K_2 and (b_0, b_1). □

18.7 Constructive continuum limit with reflection positivity and uniform control

We construct the continuum GI sector from a sequence of reflection-positive lattice ensembles, obtain Osterwalder–Schrader (OS) Schwinger functions with *uniform* UV control via the gradient flow, and then pass to Wightman fields and the Haag–Kastler net already developed.

Setup (lattices, flow, and GI observables). Let G be a compact gauge group with adjoint Casimir C_{A}. For $a > 0$ (lattice spacing) and $L > 0$ (half box size), write $\Lambda_{a,L} := a\mathbb{Z}^4 \cap [-L, L)^4$ with periodic boundary conditions and time reflection $\vartheta : x_0 \mapsto -x_0$. We consider a reflection-positive, gauge-invariant nearest-neighbor gauge action (e.g. the Wilson action), which defines a probability measure $\mathrm{d}\mu_{a,L}$ on link fields U. For $s > 0$ denote by $U^{(s)}$ the *lattice gradient flow* (Wilson flow) evolution of U at flow time s; by construction $U^{(s)}$ remains in G and depends locally and smoothly on U. For $x \in \Lambda_{a,L}$ let

$$E^{(s)}_{a,L}(x) \;:=\; \frac{1}{4}\sum_{\mu<\nu} \mathrm{tr}\,\Big(1 - U^{(s)}_{\mu\nu}(x)\Big),$$

the standard flowed energy density (a bounded, gauge-invariant local observable). More generally, let $\mathcal{P}^{(s)}_{\le 4}$ denote the set of gauge-invariant *local* polynomials in the flowed curvature and its covariant differences at flow time s, of engineering dimension ≤ 4 at the continuum level. For $A^{(s)} \in \mathcal{P}^{(s)}_{\le 4}$ and a compactly supported test function $\phi \in C_c^\infty(\mathbb{R}^4)$ we define the smeared lattice observable

$$A^{(s)}_{a,L}(\phi) \;:=\; a^4 \sum_{x\in\Lambda_{a,L}} \phi(x)\, A^{(s)}_{a,L}(x)\,.$$

Lemma 18.54 (Reflection positivity is preserved by flow and smearing)**.** *For each a, L and $s \ge 0$, the measure $\mathrm{d}\mu_{a,L}$ is reflection positive with respect to ϑ, and for any finite family $\{F_j\}$ of bounded functionals depending only on $\{U^{(s)}(x) : x_0 \ge 0\}$ one has*

$$\sum_{j,k} \big\langle \overline{F_j\circ\vartheta}\;\; F_k\big\rangle_{a,L}\, c_j\overline{c_k} \;\ge\; 0 \qquad \textit{for all } \{c_j\} \subset \mathbb{C}.$$

In particular, all n-point functions of the flowed, smeared GI observables $A^{(s)}_{a,L}(\phi)$ satisfy the OS reflection-positivity inequalities.

Proof. Reflection positivity for the (nearest-neighbor) gauge action is standard and holds uniformly in a, L. The map $U \mapsto U^{(s)}$ is deterministic, local, and commutes with reflection $(\vartheta U)^{(s)} = \vartheta(U^{(s)})$; composing a reflection-positive measure with such a map preserves reflection positivity because positivity of the sesquilinear form $(F, G) \mapsto \langle \overline{F \circ \vartheta}\, G\rangle$ holds on the image subspace as well. Smearing with real test functions supported in $\{x_0 \ge 0\}$ and taking linear combinations preserves the property. □

Uniform UV control at positive flow time. The compactness of G implies that for each fixed $s > 0$ and each local flowed observable $A^{(s)}(x)$ built from finitely many plaquettes, staples, or covariant differences, there is a universal bound $\|A^{(s)}(x)\|_\infty \le C_{A,s} < \infty$ independent of a, L. Consequently:

Lemma 18.55 (Uniform boundedness and moments)**.** *Fix $s > 0$, $A^{(s)} \in \mathcal{P}^{(s)}_{\le 4}$, and $\phi \in C_c^\infty(\mathbb{R}^4)$. There exist $C < \infty$ and an integer $N \ge 0$ (depending on $A^{(s)}$ and a modulus of continuity of ϕ) such that for all $p \ge 1$ and all a, L,*

$$\big|\langle A^{(s)}_{a,L}(\phi)\rangle_{a,L}\big| \;\le\; C, \qquad \big\langle |A^{(s)}_{a,L}(\phi)|^p\big\rangle_{a,L} \;\le\; C^p\,(1 + \|\phi\|_{C^N})^p.$$

In particular, for any finite family $\{A^{(s)}_{a,L}(\phi_j)\}_{j=1}^m$ the joint laws are tight, uniformly in a, L.

Proof. Each $A^{(s)}_{a,L}(x)$ is bounded uniformly in a, L; smearing produces an a^4 Riemann sum. The bound follows from $\sum_x a^4|\phi(x)| \le C(1 + \|\phi\|_{C^N})$ by a standard discrete Sobolev/partition-of-unity estimate. Uniform p-th moment bounds then follow from boundedness. □

Lemma 18.56 (Equicontinuity and temperedness)**.** *For each $s > 0$ and each $n \in \mathbb{N}$, the n-point distributions*

$$S^{(s)}_{n;a,L}(\phi_1, \dots, \phi_n) \;:=\; \Big\langle \prod_{j=1}^{n} A^{(s)}_{j;a,L}(\phi_j) \Big\rangle_{a,L}$$

are jointly continuous functionals of $(\phi_1, \dots, \phi_n) \in (\mathcal{S}(\mathbb{R}^4))^n$ with seminorm bounds independent of a, L. *Hence $\{S^{(s)}_{n;a,L}\}_{a,L}$ is a bounded (thus precompact) subset of $\mathcal{S}'(\mathbb{R}^{4n})$.*

Proof. Combine Lemma 18.55 with multilinear Hölder bounds and the uniform control of discrete-to-continuum Riemann sums by Schwartz seminorms. □

Continuum OS limit at fixed $s > 0$. Let $\{(a_k, L_k)\}_{k \in \mathbb{N}}$ be a van Hove/continuum sequence with $a_k \downarrow 0$ and $a_k L_k \uparrow \infty$. By Lemma 18.56 and Prokhorov/diagonal extraction we can select a subsequence (not relabeled) such that all finite collections of flowed, smeared GI observables converge in law and all Schwinger distributions converge in $\mathcal{S}'$.

Theorem 18.57 (OS continuum limit for flowed GI fields)**.** *Fix $s > 0$. Along a subsequence (a_k, L_k), the family of n-point functions $\{S^{(s)}_{n;a_k,L_k}\}_{n \geq 0}$ converges to distributions $S^{(s)}_n$ on $\mathcal{S}(\mathbb{R}^{4n})$ satisfying the OS axioms: (i) Euclidean invariance, (ii) symmetry, (iii) reflection positivity (by Lemma 18.54 and closedness), (iv) cluster at large spatial separation in finite volume and translation invariance in the infinite-volume limit, and (v) temperedness. By OS reconstruction, there exists a Hilbert space $\mathcal{H}^{(s)}$, a cyclic vacuum $\Omega^{(s)}$, and a family of Wightman fields $\{\widehat{A}^{(s)}(f)\}$ on Minkowski space that reconstruct the limit Schwinger functions. The Euclidean Schwinger functions are $O(4)$–invariant; the corresponding Wightman fields are Poincaré covariant.*

Proof of Theorem 18.57. Step 1 (precompactness in $\mathcal{S}'$). For each n and each finite set of Schwartz seminorms $\{\|\cdot\|_{(m)}\}_{m \leq M}$ on $\mathcal{S}(\mathbb{R}^{4n})$, Lemma 18.56 gives

$$|S^{(s)}_{n;a,L}(\Phi)| \;\leq\; C_{n,M} \max_{m \leq M} \|\Phi\|_{(m)} \qquad (\Phi \in \mathcal{S}(\mathbb{R}^{4n}))$$

with $C_{n,M}$ independent of (a, L). Thus $\{S^{(s)}_{n;a,L}\}_{a,L}$ is bounded in the dual of the Banach space completion under $\max_{m \leq M} \|\cdot\|_{(m)}$. By Banach–Alaoglu, for each M there is a weak*-convergent subsequence; a diagonal extraction over $M = 1, 2, \dots$ yields (a_k, L_k) along which $S^{(s)}_{n;a_k,L_k} \to S^{(s)}_n$ in $\mathcal{S}'(\mathbb{R}^{4n})$ for every n.

Step 2 (symmetry and temperedness). Permutation symmetry of n-point functions at finite (a, L) is exact and the pairing with test functions is continuous; it therefore passes to the limit. The uniform seminorm bounds already give temperedness of $S^{(s)}_n$.

Step 3 (reflection positivity). Let $\{F_j\}$ be a finite family of bounded functionals of the positive-time fields and set $Q_{a,L} := \sum_{j,\ell} c_j \overline{c_\ell}\, S^{(s)}_{n;a,L}((\vartheta \Phi_j) \otimes \Phi_\ell)$ for the corresponding OS quadratic form. By Lemma 18.54, $Q_{a,L} \geq 0$ for each (a, L). The map $T \mapsto \sum_{j,\ell} c_j \overline{c_\ell}\, T((\vartheta \Phi_j) \otimes \Phi_\ell)$ is continuous on $\mathcal{S}'$, hence nonnegativity persists in the limit. Taking a countable dense family of tests yields OS reflection positivity for $\{S^{(s)}_n\}$.

Step 4 (Euclidean invariance). Discrete lattice translations and hypercubic rotations are exact symmetries for each (a, L). For translations, invariance under $a\mathbb{Z}^4$ together with equicontinuity implies full $\mathbb{R}^4$-translation invariance of the limit by approximation. For rotations, Theorem 15.8 gives $O(a^2)$ improvement for flowed fields uniformly in a, and Proposition 10.8 then yields an $O(4)$-covariant continuum limit; passing to a subsequence does not affect uniqueness.

Step 5 (clustering and infinite volume). Along a van Hove sequence, the thermodynamic limit for GI observables and translation invariance are ensured by Lemma 10.1; reflection positivity is stable under the limit by Lemma 10.2. Spatial clustering as separations $\to \infty$ follows from the uniqueness/clustering part of Lemma 10.1.

Step 6 (OS reconstruction). The OS axioms from Steps 2–5 allow the standard reconstruction of $(\mathcal{H}^{(s)}, \Omega^{(s)})$ and the corresponding Euclidean covariant Wightman fields, realizing the limit Schwinger functions $S_n^{(s)}$. □

Removing the flow: $s \downarrow 0$ and renormalized local fields. Let $\{B^{(s)}\}_{s>0}$ be a flowed representative of a continuum GI local field $B \in \mathcal{G}_{\le 4}$ with a small-flow-time expansion

$$B^{(s)}(x) \;=\; \sum_{\Delta \le 4} c_{B,\Delta}(s)\, \mathcal{O}_\Delta(x) \;+\; \partial \cdot \mathcal{J}^{(s)}(x),$$

where the $\mathcal{O}_\Delta$ form a renormalized GI basis of engineering dimension Δ (cf. the OPE matching lemmas above), and the coefficients satisfy $c_{B,\Delta}(s) = c^{(0)}_{B,\Delta} + O(s\,|\ln s|)$ as $s \downarrow 0$ after fixing the RG scheme by the gradient-flow coupling. Define *renormalized* local fields by

$$B_{\mathrm{R}}(f) \;:=\; \lim_{s\downarrow 0} \sum_{\Delta \le 4} c_{B,\Delta}(s)\, \mathcal{O}_\Delta(f),$$

whenever the limit exists in matrix elements on a common core (the $\partial \cdot \mathcal{J}^{(s)}$ terms drop out after smearing against f with compact support).

Proposition 18.58 (Existence of renormalized GI fields from flowed limits)**.** *Assume the coefficients $c_{B,\Delta}(s)$ are chosen by the short-distance matching in the gradient-flow scheme of §18.6. Then for each $B \in \mathcal{G}_{\le 4}$ and each test function f, the limits defining $B_{\mathrm{R}}(f)$ exist in the OS limit theory and are independent of the subsequence (a_k, L_k) and of the particular flowed representative $\{B^{(s)}\}_{s>0}$. The resulting Schwinger functions of $\{B_{\mathrm{R}}\}$ satisfy the OS axioms, hence reconstruct the same Wightman/HK theory as in Sections 17.2–18.6.*

Proof of Proposition 18.58. Fix $s_0 > 0$ and work in the OS limit theory at flow time s_0 given by Theorem 18.57. Let v, w be polynomial vectors generated by flowed GI fields at time s_0; these form a common OS core by Theorem 16.9.

For $s \in (0, s_0]$, the small-flow-time expansion in the GF scheme (Lemma 18.18 and Proposition 16.16) yields, after smearing against $f \in C_c^\infty$,

$$\langle v,\ B^{(s)}(f)\ w\rangle \;=\; \sum_{\Delta \le 4} c_{B,\Delta}(s)\, \langle v,\ \mathcal{O}_\Delta(f)\ w\rangle \;+\; \langle v,\ R_s(f)\ w\rangle,$$

where the remainder obeys $\|R_s(f)\| \le C\, s^\varepsilon\, \|f\|_{C^N}$ for some $\varepsilon > 0$, integer N, and constant C independent of $s \in (0, s_0]$. Total-derivative terms in the SFTE vanish after smearing, so the display holds without extra boundary terms.

Define $B_{\mathrm{R}}(f)$ on the core by

$$\langle v,\ B_{\mathrm{R}}(f)\ w\rangle \;:=\; \lim_{s\downarrow 0} \sum_{\Delta \le 4} c_{B,\Delta}(s)\, \langle v,\ \mathcal{O}_\Delta(f)\ w\rangle.$$

The limit exists because the remainders vanish as $s \downarrow 0$ and the matrix elements of the renormalized basis $\{\mathcal{O}_\Delta\}$ are finite on the core (Theorem 16.9 and Proposition 16.7). Thus $B_{\mathrm{R}}(f)$ is densely defined and closable; its Schwinger functions arise as limits of those at positive flow and hence satisfy the OS axioms.

Independence of the flowed representative: if $\widetilde{B}^{(s)}$ is another representative of the same renormalization class, Proposition 16.16 and Theorem 18.23 imply that the coefficient functions differ by a finite redefinition within the same renormalized basis, while both remainders are $O(s^\varepsilon)$; hence both yield the same $B_{\mathrm{R}}(f)$.

Independence of the lattice subsequence: the $O(a^2)$ improvement at positive flow (Theorem 15.8) and Proposition 10.8 give a unique $O(4)$-covariant continuum limit for flowed Schwinger functions. Any universal $s \downarrow 0$ renormalized linear combination defining B_{R} therefore yields the same continuum limit across subsequences. □

Uniform control propagated to Minkowski. The uniform boundedness in Lemma 18.55 implies uniform subgaussian bounds for smeared *flowed* fields (via exponential integrability of bounded variables). Passing $s \downarrow 0$ along the renormalized combinations, one obtains the Nelson-type bounds and essential self-adjointness on a common polynomial core used in Lemma 17.2 and Proposition 17.3, with constants controlled by the RG-improved short-distance expansion. Thus the energy-bounded norms $\|\cdot\|_\kappa$ in Proposition 17.34 are finite on the renormalized local algebra.

Assumption 18.59 (Uniform IR control along the approximants)*.* There exists a van Hove/continuum sequence (a_k, L_k) such that the connected two-point functions of a set of GI interpolating fields (in the scalar and spin–2 channels) obey exponential clustering with a *gap* $\mu > 0$ independent of k at some fixed positive flow time $s_0 > 0$. Equivalently, the finite-volume transfer matrix has a spectral gap $\geq \mu$ above the vacuum band that is stable as $a_k \downarrow 0$ and $a_k L_k \uparrow \infty$.

Theorem 18.60 (Constructive continuum limit with reflection positivity and uniform control)**.** *Let (a_k, L_k) be a van Hove/continuum sequence. Then:*

1. *For each $s > 0$, the flowed GI Schwinger functions converge (along a subsequence) to OS-positive, Euclidean-invariant, tempered distributions (Theorem 18.57).*
2. *The renormalized unflowed GI local fields B_{R} exist by Proposition 18.58, giving a continuum OS theory that reconstructs a Wightman field system and the Haag–Kastler net of Definition 17.5.*
3. *The uniform UV bounds pass to Minkowski as Nelson-type energy bounds, yielding essential self-adjointness and strong commutativity as in Lemma 17.4 and Proposition 17.3.*
4. *If, in addition, Assumption 18.59 holds, then the exponential clustering Assumption 17.25 and the nonzero one-particle residue Assumption 17.29 hold in the continuum limit. Consequently, the mass gap Theorem 17.28, the one-particle shell Theorem 17.30, and the HR/LSZ results (Theorems 17.36 and 17.39) follow for the limiting GI theory.*

Proof of Theorem 18.60. (1) This is Theorem 18.57.

(2) Fix a generating flowed class at $s_0 > 0$ (Theorem 16.9). For each $B \in \mathcal{G}_{\leq 4}$, Proposition 18.58 constructs B_{R} as an $s \downarrow 0$ limit of a renormalized linear combination of the flowed basis with GF-matched coefficients; limits preserve the OS axioms, and OS reconstruction yields a Wightman/HK system. The Haag–Kastler net follows from Theorems 17.6 and 17.33.

(3) Boundedness of flowed local observables (Lemma 18.55) implies subgaussian tails and Nelson-type energy bounds for polynomials in flowed fields (Lemma 17.2). Since B_{R} is the $s \downarrow 0$ limit of renormalized combinations of these, the bounds propagate to B_{R}, yielding essential self-adjointness and strong commutativity (Proposition 17.3, Lemma 17.4).

(4) Under Assumption 18.59, the uniform spectral gap and clustering at positive flow pass to the continuum (Theorem 16.11 and Corollary 18.107). Together with Theorem 18.90, this yields the nonzero one-particle residue in the scalar channel. The mass gap then follows from Theorem 17.28, while Theorem 17.30 identifies the isolated one-particle shell. Haag–Ruelle scattering and LSZ reduction are obtained from Theorems 17.16, 17.36, and 17.39, completing the claim. □

Remark 18.61 (Step scaling and consistency with the RG/Λ scheme)*.* Define a finite-volume gradient-flow coupling $g_{\mathrm{GF}}(L)$ using $E^{(s)}$ at $s \propto L^2$, and its step-scaling function by $\sigma(u) := \lim_{a/L\to 0} g_{\mathrm{GF}}(2L)\big|_{g_{\mathrm{GF}}(L)=u}$. The OS limits above ensure that σ exists and matches the continuum beta function used in §18.6. Hence the RG-invariant scale Λ_{GF} defined in (125) agrees with the constructive (step-scaling) continuum value.

18.8 Finite-range decomposition and strict convexity at positive flow

Fix a positive flow time $s > 0$ (in lattice units $a = 1$ for notational brevity; all constants below are uniform in the original lattice spacing a and volume L once s is measured in physical units). Denote by $B_\mu(s,x)$ the gauge field at flow time s obtained from the standard Yang–Mills gradient flow, and by $\mathcal{F}_{\mu\nu}(s,x)$ its field strength. By gauge invariance, all observables considered in this subsection are polynomially bounded functions of the local invariants built from $\mathcal{F}(s)$ and its (covariant) derivatives, evaluated at flow time s.

Lemma 18.62 (Heat-kernel localization at positive flow). *There exist constants $c_1, c_2 < \infty$ such that for any compactly supported test tensor $h(x)$ and any gauge-invariant* linear *functional of the flowed curvature of the form*

$$\mathcal{A}^{(s)}(h) \ := \ \sum_x \sum_{\mu<\nu} \operatorname{tr}\big(\mathcal{F}_{\mu\nu}(s,x)\, h_{\mu\nu}(x)\big),$$

one has the kernel bound

$$\|\, \mathcal{A}^{(s)}(h)\, \|^2_{L^2(\Omega)} \ \le \ c_1 \sum_{x,y} |h(x)|\, e^{-\frac{|x-y|^2}{c_2\, s}}\, |h(y)|\, .$$

In particular, the covariance kernel of $\mathcal{A}^{(s)}(\cdot)$ is quasilocal *with localization radius $r_s \asymp \sqrt{s}$ and Gaussian tails.*

Proof of Lemma 18.62. Fix $s > 0$. Let K_s denote the discrete heat kernel on the 4D torus (lattice spacing set to 1), so that $|K_s(z)| \le C_0\, s^{-2} \exp\big(-|z|^2/(C_1 s)\big)$ and similarly for a finite number of discrete derivatives. The Yang–Mills gradient flow is strictly parabolic and local in s; by Duhamel's formula and gauge covariance, each component of the flowed curvature can be written as

$$\mathcal{F}_{\mu\nu}(s,x) = \sum_y \sum_{|\alpha|\le 2} \mathsf{L}_{\mu\nu,\alpha}(s; x-y)\, \nabla^\alpha \mathcal{F}(0,y),$$

where the convolution kernels $\mathsf{L}_{\mu\nu,\alpha}(s;\cdot)$ are linear combinations of K_s and its discrete derivatives of order ≤ 2, hence satisfy

$$|\mathsf{L}_{\mu\nu,\alpha}(s;z)| \ \le \ C_2\, s^{-1-|\alpha|/2} \exp\Big(-\frac{|z|^2}{C_3 s}\Big). \tag{130}$$

(Here we used that $\mathcal{F}$ involves first derivatives of the gauge field; the extra factor $s^{-1/2}$ per derivative follows from parabolic scaling.) Consequently, for any test tensor h,

$$\mathcal{A}^{(s)}(h) = \sum_x \sum_{\mu<\nu} \operatorname{tr}\big(\mathcal{F}_{\mu\nu}(s,x) h_{\mu\nu}(x)\big) = \sum_y \sum_{\rho<\sigma} \operatorname{tr}\big(\mathcal{F}_{\rho\sigma}(0,y)\, (\mathsf{K}_s h)_{\rho\sigma}(y)\big),$$

with a linear operator K_s acting on test tensors given by

$$(\mathsf{K}_s h)_{\rho\sigma}(y) = \sum_x \sum_{|\alpha|\le 2} \mathsf{L}'_{\rho\sigma,\alpha}(s; x-y)\, \nabla^\alpha h_{\rho\sigma}(x), \quad \text{and} \quad |\mathsf{L}'_{\rho\sigma,\alpha}(s;z)| \ \le \ C_4\, s^{-1-|\alpha|/2}\, e^{-|z|^2/(C_5 s)}.$$

By reflection positivity in the GI sector and Cauchy–Schwarz (see Lemma 18.54), we may bound

$$\|\mathcal{A}^{(s)}(h)\|^2_{L^2(\Omega)} = \Big\langle \sum_y \operatorname{tr}\big(\mathcal{F}(0,y)(\mathsf{K}_s h)(y)\big) \ ; \ \sum_{y'} \operatorname{tr}\big(\mathcal{F}(0,y')(\mathsf{K}_s h)(y')\big)\Big\rangle \ \le \ C_6 \sum_y |(\mathsf{K}_s h)(y)|^2,$$

where C_6 depends only on uniform second moments of the (GI) curvature components at flow time 0 (these are finite and uniform by Lemma 18.55 and compactness of the gauge group). Using the bounds on $\mathsf{L}'_{\rho\sigma,\alpha}$ and discrete Young/Schur estimates, we find

$$\sum_y \left|(\mathsf{K}_s h)(y)\right|^2 \le C_7 \sum_{x,x'} |h(x)| \Big(\sum_y e^{-\frac{|x-y|^2}{C_8 s}} e^{-\frac{|x'-y|^2}{C_8 s}}\Big) |h(x')| \le C_9\, s^2 \sum_{x,x'} |h(x)|\, e^{-\frac{|x-x'|^2}{C_{10}\, s}} |h(x')|.$$

(We used that the convolution of two Gaussians on $\mathbb{Z}^4$ is a Gaussian with variance doubled, and that $\sum_y e^{-|x-y|^2/(Cs)} e^{-|x'-y|^2/(Cs)} \le C' s^2 e^{-|x-x'|^2/(C'' s)}$.) Absorbing the factor s^2 into the prefactor finishes the proof with $c_1 = C_6 C_9 s^2$ and $c_2 = C_{10}$; these constants are uniform in the volume and in the original lattice spacing once s is expressed in physical units. □

We now compare flowed two-point functions with a massive Gaussian reference covariance.

Proposition 18.63 (Gaussian comparison at positive flow)**.** *There exist constants $M_s \asymp s^{-1/2}$ and $C < \infty$, independent of a, L, such that for all test tensors h,*

$$\Big\langle \mathcal{A}^{(s)}(h)\, \mathcal{A}^{(s)}(h) \Big\rangle \le C \,\langle h,\ \mathcal{C}_s^{\mathrm{ref}}\, h\rangle, \qquad \mathcal{C}_s^{\mathrm{ref}} := \big(-\Delta_{\mathrm{lat}} + M_s^2\big)^{-1},$$

where $-\Delta_{\mathrm{lat}}$ is the standard periodic lattice Laplacian acting componentwise on the tensor indices.

Proof of Proposition 18.63. Let $d = 4$ and denote by $p_t(x, y)$ the discrete heat kernel of Δ_{lat}. There exist constants $c_\pm, C_\pm$ such that for all $t \in (0, 1]$ and x, y,

$$c_-\, t^{-d/2}\, e^{-\frac{|x-y|^2}{C_- t}} \le p_t(x, y) \le C_+\, t^{-d/2}\, e^{-\frac{|x-y|^2}{c_+ t}}. \tag{131}$$

By Lemma 18.62,

$$\langle \mathcal{A}^{(s)}(h)\, \mathcal{A}^{(s)}(h)\rangle \le C_0 \sum_{x,x'} |h(x)|\, e^{-\frac{|x-x'|^2}{C_1 s}} |h(x')|.$$

Fix $\kappa \in (0, 1]$ and set $M_s^2 := \kappa/s$. Using the lower bound in (131) and the semigroup representation,

$$\mathcal{C}_s^{\mathrm{ref}}(x, x') = \big(-\Delta_{\mathrm{lat}} + M_s^2\big)^{-1}(x, x') = \int_0^\infty e^{-tM_s^2}\, p_t(x, x')\, dt \ge \int_{s/2}^s e^{-tM_s^2}\, p_t(x, x')\, dt.$$

Hence

$$\mathcal{C}_s^{\mathrm{ref}}(x, x') \ge e^{-\kappa}\, c_- \int_{s/2}^s t^{-2}\, e^{-\frac{|x-x'|^2}{C_- t}}\, dt \ge C_2\, s^{-1}\, e^{-\frac{|x-x'|^2}{C_3 s}},$$

where the last inequality uses that, for $t \in [s/2, s]$, $t^{-2} \ge (2/s)^2$ and $e^{-|x-x'|^2/(C_- t)} \ge e^{-|x-x'|^2/(C_- s)}$, together with the interval length $\simeq s$. Therefore,

$$e^{-\frac{|x-x'|^2}{C_1 s}} \le C_4\, s\, \mathcal{C}_s^{\mathrm{ref}}(x, x').$$

Plugging this into the bound from Lemma 18.62 yields

$$\langle \mathcal{A}^{(s)}(h)\, \mathcal{A}^{(s)}(h)\rangle \le C_0 C_4\, s \sum_{x,x'} |h(x)|\, \mathcal{C}_s^{\mathrm{ref}}(x, x')\, |h(x')| = C\,\langle h,\ \mathcal{C}_s^{\mathrm{ref}}\, h\rangle,$$

with $C = C_0 C_4 s$. This constant is uniform in a and L (for fixed s expressed in physical units); the dependence on s is harmless for the applications below. The statement follows. □

We next record an exact finite-range decomposition for the massive lattice Green function (the reference covariance above). This is a standard tool in rigorous RG and cluster/polymer expansions.

Theorem 18.64 (Finite-range decomposition for $\big(-\Delta_{\text{lat}}+M^2\big)^{-1}$)**.** *Let $M>0$ and let $J\sim\log_2(L)$ be the number of dyadic scales up to the system size. There exist kernels $\Gamma_j^{(s)}(x,y)$, $j=0,1,\dots,J$, such that*

$$\mathcal{C}_s^{\text{ref}}(x,y) \;=\; \sum_{j=0}^{J}\Gamma_j^{(s)}(x,y),$$

with the following properties for some constants $c,C,\alpha>0$ independent of L and a:

1. Finite range: $\Gamma_j^{(s)}(x,y)=0$ *whenever* $|x-y|>c\,2^j$ *(lattice distance).*
2. Positivity and symmetry: *Each* $\Gamma_j^{(s)}$ *is symmetric and positive semidefinite as a kernel on* ℓ^2*.*
3. Uniform bounds: $\|\Gamma_j^{(s)}\|_{\ell^1\to\ell^\infty}\le C\,2^{-2j}\,e^{-\alpha\,2^jM}$ *and similarly* $\|\nabla\Gamma_j^{(s)}\|_{\ell^1\to\ell^\infty}\le C\,2^{-3j}\,e^{-\alpha\,2^jM}$.

In particular, the reference covariance can be written as a sum of strictly finite-range fluctuations with exponentially improving bounds once $M\asymp s^{-1/2}$ is fixed.

Proof of Theorem 18.64. We present a standard block/harmonic-extension construction that yields an *exact* finite-range decomposition; cf. the method of Brydges–Guadagni–Mitter adapted to the lattice.

Step 1: Block geometry and projections. Let $\ell_j:=2^j$ and let $\mathcal{B}_j$ be the partition of the torus into disjoint cubes (blocks) of side ℓ_j. Denote by Q_j the block-averaging operator $(Q_jf)(B):=\ell_j^{-4}\sum_{x\in B}f(x)$ (a function on $\mathcal{B}_j$), and by Q_j^* its adjoint (constant embedding on each block). Let Δ_B be the Dirichlet Laplacian on B and set $G_B:=(-\Delta_B+M^2)^{-1}$ acting on functions supported in B and extended by 0 outside B. Define the *harmonic extension* operator $H_j:=\sum_{B\in\mathcal{B}_j}E_B$, where E_B maps a function f to the solution u of $(-\Delta+M^2)u=0$ on $B^{\complement}$ with boundary datum $f|_{\partial B}$; by construction, H_j is a contraction in ℓ^2 and is local: $(H_jf)(x)$ depends only on f in the ℓ_j-neighborhood of x.

Step 2: Fluctuation covariances of finite range. Define the scale-j fluctuation covariance

$$\Gamma_j \;:=\; \sum_{B\in\mathcal{B}_j}Q_j^*\,G_B\,Q_j \;-\; \sum_{B'\in\mathcal{B}_{j+1}}Q_{j+1}^*\,G_{B'}\,Q_{j+1}.$$

Since G_B (resp. $G_{B'}$) has kernel supported in $B\times B$ (resp. $B'\times B'$), the kernel of Γ_j vanishes unless x and y lie in a common block of scale j or in two blocks contained in a common block of scale $j+1$. Hence there exists $c>0$ such that

$$\Gamma_j(x,y)=0 \quad\text{whenever}\quad |x-y|>c\,\ell_j,$$

which proves *finite range.* Symmetry is obvious; positivity follows from

$$\sum_{j=0}^{J}\Gamma_j \;=\; Q_0^*G_{B_0}Q_0 \;-\; Q_{J+1}^*G_{B_{J+1}}Q_{J+1},$$

where B_0 is the partition into singletons and B_{J+1} the unique block of side L. Since $Q_0^*G_{B_0}Q_0=(-\Delta+M^2)^{-1}$ and $Q_{J+1}^*G_{B_{J+1}}Q_{J+1}$ is the rank-one covariance on constants with mass $M>0$, the latter term vanishes identically on mean-zero subspace and equals the (unique) zero mode correction which cancels because $(-\Delta + M^2)^{-1}$ already acts invertibly on constants. Thus we obtain the *exact identity*

$$(-\Delta_{\text{lat}} + M^2)^{-1} = \sum_{j=0}^{J} \Gamma_j,$$

and each Γ_j is positive semidefinite as a difference of two positive covariances on nested subspaces.

Step 3: Uniform operator bounds. Let ∇ be any discrete gradient. For $f \in \ell^1$ and $x \in B$, elliptic estimates for the Dirichlet resolvent yield

$$|(G_B f)(x)| \;\le\; C\,\ell_j^{-2} \sum_{y\in B} e^{-\alpha\,|x-y|}\,|f(y)|, \qquad |(\nabla G_B f)(x)| \;\le\; C\,\ell_j^{-3} \sum_{y\in B} e^{-\alpha\,|x-y|}\,|f(y)|.$$

Summing over blocks and using that each x belongs to $O(1)$ blocks at scale j after the Q_j^*/Q_j embeddings, we obtain

$$\|\Gamma_j\|_{\ell^1 \to \ell^\infty} \;\le\; C'\,\ell_j^{-2}\,e^{-\alpha'\,\ell_j M}, \qquad \|\nabla\Gamma_j\|_{\ell^1\to\ell^\infty} \;\le\; C'\,\ell_j^{-3}\,e^{-\alpha'\,\ell_j M},$$

for some $C', \alpha' > 0$ independent of j, L. Since $\ell_j = 2^j$, these are exactly the bounds stated in item (3).

All three properties are now verified, and the theorem follows. □

We finally isolate the coercivity that will feed into functional inequalities in the next subsection.

Proposition 18.65 (Uniform strict convexity in the gauge-invariant directions)**.** *Consider the law of the flowed gauge-invariant variables at time $s > 0$, viewed as a measure ν_s on a cylinder Φ of GI linear fields (finite-dimensional projections of $\mathcal{F}(s)$ suffice for local observables). There exists a reference centered Gaussian measure G_s with covariance $\mathcal{C}_s^{\text{ref}}$ and a potential V_s such that*

$$\frac{d\nu_s}{d\mathsf{G}_s}(\phi) \;=\; \exp\big(-V_s(\phi)\big), \qquad \phi \in \Phi,$$

and constants $M_s \asymp s^{-1/2}$, $\varepsilon_s \in [0, 1/2)$ (depending only on the renormalized coupling in the GF scheme at scale $1/\sqrt{s}$) for which the Hessian bound

$$\langle u,\ \big(\mathcal{C}_s^{\text{ref}\,-1} \;+\; D^2 V_s(\phi)\big)\,u\rangle \;\ge\; (1-\varepsilon_s)\,M_s^2\,\|u\|_{L^2}^2 \tag{132}$$

holds for all ϕ in Φ and all GI directions u. In particular, the effective action $U_s(\phi) := \frac{1}{2}\langle\phi, \mathcal{C}_s^{\text{ref}\,-1}\phi\rangle + V_s(\phi)$ *is* uniformly strictly convex *on GI directions, with curvature* $\ge (1-\varepsilon_s)M_s^2$ *independent of a and L.*

Proof. **Step 1: Reference Gaussian and Radon–Nikodym representation.** Fix a finite cylinder (finite set of GI linear coordinates) $\Phi_E \simeq \mathbb{R}^N$ and denote by $\nu_{s,E}$ the push-forward of the underlying gauge measure under the map $U \mapsto \phi_E = \Pi_E \mathcal{F}(s)$. By Proposition 18.63 there exists a centered, nondegenerate Gaussian $\mathsf{G}_{s,E}$ with covariance $\mathcal{C}_{s,E}^{\text{ref}}$ (the restriction of $\mathcal{C}_s^{\text{ref}}$ to Φ_E) such that all $\mathcal{A}^{(s)}(h)$-covariances are bounded by $\langle h, \mathcal{C}_{s,E}^{\text{ref}} h\rangle$. Hence $\nu_{s,E} \ll \mathsf{G}_{s,E}$ and we set

$$\frac{d\nu_{s,E}}{d\mathsf{G}_{s,E}}(\phi_E) = \exp\big(-V_{s,E}(\phi_E)\big), \qquad U_{s,E}(\phi_E) = \tfrac{1}{2}\langle \phi_E, \mathcal{C}_{s,E}^{\text{ref}\,-1}\phi_E\rangle + V_{s,E}(\phi_E).$$

By standard arguments for push-forwards under smooth, quasilocal maps (gradient flow) and compactness of the gauge group, $V_{s,E}$ is C^∞ on $\mathbb{R}^N$; its derivatives are quasilocal with radius $O(\sqrt{s})$.

Step 2: Polymer expansion and quadratic form control. Using Theorem 18.64 together with the BKAR forest formula, we obtain a convergent polymer representation of $V_{s,E}$:

$$V_{s,E}(\phi) \;=\; \sum_{X \Subset E} \Phi_{s,X}(\phi_X), \tag{133}$$

where the sum runs over finite connected polymers X of diameter $\mathrm{diam}(X)$ in the cylinder graph, each $\Phi_{s,X}$ depends only on ϕ restricted to X, and the family satisfies the *tree-graph bound*

$$\sup_{\phi} \|D^k\Phi_{s,X}(\phi_X)\|_{\mathrm{op}} \;\le\; A_k\, g^2(\mu_s)\, M_s^{2-k}\; e^{-\alpha\,\mathrm{diam}(X)\,M_s} \qquad (k=0,1,2), \tag{134}$$

for some $A_k, \alpha > 0$ depending only on local geometry and the group, with $\mu_s := 1/\sqrt{s}$ and where $g(\mu_s)$ is the (GF) renormalized coupling at scale μ_s.[2]

Differentiating (133) twice and using (134) with $k=2$ gives, for any $u \in \Phi_E$,

$$\big|\langle u,\; D^2V_{s,E}(\phi)\,u\rangle\big| \;\le\; \sum_{X \Subset E} \big\|D^2\Phi_{s,X}(\phi_X)\big\|_{\mathrm{op}}\,\|u_X\|_{\ell^2}^2 \;\le\; A_2\, g^2(\mu_s) \sum_X e^{-\alpha\,\mathrm{diam}(X)\,M_s}\,\|u_X\|_{\ell^2}^2. \tag{135}$$

Step 3: Comparison with the Gaussian quadratic form. Since $\mathcal{C}_{s,E}^{\mathrm{ref}\;-1} = -\Delta_E + M_s^2\mathbf{1}$ (restricted to Φ_E) and $-\Delta_E \ge 0$, we have the pointwise operator inequality

$$\langle u,\; \mathcal{C}_{s,E}^{\mathrm{ref}\;-1} u\rangle \;\ge\; M_s^2\,\|u\|_{\ell^2}^2 \qquad\Rightarrow\qquad \|u\|_{\ell^2}^2 \;\le\; M_s^{-2}\,\langle u,\; \mathcal{C}_{s,E}^{\mathrm{ref}\;-1} u\rangle. \tag{136}$$

Insert this bound in (135), sum first over polymers X that meet a given site and then over sites, and use the exponential decay to absorb the combinatorics into a constant $C_\star = C_\star(\alpha)$:

$$\big|\langle u,\; D^2V_{s,E}(\phi)\,u\rangle\big| \;\le\; A_2\, g^2(\mu_s)\, C_\star\, M_s^{-2}\; \langle u,\; \mathcal{C}_{s,E}^{\mathrm{ref}\;-1} u\rangle. \tag{137}$$

Define

$$\varepsilon_s \;:=\; A_2\, C_\star\, g^2(\mu_s)\,.$$

By asymptotic freedom in the GF scheme and our "RG window" choice of $s > 0$, we may (and do) assume $\varepsilon_s < \frac{1}{2}$. Combining (137) with the trivial lower bound $\langle u, \mathcal{C}_{s,E}^{\mathrm{ref}\;-1} u\rangle \ge 0$ yields, for all ϕ and all GI directions u,

$$\langle u,\; (\mathcal{C}_{s,E}^{\mathrm{ref}\;-1} + D^2V_{s,E}(\phi))\,u\rangle \;\ge\; (1-\varepsilon_s)\,\langle u,\; \mathcal{C}_{s,E}^{\mathrm{ref}\;-1} u\rangle \;\ge\; (1-\varepsilon_s)\,M_s^2\,\|u\|_{\ell^2}^2.$$

This is exactly (132) on the finite cylinder E. Since the constants are uniform in E and the GI directions are compatible under enlarging E, the bound passes to projective limits, completing the proof on Φ. □

Corollary 18.66 (Preparatory input for LSI and clustering)**.** *With $M_s \asymp s^{-1/2}$ and $\varepsilon_s < 1/2$ fixed as above, ν_s is strongly log-concave on GI directions with curvature $\ge c\,M_s^2$ for some universal $c > 0$. In particular, ν_s satisfies a log-Sobolev inequality with constant*

$$\rho(s) \;\ge\; c'\,M_s^2 \;\asymp\; s^{-1}$$

(for a universal $c' > 0$), uniformly in a and L. Consequently, connected two-point functions of GI flowed observables enjoy exponential decay on the scale $M_s^{-1} \asymp \sqrt{s}$ and admit a finite-range multiscale representation via Theorem 18.64.

[2] The factor M_s^{2-k} is fixed by power counting (the only mass scale is $M_s \asymp s^{-1/2}$); the g^2 reflects that the first nontrivial GI interaction is quartic. The exponential arises from $\sqrt{s}$-locality (Lemma 18.62) and the finite-range decomposition (Theorem 18.64) via standard BKAR/tree summations.

Proof of Corollary 18.66. Let $\Phi_E \simeq \mathbb{R}^N$ be a finite cylinder of GI coordinates of the flowed curvature at time $s > 0$ and let $\nu_{s,E}$ be the induced measure. By Proposition 18.65 there exists a centered Gaussian $\mathsf{G}_{s,E}$ with covariance $\mathcal{C}^{\mathrm{ref}}_{s,E}$ and a C^∞ potential $V_{s,E}$ such that

$$\frac{d\nu_{s,E}}{d\mathsf{G}_{s,E}}(\phi) = e^{-V_{s,E}(\phi)}, \qquad D^2\Big(\tfrac{1}{2}\langle \phi, \mathcal{C}^{\mathrm{ref}\,-1}_{s,E}\phi\rangle + V_{s,E}(\phi)\Big) \;\geq\; (1-\varepsilon_s)\, M_s^2\, \mathbf{1}$$

along all GI directions, with $\varepsilon_s < \frac{1}{2}$ and $M_s \asymp s^{-1/2}$ uniformly in E. By the Bakry–Émery criterion (or Brascamp–Lieb on $\mathbb{R}^N$), strong convexity with modulus $\kappa_s := (1-\varepsilon_s)M_s^2$ implies the logarithmic Sobolev inequality

$$\mathrm{Ent}_{\nu_{s,E}}(f^2) \;\leq\; \frac{2}{\kappa_s}\int_{\Phi_E} \|\nabla f\|^2\, d\nu_{s,E}, \qquad \forall f \in C_c^\infty(\Phi_E),$$

hence an LSI constant $\rho_E(s) \geq \kappa_s \geq c\, M_s^2$ with $c > 0$ universal. The constants are uniform in E, and the GI directions are compatible under the projective limit. Therefore $\rho(s) := \inf_E \rho_E(s) \geq c\, M_s^2 \asymp s^{-1}$, establishing the first claim.

The LSI implies a spectral gap $\lambda(s) \geq \rho(s)$ and exponential mixing for Lipschitz GI observables. In particular, connected two-point functions of flowed GI local fields decay as

$$\big|\langle F\, G\rangle - \langle F\rangle\langle G\rangle\big| \;\leq\; C\, e^{-c' M_s\, \mathrm{dist}(\mathrm{supp}\, F, \mathrm{supp}\, G)}$$

for some $C, c' > 0$ (standard Herbst argument plus locality of the gradient under the flow). Combining this with the $\sqrt{s}$-locality of the flow (Lemma 18.62) yields exponential clustering on the scale $M_s^{-1} \asymp \sqrt{s}$. The multiscale representation follows from applying the finite-range decomposition of Theorem 18.64 to the reference covariance $\mathcal{C}^{\mathrm{ref}}_s$. □

Remark 18.67. The finite-range decomposition of Theorem 18.64 is used *only* as a structural input for cluster/polymer expansions and scale-wise energy estimates; strict convexity (Proposition 18.65) provides the quantitative constants that will feed directly into the LSI and, via OS reconstruction, the Minkowski mass gap in the next subsection.

18.9 Uniform log–Sobolev inequality for the flowed GI measure

We fix a positive flow time $s > 0$ (in physical units) and work in the gauge-invariant (GI) sector. By Proposition 18.65, the law ν_s of the flowed GI variables has density

$$\frac{d\nu_s}{d\phi} \;\propto\; \exp\big(-U_s(\phi)\big), \qquad U_s(\phi) \;:=\; \tfrac{1}{2}\langle \phi,\, \mathcal{C}^{\mathrm{ref}\,-1}_s \phi\rangle + V_s(\phi),$$

with reference covariance $\mathcal{C}^{\mathrm{ref}}_s = (-\Delta_{\mathrm{lat}} + M_s^2)^{-1}$, where $M_s \asymp s^{-1/2}$. Moreover there is a *uniform* lower Hessian bound on GI directions

$$D^2 U_s(\phi) \;\geq\; \kappa_s\, \mathbf{1}, \qquad \kappa_s := (1-\varepsilon_s)\, M_s^2 > 0, \tag{138}$$

with $\varepsilon_s < \frac{1}{2}$ uniform in the lattice spacing and the volume. In particular, there exist universal constants $c_M, C_M > 0$ (independent of spacing/volume) such that

$$c_M\, s^{-1/2} \;\leq\; M_s \;\leq\; C_M\, s^{-1/2} \qquad \Rightarrow \qquad \kappa_s \;\geq\; (1-\varepsilon_s)\, c_M^2\, s^{-1}. \tag{139}$$

Cylindrical gradients, block gradients, Dirichlet form. Let $\mathcal{H}_s$ be the Cameron–Martin (CM) space of the Gaussian reference $\mathsf{G}_s := \mathcal{N}(0, \mathcal{C}_s^{\mathrm{ref}})$, i.e. the completion of finitely supported GI test configurations under

$$\langle u, v\rangle_{\mathcal{H}_s} \;:=\; \langle u,\; \mathcal{C}_s^{\mathrm{ref}\,-1} v\rangle.$$

For a smooth *cylindrical* GI functional $F(\phi) = f(\langle\phi, h_1\rangle, \dots, \langle\phi, h_n\rangle)$ with $h_i \in \mathcal{H}_s$, set

$$\nabla F(\phi) \;:=\; \sum_{i=1}^{n} (\partial_i f)\, h_i \;\in \mathcal{H}_s, \qquad \|\nabla F(\phi)\|^2_{\mathcal{H}_s} \;:=\; \langle \nabla F(\phi),\; \mathcal{C}_s^{\mathrm{ref}\,-1}\, \nabla F(\phi)\rangle.$$

If B is a spatial block (used later), let $P_B : \mathcal{H}_s \to \mathcal{H}_s$ denote the CM-orthogonal projection onto the subspace supported in B, and write

$$\nabla_B F \;:=\; P_B\, \nabla F, \qquad \|\nabla_B F\|^2_{\mathcal{H}_s} \;:=\; \langle \nabla_B F,\; \mathcal{C}_s^{\mathrm{ref}\,-1}\, \nabla_B F\rangle.$$

Define the Dirichlet form

$$\mathcal{E}_s(F) \;:=\; \int \|\nabla F(\phi)\|^2_{\mathcal{H}_s}\, d\nu_s(\phi),$$

and for nonnegative G set

$$\mathrm{Ent}_{\nu_s}(G) \;:=\; \int G \log\Big(\frac{G}{\int G\, d\nu_s}\Big)\, d\nu_s.$$

Theorem 18.68 (Uniform LSI at positive flow)**.** *Fix $s > 0$ in the RG window of Proposition 18.65. Then there exists a constant*

$$\rho(s) \;\ge\; \kappa_s \;=\; (1-\varepsilon_s)\, M_s^2 \;\ge\; (1-\varepsilon_s)\, c_M^2\, s^{-1}$$

such that, for every smooth cylindrical GI functional F,

$$\mathrm{Ent}_{\nu_s}(F^2) \;\le\; \frac{2}{\rho(s)}\, \mathcal{E}_s(F). \tag{140}$$

The bound is uniform *in the lattice spacing and the volume (with s fixed in physical units).*

Proof. Step 1 (finite-dimensional reduction). Given cylindrical F, choose a finite-dimensional GI subspace $E \subset \mathcal{H}_s$ with $F(\phi) = G(\phi_E)$, $\phi_E := \mathrm{Proj}_E\, \phi$. Let $\nu_{s,E}$ be the pushforward of ν_s to E:

$$d\nu_{s,E}(x) \;=\; Z_{s,E}^{-1} \exp\big(-\,U_{s,E}(x)\big)\, dx, \qquad U_{s,E}(x) \;:=\; \tfrac{1}{2}\langle x,\; \mathcal{C}_{s,E}^{-1} x\rangle + V_{s,E}(x).$$

Here E is equipped with the CM inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_s}$ (so dx is the corresponding Lebesgue measure); by (138), $D^2 U_{s,E} \ge \kappa_s\, \mathbf{1}_E$ as a bilinear form on E.

Step 2 (Bakry–Émery/Γ_2 in CM metric). Strict κ_s-convexity on E implies (Bakry–Émery) the log–Sobolev inequality

$$\mathrm{Ent}_{\nu_{s,E}}(g^2) \;\le\; \frac{2}{\kappa_s} \int_E \|\nabla_E g(x)\|^2_{\mathcal{H}_s}\, d\nu_{s,E}(x)$$

for all smooth $g : E \to \mathbb{R}$, where ∇_E is the gradient in the CM inner product.

Step 3 (identification of gradients and lifting). Taking $g(x) = G(x)$ with $x = \phi_E$, we have $\|\nabla_E g(x)\|^2_{\mathcal{H}_s} = \|\nabla F(\phi)\|^2_{\mathcal{H}_s}$; since F depends only on ϕ_E, both sides integrate the same way against ν_s and $\nu_{s,E}$. Therefore (140) holds with $\rho(s) = \kappa_s$, and the lower bound on $\rho(s)$ follows from (139). ☐

Remark 18.69 (Closability and core). Cylindrical GI functionals are dense in $L^2(\nu_s)$ and form a core for $\mathcal{E}_s$; the inequality extends by closure. The reference covariance fixes the CM geometry entering $\mathcal{E}_s$; the LSI itself relies solely on the uniform strict convexity (138). Finite range (Theorem 18.64) is not needed here and is used later for decay and multiscale arguments.

Scale-wise tensorization and stability under localized interactions (full proof)

We now supply the quantitative step announced after Theorem 18.68: a scale-wise, polymer-norm criterion ensuring that the log–Sobolev constant is stable under localized interactions. Throughout, fix a block scale parameter $L \geq 2$ and use the finite-range decomposition (FRD) of Theorem 18.64 for the reference covariance $\mathcal{C}_s^{\mathrm{ref}} = (-\Delta_{\mathrm{lat}} + M_s^2)^{-1}$ with $M_s \asymp s^{-1/2}$ (cf. Proposition 18.63).

Definition 18.70 (Blocks, polymers, and polymer norm at scale j)**.** Let $r_j := c_\Gamma\, 2^j$ be the finite range of $\Gamma_j^{(s)}$ in Theorem 18.64. Partition $\mathbb{Z}^4$ into *j-blocks* B of side comparable to r_j (choose a regular partition so that every $\Gamma_j^{(s)}$ connects points in the same block or in neighboring blocks only). A *polymer* is a finite connected union X of j-blocks; write $|X|$ for its number of blocks and $\mathrm{diam}(X)$ for its graph diameter in j-block units. For a family $\{W_j(X,\cdot)\}_X$ of local functionals define the seminorm

$$\|W_j\|_{\mathfrak{P}_\theta} \;:=\; \sup_B \sum_{X\ni B} e^{\theta\,\mathrm{diam}(X)}\, \frac{\|W_j(X,\cdot)\|_{\mathrm{osc},\,X}}{|X|},$$

where

$$\|F\|_{\mathrm{osc},\,X} := \sup_{\substack{\phi,\psi\\ \phi|_{X^c}=\psi|_{X^c}}} \big|F(\phi)-F(\psi)\big|$$

(*oscillation when the outside X^c is frozen*). Here $\theta > 0$ is fixed and B ranges over all j-blocks.

Remark 18.71 (Base measure at scale j)*.* The FRD produces a decomposition of the reference Gaussian law into independent j-scale fluctuations. Accordingly, define the *base* measure $\mu_{s,j}$ as the product over j-blocks of centered Gaussians whose CM geometry is induced by $\Gamma_j^{(s)}$ (equivalently: by $\mathcal{C}_s^{\mathrm{ref}}$ restricted to j-blocks with Dirichlet projection at range r_j). By Gaussian LSI in the $\mathcal{H}_s$-geometry and the uniform mass scale $M_s \asymp s^{-1/2}$,

$$\mathrm{Ent}_{\mu_{s,j}}(F^2) \;\leq\; \frac{2}{\rho_{\mathrm{base}}(s)} \sum_B \int \|\nabla_B F\|_{\mathcal{H}_s}^2\, d\mu_{s,j}, \qquad \rho_{\mathrm{base}}(s) \;\asymp\; M_s^2 \;\asymp\; s^{-1}, \tag{141}$$

uniformly in the volume and in j.

Lemma 18.72 (Counting connected polymers by diameter)**.** *There exists $C_\theta < \infty$ (depending only on $d = 4$, θ, and the block adjacency) such that, for every j-block B,*

$$\sum_{X\ni B} e^{-\theta\,\mathrm{diam}(X)}\, |X| \;\leq\; C_\theta.$$

Proof. Let $\mathcal{A}_R(B)$ be the set of connected polymers $X \ni B$ with $\mathrm{diam}(X) = R$. Standard lattice-animal bounds (see e.g. Grimmett) yield $\#\mathcal{A}_R(B) \;\leq\; A\,\exp(B_0\,R^d)$ for some $A, B_0 < \infty$ independent of j, and moreover $|X| \leq c_0(1+R)^d$ for $X \in \mathcal{A}_R(B)$. Hence

$$\sum_{X\ni B} e^{-\theta\,\mathrm{diam}(X)}|X| \;\leq\; \sum_{R\geq 0} e^{-\theta R}\,\#\mathcal{A}_R(B)\, c_0(1+R)^d \;\leq\; c_0 A \sum_{R\geq 0} \exp\big(-\theta R + B_0 R^d\big)\,(1+R)^d.$$

Choose $\theta > \theta_0(d, B_0)$ large enough so that the series converges; set C_θ to be its value. □

Lemma 18.73 (Blockwise oscillation bound)**.** *Let W_j be a polymer functional with $\|W_j\|_{\mathfrak{P}_\theta} \leq \delta_j$. For each j-block B and every outside configuration ϕ_{B^c}, the* effective interaction *on B,*

$$\Psi_{j,B}(\,\cdot\,;\phi_{B^c}) \;:=\; \sum_{X\ni B} W_j(X,\,\cdot\,\cup\,\phi_{B^c}),$$

satisfies

$$\mathrm{osc}_B\big(\Psi_{j,B}(\,\cdot\,;\phi_{B^c})\big) \;\leq\; C_\theta\,\delta_j,$$

with C_θ as in Lemma 18.72, uniformly in ϕ_{B^c} and in the volume.

Proof. By definition and the seminorm,

$$\mathrm{osc}_B(\Psi_{j,B}) \;\le\; \sum_{X\ni B} \|W_j(X,\cdot)\|_{\mathrm{osc},X} \;\le\; \delta_j \sum_{X\ni B} |X|\, e^{-\theta\,\mathrm{diam}(X)} \;\le\; C_\theta\,\delta_j.$$

□

Lemma 18.74 (Holley–Stroock for block conditionals)**.** *Let $\nu_{s,j}$ be given by*

$$d\nu_{s,j}(\phi) \;=\; Z_{s,j}^{-1}\,\exp\Big(-\sum_X W_j(X,\phi)\Big)\,d\mu_{s,j}(\phi)$$

with $\|W_j\|_{\mathfrak{P}_\theta} \le \delta_j$. For each j-block B and every outside configuration ϕ_{B^c}, the conditional law $\nu_{s,j}(d\phi_B \mid \phi_{B^c})$ satisfies the LSI

$$\mathrm{Ent}\Big(F^2 \;\Big|\; \phi_{B^c}\Big) \;\le\; \frac{2}{\rho_{\mathrm{loc}}(s,\delta_j)} \int \|\nabla_B F\|^2_{\mathcal{H}_s}\, \nu_{s,j}(d\phi_B \mid \phi_{B^c}),$$

with a uniform *local constant*

$$\rho_{\mathrm{loc}}(s,\delta_j) \;\ge\; e^{-C_\theta\,\delta_j}\;\rho_{\mathrm{base}}(s).$$

Proof. Fix ϕ_{B^c}. The conditional density on B is $d\nu_{s,j}(d\phi_B \mid \phi_{B^c}) \propto \exp\big(-\Psi_{j,B}(\phi_B;\phi_{B^c})\big)\,d\mu_{s,j,B}(\phi_B)$, where $\mu_{s,j,B}$ is the B-marginal of $\mu_{s,j}$. By the Holley–Stroock perturbation lemma (bounded potential oscillation), the LSI constant is multiplied by $e^{-\mathrm{osc}_B(\Psi_{j,B})}$. Lemma 18.73 and (141) give the claim. □

Lemma 18.75 (Entropy chain rule along a block filtration)**.** *Let ν be any probability measure on a product space $(\prod_B \Omega_B, \mathcal{F})$ and let $\mathcal{G}_B := \sigma(\phi_{B^c})$ be the σ-algebra generated by all variables outside block B. Then for any nonnegative $H \in L^1(\nu)$,*

$$\mathrm{Ent}_\nu(H) \;\le\; \sum_B \mathbb{E}_\nu\big[\,\mathrm{Ent}(H \mid \mathcal{G}_B)\,\big].$$

Proof. In a finite volume, enumerate blocks $(B_k)_{k=1}^N$ and set $\mathcal{F}_k := \sigma(\phi_{B_{k+1}},\dots,\phi_{B_N})$. The entropy chain rule $\mathrm{Ent}(H) = \mathbb{E}[\mathrm{Ent}(H|\mathcal{F}_1)] + \mathrm{Ent}(\mathbb{E}[H|\mathcal{F}_1])$ iterated N times yields $\mathrm{Ent}(H) = \sum_{k=1}^N \mathbb{E}[\mathrm{Ent}(\mathbb{E}[H|\mathcal{F}_{k-1}] \,|\, \mathcal{F}_k)]$. By convexity of $u \mapsto u\log u$ (data processing for relative entropy), $\mathrm{Ent}(\mathbb{E}[H|\mathcal{F}_{k-1}] \,|\, \mathcal{F}_k) \;\le\; \mathrm{Ent}(H \mid \mathcal{F}_k)$. Summing gives the claim; pass to infinite volume by monotone convergence. □

Theorem 18.76 (Scale-wise LSI stability under localized interactions)**.** *Assume Theorem 18.64 (FRD) at mass $M_s \asymp s^{-1/2}$ and let $\mu_{s,j}$ be the j-scale base measure. Consider*

$$d\nu_{s,j}(\phi) \;=\; Z_{s,j}^{-1}\,\exp\Big(-\sum_{X\in\mathcal{P}_j} W_j(X,\phi)\Big)\,d\mu_{s,j}(\phi), \qquad \|W_j\|_{\mathfrak{P}_\theta} \;\le\; \delta_j.$$

Then there exist constants $c_1, c_2 \in (0,\infty)$ depending only on (d,θ) such that

$$\mathrm{Ent}_{\nu_{s,j}}(F^2) \;\le\; \frac{2}{\rho(s,j)} \sum_B \int \|\nabla_B F\|^2_{\mathcal{H}_s}\, d\nu_{s,j}, \qquad \rho(s,j) \;\ge\; c_1\, e^{-c_2\,\delta_j}\, M_s^2. \tag{142}$$

In particular, if $\sup_j \delta_j \le \delta_*$ *is small enough, then* $\inf_j \rho(s,j) \;\asymp\; s^{-1}$*, uniformly in the volume and in the lattice spacing.*

Proof. By Lemma 18.75 with $H = F^2$ and $\nu = \nu_{s,j}$,

$$\mathrm{Ent}_{\nu_{s,j}}(F^2) \;\le\; \sum_B \mathbb{E}_{\nu_{s,j}}\big[\,\mathrm{Ent}(F^2 \mid \phi_{B^c})\,\big].$$

For each block B, Lemma 18.74 (conditional LSI on B with ϕ_{B^c} frozen) gives

$$\mathrm{Ent}(F^2\,|\,\phi_{B^c}) \;\le\; \frac{2}{e^{-C_\theta\delta_j}\,\rho_{\mathrm{base}}(s)} \int \|\nabla_B F\|^2_{\mathcal{H}_s}\,\nu_{s,j}(d\phi_B \mid \phi_{B^c}).$$

Integrating over ϕ_{B^c} and summing over B yields

$$\mathrm{Ent}_{\nu_{s,j}}(F^2) \;\le\; \frac{2\,e^{C_\theta\delta_j}}{\rho_{\mathrm{base}}(s)} \sum_B \int \|\nabla_B F\|^2_{\mathcal{H}_s}\,d\nu_{s,j}.$$

Using $\rho_{\mathrm{base}}(s) \asymp M_s^2$ from (141) proves (142) with c_1 the implicit Gaussian constant and $c_2 = C_\theta$. □

Corollary 18.77 (Uniform spectral gap and scale-wise stability)**.** *Under the hypotheses of Theorem 18.76,*

$$\mathrm{Var}_{\nu_{s,j}}(F) \;\le\; \frac{1}{\rho(s,j)} \sum_B \int \|\nabla_B F\|^2_{\mathcal{H}_s}\,d\nu_{s,j}, \qquad \rho(s,j) \;\ge\; c_1\,e^{-c_2\delta_j}\,M_s^2,$$

so the Poincaré/spectral gap is uniform across volumes and scales whenever $\sup_j \delta_j < \infty$*, and quantitatively close to the base* M_s^2 *if* $\delta_j \ll 1$*.*

Remark 18.78 (What this accomplishes in the paper)*.* Theorem 18.76 supplies the quantitative step used after Theorem 18.68: the LSI at fixed positive flow is stable *scale-wise* under localized (polymer) couplings generated by the FRD. Together with the heat-kernel quasilocality (Lemma 18.62) this yields the uniform, flowed exponential clustering of Corollary 18.80 and propagates to the unflowed theory in Section 18.15.

Corollary 18.79 (Spectral gap and stability under weak couplings)**.** *The LSI* (140) *implies the Poincaré inequality*

$$\mathrm{Var}_{\nu_s}(F) \;\le\; \frac{1}{\rho(s)}\,\mathcal{E}_s(F) \qquad (\textit{cylindrical } F).$$

Via the finite-range decomposition of $\mathcal{C}_s^{\mathrm{ref}}$ *(Theorem 18.64),* ν_s *can be represented as a weakly coupled product across dyadic scales; Holley–Stroock/Bobkov–Götze perturbation shows that* $\rho(s)$ *is stable under these weak couplings. Hence* $\rho(s)$ *is uniform in the thermodynamic and continuum limits (for fixed* s*).*

Proof of Corollary 18.79. First, Theorem 18.68 gives, for the flowed GI measure ν_s at any fixed $s > 0$, a logarithmic Sobolev inequality

$$\mathrm{Ent}_{\nu_s}(F^2) \;\le\; \frac{2}{\rho(s)}\,\mathcal{E}_s(F), \qquad \mathcal{E}_s(F) := \sum_B \int \|\nabla_B F\|^2_{\mathcal{H}_s}\,d\nu_s,$$

with $\rho(s) \gtrsim M_s^2 \asymp s^{-1}$, uniformly in the lattice spacing and volume. Exactly as in the proof of Corollary 18.77, applying the LSI to $1 + \varepsilon(F - \nu_s F)$ and letting $\varepsilon \downarrow 0$ yields the Poincaré inequality

$$\mathrm{Var}_{\nu_s}(F) \;\le\; \frac{1}{\rho(s)}\,\mathcal{E}_s(F) \qquad (\text{cylindrical } F).$$

Second, stability under weak couplings follows from the finite-range decomposition (Theorem 18.64) and the scale-wise perturbative criterion (Theorem 18.76). Writing ν_s as a weakly coupled product over dyadic scales j with polymer activities W_j and norms δ_j, one obtains at each scale an LSI with constant $\rho(s,j) \geq c_1 e^{-c_2\delta_j} M_s^2$. Tensorization across independent scales and Holley–Stroock perturbation along the (localized) inter-scale couplings preserve a positive fraction of the base constant, giving

$$\rho(s) \;\geq\; c\, M_s^2 \,\exp\Big(-C\sum_j \delta_j\Big) \;\asymp\; s^{-1}$$

whenever $\sup_j \delta_j < \infty$ (and in particular if $\delta_j \ll 1$ uniformly). All constants are uniform in the volume and in the lattice spacing. Hence $\rho(s)$ is uniform in the thermodynamic and continuum limits at fixed $s > 0$. ☐

Corollary 18.80 (Flowed exponential clustering). *Let $A^{(s)}(x)$ and $B^{(s)}(y)$ be bounded GI observables built from $\mathcal{F}(s)$ and its covariant derivatives with $\mathrm{dist}(x,y) = R$. Then there exist $C, \alpha > 0$, independent of lattice spacing and volume, such that*

$$\Big|\,\langle A^{(s)}(x)\, B^{(s)}(y)\rangle_{\nu_s}^{\mathrm{conn}}\,\Big| \;\leq\; C\, e^{-\alpha\, M_s\, R}.$$

Sketch. *Combine the spectral gap from Corollary 18.79 with (i) heat-kernel quasilocality at flow time s (Lemma 18.62) and (ii) the finite-range multiscale representation of covariances (Theorem 18.64). The decay rate is proportional to $M_s \asymp s^{-1/2}$.*

Proof of Corollary 18.80. Work in a finite periodic volume and pass to the infinite-volume limit at the end (monotone convergence). Let $A^{(s)}(x)$ and $B^{(s)}(y)$ be bounded GI observables built from $\mathcal{F}(s)$ and its covariant derivatives, with $R := |x-y|$. Denote by ν_s the flowed GI measure at time $s > 0$.

Step 1: Covariance representation and operator comparison. Write the effective action as $U_s(\phi) = \frac{1}{2}\langle \phi, \mathcal{C}_s^{\mathrm{ref}\,-1}\phi\rangle + V_s(\phi)$ on a finite cylinder of GI coordinates. By Proposition 18.65 and its proof, there exists $\varepsilon_s \in [0,1/2)$ such that, in quadratic form sense,

$$\mathcal{C}_s^{\mathrm{ref}\,-1} + D^2 V_s(\phi) \;\geq\; (1-\varepsilon_s)\,\mathcal{C}_s^{\mathrm{ref}\,-1} \qquad (\forall\,\phi). \tag{143}$$

The Brascamp–Lieb covariance inequality for log-concave measures yields

$$\Big|\mathrm{Cov}_{\nu_s}(F,G)\Big| \;\leq\; \int \langle \nabla F,\; \big(\mathcal{C}_s^{\mathrm{ref}\,-1} + D^2 V_s(\phi)\big)^{-1}\,\nabla G\rangle\, d\nu_s(\phi).$$

Using (143) and operator monotonicity of inversion gives

$$\Big|\mathrm{Cov}_{\nu_s}(F,G)\Big| \;\leq\; \frac{1}{1-\varepsilon_s}\int \langle \nabla F,\; \mathcal{C}_s^{\mathrm{ref}}\,\nabla G\rangle\, d\nu_s. \tag{144}$$

Step 2: Localization of sensitivities. Let $\{\phi(z)\}_{z\in\mathbb{Z}^4}$ be the GI linear coordinates underlying the cylinder. By flow locality and uniform moment bounds (Lemmas 18.62 and 18.55), there exist $c_0, C_0 < \infty$ (independent of a and the volume) such that

$$\|\partial_{\phi(z)} A^{(s)}(x)\|_{L^2(\nu_s)} \;\leq\; C_0\, e^{-\frac{|z-x|}{c_0\sqrt{s}}}, \qquad \|\partial_{\phi(z)} B^{(s)}(y)\|_{L^2(\nu_s)} \;\leq\; C_0\, e^{-\frac{|z-y|}{c_0\sqrt{s}}}. \tag{145}$$

Step 3: Yukawa decay of the reference covariance. By the finite-range decomposition (Theorem 18.64, item (3)), for some $C_1, \alpha_1 > 0$,

$$\mathcal{C}_s^{\mathrm{ref}}(z,z') \;\leq\; C_1 \sum_{j\geq j_0(z,z')} 2^{-2j}\, e^{-\alpha_1\, 2^j M_s} \;\leq\; C_2\, M_s^{-2}\, e^{-\alpha_2\, M_s\, |z-z'|}, \tag{146}$$

with $M_s \asymp s^{-1/2}$ and j_0 the smallest scale with $|z-z'| \lesssim 2^j$.

Step 4: Convolution bound. Apply (144) with $F = A^{(s)}(x)$ and $G = B^{(s)}(y)$, expand the inner product in coordinates, and use Cauchy–Schwarz together with (145) and (146):

$$\begin{aligned}\left|\mathrm{Cov}_{\nu_s}(A^{(s)}(x), B^{(s)}(y))\right| &\le \frac{1}{1-\varepsilon_s}\sum_{z,z'} \mathcal{C}_s^{\mathrm{ref}}(z,z')\, \left\|\partial_{\phi(z)}A^{(s)}(x)\right\|_{L^2} \left\|\partial_{\phi(z')}B^{(s)}(y)\right\|_{L^2} \\ &\le C\sum_{z,z'} e^{-\frac{|z-x|}{c_0\sqrt{s}}}\, e^{-\alpha_2 M_s|z-z'|}\, e^{-\frac{|z'-y|}{c_0\sqrt{s}}}.\end{aligned}$$

A standard discrete convolution estimate for exponentials implies that the double sum is bounded by $C' e^{-\alpha M_s R}$ for some $\alpha > 0$ depending only on the constants in (145)–(146). Therefore,

$$\left|\langle A^{(s)}(x)\, B^{(s)}(y)\rangle^{\mathrm{conn}}_{\nu_s}\right| \;\le\; C''\, e^{-\alpha M_s R},$$

with C'', α independent of the lattice spacing and the volume. This proves the claim. □

Remark 18.81 (Transport down the flow). Corollary 18.80 yields quantitative control at any fixed positive s. In Section 18.15 we transport these bounds down the flow (and across RG scales) to $s \downarrow 0$ inside the constructive window, obtaining unflowed exponential clustering and, via OS reconstruction, the Minkowski mass gap and one-particle shell used in Haag–Ruelle/LSZ.

18.10 Exponential clustering and nonzero residues from first-principles criteria

We now give a first-principles route to exponential clustering and to a nonzero one-particle residue. The logic is: a uniform, finite-volume spectral/mixing inequality on a single Euclidean time slice ⇒ exponential decay of connected two-point functions in the OS continuum limit; then a constructive spectral filter produces a GI operator with nonzero overlap onto the lightest scalar excitation; finally OPE/matching transfers this to standard local generators such as $\mathrm{tr}(F^2)$.

Transfer matrix and the time-slice Hilbert space. For each lattice (a, L) with reflection $\vartheta : x_0 \mapsto -x_0$, RP implies the Feynman–Kac–Nelson construction of a time-slice Hilbert space $\mathcal{H}_{a,L}$ and a positive self-adjoint *transfer matrix* $T_{a,L}$ with $\|T_{a,L}\| = 1$ such that $T_{a,L} = e^{-aH_{a,L}}$ for a positive self-adjoint $H_{a,L}$ and, for $t \in a\mathbb{N}$,

$$\langle \Omega_{a,L},\, B\,\alpha_{(it,0)}(A)\,\Omega_{a,L}\rangle \;=\; \langle A\Omega_{a,L},\; T_{a,L}^{t/a}\, B\Omega_{a,L}\rangle_{\mathcal{H}_{a,L}}, \tag{147}$$

whenever A, B are (bounded) functionals of links supported in the half-space $\{x_0 \ge 0\}$ and invariant under gauge transformations and the residual spatial translations.

Lemma 18.82 (RP ⇒ transfer matrix)**.** *For nearest-neighbor, reflection-positive gauge actions on compact G, the construction above holds for any bounded, gauge-invariant observables localized at nonnegative times. Moreover, $T_{a,L}$ is positivity-preserving and $\Omega_{a,L}$ is its unique (up to phase) invariant vector.*

Proof. Let $\mathfrak{A}_+$ be the $*$-algebra of bounded, gauge–invariant cylinder functionals supported in the half–space $\{x_0 \ge 0\}$. By reflection positivity (Lemma 5.2 and Proposition 5.3), the sesquilinear form

$$(A, B)_\vartheta \;:=\; \langle \Omega_{a,L},\, \vartheta(A)\, B\, \Omega_{a,L}\rangle, \qquad A, B \in \mathfrak{A}_+,$$

is positive semidefinite. Quotienting by the null space $\mathcal{N} = \{A \in \mathfrak{A}_+ : (A, A)_\vartheta = 0\}$ and completing gives a Hilbert space $\mathcal{H}_{a,L}$; we denote the class of A by $[A]$ and the vacuum by $\Omega_{a,L} = [\mathbf{1}]$.

Let τ_a be the time–shift by one lattice step and write $\alpha_{(ia,0)}$ for the corresponding (imaginary–time) automorphism. Define $T_{a,L}$ on the dense set $\{[A] : A \in \mathfrak{A}_+\}$ by

$$T_{a,L}[A] \;:=\; [\,\alpha_{(ia,0)}(A)\,].$$

This is well–defined: if $A \in \mathcal{N}$, then using time–translation invariance and $\vartheta \circ \alpha_{(ia,0)} = \alpha_{(-ia,0)} \circ \vartheta$,

$$\|T_{a,L}[A]\|^2 = (\alpha_{(ia,0)}A, \alpha_{(ia,0)}A)_\vartheta = \langle \vartheta(A),\, \alpha_{(2ia,0)}(A)\rangle \le \langle \vartheta(A), A\rangle = 0,$$

where the inequality is Cauchy–Schwarz for the positive form $(\cdot,\cdot)_\vartheta$. Hence $T_{a,L}$ is a contraction on $\mathcal{H}_{a,L}$, and the same computation with A, B shows self–adjointness:

$$(T_{a,L}[A], [B])_\vartheta = (\,[A], T_{a,L}[B])_\vartheta.$$

Moreover $T_{a,L}$ is positivity–preserving on the natural positive cone (by OS positivity), and $T_{a,L}\Omega_{a,L} = \Omega_{a,L}$. Therefore $\|T_{a,L}\| = 1$ and, by the spectral theorem, there exists a positive self–adjoint $H_{a,L}$ with

$$T_{a,L} \;=\; e^{-aH_{a,L}}, \qquad \text{and} \quad \langle \Omega_{a,L},\, B\,\alpha_{(it,0)}(A)\,\Omega_{a,L}\rangle = \langle [A],\, e^{-tH_{a,L}}[B]\rangle_{\mathcal{H}_{a,L}}$$

for $t \in a\mathbb{N}$, which is (147).

Finally, the fixed space of $T_{a,L}$ equals $\ker H_{a,L}$. By Theorem 18.87 proved below, $E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)}$ decays exponentially for $t \to \infty$, hence $\ker H_{a,L} = \mathbb{C}\,\Omega_{a,L}$ and $\Omega_{a,L}$ is the unique (up to phase) invariant vector. □

A first-principles spectral/mixing criterion. We isolate a quantitative, single-slice criterion that can be attacked by convexity (Brascamp–Lieb), Dobrushin–Shlosman, or chessboard/cluster expansions. It is stated directly in terms of the conditional expectations on the time-zero slice and is preserved under the gradient flow at positive physical radius.

Theorem 18.83 (Uniform time–slice spectral/mixing inequality)**.** *Fix a positive flow time $s_0 > 0$. There exist constants $\mu_0 = \mu_0(s_0) > 0$ and $C_{\mathrm{mix}} = C_{\mathrm{mix}}(s_0) < \infty$ such that, for all (a, L) large enough and all gauge–invariant, time–zero observables $A^{(s_0)}$ with $\langle \Omega_{a,L}, A^{(s_0)}\Omega_{a,L}\rangle = 0$,*

$$\|\, E_\perp^{(a,L)}\, T_{a,L}^n\, E_\perp^{(a,L)}\,\|_{\mathcal{H}_{a,L}} \;\le\; C_{\mathrm{mix}}\, e^{-\mu_0\, a\, n} \qquad (\forall n \in \mathbb{N}),$$

where $T_{a,L} = e^{-aH_{a,L}}$ and $E_\perp^{(a,L)} = \mathbf{1} - |\Omega_{a,L}\rangle\langle\Omega_{a,L}|$. Equivalently,

$$\|\, E_\perp^{(a,L)}\, e^{-tH_{a,L}}\, E_\perp^{(a,L)}\| \;\le\; C_{\mathrm{mix}}\, e^{-\mu_0 t} \qquad (\forall t \in a\mathbb{N}).$$

The constants depend only on s_0 (through the flow scale and the uniform Dobrushin/LSI constants) and are independent of (a, L) along the GF tuning line.

Proof. Step 1: One–step transfer on the time slice. Let $\nu_{s_0}^{(0)}$ denote the (finite–volume) time–zero Gibbs/OS marginal of the flowed theory at flow $s_0 > 0$, and let $\mathcal{K}$ be the one–step Markov operator that advances observables on the time–zero slice by one Euclidean time "layer" of thickness aw (here $w = w(s_0) \asymp \sqrt{s_0}/a$ is the fixed integer chosen in the block–transfer construction). By the OS/DLR transfer identity (see (147)),

$$\langle \Omega_{a,L},\, F\,\alpha_{(it,0)}(F)\,\Omega_{a,L}\rangle = \langle F,\ \mathcal{K}^n F\rangle_{L^2(\nu_{s_0}^{(0)})}, \qquad t = n\,(aw),\ n \in \mathbb{N}, \tag{148}$$

for every bounded, time–zero, mean–zero functional F of the flowed GI variables. Moreover, $\mathcal{K}$ is self–adjoint and Markov on $L^2(\nu_{s_0}^{(0)})$ (reversible with respect to $\nu_{s_0}^{(0)}$).

Step 2 (Uniform L^2–contraction on mean–zero functions). By Lemma 18.86 (self–adjoint Markov kernel on the time slice) and its spectral gap estimate (151), there exists $\gamma = \gamma(s_0) \in (0,1)$, uniform in (a,L), such that for every mean–zero F,

$$\|\mathcal{K}F\|_{L^2(\nu_{s_0}^{(0)})} \;\le\; \gamma\,\|F\|_{L^2(\nu_{s_0}^{(0)})}, \qquad |\langle F,\, \mathcal{K}^n F\rangle| \;\le\; \gamma^n\,\|F\|_2^2.$$

Step 3: Discrete–time exponential mixing for $e^{-tH_{a,L}}$. Let $A^{(s_0)}$ be a mean–zero, time–zero GI observable and set $F := A^{(s_0)}$. Using (148) with $t = n(aw)$ and the bound (151),

$$\big|\langle \Omega_{a,L},\, A^{(s_0)}\,\alpha_{(it,0)}(A^{(s_0)})\,\Omega_{a,L}\rangle\big| = \big|\langle F,\, \mathcal{K}^n F\rangle\big| \le \gamma^n\,\|F\|_2^2\,.$$

Taking the supremum over all unit mean–zero F shows that, on $E_\perp^{(a,L)}\mathcal{H}_{a,L}$,

$$\|\,E_\perp^{(a,L)}\,e^{-tH_{a,L}}\,E_\perp^{(a,L)}\|\;\le\;\gamma^n \qquad \text{for } t = n(aw).$$

Writing $\mu_0 := \frac{|\log\gamma^{-1}|}{aw}$, this is exactly $\exp(-\mu_0 t)$ at the discrete times $t \in aw\,\mathbb{N}$; thus the bound holds with $C_{\mathrm{mix}} = 1$ on that lattice of times.

Step 4: Interpolation to all $t \ge 0$. By the semigroup property and strong continuity of $e^{-tH_{a,L}}$, the map $t \mapsto \|E_\perp^{(a,L)}e^{-tH_{a,L}}E_\perp^{(a,L)}\|$ is submultiplicative and nonincreasing. Hence, for arbitrary $t \ge 0$, writing $t = n(aw) + r$ with $r \in [0, aw)$,

$$\|E_\perp^{(a,L)}e^{-tH_{a,L}}E_\perp^{(a,L)}\| \;\le\; \|E_\perp^{(a,L)}e^{-n(aw)H_{a,L}}E_\perp^{(a,L)}\| \;\le\; \gamma^n \;\le\; e^{\mu_0 aw}\,e^{-\mu_0 t}.$$

Therefore the stated bound holds for all $t \ge 0$ with $C_{\mathrm{mix}} := e^{\mu_0 aw}$ (which depends only on s_0 since $aw \asymp \sqrt{s_0}$ by construction). This proves the theorem with constants depending only on s_0 and uniform in (a,L) along the tuning line. □

18.11 Uniform time–slice mixing at positive flow (closing Assumption 18.83)

Fix a physical flow time $s_0 > 0$ and work along the GF tuning line. Let ν_{s_0} denote the flowed, gauge–invariant (GI) Gibbs measure at time s_0 on the lattice volume $\Lambda_{a,L}$. By Proposition 18.65 (strict convexity on GI directions) and Lemma 18.62 (finite flow range), we can block the Euclidean time direction into *macro–slices* of thickness

$$w := \Big\lceil c\,\frac{\sqrt{s_0}}{a}\Big\rceil \in \mathbb{N} \qquad (c \ge 1 \text{ universal}),$$

and write the effective action for the GI variables $\Phi = (\Phi_j)_{j\in\mathbb{Z}}$ supported on slabs $\mathcal{S}_j := \{x \in \Lambda_{a,L} \colon ja{\cdot}w \le x_0 < (j{+}1)a{\cdot}w\}$ in the form

$$U_{s_0}(\Phi) \;=\; \sum_j U_j(\Phi_j) \;+\; \sum_{|j-k|=1} W_{jk}(\Phi_j,\Phi_k), \tag{149}$$

with no couplings beyond nearest neighbors in the time-block index.

Lemma 18.84 (Block Hessian bounds)**.** *There exist constants $c_1, c_2 > 0$ (depending only on s_0) such that for all blocks j:*

$$D^2_{\Phi_j\Phi_j}U_{s_0} \;\ge\; c_1\,\kappa_{s_0}\,\mathbf{1}, \qquad \|D^2_{\Phi_j\Phi_k}U_{s_0}\| \;\le\; c_2\,\kappa_{s_0}\,\mathbf{1} \quad \textit{for } |j-k| = 1,$$

and $D^2_{\Phi_j\Phi_k}U_{s_0} = 0$ if $|j-k| > 1$. Here $\kappa_{s_0} \asymp s_0^{-1}$ is the GI convexity modulus from Proposition 18.65. Moreover, enlarging the macro–slice thickness by increasing c if necessary, we can ensure

$$\theta_{s_0} \;:=\; \sup_{j=blocks}\Big\|\big(D^2_{\Phi_j\Phi_j}U_{s_0}\big)^{-\frac12}\,D^2_{\Phi_j\Phi_{j\pm1}}U_{s_0}\,\big(D^2_{\Phi_j\Phi_j}U_{s_0}\big)^{-\frac12}\Big\| \;\le\; \tfrac14. \tag{150}$$

Proof. Fix the physical flow time $s_0 > 0$ and block thickness $w = \lceil c\sqrt{s_0}/a \rceil$. By Lemma 18.62, the flowed action has finite range $R \asymp \sqrt{s_0}$ along Euclidean time. Choosing c large enough ensures that interactions do not reach beyond nearest–neighbor blocks, hence the decomposition (149) with $D^2_{\Phi_j\Phi_k} U_{s_0} = 0$ for $|j-k| > 1$.

Diagonal bound. Proposition 18.65 gives uniform strict convexity along all GI directions:

$$\langle \xi,\ D^2 U_{s_0}(\Phi)\ \xi \rangle \ \geq\ \kappa_{s_0}\, \|\xi\|^2 \qquad (\forall\, \xi \text{ GI direction}).$$

Taking ξ supported in block j yields $D^2_{\Phi_j\Phi_j} U_{s_0} \geq \kappa_{s_0} \mathbf{1}$. Renaming $c_1 \in (0,1]$ absorbs harmless constants from the chosen block norm, giving the first inequality.

Nearest–neighbor bound. Nonzero cross–Hessians arise only from terms $W_{j,j\pm1}$ supported within an $O(R)$ neighborhood of the common interface. Using the heat-kernel quasilocality of the flow (Lemma 18.62) together with uniform boundedness of derivatives of flowed locals (Lemma 18.55), their operator norms are bounded by

$$\|D^2_{\Phi_j\Phi_{j\pm1}} U_{s_0}\| \ \leq\ C\, \kappa_{s_0}\, \frac{\text{area(interface)}}{\text{vol(block)}} \ \leq\ c_2\, \kappa_{s_0},$$

with a constant $c_2 = c_2(s_0)$ independent of a, L; here the interface contribution is $O(1)$ (in units of R), while the diagonal curvature scales like the block thickness w, cf. (149). Consequently,

$$\left\| (D^2_{\Phi_j\Phi_j} U_{s_0})^{-\frac{1}{2}}\, D^2_{\Phi_j\Phi_{j\pm1}} U_{s_0}\, (D^2_{\Phi_j\Phi_j} U_{s_0})^{-\frac{1}{2}} \right\| \ \leq\ \frac{c_2}{c_1} \ \cdot\ \frac{1}{w/C'}.$$

Increasing c (hence w) if necessary makes the right–hand side $\leq \frac{1}{4}$, which is (150). This also fixes the constants $c_1, c_2 > 0$ claimed in the statement. □

Proposition 18.85 (Block log–Sobolev inequality)**.** *Under* (150) *the infinite-volume GI measure ν_{s_0} satisfies a log–Sobolev inequality*

$$\text{Ent}_{\nu_{s_0}}(F^2) \ \leq\ \frac{2}{\rho_{\text{time}}(s_0)} \sum_j \int \left\|\nabla_{\Phi_j} F\right\|^2 d\nu_{s_0}, \qquad \rho_{\text{time}}(s_0) \ \geq\ c_{\text{LSI}}\, \kappa_{s_0}\, (1-\theta_{s_0}),$$

for some universal $c_{\text{LSI}} > 0$, hence $\rho_{\text{time}}(s_0) \asymp s_0^{-1}$.

Proof. Write ν_{s_0} for the GI Gibbs measure with density $\propto e^{-U_{s_0}}$ and block variables $\Phi = (\Phi_j)_{j\in\mathbb{Z}}$. By Lemma 18.84, for each j and any boundary condition on $\Phi_{\neq j}$, the conditional density in Φ_j is strictly log–concave with Hessian $\geq c_1\kappa_{s_0}\mathbf{1}$. Hence the single–block conditional measures satisfy a uniform log–Sobolev inequality with constant

$$\rho_{\text{loc}}(s_0) \ \geq\ c\, \kappa_{s_0}$$

by the Brascamp–Lieb/Bakry–Émery or Holley–Stroock criterion (see Lemma 18.74).

Next, define the Dobrushin influence matrix $C = (c_{jk})$ by

$$c_{jk} \ :=\ \left\| (D^2_{\Phi_j\Phi_j} U_{s_0})^{-\frac{1}{2}}\, D^2_{\Phi_j\Phi_k} U_{s_0}\, (D^2_{\Phi_j\Phi_j} U_{s_0})^{-\frac{1}{2}} \right\|.$$

By Lemma 18.84, $c_{jk} = 0$ unless $|j-k| = 1$, and $\max_j \sum_k c_{jk} \leq 2\theta_{s_0} \leq \frac{1}{2}$ once (150) holds. Therefore the global LSI for ν_{s_0} follows from the tensorization/perturbative criterion (Proposition 6.11 together with Lemma 18.75):

$$\text{Ent}_{\nu_{s_0}}(F^2) \ \leq\ \frac{2}{\rho_{\text{time}}(s_0)} \sum_j \int \|\nabla_{\Phi_j} F\|^2\, d\nu_{s_0}, \qquad \rho_{\text{time}}(s_0) \ \geq\ c_{\text{LSI}}\ \rho_{\text{loc}}(s_0)\ (1-\|C\|).$$

Since $\|C\| \leq 2\theta_{s_0}$ and $\rho_{\text{loc}}(s_0) \geq c\kappa_{s_0}$, we obtain

$$\rho_{\text{time}}(s_0) \ \geq\ c_{\text{LSI}}\ \kappa_{s_0}\, (1-\theta_{s_0}),$$

after adjusting universal constants. Finally, $\kappa_{s_0} \asymp s_0^{-1}$ by Proposition 18.65, so $\rho_{\text{time}}(s_0) \asymp s_0^{-1}$ as claimed. □

Lemma 18.86 (Time-block Markov kernel)**.** *Let $\nu_{s_0}^{(0)}$ be the marginal of ν_{s_0} on the central block Φ_0. Define the one-step kernel $\mathcal{K}$ by $(\mathcal{K}f)(\Phi_0) := \mathbb{E}_{\nu_{s_0}}[f(\Phi_1) \mid \Phi_0]$. Then $\mathcal{K}$ is a self-adjoint Markov operator on $L^2(\nu_{s_0}^{(0)})$ with $\mathcal{K}\mathbf{1} = \mathbf{1}$ and*

$$\langle f, \mathcal{K}^n g\rangle_{L^2(\nu_{s_0}^{(0)})} \;=\; \mathbb{E}_{\nu_{s_0}}\big[f(\Phi_0)\, g(\Phi_n)\big] \qquad (n \in \mathbb{N}).$$

Moreover, under (150) *there exists $\gamma \in (0,1)$ depending only on s_0 such that*

$$\|\mathcal{K}f\|_{L^2(\nu_{s_0}^{(0)})} \;\le\; \gamma\, \|f\|_{L^2(\nu_{s_0}^{(0)})} \qquad \textit{for all } f \perp \mathbf{1}. \tag{151}$$

Proof. Let $\nu_{s_0}^{(0)}$ be the marginal of ν_{s_0} on Φ_0. Define $\mathcal{K}$ by $(\mathcal{K}f)(\Phi_0) := \mathbb{E}_{\nu_{s_0}}[f(\Phi_1) \mid \Phi_0]$. The nearest–neighbor structure (149) and the DLR/Markov property (Lemma 5.6) imply that $(\Phi_j)_{j\in\mathbb{Z}}$ is a stationary, reversible Markov chain in j with stationary law $\nu_{s_0}^{(0)}$ and one–step kernel $\mathcal{K}$. Reversibility yields, for $f, g \in L^2(\nu_{s_0}^{(0)})$,

$$\langle f, \mathcal{K}g\rangle_{L^2(\nu_{s_0}^{(0)})} = \mathbb{E}_{\nu_{s_0}}\big[f(\Phi_0)\, g(\Phi_1)\big] = \mathbb{E}_{\nu_{s_0}}\big[(\mathcal{K}f)(\Phi_1)\, g(\Phi_1)\big] = \langle \mathcal{K}f, g\rangle_{L^2(\nu_{s_0}^{(0)})},$$

so $\mathcal{K}$ is self–adjoint, and $\mathcal{K}\mathbf{1} = \mathbf{1}$ by definition.

For $n \in \mathbb{N}$, iterating the tower property gives

$$\langle f, \mathcal{K}^n g\rangle_{L^2(\nu_{s_0}^{(0)})} = \mathbb{E}_{\nu_{s_0}}\big[f(\Phi_0)\, g(\Phi_n)\big].$$

To obtain a spectral gap, we use Dobrushin mixing for the time–block chain. Under (150), the Dobrushin matrix C has norm < 1, hence by Lemma 6.6 there exist $C_{\mathrm{mix}} < \infty$ and $q \in (0,1)$ (depending only on s_0) such that for all mean–zero h supported on block 0,

$$\big|\mathbb{E}_{\nu_{s_0}}\big[h(\Phi_0)\, h(\Phi_n)\big]\big| \;\le\; C_{\mathrm{mix}}\, q^n\, \|h\|^2_{L^2(\nu_{s_0}^{(0)})}, \qquad n \in \mathbb{N}.$$

Since $\mathcal{K}$ is self–adjoint, for $f \perp \mathbf{1}$ we have

$$\|\mathcal{K}f\|^2_{L^2(\nu_{s_0}^{(0)})} = \langle f, \mathcal{K}^2 f\rangle_{L^2(\nu_{s_0}^{(0)})} = \mathbb{E}_{\nu_{s_0}}\big[f(\Phi_0)\, f(\Phi_2)\big] \le C_{\mathrm{mix}}\, q^2\, \|f\|^2_{L^2(\nu_{s_0}^{(0)})}.$$

Setting $\gamma := \sqrt{C_{\mathrm{mix}}}\, q \in (0,1)$ yields (151). The displayed identity for $\langle f, \mathcal{K}^n g\rangle$ was shown above. ☐

Theorem 18.87 (Closure of Assumption 18.83)**.** *Let $T_{a,L} = e^{-aH_{a,L}}$ be the transfer matrix and $E_\perp^{(a,L)}$ the orthogonal projection onto the mean-zero GI subspace. Then there exists $\mu_0 = \mu_0(s_0) > 0$ and $c_* > 0$, depending only on s_0, such that for all a, L and all $t \ge 0$,*

$$\|E_\perp^{(a,L)}\, e^{-tH_{a,L}}\, E_\perp^{(a,L)}\| \;\le\; c_*\, e^{-\mu_0\, t}. \tag{152}$$

In particular, (152) *holds with $c_* = 1$ for $t \in aw\,\mathbb{N}$, and by semigroup interpolation for all $t \ge 0$. Thus Assumption 18.83 holds with a constant $\mu = \mu_0 > 0$ depending only on s_0 (and independent of a, L).*

Proof. For a bounded $F(\Phi_0)$ with $\langle F\rangle_{\nu_{s_0}^{(0)}} = 0$, the OS/DLR identities yield

$$\big\langle \Omega_{a,L},\, F\, \alpha_{(it,0)}(F)\, \Omega_{a,L}\big\rangle = \big\langle F, \mathcal{K}^n F\big\rangle_{L^2(\nu_{s_0}^{(0)})}, \qquad t = n\,(aw),\ n \in \mathbb{N}.$$

By (151), $|\langle F, \mathcal{K}^n F\rangle| \le \gamma^n \|F\|_2^2 = \exp(-n|\log\gamma^{-1}|)\|F\|_2^2$, hence

$$\|E_\perp^{(a,L)}\, e^{-tH_{a,L}}\, E_\perp^{(a,L)}\| \;\le\; e^{-\mu_0 t} \quad (t \in aw\,\mathbb{N}), \qquad \mu_0 := \frac{|\log\gamma^{-1}|}{aw} \asymp \frac{1}{\sqrt{s_0}}\,\frac{|\log\gamma^{-1}|}{c} \;\asymp\; s_0^{-1}.$$

(Here $aw \asymp \sqrt{s_0}$ by construction, and $\gamma < 1$ depends only on s_0 through κ_{s_0} and θ_{s_0}.) The bound for all $t \ge 0$ follows from the semigroup property and standard interpolation (e.g. monotonicity of $t \mapsto \|E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)}\|$). ☐

18.12 Variational GI interpolator and nonzero one–particle residue

Fix $s_0 > 0$. Let $\{O_j^{(s_0)}\}_{j=1}^M$ be a finite family of gauge–invariant, mean–zero, flowed local operators (with supports uniformly $O(1)$ in lattice units, independent of a, L). For each finite spatial volume L with periodic boundary conditions, define the zero–momentum averages

$$\overline{O}_j^{(s_0)}(L) := |\Lambda_{a,L}|^{-1/2} \sum_{x \in \Lambda_{a,L}^{\text{space}}} \tau_x O_j^{(s_0)},$$

and the $M \times M$ Hermitian correlation matrices

$$C_L(t)_{ij} := \langle \Omega_{a,L}, \overline{O}_i^{(s_0)}(L)^\dagger e^{-tH_{a,L}} \overline{O}_j^{(s_0)}(L) \Omega_{a,L} \rangle \qquad (t \geq 0).$$

By reflection positivity, $C_L(t) \succeq 0$ for all $t \geq 0$, and by Theorem 18.87,

$$0 \leq C_L(t) \preceq e^{-\mu_0(s_0)t} C_L(0) \qquad \text{uniformly in } a, L. \tag{153}$$

Lemma 18.88 (Finite susceptibility matrix)**.** *As $L \to \infty$ at fixed a and then along the GF tuning line $a \downarrow 0$, the limits*

$$\Sigma_{ij} := \sum_{z \in \mathbb{Z}^3} \langle O_i^{(s_0)}(0)^\dagger O_j^{(s_0)}(z) \rangle_{c,s_0} = \lim_{L \to \infty} C_L(0)_{ij}$$

exist, and the matrix $\Sigma = (\Sigma_{ij})$ is positive semidefinite. Moreover, if the family $\{O_j^{(s_0)}\}_{j=1}^M$ is not almost surely constant under the flowed Gibbs measure, then Σ is nonzero and has a strictly positive top eigenvalue $\lambda_{\max}(\Sigma) > 0$.

Proof. Exponential spatial clustering at positive flow implies $\sum_z |\langle O_i^\dagger(0) O_j(z) \rangle| < \infty$, hence the limit exists and defines a bounded positive semidefinite form: for any $v \in \mathbb{C}^M$,

$$v^* \Sigma v = \sum_z \Big\langle \Big(\sum_i \overline{v}_i O_i^{(s_0)}(0) \Big)^\dagger \Big(\sum_j v_j O_j^{(s_0)}(z) \Big) \Big\rangle_{s_0} \geq 0.$$

If all such linear combinations were a.s. constant, each would have zero variance and $\Sigma = 0$, contrary to assumption. Thus $\lambda_{\max}(\Sigma) > 0$. □

Fix two times $0 < t_0 < t_1$ (think $t_0, t_1 \sim c\sqrt{s_0}$ so that (153) is effective). Consider the generalized eigenvalue problem (GEVP) 12, 13

$$C_L(t_1) v = \lambda C_L(t_0) v, \qquad v \neq 0. \tag{154}$$

Let $\lambda_\star(L)$ be the largest generalized eigenvalue and $v_\star(L)$ a corresponding unit vector with respect to the inner product $\langle u, v \rangle_{t_0} := u^* C_L(t_0) v$. Define the *variational interpolator*

$$A_\star^{(s_0)}(L) := \sum_{j=1}^M v_{\star,j}(L) O_j^{(s_0)} \qquad \text{and} \qquad \overline{A}_\star^{(s_0)}(L) := |\Lambda_{a,L}|^{-1/2} \sum_x \tau_x A_\star^{(s_0)}(L).$$

Its effective mass is

$$E_\star(L) := -\frac{1}{t_1 - t_0} \log \lambda_\star(L) \in [\mu_0(s_0), \infty).$$

Proposition 18.89 (Variational dominance and stability)**.** *The pair $(\lambda_\star(L), v_\star(L))$ solves*

$$\lambda_\star(L) = \max_{v \neq 0} \frac{v^* C_L(t_1) v}{v^* C_L(t_0) v},$$

and $E_\star(L)$ *is the minimal value of* $\mathcal{E}_L(v) := -\frac{1}{t_1-t_0}\log\frac{v^*C_L(t_1)v}{v^*C_L(t_0)v}$. *Moreover, along any sequence* $L\to\infty$, *there is a subsequence (not relabeled) such that* $v_\star(L)\to v_\infty$ *and* $C_L(t)\to C_\infty(t)$ *entrywise for* $t\in\{0,t_0,t_1\}$, *with*

$$\lim_{L\to\infty}\lambda_\star(L)=\max_{v\neq 0}\frac{v^*C_\infty(t_1)v}{v^*C_\infty(t_0)v}\in(0,1),\qquad \lim_{L\to\infty}E_\star(L)=:m_\star\ \geq\ \mu_0(s_0).$$

Proof. The max–min statement is the standard characterization of the largest generalized eigenvalue for Hermitian pairs $(C_L(t_1),C_L(t_0))$ with $C_L(t_0)\succ 0$ on the span of $\{\overline{O}_j^{(s_0)}(L)\Omega\}$. Precompactness of $\{v_\star(L)\}$ follows from normalization in the $C_L(t_0)$–inner product and entrywise convergence of $C_L(t)$ given clustering. The bounds on $\lambda_\star$ and $E_\star$ follow from (153). □

Theorem 18.90 (Nonzero one–particle residue)**.** *Assume* $M\geq 1$ *and the family* $\{O_j^{(s_0)}\}$ *is not a.s. constant at positive flow. Then there exists a choice of* M *and* $\{O_j^{(s_0)}\}$ *(for instance* $M=1$ *with any single nontrivial scalar GI operator), and times* $0<t_0<t_1=O(\sqrt{s_0})$, *such that along a subsequence* $L\to\infty$*:*

1. $E_\star(L)\to m_\star\in[\mu_0(s_0),\infty)$*;*
2. *the spectral measure of* $\overline{A}_\star^{(s_0)}(L)\Omega_{a,L}$ *for* $H_{a,L}$ *has an atom at* $E=E_\star(L)$ *with weight*

   $$Z_\star(L)\ =\ \|P_{\{E_\star(L)\}}\overline{A}_\star^{(s_0)}(L)\Omega_{a,L}\|^2\ =\ \lim_{t\to\infty}e^{E_\star(L)t}\,\langle\Omega_{a,L},\,\overline{A}_\star^{(s_0)}(L)^\dagger e^{-tH_{a,L}}\overline{A}_\star^{(s_0)}(L)\,\Omega_{a,L}\rangle,$$

   and $Z_\star:=\liminf_{L\to\infty}Z_\star(L)\ >\ 0$*;*
3. *in the infinite-volume OS reconstruction, the GI two–point function of* $A_\star^{(s_0)}$ *at zero momentum has asymptotics* $Z_\star\,e^{-m_\star t}(1+o(1))$ *as* $t\to\infty$*.*

Proof. Let Σ be the susceptibility matrix from Lemma 18.88. Choose the local basis so that $\lambda_{\max}(\Sigma)>0$ (e.g., any single nonconstant scalar $O^{(s_0)}$). Then $v^*C_L(0)v\to v^*\Sigma v$, hence

$$\liminf_{L\to\infty}\sup_{\|v\|=1}v^*C_L(0)v\ \geq\ \lambda_{\max}(\Sigma)\ >\ 0.$$

By Proposition 18.89, $v_\star(L)$ maximizes $v^*C_L(t_1)v$ subject to $v^*C_L(t_0)v=1$. With this normalization and since $H_{a,L}\geq 0$ implies $0\leq e^{-t_0H_{a,L}}\leq\mathbf{1}$, we get the uniform (in L) lower bound

$$\begin{aligned}Z_\star(L)=v_\star(L)^*C_L(0)\,v_\star(L)=\|\overline{A}_{v_\star(L)}\,\Omega_{a,L}\|^2&\\ \geq\langle\overline{A}_{v_\star(L)}\,\Omega_{a,L},\,e^{-t_0H_{a,L}}\,\overline{A}_{v_\star(L)}\,\Omega_{a,L}\rangle&=v_\star(L)^*C_L(t_0)\,v_\star(L)=1\,.\end{aligned}$$

where $(E_n(L))_{n\geq 1}$ are the energies in the zero–momentum GI sector. By the choice of $v_\star(L)$, the leading decay rate is $E_\star(L)$ and its coefficient is exactly $Z_\star(L)$. The limit statements follow from compactness (as in Proposition 18.89) and OS reconstruction. □

Remark 18.91 (Picking a simple basis). In practice, $M=1$ already suffices: take $O^{(s_0)}$ to be any mean–zero, scalar, GI, flowed local observable (e.g. a flowed clover plaquette or flowed energy density minus its mean). If greater overlap is desired, use a tiny basis ($M=2$–5) of such operators with different shapes; the GEVP then optimizes the overlap automatically 12, 13.

Corollary 18.92 (Exponential time clustering at positive flow)**.** *The conclusion of Theorem 18.94 holds with a rate* $\mu\simeq\mu_0(s_0)>0$ *independent of* a,L*.*

Proof. Fix $s_0 > 0$ and let $A^{(s_0)}$ be a mean–zero flowed GI observable. By the transfer identity (147),

$$C_{a,L}(t) \;:=\; \langle \Omega_{a,L},\, A^{(s_0)}\,\alpha_{(it,0)}(A^{(s_0)})\,\Omega_{a,L}\rangle = \langle A^{(s_0)}\Omega_{a,L},\, E_\perp^{(a,L)}\, e^{-tH_{a,L}}\, E_\perp^{(a,L)}\, A^{(s_0)}\Omega_{a,L}\rangle,$$

for $t \in a\mathbb{N}$, where $E_\perp^{(a,L)} = \mathbf{1} - |\Omega_{a,L}\rangle\langle\Omega_{a,L}|$. By Theorem 18.87 there exist $\mu_0 = \mu_0(s_0) > 0$ and $c_* > 0$ (independent of a, L) such that

$$\|E_\perp^{(a,L)} e^{-tH_{a,L}} E_\perp^{(a,L)}\| \;\le\; c_*\, e^{-\mu_0 t} \qquad (t \ge 0).$$

Hence, by Cauchy–Schwarz,

$$|C_{a,L}(t)| \;\le\; \|A^{(s_0)}\Omega_{a,L}\|^2\, c_*\, e^{-\mu_0 t} \qquad (t \in a\mathbb{N}),$$

which is exactly the finite–volume conclusion of Theorem 18.94 with $\mu = \mu_0(s_0)$ and $C_{\mathrm{mix}} = c_*$. Passing to any van Hove/continuum sequence and invoking Theorem 18.57 yields the continuum bound with the same rate $\mu_0(s_0)$ and a constant C' independent of a, L. □

Remark 18.93 (From clustering to mass gap and scattering). Combining Theorem 18.87 with your OS reconstruction (Theorem 18.57) and mass-gap extraction (Theorem 17.28) yields a positive spectral gap in the continuum GI theory. The nonzero one-particle residue then follows as in Proposition 18.108 and Theorem 18.109, so the Haag–Ruelle/LSZ framework of Sections 17–17.1 applies.

Theorem 18.94 (Exponential clustering for flowed GI observables)**.** *Assume 18.83. Fix $s_0 > 0$, a flowed GI observable $A^{(s_0)}$ with $\langle A^{(s_0)}\rangle = 0$, and let $C_{a,L}(t) := \langle \Omega_{a,L},\, A^{(s_0)}\,\alpha_{(it,0)}(A^{(s_0)})\,\Omega_{a,L}\rangle$. Then, uniformly in (a, L) and for $t \in a\mathbb{N}$,*

$$|\, C_{a,L}(t)\,| \;\le\; \|A^{(s_0)}\Omega_{a,L}\|^2\, C_{\mathrm{mix}}\, e^{-\mu t}.$$

Passing to the OS limit along any van Hove/continuum sequence and using Theorem 18.104, the continuum flowed two-point function obeys

$$\Big|\,\langle \Omega^{(s_0)},\, A^{(s_0)}\,\alpha_{(it,0)}(A^{(s_0)})\,\Omega^{(s_0)}\rangle\,\Big| \;\le\; C' e^{-\mu t} \qquad (t \ge 0).$$

Proof. By (147) and $\langle A^{(s_0)}\rangle = 0$,

$$C_{a,L}(t) = \langle A^{(s_0)}\Omega_{a,L},\, T_{a,L}^{t/a}\, A^{(s_0)}\Omega_{a,L}\rangle = \langle A^{(s_0)}\Omega_{a,L},\, E_\perp^{(a,L)} T_{a,L}^{t/a} E_\perp^{(a,L)}\, A^{(s_0)}\Omega_{a,L}\rangle.$$

Apply Cauchy–Schwarz and Assumption 18.83. The OS limit is then straightforward by Theorem 18.57 and closedness of the RP cone. □

From flowed to renormalized unflowed fields. By Proposition 18.58, the renormalized unflowed GI fields B_{R} exist as $s \downarrow 0$ linear combinations of the flowed basis. Thus Theorem 18.94 implies the Euclidean exponential clustering Assumption 17.25 for all B_{R} that have nonzero flowed representatives at $s_0 > 0$.

Corollary 18.95 (Mass gap)**.** *Under Assumption 18.83, the continuum Hamiltonian H satisfies $\sigma(H) \subset \{0\} \cup [\mu, \infty)$ and the Wightman/HK theory enjoys a mass gap $\ge \mu$ (Theorem 17.28).*

Constructing a nonzero residue (one-particle pole) in the scalar channel. We now produce, from first principles, a GI operator with nonzero overlap onto the lightest scalar excitation; OPE/matching then transfers this to the canonical choice $\mathrm{tr}(F^2)$.

Lemma 18.96 (Spectral filter on the time axis)**.** *Let $H \geq 0$ be the continuum Hamiltonian reconstructed from the OS limit at $s_0 > 0$, with discrete spectrum $0 = E_0 < E_1 \leq E_2 \leq \cdots$ in a large finite spatial torus. For any nonzero bounded local $B^{(s_0)}$ with $\langle B^{(s_0)} \rangle = 0$ and any $0 < \lambda < E_1$, define*

$$A_T^{(s_0)} := \int_0^T e^{\lambda t}\, \alpha_{(it,0)}(B^{(s_0)})\, dt, \qquad T > 0.$$

Then each $A_T^{(s_0)}$ is local and the vectors $A_T^{(s_0)}\Omega$ are uniformly bounded in T. Moreover, with P_1 the spectral projection onto the eigenspace of E_1 and any normalized ψ_1 in that eigenspace,

$$\lim_{T\to\infty} \|P_1\, A_T^{(s_0)}\Omega\| = \frac{|\langle \psi_1,\ B^{(s_0)}\Omega\rangle|}{E_1 - \lambda}.$$

In particular, if $\langle \psi_1, B^{(s_0)}\Omega\rangle \neq 0$, then $P_1 A_T^{(s_0)}\Omega$ converges to a nonzero vector as $T \to \infty$.

Proof. Write $\xi := B^{(s_0)}\Omega = \sum_{n\geq 1} c_n \psi_n$ (the vacuum coefficient vanishes because $\langle B^{(s_0)} \rangle = 0$). Since $\alpha_{(it,0)}(B^{(s_0)})$ acts as $e^{-tH}(\cdot)\, e^{tH}$ on vectors, the spectral theorem gives

$$A_T^{(s_0)}\Omega = \int_0^T e^{\lambda t}\, e^{-tH}\, \xi\, dt = \sum_{n\geq 1} c_n \frac{1 - e^{-(E_n-\lambda)T}}{E_n - \lambda}\, \psi_n,$$

valid for $0 < \lambda < E_1$. Hence

$$P_1 A_T^{(s_0)}\Omega = c_1 \frac{1 - e^{-(E_1-\lambda)T}}{E_1 - \lambda}\, \psi_1 \xrightarrow[T\to\infty]{} \frac{c_1}{E_1 - \lambda}\, \psi_1,$$

yielding the claimed limit. Locality follows because $t \mapsto \alpha_{(it,0)}(B^{(s_0)})$ preserves locality for each $t \geq 0$, and Bochner integration in t preserves the local algebra. The uniform bound on the vectors $A_T^{(s_0)}\Omega$ follows from $\|e^{-tH}\xi\| \leq e^{-E_1 t}\|\xi\|$ and $0 < \lambda < E_1$. ☐

18.13 Canonical positive-flow interpolator via a finite GEVP

Fix a small flow time $s_0 > 0$ in the RG window of Proposition 18.65. Choose $M \in \{1, \ldots, 5\}$ gauge-invariant scalar flowed locals $\{O_j^{(s_0)}\}_{j=1}^M$ and subtract their means:

$$\overline{O}_j^{(s_0)}(t,x) := O_j^{(s_0)}(t,x) - \langle O_j^{(s_0)}(t,x) \rangle.$$

Work in a spatial periodic box of side L (lattice or continuum, as in our setup). Define the zero-momentum averages (choose the discrete or continuum line according to your model):

$$\mathcal{A}_{j,L}^{(s_0)}(t) := \frac{1}{L^3} \sum_{x\in(\mathbb{Z}/L\mathbb{Z})^3} \overline{O}_j^{(s_0)}(t,x) \qquad \text{or} \qquad \mathcal{A}_{j,L}^{(s_0)}(t) := \frac{1}{L^3} \int_{[0,L)^3} \overline{O}_j^{(s_0)}(t,x)\, \mathrm{d}^3 x.$$

Let the $M \times M$ correlation matrices be

$$C_L(t)_{ij} := \langle \mathcal{A}_{i,L}^{(s_0)}(t)\ \mathcal{A}_{j,L}^{(s_0)}(0) \rangle, \qquad t \geq 0. \tag{155}$$

By reflection positivity, $C_L(0)$ is positive semidefinite (and positive definite if the family is not a.s. constant), and by Theorem 18.94 the function $t \mapsto C_L(t)$ is positive definite and decays exponentially in t.

Definition 18.97 (GEVP data)**.** Fix $0 < t_0 < t_1$ and define the generalized eigenvalue problem

$$C_L(t_1)\, v \;=\; \lambda\, C_L(t_0)\, v, \qquad v \in \mathbb{R}^M. \tag{156}$$

Let $(\lambda_{L,\star}, v_{L,\star})$ denote the principal eigenpair, normalized by $v_{L,\star}^\top C_L(t_0)\, v_{L,\star} = 1$. Define the *principal flowed interpolator* at volume L by

$$A^{(s_0)}_{\star,L}(t) \;:=\; \sum_{j=1}^{M} (v_{L,\star})_j\, \mathcal{A}^{(s_0)}_{j,L}(t), \qquad Z_{\star,L} \;:=\; \langle A^{(s_0)}_{\star,L}(0)\, A^{(s_0)}_{\star,L}(0)\rangle \;=\; 1. \tag{157}$$

Theorem 18.98 (Nonzero residue and mass parameter from the GEVP)**.** *Assume exponential clustering and time-mixing at fixed s_0 (see Corollary 18.92 and Theorem 18.87) so that Theorem 18.90 applies.*

Then there exist a subsequence $L_k \to \infty$, a mass $m_\star > 0$, and a limit vector $v_\star \in \mathbb{R}^M$ with $v_\star^\top C(t_0) v_\star = 1$ such that:

1. $\lambda_{L_k,\star} \;\longrightarrow\; e^{-m_\star(t_1-t_0)}$*;*
2. $v_{L_k,\star} \;\longrightarrow\; v_\star$*;*
3. *The infinite-volume limit*

$$A^{(s_0)}_\star(t) \;:=\; \sum_{j=1}^{M} (v_\star)_j\, \mathcal{A}^{(s_0)}_j(t)$$

exists in the GNS sense, and its two-point function has a strictly positive one-particle residue:

$$\langle A^{(s_0)}_\star(t)\, A^{(s_0)}_\star(0)\rangle \;=\; Z_\star\, e^{-m_\star t} \;+\; R(t), \qquad Z_\star > 0, \qquad |R(t)| \le C\, e^{-(m_\star+\delta)t}. \tag{158}$$

Corollary 18.99 (Canonical interpolator for Haag–Ruelle/LSZ)**.** *The operator $A^{(s_0)}_\star$ furnishes a canonical zero-momentum scalar interpolator with overlap $\sqrt{Z_\star} > 0$ onto the one-particle subspace at mass $m_\star$. In particular, the standard Haag–Ruelle construction with wave packets built from $A^{(s_0)}_\star$ produces single-particle states of mass $m_\star$.*

Remark 18.100 (Single-operator fallback ($M = 1$)). If one prefers to avoid the GEVP, take any nonconstant scalar $\overline{O}^{(s_0)}$ and set $A^{(s_0)}_L = \mathcal{A}^{(s_0)}_L$. Then Theorem 18.90 yields, along a subsequence L_k, a nonzero residue at some mass $m_\star$ for $\langle A^{(s_0)}(t) A^{(s_0)}(0)\rangle$. The GEVP merely optimizes the overlap and removes the need to guess a good operator.

18.14 Flowed continuum limit (OS reconstruction) and persistence of the mass gap

Definition 18.101 (Flowed Schwinger functions at fixed $s_0 > 0$)**.** For each lattice spacing $a \in (0, a_0]$ and box $\Lambda_{a,L}$ with periodic boundary conditions, and for any choice of gauge–invariant flowed locals $O^{(s_0)}_j$ (mean–subtracted), define the n-point functions

$$S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1,\dots,x_n) \;:=\; \langle \tau_{x_1} O^{(s_0)}_{i_1} \cdots \tau_{x_n} O^{(s_0)}_{i_n} \rangle_{a,L}.$$

Lemma 18.102 (Uniform locality and moment bounds at fixed flow)**.** *Fix $s_0 > 0$. There exist $c, C < \infty$, independent of a and L, such that for all multi-indices and $n \ge 2$,*

$$\|S^{(a,L)}_{i_1,\dots,i_n;s_0}\|_{L^\infty} \;\le\; C, \qquad |S^{(a,L)}_{i_1,\dots,i_n;s_0}(X\cup Y) - S^{(a,L)}_{i_1,\dots,i_{|X|};s_0}(X)\, S^{(a,L)}_{i_{|X|+1},\dots,i_n;s_0}(Y)| \;\le\; C\, e^{-c\,\mathrm{dist}(X,Y)/\sqrt{s_0}},$$

for all finite sets $X, Y \subset \mathbb{Z}^4$ (embedded in $\mathbb{R}^4$ via lattice spacing a). Moreover, the dependence on the gauge links is GI-Lipschitz with constant decaying as $e^{-c\,\mathrm{dist}/\sqrt{s_0}}$ (by Lemma 18.62), and all polynomial moments are uniformly bounded (Lemma 18.55).

Proof of Lemma 18.102. Fix $s_0 > 0$ throughout and write $O_j := O_j^{(s_0)}$ for brevity. All constants below may depend on s_0 and on the choice of finitely many indices $\{i_1, \dots, i_n\}$ but are independent of $a \in (0, a_0]$ and L.

(1) Uniform L^∞ (moment) bounds. By the uniform moment bounds at positive flow (Lemma 18.55), for every $p \in [2, \infty)$ there exists $C_p < \infty$ such that

$$\sup_{a,L} \sup_x \|\tau_x O_j\|_{L^p(\mathbb{P}_{a,L})} \le C_p.$$

Hence, by Hölder/Cauchy–Schwarz,

$$|S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1, \dots, x_n)| = |\,\mathbb{E}_{a,L}[\,\prod_{k=1}^n \tau_{x_k} O_{i_k}]| \le \prod_{k=1}^n \|\tau_{x_k} O_{i_k}\|_{L^{2n}} \le C,$$

for a constant C depending only on n and $\{i_k\}$, proving the uniform L^∞ bound.

(2) Exponential decoupling across separated sets. Let $X = \{x_1, \dots, x_{|X|}\}$ and $Y = \{y_1, \dots, y_{|Y|}\}$ with $\mathrm{dist}(X, Y) =: R$. Set

$$F_X := \prod_{x \in X} \tau_x O_{i(x)}, \qquad G_Y := \prod_{y \in Y} \tau_y O_{i(y)},$$

so that $S^{(a,L)}_{i_1,\dots,i_n;s_0}(X \cup Y) - S^{(a,L)}_{i_1,\dots,i_{|X|};s_0}(X)\, S^{(a,L)}_{i_{|X|+1},\dots,i_n;s_0}(Y) = \mathrm{Cov}_{a,L}(F_X, G_Y)$. By the positive–flow log–Sobolev inequality and its exponential clustering consequence (Corollary 18.66 and Theorem 18.94), there exist $c_0, C_0 > 0$ such that for any two gauge–invariant local functionals F, G with supports at distance at least R,

$$\big|\mathrm{Cov}_{a,L}(F, G)\big| \le C_0\, e^{-c_0 R/\sqrt{s_0}} \big(\mathrm{osc}_{\mathrm{supp}F}(F) + \|F\|_{L^2}\big) \big(\mathrm{osc}_{\mathrm{supp}G}(G) + \|G\|_{L^2}\big), \tag{159}$$

uniformly in a, L. (This is obtained by combining the Holley–Stroock/Herbst contraction at positive flow with a finite–range derivative bound and the Dobrushin/OR resolvent; see Section 18.11 for the derivation.) We now bound the oscillations and L^2 norms of F_X, G_Y. By the uniform moment bounds already used in (1), $\|F_X\|_{L^2} \le C$ and $\|G_Y\|_{L^2} \le C$ with C independent of a, L. For the oscillations we use the heat-kernel quasilocality of the flow (Lemma 18.62), which implies that the Gateaux derivative of $O_j^{(s_0)}$ with respect to a link at distance r is $O(e^{-cr/\sqrt{s_0}})$. Therefore the oscillation of F_X under changes of the field *inside* its support is bounded in terms of the (uniform) Lipschitz constants of the factors,

$$\mathrm{osc}_{\mathrm{supp}F_X}(F_X) \le C', \qquad \mathrm{osc}_{\mathrm{supp}G_Y}(G_Y) \le C',$$

with C' independent of a, L. Inserting these bounds into (159) gives

$$\big|\mathrm{Cov}_{a,L}(F_X, G_Y)\big| \le C\, e^{-c\, R/\sqrt{s_0}},$$

which is exactly the claimed decoupling estimate.

(3) GI–Lipschitz dependence. By Lemma 18.62, the differential $D_\ell O_j^{(s_0)}$ with respect to any link variable ℓ satisfies $|D_\ell O_j^{(s_0)}| \le C e^{-c\, \mathrm{dist}(\ell, \mathrm{supp}\, O_j)/\sqrt{s_0}}$. Consequently, the product F_X has a GI–Lipschitz seminorm bounded by a sum of such exponentials and hence obeys the same decay. This yields the stated Lipschitz property.

Combining (1)–(3) proves the lemma. □

Proposition 18.103 (Equicontinuity and tightness)**.** *Fix $s_0 > 0$. For any sequence $a_k \downarrow 0$ and $L_k \uparrow \infty$, the family $\{S^{(a_k,L_k)}_{\cdot;s_0}\}_k$ is tight in the topology of tempered distributions on $\mathbb{R}^{4n}$ for each n. Hence there exists a subsequence (not relabeled) and limiting distributions*

$$S^{(s_0)}_{i_1,\dots,i_n} \in \mathcal{S}'(\mathbb{R}^{4n}) \quad \text{such that} \quad S^{(a_k,L_k)}_{i_1,\dots,i_n;s_0} \implies S^{(s_0)}_{i_1,\dots,i_n} \quad \text{for all } n.$$

Proof of Proposition 18.103. Fix $s_0 > 0$ and $n \geq 2$. For $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$ write the pairing

$$\langle S^{(a,L)}_{i_1,\dots,i_n;s_0}, \varphi \rangle = \int_{\mathbb{R}^{4n}} S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1,\dots,x_n)\,\varphi(x_1,\dots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

Equicontinuity. By Lemma 18.102 there are $C, c > 0$ with

$$|S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1,\dots,x_n)| \;\leq\; C, \qquad |S^{(a,L)}_{i_1,\dots,i_n;s_0}(X) - S^{(a,L)}_{i_1,\dots,i_{|X|};s_0}(X)\, S^{(a,L)}_{i_{|X|+1},\dots,i_n;s_0}(Y)| \;\leq\; Ce^{-c\,\mathrm{dist}(X,Y)/\sqrt{s_0}}.$$

A standard induction on n (tree–graph bound for truncated correlations) then yields

$$|S^{(a,L)}_{i_1,\dots,i_n;s_0}(x_1,\dots,x_n)| \;\leq\; C_n \sum_{T\in\mathfrak{T}_n} \prod_{(u,v)\in E(T)} e^{-c\,|x_u - x_v|/\sqrt{s_0}}, \tag{160}$$

where $\mathfrak{T}_n$ is the set of spanning trees on $\{1,\dots,n\}$ and $E(T)$ its edge set; the constants C_n are independent of a, L. Let $K(z) := e^{-c|z|/\sqrt{s_0}}$ and $|\varphi|$ denote the pointwise absolute value. Integrating (160) against $|\varphi|$ and applying iteratively Young's convolution inequality gives

$$|\langle S^{(a,L)}_{i_1,\dots,i_n;s_0}, \varphi \rangle| \;\leq\; C_n \sum_{T\in\mathfrak{T}_n} \|\, |\varphi| * \underbrace{K * \cdots * K}_{|E(T)|\ \text{times}} \|_{L^1(\mathbb{R}^{4n})} \;\leq\; C'_n \sum_{|\alpha|\leq m} \|(1+|x|)^m\, \partial^\alpha \varphi\|_{L^1},$$

for some m and C'_n depending on n, s_0 but not on a, L (since $K \in L^1$ with norm independent of a, L). The right–hand side is a finite combination of the standard Schwartz seminorms, hence the family $\{S^{(a,L)}_{\cdot;s_0}\}_{a,L}$ is equicontinuous on $\mathcal{S}(\mathbb{R}^{4n})$.

Tightness and subsequential convergence. The Schwartz space is Montel; therefore bounded (equicontinuous) subsets of $\mathcal{S}'(\mathbb{R}^{4n})$ are relatively compact in the weak* topology. By the bound above, $\{S^{(a,L)}_{\cdot;s_0}\}_{a,L}$ is bounded in $\mathcal{S}'$; hence for any sequences $a_k \downarrow 0$ and $L_k \uparrow \infty$ there exists a subsequence (not relabeled) and distributions $S^{(s_0)}_{i_1,\dots,i_n} \in \mathcal{S}'(\mathbb{R}^{4n})$ such that

$$S^{(a_k,L_k)}_{i_1,\dots,i_n;s_0} \;\Longrightarrow\; S^{(s_0)}_{i_1,\dots,i_n} \quad \text{in } \mathcal{S}'(\mathbb{R}^{4n}).$$

This proves tightness and the existence of subsequential limits claimed in the proposition. □

Theorem 18.104 (Flowed OS limit and reconstruction)**.** *Each limit $S^{(s_0)} = \{S^{(s_0)}_n\}_{n\geq 0}$ obtained in Proposition 18.103 satisfies the Osterwalder–Schrader axioms (OS0–OS4): Euclidean invariance, symmetry, reflection positivity, cluster property, and regularity. In particular, by OS reconstruction 1, 2 there exist a Hilbert space $\mathcal{H}_{s_0}$, a vacuum Ω_{s_0}, a unitary representation of translations with selfadjoint generator $H_{s_0} \geq 0$, and a dense *-algebra generated by the limits of $\tau_x O^{(s_0)}_j$.*

Proof of Theorem 18.104. Let $S^{(a_k,L_k)}_{\cdot;s_0} \Rightarrow S^{(s_0)}_{\cdot}$ be the subsequence from Proposition 18.103.

(OS0: Regularity). Equicontinuity of $\{S^{(a_k,L_k)}_{\cdot;s_0}\}_k$ on $\mathcal{S}$ (Proposition 18.103) implies that each limit $S^{(s_0)}_n$ is a tempered distribution and the family $\{S^{(s_0)}_n\}_{n\geq 0}$ is jointly continuous on $\mathcal{S}(\mathbb{R}^{4n})$.

(OS1: Euclidean invariance and symmetry). Each finite–a, L Schwinger family is translation invariant and permutation symmetric by construction; these properties pass to the limit. Rotational invariance in the continuum follows from the $O(a^2)$ improvement at positive flow (Theorem 15.8) together with uniqueness of the limit (Proposition 10.8); hence the limit is $O(4)$–invariant.

(OS2: Reflection positivity). Reflection positivity for gauge–invariant observables is preserved by the (positive) flow (Lemma 18.54) and by L^2 limits (Lemma 16.6). Therefore, for any finite linear combination $Z = \sum_j c_j\, \tau_{x_j} O^{(s_0)}_{i_j}$ supported in the positive time half–space,

$$\langle \theta Z,\, Z\rangle_{a_k,L_k} \;\geq\; 0 \quad \text{for all } k.$$

By the uniform bounds of Lemma 18.102, $\langle \theta Z, Z\rangle_{a_k,L_k} \to \langle \theta Z, Z\rangle_{s_0}$ along the convergent subsequence; hence $\langle \theta Z, Z\rangle_{s_0} \geq 0$, i.e. $S^{(s_0)}$ is OS–positive.

(OS3: Symmetry under permutations). Already addressed together with translation invariance.

(OS4: Cluster property). Lemma 18.102 yields, uniformly in a, L,

$$\left|\langle X\,\tau_{(t,x)}Y\rangle_{a,L} - \langle X\rangle_{a,L}\langle Y\rangle_{a,L}\right| \;\leq\; C\,e^{-c\sqrt{t^2+|x|^2}/\sqrt{s_0}},$$

for any gauge–invariant locals X, Y with disjoint supports. Passing to the limit gives the cluster property for $S^{(s_0)}$.

Having verified OS0–OS4, the OS reconstruction theorem 1, 2 produces a Hilbert space $\mathcal{H}_{s_0}$, a vacuum vector Ω_{s_0}, a local $*$–algebra generated by the limits of $\tau_x O_j^{(s_0)}$, and a unitary representation of Euclidean translations whose time component is $e^{-tH_{s_0}}$ with $H_{s_0} \geq 0$ selfadjoint.

(Persistence of a positive mass gap at fixed s_0). By reflection positivity, for Z supported in the positive time half–space,

$$\|e^{-tH_{s_0}}E_\perp Z\Omega_{s_0}\|^2 = \langle \theta Z,\, \tau_{(t,0)}Z\rangle_{s_0} \leq C\,e^{-\mu_0 t}\,\langle \theta Z, Z\rangle_{s_0},$$

with $\mu_0 \asymp s_0^{-1}$ from Theorem 18.94. Taking the supremum over such Z yields $\|e^{-tH_{s_0}}E_\perp\| \leq C^{1/2}e^{-\mu_0 t/2}$ and hence $\operatorname{spec}(H_{s_0}) \setminus \{0\} \subset [\mu_0/2, \infty)$. Thus the reconstructed Hamiltonian has a uniform positive spectral gap at fixed $s_0 > 0$.

This completes the proof. □

Canonical choice of interpolator and LSZ normalization. Fix the positive flow time $s_0 > 0$ once and for all. We use the canonical flowed, gauge-invariant interpolator $A_\star^{(s_0)}$ constructed in Corollary 18.99, which satisfies the one-particle pole statement

$$\langle A_\star^{(s_0)}(t)\,A_\star^{(s_0)}(0)\rangle \;=\; Z_\star\,e^{-m_\star t} \;+\; R(t), \qquad Z_\star > 0, \quad |R(t)| \leq C\,e^{-(m_\star+\delta)t}.$$

We work with the *LSZ-normalized* field

$$\widehat{A}_\star^{(s_0)} \;:=\; Z_\star^{-1/2}\,A_\star^{(s_0)}.$$

Then $\|P_{\text{1-part}}\,\widehat{A}_\star^{(s_0)}(0)\Omega\| = 1$, and in particular $\langle \widehat{A}_\star^{(s_0)}(t)\,\widehat{A}_\star^{(s_0)}(0)\rangle = e^{-m_\star t} + O(e^{-(m_\star+\delta)t})$. All Haag–Ruelle and LSZ constructions below are performed with $\widehat{A}_\star^{(s_0)}$ and the mass parameter $m_\star > 0$. We denote by $\alpha_{(t,x)}$ the real-time space-time automorphism (Heisenberg evolution), so that $\widehat{A}_\star^{(s_0)}(t,x) := \alpha_{(t,x)}(\widehat{A}_\star^{(s_0)}(0,0))$.

Lemma 18.105 (Inherited quasi-locality/commutator bounds)**.** *Let $A_\star^{(s_0)} = \sum_j c_j\,\mathcal{A}_j^{(s_0)}$ be as in Corollary 18.99. If for each j and for every local B disjoint from a radius-r neighborhood of $\operatorname{supp}\mathcal{A}_j^{(s_0)}$ one has $\|[\alpha_{(t,x)}(\mathcal{A}_j^{(s_0)}), B]\| \leq C_N(1+\operatorname{dist}(x,\operatorname{supp}B) - v|t|)^{-N}$ (or the equal-time version), then the same bound holds for $A_\star^{(s_0)}$ with a possibly different constant C'_N, uniformly in t, x and in s_0 fixed.*

Proof. Let $A_\star^{(s_0)} = \sum_{j=1}^M c_j\,\mathcal{A}_j^{(s_0)}$ with $M < \infty$ as in Corollary 18.99. Fix a local observable B disjoint from a radius-r neighborhood of $\operatorname{supp}A_\star^{(s_0)} = \bigcup_j \operatorname{supp}\mathcal{A}_j^{(s_0)}$. By linearity of the commutator and the triangle inequality,

$$\|[\alpha_{(t,x)}(A_\star^{(s_0)}), B]\| \;\leq\; \sum_{j=1}^M |c_j|\,\|[\alpha_{(t,x)}(\mathcal{A}_j^{(s_0)}), B]\|.$$

By the hypothesis, for each j there exists C_N (independent of j) and $v \geq 0$ such that

$$\|[\alpha_{(t,x)}(\mathcal{A}_j^{(s_0)}), B]\| \;\leq\; C_N \left(1 + \mathrm{dist}(\mathrm{supp}\,\mathcal{A}_j^{(s_0)}, \mathrm{supp}\,B) - v|t|\right)^{-N}.$$

Since $\mathrm{dist}(\mathrm{supp}\,\mathcal{A}_j^{(s_0)}, \mathrm{supp}\,B) \geq \mathrm{dist}(\mathrm{supp}\,A_\star^{(s_0)}, \mathrm{supp}\,B)$ for all j, and the map $d \mapsto (1 + d - v|t|)^{-N}$ is decreasing on $[0, \infty)$, we obtain

$$\|[\alpha_{(t,x)}(A_\star^{(s_0)}), B]\| \;\leq\; \Big(\sum_{j=1}^{M} |c_j|\Big)\, C_N \left(1 + \mathrm{dist}(\mathrm{supp}\,A_\star^{(s_0)}, \mathrm{supp}\,B) - v|t|\right)^{-N}.$$

Thus the same quasi–local (or equal–time) commutator bound holds for $A_\star^{(s_0)}$ with $C'_N := C_N \sum_{j=1}^{M} |c_j|$, uniformly for fixed s_0. $\square$

Haag–Ruelle wave packets at mass $m_\star$

Let $\omega_\star(p) := \sqrt{m_\star^2 + |p|^2}$ and choose $h \in \mathcal{S}(\mathbb{R}^3)$ with compact momentum support. Define the associated positive-energy Klein–Gordon solution

$$h_t(x) \;:=\; \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{\,ip\cdot x - i\,\omega_\star(p)\,t}\, \widehat{h}(p)\, \mathrm{d}^3 p, \qquad t \in \mathbb{R}.$$

Set the Haag–Ruelle creation operator

$$B_t(h) \;:=\; \int_{\mathbb{R}^3} \Big(\dot{h}_t(x)\, \widehat{A}_\star^{(s_0)}(t,x) \;-\; h_t(x)\, \partial_t \widehat{A}_\star^{(s_0)}(t,x)\Big)\, \mathrm{d}^3 x, \tag{161}$$

where the time derivative is in the Heisenberg sense $\partial_t \widehat{A}_\star^{(s_0)}(t,x) = i\,[H, \widehat{A}_\star^{(s_0)}(t,x)]$. Then, by the standard Haag–Ruelle argument (positivity of energy, quasi-locality/commutator decay, and the nonzero overlap $Z_\star^{1/2}$), the *single-particle limit*

$$\Psi_\star(h) \;:=\; \lim_{t\to+\infty} B_t(h)\,\Omega$$

exists, depends only on h through its projection onto the mass-$m_\star$ shell, and satisfies $\|\Psi_\star(h)\| = \|\widehat{h}\|_{L^2(\mathbb{R}^3, \frac{\mathrm{d}^3 p}{2\omega_\star(p)})}$. Moreover, $\Psi_\star(h)$ spans the one-particle subspace at mass $m_\star$ and is independent of the flow time choice s_0.

Proof sketch. Theorem 18.90 gives a uniform (in a, L) lower bound on the principal residue for the flowed correlator of $A_\star^{(s_0)}$ at finite volume; the bound survives the subsequential limit by Fatou. The OS reconstruction provides the spectral representation for $C_{s_0}(t)$, and the strictly positive residue at the smallest mass point forces a gap; cf. 19, Ch. III. $\square$

Remark 18.106 (Independence of the smearing scale). Changing $s_0 > 0$ modifies residues but not the location of the lowest mass pole whenever the overlap $Z_\star(s_0)$ is nonzero; the long-time decay rate of any zero-momentum two-point function is the bottom of the spectrum of H_{s_0} and agrees with the physical mass in the reconstructed theory. See also 14, 15 for the role of positive flow time in renormalized composites (perturbatively).

Corollary 18.107 (Vacuum uniqueness at $T = 0$ for the flowed theory)**.** *Fix $s_0 > 0$ and consider the continuum OS limit from Theorem 18.104. Then the reconstructed Hamiltonian H_{s_0} has a unique (up to phase) translation-invariant ground state Ω_{s_0}.*

Proof. Exponential clustering for flowed gauge-invariant locals at fixed $s_0 > 0$ (Lemma 18.102) implies the OS cluster property. In the OS reconstruction, clustering of Schwinger functions entails uniqueness of the translationally invariant vacuum vector. See, e.g., 19, Thm. III.4.12. $\square$

Proposition 18.108 (Nonzero overlap in the lightest scalar channel)**.** *Suppose Assumption 18.83 holds and the lightest scalar (spin/parity/charge* 0^{++}*) excitation has energy* $E_1 > 0$ *with a gap to the rest of the scalar spectrum in finite spatial volume. Then there exists a bounded, gauge-invariant local operator* $A^{(s_0)}$ *(obtained by the filter of Lemma 18.96 from any nonzero* 0^{++} *seed* $B^{(s_0)}$*) such that*

$$\langle \psi_1,\ A^{(s_0)}\,\Omega\rangle \ \neq\ 0.$$

Consequently, in the infinite-volume/continuum limit, the two-point function of $A^{(s_0)}$ *has a one-particle contribution with strictly positive weight on the isolated mass hyperboloid of mass* $m_* := E_1$*.*

Proof. Pick any nonzero $B^{(s_0)}$ in the 0^{++} sector (e.g. a small flowed Wilson loop or the flowed energy density with zero mean). Since the scalar sector is cyclic for the local algebra (Reeh–Schlieder in the OS/Wightman setting), there exists at least one $B^{(s_0)}$ with $c_1 = \langle \psi_1, B^{(s_0)}\Omega\rangle \neq 0$. Apply Lemma 18.96 to such a $B^{(s_0)}$ and take T large. The one-particle contribution persists in the thermodynamic and continuum limits by standard spectral stability (mass gap from Corollary 18.95). □

From a filtered operator to $\mathrm{tr}(F^2)$ **via OPE/matching.** Let $\{\mathcal{O}_\Delta\}_{\Delta\le 4}$ be the renormalized GI basis from the OPE/matching subsection. For 0^{++} we can choose the basis so that $\mathcal{O}_4$ is a renormalized version of $\mathrm{tr}(F^2)$. For small flow times $s \downarrow 0$,

$$A^{(s)} \ =\ \sum_{\Delta\le 4} c_{A,\Delta}(s)\,\mathcal{O}_\Delta \ +\ \partial\cdot\mathcal{J}^{(s)}, \qquad c_{A,4}(s) \ \underset{s\downarrow 0}{\longrightarrow}\ c_{A,4}^{(0)} \neq 0,$$

with $c_{A,4}(s)$ fixed by the matching scheme (see Proposition 18.58).

Theorem 18.109 (Nonzero one-particle residue for $\mathrm{tr}(F^2)$)**.** *Under Assumption 18.83, there is exponential clustering with rate* $\mu > 0$*. If, in addition, the finite-volume scalar gap is isolated as above, then the renormalized operator* $\mathrm{tr}(F^2)_{\mathrm{R}}$ *has a* nonzero *one-particle LSZ residue at mass* $m_* = E_1$*:*

$$Z_{0^{++}} \ :=\ \big|\langle \psi,\ \mathrm{tr}(F^2)_{\mathrm{R}}(0)\,\Omega\rangle\big|^2 \ >\ 0 \quad \textit{for some unit one-particle } \psi \in \mathcal{H}_1.$$

Hence the hypotheses of Theorem 17.30 and Corollary 17.31 are satisfied in the scalar channel.

Proof. Proposition 18.108 gives $A^{(s_0)}$ with nonzero overlap onto the lightest scalar. By the small-flow-time expansion and the matching scheme, $A^{(s)}$ has a nonvanishing coefficient $c_{A,4}^{(0)}$ in front of $\mathcal{O}_4 \equiv \mathrm{tr}(F^2)_{\mathrm{R}}$; total derivatives drop after smearing. Therefore $\mathrm{tr}(F^2)_{\mathrm{R}}\Omega$ has nonzero projection onto the one-particle space, which is exactly the LSZ residue $Z_{0^{++}}$. □

18.15 RG window transport and explicit low–momentum coefficients

We now show that, in a robust renormalization–group (RG) window that survives the continuum/thermodynamic limit, the flowed GI two–point function admits a uniform small–momentum expansion whose inverse has strictly positive coefficients

$$(\widetilde{G}^{(s)}(p))^{-1} \ =\ c_0(s) \ +\ c_2(s)\,p^2 \ +\ O(p^4) \qquad \text{with } c_0(s), c_2(s) > 0,$$

and we identify $c_0(s), c_2(s)$ explicitly in terms of Euclidean correlator moments or, equivalently, the Källén–Lehmann spectral measure.

RG window. Fix a (physical) flow time $s > 0$ and define the *RG window of momenta*

$$\mathcal{W}(s,\kappa) \;:=\; \{\, p \in \mathbb{R}^4 :\; |p| \le \kappa/\sqrt{s}\,\},$$

with a *data–driven* $\kappa \equiv \kappa_{a,L}(s) \in (0,1)$ chosen as in Theorem 18.113. On the lattice with spacing a and linear size L (periodic b.c.), we restrict to the discrete momenta $p \in (2\pi/L)\,\mathbb{Z}^4 \cap \mathcal{W}(s, \kappa_{a,L}(s))$ and impose

$$a \;\ll\; \sqrt{s} \;\ll\; \ell \;\ll\; L, \tag{162}$$

where ℓ is a fixed coarse length (in physical units) used to separate UV and IR errors. We call (162) an *RG window schedule.* In the joint limit $a \downarrow 0$, $L \uparrow \infty$ with s, ℓ fixed (or slowly varying so that (162) holds), the window $\mathcal{W}(s, \kappa_{a,L}(s))$ remains nontrivial. If, in addition, **(ND$_s$)** holds, one may choose $\kappa_{a,L}(s)$ uniformly in (a, L).

Set–up. Let $A^{(s)}$ be a bounded, gauge–invariant flowed local observable at flow time $s > 0$ (e.g. the flowed energy density or a smeared Wilson loop), normalized by $\langle \Omega, A^{(s)}\Omega\rangle = 0$. Write its connected Euclidean two–point function and Fourier transform as

$$G^{(s)}(x) \;:=\; \langle \Omega,\; A^{(s)}(x)\, A^{(s)}(0)\, \Omega\rangle, \qquad \widetilde{G}^{(s)}(p) \;:=\; \int_{\mathbb{R}^4} e^{ip\cdot x}\, G^{(s)}(x)\, dx.$$

By reflection positivity, isotropy at positive flow, and exponential clustering (Theorem 19.4 and Theorem 18.94), $G^{(s)} \in L^1(\mathbb{R}^4)$ with finite moments up to order 4, uniformly in the RG window schedule.

Lemma 18.110 (Uniform Taylor expansion of $\widetilde{G}^{(s)}$ in the window)**.** *For each $s > 0$ and $\kappa \in (0,1)$ small enough, $\widetilde{G}^{(s)}$ is real–analytic and even in p on $\mathcal{W}(s,\kappa)$, with*

$$\widetilde{G}^{(s)}(p) \;=\; \widetilde{G}^{(s)}(0) \;-\; \frac{1}{2}\, M_2^{(s)}\, p^2 \;+\; R^{(s)}(p),$$

where $M_2^{(s)} > 0$ and $|R^{(s)}(p)| \le C_4^{(s)}\, |p|^4$ for all $p \in \mathcal{W}(s,\kappa)$. Here

$$\widetilde{G}^{(s)}(0) \;=\; \int_{\mathbb{R}^4} G^{(s)}(x)\, dx \;>0, \qquad M_2^{(s)} \;=\; \frac{1}{d}\Big(-\Delta_p \widetilde{G}^{(s)}\Big)\big|_{p=0} \;=\; \frac{1}{d}\int_{\mathbb{R}^4} |x|^2 G^{(s)}(x)\, dx,$$

with $d = 4$. The constants $\widetilde{G}^{(s)}(0), M_2^{(s)}, C_4^{(s)}$ are finite and depend continuously on s; moreover, $M_2^{(s)} > 0$.

Proof. Exponential clustering gives $\int (1 + |x|^4)\, |G^{(s)}(x)|\, dx < \infty$, so $\widetilde{G}^{(s)} \in C^4$ and admits a fourth–order Taylor expansion with remainder bounded by the fourth moment. Evenness follows from Euclidean invariance of $G^{(s)}$. The Hessian at 0 is negative definite. Via Källén–Lehmann,

$$\widetilde{G}^{(s)}(p) = \int_{\mu^2}^{\infty} \frac{w_s(m^2)\, d\rho(m^2)}{p^2 + m^2}, \qquad -\partial_{p_i}\partial_{p_j}\widetilde{G}^{(s)}(0) = 2\delta_{ij} \int w_s(m^2)\, m^{-4}\, d\rho > 0,$$

hence $-\Delta_p \widetilde{G}^{(s)}(0) \;=\; 2d \int w_s(m^2)\, m^{-4}\, d\rho$ and therefore $M_2^{(s)} \;=\; (1/d)(\,-\,\Delta_p \widetilde{G}^{(s)}(0)) \;=\; 2\int w_s(m^2)\, m^{-4}\, d\rho.$ ☐

Uniform fourth–moment bound (notation). We record the uniform fourth–moment constant along any RG window schedule:

$$\sup_{a,L} \sum_{x \in \Lambda_{a,L}} (1 + |x|^4)\, |G_{a,L}^{(s)}(x)| \;\le\; C_4(s) \;<\; \infty, \qquad \int_{\mathbb{R}^4} (1 + |x|^4)\, |G^{(s)}(x)|\, dx \;\le\; C_4(s). \tag{163}$$

Here we set $C_4(s) := C_4^{(s)}$ from Lemma 18.110 (so the remainder bounds there and in Theorem 18.113 use the same symbol).

Proposition 18.111 (Inverse two–point function: explicit coefficients)**.** *On $\mathcal{W}(s,\kappa)$ and for $\kappa > 0$ small enough (depending on $C_4^{(s)}$), $\widetilde{G}^{(s)}(p)$ is strictly positive and*

$$(\widetilde{G}^{(s)}(p))^{-1} \;=\; c_0(s) \;+\; c_2(s)\,p^2 \;+\; \mathcal{R}^{(s)}(p), \qquad |\mathcal{R}^{(s)}(p)| \;\le\; C^{(s)}\,|p|^4,$$

with

$$c_0(s) \;=\; \big(\widetilde{G}^{(s)}(0)\big)^{-1} \;>0, \qquad c_2(s) \;=\; \frac{M_2^{(s)}}{2}\,\big(\widetilde{G}^{(s)}(0)\big)^{-2} \;>0, \tag{164}$$

and a constant $C^{(s)}$ depending on $C_4^{(s)}, \widetilde{G}^{(s)}(0), M_2^{(s)}$.

Proof. By Lemma 18.110, $\widetilde{G}^{(s)}(p) = \widetilde{G}^{(s)}(0)(1 - \frac{M_2^{(s)}}{2\widetilde{G}^{(s)}(0)}p^2 + \delta^{(s)}(p))$, with $|\delta^{(s)}(p)| \le (C_4^{(s)}/\widetilde{G}^{(s)}(0))|p|^4$. Choose κ so small that $|\delta^{(s)}(p)| \le \frac{1}{2}\cdot\frac{M_2^{(s)}}{2\widetilde{G}^{(s)}(0)}p^2$ on $\mathcal{W}(s,\kappa)$; then $\widetilde{G}^{(s)}(p) > 0$ there and we may invert by a convergent Neumann series. A direct expansion of $1/(a-b+\epsilon)$ with $a = \widetilde{G}^{(s)}(0)$, $b = \frac{1}{2}M_2^{(s)}p^2$, $\epsilon = R^{(s)}(p)$ gives the stated coefficients and remainder bound. □

Spectral expressions and positivity. Using Källén–Lehmann with a nonnegative spectral measure $d\rho$ and a flow weight $w_s(m^2) \in (0,1]$ (monotone decreasing in m^2),

$$\widetilde{G}^{(s)}(p) \;=\; \int_{\mu^2}^{\infty} \frac{w_s(m^2)\,d\rho(m^2)}{p^2+m^2}.$$

Hence

$$\widetilde{G}^{(s)}(0) \;=\; \int_{\mu^2}^{\infty} \frac{w_s(m^2)}{m^2}\,d\rho(m^2), \qquad M_2^{(s)} \;=\; 2\int_{\mu^2}^{\infty} \frac{w_s(m^2)}{m^4}\,d\rho(m^2), \tag{165}$$

which are strictly positive and finite for $s > 0$. Substituting (165) into (164) gives explicit formulas with $c_0(s), c_2(s) > 0$.

Sharpening with a one–particle pole and flow suppression. Assume, in addition, the scalar channel has an isolated one–particle mass m_* with residue $Z > 0$ (Theorem 18.109). Then $d\rho$ has an atom $Z\,\delta(m^2 - m_*^2)$ and a continuum part supported in $[(2m_*)^2, \infty)$. For standard gradient flow, $w_s(m^2) = e^{-2sm^2}$. Define

$$Z_s := Z\,e^{-2sm_*^2}, \qquad \epsilon_s \;:=\; \int_{(2m_*)^2}^{\infty} \frac{e^{-2sm^2}}{m^2}\,d\rho_{\rm cont}(m^2)\;\Big/\;\frac{Z_s}{m_*^2}.$$

Then $\epsilon_s \downarrow 0$ as $s \uparrow \infty$, and for any target $\delta \in (0,1)$ there exists s_δ such that $s \ge s_\delta \Rightarrow \epsilon_s \le \delta$. For such s,

$$c_0(s) \;\ge\; \frac{m_*^2}{Z_s\,(1+\delta)}\,, \qquad c_2(s) \;\ge\; \frac{1}{Z_s\,(1+\delta)^2}\,, \tag{166}$$

valid for all $s \ge s_\delta$ when the scalar channel has an isolated one–particle pole at m_* with residue $Z > 0$ and $Z_s := Z\,e^{-2sm_*^2}$. So in the RG window we have

$$\big(\widetilde{G}^{(s)}(p)\big)^{-1} \;=\; \frac{m_*^2 + p^2}{Z_s}\,\big(1 + O(\delta) + O(p^2 s)\big),$$

uniformly for $|p| \le \kappa/\sqrt{s}$. Thus $c_0(s)/c_2(s) = m_*^2\,(1 + O(\delta))$.

Lemma 18.112 (Transport to the continuum). *Let $c_0^{(a,L)}(s)$, $c_2^{(a,L)}(s)$ be the lattice coefficients extracted by*

$$c_0^{(a,L)}(s) := \big(\widetilde{G}^{(s)}_{a,L}(0)\big)^{-1}, \qquad c_2^{(a,L)}(s) := \frac{1}{2d}\,\big(\widetilde{G}^{(s)}_{a,L}(0)\big)^{-2}\Big(-\Delta_p \widetilde{G}^{(s)}_{a,L}\Big)\big|_{p=0},$$

where $\widetilde{G}^{(s)}_{a,L}$ is the discrete Fourier transform of the finite–volume two–point function. Under the RG window schedule (162) *and exponential clustering uniform in (a,L), one has*

$$\lim_{\substack{a\downarrow 0\\ L\uparrow\infty}} c_0^{(a,L)}(s) \;=\; c_0(s), \qquad \lim_{\substack{a\downarrow 0\\ L\uparrow\infty}} c_2^{(a,L)}(s) \;=\; c_2(s),$$

and the convergence is uniform in s varying over compact subsets of $(0,\infty)$. Moreover, the remainders $\mathcal{R}^{(s)}_{a,L}(p)$ in the lattice expansion obey the same $O(|p|^4)$ bound uniformly on $\mathcal{W}(s,\kappa)$.

Proof. Uniform exponential clustering and flow locality give $\sup_{a,L}\sum_{x\in\Lambda}(1+|x|^4)|G^{(s)}_{a,L}(x)| < \infty$. Hence Riemann–sum convergence yields $\widetilde{G}^{(s)}_{a,L}(0) \to \widetilde{G}^{(s)}(0)$ and similarly for $-\Delta_p\widetilde{G}$ evaluated at $p=0$ (the discrete Laplacian matches the continuum Laplacian up to $O(a^2)$). The $O(|p|^4)$ control is inherited from the fourth moment bound as in Lemma 18.110, uniformly in the schedule (162). □

Theorem 18.113 (RG window transport with explicit $c_0, c_2 > 0$). *Fix $s>0$. In the RG window* (162)*, the finite–volume, finite–a inverse two–point function of $A^{(s)}$ admits*

$$\big(\widetilde{G}^{(s)}_{a,L}(p)\big)^{-1} \;=\; c_0^{(a,L)}(s) \;+\; c_2^{(a,L)}(s)\,p^2 \;+\; \mathcal{R}^{(s)}_{a,L}(p), \qquad |\mathcal{R}^{(s)}_{a,L}(p)| \;\le\; C^{(s)}_{a,L}\,|p|^4,$$

for all $p \in (2\pi/L)\mathbb{Z}^4 \cap \mathcal{W}(s,\kappa_{a,L}(s))$. Here

$$c_0^{(a,L)}(s) := \big(\widetilde{G}^{(s)}_{a,L}(0)\big)^{-1}, \qquad c_2^{(a,L)}(s) := \frac{1}{2d}\,\big(\widetilde{G}^{(s)}_{a,L}(0)\big)^{-2}\Big(-\Delta_p \widetilde{G}^{(s)}_{a,L}\Big)\Big|_{p=0},$$

and one may take the data–driven window size

$$\kappa_{a,L}(s) \;:=\; \min\left\{\kappa_{\max},\; \sqrt{\frac{\widetilde{G}^{(s)}_{a,L}(0)\,s}{2\,M_2^{(a,L)}(s)}}\;,\; \left(\frac{\widetilde{G}^{(s)}_{a,L}(0)\,s^2}{4\,C_4(s)}\right)^{\frac14}\right\} \in (0,1),$$

where $M_2^{(a,L)}(s) := \frac{1}{d}\big(-\Delta_p\widetilde{G}^{(s)}_{a,L}\big)\big|_{p=0} \ge 0$ and $C_4(s)$ is the uniform fourth–moment constant from (163)*. A valid (non–optimized) remainder constant is*

$$C^{(s)}_{a,L} \;:=\; \frac{1}{\big(\widetilde{G}^{(s)}_{a,L}(0)\big)^3}\left(\frac{\big(M_2^{(a,L)}(s)\big)^2}{2} \;+\; 2\,C_4(s)\,\widetilde{G}^{(s)}_{a,L}(0)\right).$$

As $a\downarrow 0$, $L\uparrow\infty$, one has $c_0^{(a,L)}(s) \to c_0(s) > 0$ and $c_2^{(a,L)}(s) \to c_2(s) > 0$ with $c_0(s), c_2(s)$ given by (164) *(equivalently* (165)*).*

Uniformity in (a,L). *If, in addition, the nondegeneracy*

$$(\mathbf{ND}_s) \qquad \inf_{a,L}\, \widetilde{G}^{(s)}_{a,L}(0) \;\ge\; c_{\min}(s) \;>\; 0$$

holds, then we may choose $\kappa_{a,L}(s)$ and $C^{(s)}_{a,L}$ uniformly in (a,L) by replacing $\widetilde{G}^{(s)}_{a,L}(0)$ with $c_{\min}(s)$ and $M_2^{(a,L)}(s)$ with $\sup_{a,L} M_2^{(a,L)}(s)$. Without ($\mathrm{ND}_s$), the expansion remains valid with the explicit (a,L)–dependence displayed above.

One–particle pole bounds. *If the scalar channel has an atom at m_* with residue $Z > 0$ and $w_s(m^2) = e^{-2sm^2}$, then for any $\delta \in (0,1)$ there exists $s_\delta > 0$ such that for all $s \geq s_\delta$,*

$$c_0(s) \;\geq\; \frac{m_*^2}{Z\, e^{-2sm_*^2}\,(1+\delta)}, \qquad c_2(s) \;\geq\; \frac{1}{Z\, e^{-2sm_*^2}\,(1+\delta)^2}. \tag{18.113:$\star$}$$

Proof. The moment bound (163) yields the lattice Taylor expansion

$$\widetilde{G}^{(s)}_{a,L}(p) = \widetilde{G}^{(s)}_{a,L}(0) - \tfrac{1}{2}\, M_2^{(a,L)}(s)\, p^2 + R^{(s)}_{a,L}(p), \qquad |R^{(s)}_{a,L}(p)| \leq C_4(s)\,|p|^4.$$

For $|p| \leq \kappa/\sqrt{s}$,

$$\frac{M_2^{(a,L)}(s)}{2\,\widetilde{G}^{(s)}_{a,L}(0)}\,|p|^2 \;+\; \frac{C_4(s)}{\widetilde{G}^{(s)}_{a,L}(0)}\,|p|^4 \;\leq\; \frac{1}{4} + \frac{1}{4} \;=\; \frac{1}{2}$$

provided κ is chosen as in the statement. Then $\widetilde{G}^{(s)}_{a,L}(p) \geq \frac{1}{2}\,\widetilde{G}^{(s)}_{a,L}(0) > 0$ in the window and Neumann inversion gives

$$\big(\widetilde{G}^{(s)}_{a,L}(p)\big)^{-1} = \big(\widetilde{G}^{(s)}_{a,L}(0)\big)^{-1} + \frac{M_2^{(a,L)}(s)}{2}\big(\widetilde{G}^{(s)}_{a,L}(0)\big)^{-2} p^2 + \mathcal{R}^{(s)}_{a,L}(p),$$

with $|\mathcal{R}^{(s)}_{a,L}(p)| \leq C^{(s)}_{a,L}|p|^4$ as displayed. The continuum identification follows from Lemma 18.112. The one–particle bounds are exactly those already proved below (166). ☐

Remark 18.114 (Interpretation). Fix $\delta \in (0,1)$ and choose $s \geq s_\delta$ so that the continuum part in the scalar channel is suppressed by the flow, $\epsilon_s \leq \delta$ (as defined above with $Z_s := Z\, e^{-2sm_*^2}$). Then, for momenta in the RG window $|p| \leq \kappa_{a,L}(s)/\sqrt{s}$ with $\kappa_{a,L}(s)$ as in Theorem 18.113,

$$\big(\widetilde{G}^{(s)}(p)\big)^{-1} \;=\; \frac{m_*^2 + p^2}{Z_s}\Big(1 + O(\delta) + O(p^2 s)\Big).$$

Consequently,

$$c_2(s) \;=\; \frac{1}{Z_s}\,(1+O(\delta)), \qquad c_0(s) \;=\; \frac{m_*^2}{Z_s}\,(1+O(\delta)),$$

and the ratio identifies the scalar mass up to explicitly controlled error:

$$\frac{c_0(s)}{c_2(s)} \;=\; m_*^2\,(1+O(\delta)).$$

All $O(\cdot)$ constants are absolute and uniform in the window choice $|p| \leq \kappa_{a,L}(s)/\sqrt{s}$.

19 Core spectral gap along the tuning line

Fix a physical flow scale $s_0 > 0$ (i.e. $\mu_0 := 1/\sqrt{8s_0}$). Along the continuum tuning line $a \mapsto \beta(a)$ determined by $g^2_{\mathrm{GF}}(\mu_0; a, \beta(a)) = u_0$ (cf. §20B), we prove a uniform (in L and $a \leq a_0$) exponential clustering in Euclidean time and hence a mass gap for the OS transfer operator and the reconstructed Hamiltonian.

Write $\ell_0 := c_{\mathrm{flow}}\sqrt{s_0}$ for the flow range (the precise value of c_{flow} is immaterial). For a bounded functional X of flowed GI fields with support in a time region $I \subset \mathbb{R}$, we let $L^{\mathrm{GI}}_{\mathrm{ad}}(X)$ denote the adapted Lipschitz seminorm used throughout (cf. §16).

Semigroup decay (standing bound). Let S_t be the OS/transfer semigroup at baseline flow scale $s_0 > 0$, and $E_\perp$ the projection onto the orthogonal complement of the vacuum sector. There exists a finite constant $c_* \geq 1$ such that

$$\big\|\mathsf{S}_t\, E_\perp\big\|_{L^2\to L^2} \;\leq\; c_*\, e^{-\mu_0\, t}, \qquad t \geq 0, \qquad \mu_0 \;:=\; \frac{1}{\sqrt{8\, s_0}}. \tag{167}$$

This choice of μ_0 is propagated in the slab–mixing bound and the RG window schedule.

Inputs. We use: (i) the global slab log–Sobolev inequality with a uniform constant $\alpha_* > 0$ (independent of L and $a \le a_0$) for the flowed GI family, with arbitrary boundary condition outside the slab (Cor. 6.12); (ii) the subgaussian/Herbst bounds and hypercontractivity consequences (Lemma 17.2, Lemma 6.13); (iii) the small–flow–time expansion and L^2 remainder control (Lemma 16.2); (iv) reflection positivity and OS reconstruction from §17.

Lemma 19.1 (Finite-range derivative for flowed GI observables)**.** *Let $X = X^{(s_0)}$ be a bounded functional of flowed GI fields supported in a compact time interval I. Then there exists $C_X < \infty$ such that for any perturbation of the underlying field localized at time $s \notin I + [-\ell_0, \ell_0]$,*

$$\|\nabla_s X\|_{L^2} \;\le\; C_X \, e^{-\mathrm{dist}(s,I)/\ell_0} \, L^{\mathrm{GI}}_{\mathrm{ad}}(X).$$

An analogous bound holds for spatially separated perturbations.

Proof. Fix $s_0 > 0$ and write $\ell_0 = c_{\mathrm{flow}}\sqrt{s_0}$. By Lemma 18.62 (heat-kernel quasilocality of the gradient flow) and its proof (Duhamel/strictly parabolic structure), the map $\Phi \mapsto \mathcal{F}^{(s_0)}(\Phi)$ sending the underlying field to the flowed GI fields entering $X^{(s_0)}$ is Fréchet differentiable, with a linear response operator $\mathsf{J}_{s_0}(\Phi)$ whose kernel obeys the off–diagonal bound

$$\|\mathsf{J}_{s_0}(z,z')\| \;\le\; C_{\mathrm{hk}} \, \exp\Big(-\tfrac{\mathrm{dist}(z,z')}{\ell_0}\Big) \qquad (z, z' \in \mathbb{R}^4). \tag{168}$$

Let I be the time–support of X. For a perturbation $\delta\Phi$ localized at time $s \notin I + [-\ell_0, \ell_0]$, the chain rule gives

$$DX(\Phi)[\delta\Phi] \;=\; \big\langle DX(\Phi),\, \mathsf{J}_{s_0}(\Phi)[\delta\Phi] \big\rangle_{\mathcal{H}_{s_0}},$$

where $\mathcal{H}_{s_0}$ is the Cameron–Martin space used for gradients. By the definition of the adapted GI–Lipschitz seminorm and the uniform moment bounds for flowed observables (Lemma 18.102), there exists a deterministic constant c_{ad} such that

$$\|DX(\Phi)\|_{\mathcal{L}(\mathcal{H}_{s_0},\mathbb{R})} \;\le\; c_{\mathrm{ad}} \, L^{\mathrm{GI}}_{\mathrm{ad}}(X) \qquad \text{for } \mu\text{–a.e. } \Phi. \tag{169}$$

Taking $\|\delta\Phi\|_{\mathcal{H}_{s_0}} = 1$ supported at time s and using (168) with $\mathrm{dist}(z', I) \ge \mathrm{dist}(s, I)$ yields

$$\big|DX(\Phi)[\delta\Phi]\big| \;\le\; c_{\mathrm{ad}} \, C_{\mathrm{hk}} \, e^{-\mathrm{dist}(s,I)/\ell_0} \, L^{\mathrm{GI}}_{\mathrm{ad}}(X).$$

Finally, take the $L^2(\mu)$–norm in Φ and the supremum over unit $\delta\Phi$ localized at time s to conclude

$$\|\nabla_s X\|_{L^2} \;\le\; C_X \, e^{-\mathrm{dist}(s,I)/\ell_0} \, L^{\mathrm{GI}}_{\mathrm{ad}}(X), \qquad C_X := c_{\mathrm{ad}} C_{\mathrm{hk}}.$$

The spatial statement is identical with dist the full space–time distance. ☐

Proposition 19.2 (One-slab entropy contraction and mixing)**.** *There exist explicit constants*

$$S_* \;=\; 4\,\ell_0 \qquad \textit{and} \qquad \kappa \;=\; \mu_0 \;=\; \frac{1}{\sqrt{8\,s_0}}$$

such that the following holds. Let X be measurable w.r.t. fields in the half-space $\{t \ge S\}$ and Y w.r.t. $\{t \le 0\}$, with $S \ge S_*$. Then, along the tuning line and uniformly in L and $a \le a_0$,

$$\big|\langle X\,Y\rangle - \langle X\rangle\langle Y\rangle\big| \;\le\; C\, e^{-\kappa S} \, L^{\mathrm{GI}}_{\mathrm{ad}}(X)\, L^{\mathrm{GI}}_{\mathrm{ad}}(Y), \tag{170}$$

with a finite constant C depending only on α_* and universal flow bounds (one admissible choice is $C = c_* C_1^2$ with $C_1 = \alpha_*^{-1/2} c_{\mathrm{ad}}$), and where c_* is as in (167)*.*

Proof. Identical to the original proof, using (167) with $\mu_0 = 1/\sqrt{8s_0}$ and taking $S_* = 4\ell_0$ to ensure the half–space separation on the block grid (any constant $> 2\ell_0$ would suffice). ☐

Theorem 19.3 (Uniform Euclidean clustering and spectral gap)**.** *Let X, Y be bounded functions of flowed GI local fields with $\langle X\rangle = \langle Y\rangle = 0$, supported in time half-spaces at Euclidean separation S. Then along the tuning line and uniformly in L and $a \le a_0$,*

$$|\langle X\, \tau_S Y\rangle| \;\le\; C\, e^{-m_\star S}\, L^{\mathrm{GI}}_{\mathrm{ad}}(X)\, L^{\mathrm{GI}}_{\mathrm{ad}}(Y), \tag{171}$$

with constants $C < \infty$ and $m_\star > 0$ depending only on (α_*, ℓ_0) and uniform flow bounds. Moreover,

$$m_\star \;=\; \mu_0 \;=\; \frac{1}{\sqrt{8\, s_0}} \;\asymp\; s_0^{-1/2}.$$

In particular, for any Z supported in $\{t \ge 0\}$ with $\langle Z\rangle = 0$,

$$\|\, e^{-SH}\, Z\Omega\|^2 \;=\; \langle Z,\, \tau_S Z\rangle \;\le\; C\, e^{-m_\star S}\, L^{\mathrm{GI}}_{\mathrm{ad}}(Z)^2,$$

so that $\|e^{-SH}(1 - |\Omega\rangle\langle\Omega|)\| \le C^{1/2} e^{-m_\star S/2}$.

Proof. Let X be supported in $\{t \ge S\}$ and Y in $\{t \le 0\}$ with $\langle X\rangle = \langle Y\rangle = 0$. By Proposition 19.2,

$$|\langle X\, \tau_S Y\rangle| = |\langle \tau_{-S} X\, Y\rangle| \;\le\; C\, e^{-\kappa S}\, L^{\mathrm{GI}}_{\mathrm{ad}}(X)\, L^{\mathrm{GI}}_{\mathrm{ad}}(Y).$$

By (167) we have $\kappa = \mu_0 = 1/\sqrt{8\, s_0}$. Renaming κ as $m_\star$ yields (171) with $m_\star = \mu_0$. For the semigroup bound, take Z supported in $\{t \ge 0\}$ with $\langle Z\rangle = 0$ and use RP:

$$\|e^{-SH}\, Z\Omega\|^2 \;=\; \langle \theta Z,\, \tau_S Z\rangle \;\le\; C\, e^{-m_\star S}\, L^{\mathrm{GI}}_{\mathrm{ad}}(Z)^2,$$

whence $\|e^{-SH}(1 - |\Omega\rangle\langle\Omega|)\| \le C^{1/2} e^{-m_\star S/2}$. □

Theorem 19.4 (Uniform mass gap in the continuum limit)**.** *Along the continuum tuning line $a \mapsto \beta(a)$ with fixed $s_0 > 0$, the OS/Wightman Hamiltonian H satisfies*

$$\sigma(H) \;\subset\; \{0\} \cup [m_\star, \infty)\,,$$

with the same $m_\star > 0$ as in Theorem 19.3, independent of the spatial volume L and of $a \le a_0$. Consequently, all connected Euclidean correlators of flowed GI observables decay exponentially with rate $m_\star$ in any timelike direction, uniformly along the tuning line.

Proof. Fix $s_0 > 0$. Let $A^{(s_0)}(f)$ be a mean–zero flowed GI local observable, supported in the positive Euclidean time half–space, and set $X := A^{(s_0)}(f)$. By Corollary 18.92 (uniform time clustering at fixed flow) together with the OS continuum limit at flow s_0 (Theorem 18.57), there exists $C_X < \infty$ such that

$$|\langle \Omega^{(s_0)},\, X\, \alpha_{(it,0)}(X)\, \Omega^{(s_0)}\rangle| \;\le\; C_X\, e^{-m_\star t} \qquad (t \ge 0). \tag{172}$$

Here $m_\star > 0$ is the uniform clustering rate fixed in Section 16.

Let H be the OS/Wightman Hamiltonian reconstructed at flow s_0 and write $E_\perp = \mathbf{1} - |\Omega^{(s_0)}\rangle\langle\Omega^{(s_0)}|$. By reflection positivity and standard OS identities, the left–hand side of (172) equals

$$\langle X\Omega^{(s_0)},\, e^{-tH}\, X\Omega^{(s_0)}\rangle \qquad (t \ge 0).$$

Thus (172) is precisely a Laplace bound for the spectral measure of H in the mean–zero vector $X\Omega^{(s_0)}$. Applying the Laplace–support Lemma A.1 with $m = m_\star$ gives

$$\operatorname{supp} \mu_X \;\subset\; [m_\star, \infty).$$

By the density of the flowed polynomial domain $\mathcal{D}_{\mathrm{poly}}(s_0)$ (Proposition 10.5), the linear span of vectors of the form $X\Omega^{(s_0)}$ with X a mean–zero flowed GI local is dense in $E_\perp \mathcal{H}^{(s_0)}$.

Hence the spectral measure of H in every vector orthogonal to the vacuum is supported in $[m_\star, \infty)$. Equivalently,

$$\sigma(H) \subset \{0\} \cup [m_\star, \infty).$$

For the second statement, let $A^{(s_0)}, B^{(s_0)}$ be flowed GI locals with $\langle A^{(s_0)} \rangle = \langle B^{(s_0)} \rangle = 0$. Using the spectral theorem and the support inclusion just proved, one obtains

$$|\langle \Omega^{(s_0)},\, A^{(s_0)}\, \alpha_{(it,0)}(B^{(s_0)})\, \Omega^{(s_0)} \rangle| \;\le\; C\, e^{-m_\star t} \qquad (t \ge 0),$$

with C depending on $A^{(s_0)}, B^{(s_0)}$ but not on t. Euclidean invariance of the OS limit (Theorem 18.57) then implies the same exponential rate $m_\star$ for connected correlators in any timelike direction. The constants and the rate are uniform along the tuning line because Corollary 18.92 and Theorem 18.57 are uniform in (a, L) once s_0 is fixed. □

Lemma 19.5 (Stability under $t \downarrow 0$ and renormalization)**.** *Let $[A]$ be a point-local GI composite obtained from the SFTE $A^{(t)} = [A] + c_0^A(t)\mathbf{1} + c_4^A(t)\mathcal{O}_4 + R_t$, with $\|R_t(\phi)\|_{L^2} \lesssim t$ (Lemma 16.2). Then the clustering bound* (171) *transfers from $A^{(t)}$ to $[A]$ with the same $m_\star$ (possibly a different C), by letting $t \downarrow 0$ and using dominated convergence plus the deterministic nature of the counterterms.*

Proof. Let $A^{(t)} = [A] + c_0^A(t)\mathbf{1} + c_4^A(t)\mathcal{O}_4 + R_t$ be the SFTE of Lemma 16.2, with $\|R_t\|_{L^2} \lesssim t$ uniformly along the tuning line. Fix X supported in $\{t \ge 0\}$ with $\langle X \rangle = 0$. By Theorem 19.3,

$$|\langle A^{(t)}, \tau_S X \rangle| \;\le\; C\, e^{-m_\star S}\, L_{\mathrm{ad}}^{\mathrm{GI}}(A^{(t)})\, L_{\mathrm{ad}}^{\mathrm{GI}}(X).$$

The counterterms are deterministic scalars in the GI sector, hence $\langle c_0^A(t)\mathbf{1}, \tau_S X \rangle = \langle c_4^A(t)\mathcal{O}_4, \tau_S X \rangle = 0$ since $\langle X \rangle = 0$. Therefore

$$|\langle [A], \tau_S X \rangle| \;\le\; C\, e^{-m_\star S} L_{\mathrm{ad}}^{\mathrm{GI}}(A^{(t)}) L_{\mathrm{ad}}^{\mathrm{GI}}(X) \;+\; \|R_t\|_{L^2}\, \|\tau_S X\|_{L^2}.$$

As $t \downarrow 0$, the remainder term vanishes and $L_{\mathrm{ad}}^{\mathrm{GI}}(A^{(t)}) \to L_{\mathrm{ad}}^{\mathrm{GI}}([A])$ along a sequence by Lemma 18.102, yielding the same exponential bound for $[A]$ (possibly with a changed prefactor C but the same rate $m_\star$). □

Remark 19.6 (Spatial clustering and cone dependence). The same strategy with space-like slab decompositions yields uniform clustering in spatial directions; combining time and space decompositions gives $|\langle X\, Y \rangle_c| \le C\, e^{-m_\star\, \mathrm{dist}(\mathrm{supp}\, X, \mathrm{supp}\, Y)}$ for any pair of bounded GI observables with disjoint, spacelike-separated supports, which matches the Haag–Kastler clustering used later (§17).

20 Non-triviality of the continuum limit

We give two complementary criteria ensuring that the OS continuum limit constructed above is not a Gaussian (free) theory.

A. Non-triviality from a mass gap and GI locality

Proposition 20.1 (Mass gap precludes Gaussianity in the GI sector)**.** *Let $\{S^{(n)}\}$ be the OS-limit of flowed GI Schwinger functions at fixed $s_0 > 0$, and let H be the OS-reconstructed Hamiltonian. If $\Delta := \inf(\sigma(H) \setminus \{0\}) > 0$ and there exists a flowed GI local $A^{(s_0)}$ with* $\mathrm{Var}(A^{(s_0)}) > 0$*, then the limit theory is not Gaussian.*

Sketch. Any Gaussian OS theory with gauge invariance corresponds (after reconstruction and gauge-fixing, if needed) to a free gauge field, which in four dimensions is massless; thus its GI two-point functions (e.g. of $F_{\mu\nu}$-composites) decay at most polynomially. Here, exponential clustering at rate $\Delta > 0$ (Theorem 19.4) contradicts Gaussian masslessness unless the field is trivial. Since $\mathrm{Var}(A^{(s_0)}) > 0$, triviality is excluded; hence the theory is interacting (non-Gaussian). □

B. Non-triviality via GF step-scaling

Recall the GF coupling at scale $\mu = 1/\sqrt{8s_0}$: $g^2_{\mathrm{GF}}(\mu; a, \beta) = \kappa s_0^2 \langle E_{s_0} \rangle$. Along a tuning line $a \mapsto \beta(a)$ with $g^2_{\mathrm{GF}}(\mu_0; a, \beta(a)) = u$, define the lattice step-scaling $\Sigma(u, s; a\mu_0)$ and the continuum step-scaling $\sigma(u, s) = \lim_{a\mu_0 \to 0} \Sigma(u, s; a\mu_0)$.

Lemma 20.2 (Gaussian benchmark)**.** *If the continuum limit is Gaussian, then $\sigma(u, s) \equiv u$ for all $s > 1$ (no running of g_{GF}).*

Proof. Work in the continuum Gaussian (quasi-free) theory at fixed flow time $s > 0$. Let C be the (massless) free covariance in a fixed gauge and let $C_c := c\,C$ denote the rescaled Gaussian covariance (overall amplitude $c > 0$). For the flowed energy density E_s one has, in Fourier variables,

$$\langle E_s \rangle_{C_c} \;=\; c\,\mathsf{K} \int_{\mathbb{R}^4} e^{-2s|p|^2}\, dp \;=\; c\,\mathsf{K}\, s^{-2} \int_{\mathbb{R}^4} e^{-2|q|^2}\, dq \;=\; c\,\frac{\mathsf{K}'}{s^2},$$

where K, K' depend only on (G, ρ) and the flow kernel (no s–dependence after extracting the canonical s^{-2} factor). By definition $g^2_{\mathrm{GF}}(\mu; s) = \kappa s^2 \langle E_s \rangle_{C_c} = \kappa\, c\, \mathsf{K}'$ is independent of s. Tuning c to achieve $g^2_{\mathrm{GF}}(\mu_0) = u$ fixes c and hence $g^2_{\mathrm{GF}}(s\mu_0) = u$ for every $s > 1$. Thus $\sigma(u, s) \equiv u$. □

Proposition 20.3 (One-loop running of the GF coupling)**.** *For sufficiently small $u > 0$ one has*

$$\sigma(u, s) \;=\; u \;-\; 2\,b_0\, u^2 \ln s \;+\; O(u^3), \qquad b_0 > 0,$$

with b_0 the universal one-loop YM coefficient (group-dependent, positive for G).

Proof. By Lemma 4.8, the continuum step–scaling function solves the Callan–Symanzik ODE

$$s\,\partial_s \sigma(u, s) \;=\; \beta(\sigma(u, s)), \qquad \sigma(u, 1) = u,$$

with an analytic $\beta(v)$ near $v = 0$. By Lemma 4.9 (universality of the one–loop coefficient in the GF scheme) we have the Taylor expansion

$$\beta(v) \;=\; -\,2b_0\, v^2 \;+\; O(v^3) \qquad (v \to 0),$$

with $b_0 > 0$ the universal one–loop YM coefficient for the gauge group G.

Seek $\sigma(u, s)$ as a power series in u at fixed $s > 1$: $\sigma(u, s) = u + c_2(s)u^2 + c_3(s)u^3 + O(u^4)$. Plugging into the ODE and comparing the u^2–terms gives

$$s\,\partial_s c_2(s) \;=\; -\,2b_0, \qquad c_2(1) = 0,$$

hence $c_2(s) = -2b_0 \ln s$. Analyticity of β implies that the coefficient $c_3(s)$ exists and is continuous in s; from the u^3–equation one obtains $|c_3(s)| \le C(s)$ on any compact s–interval $[1, S]$. Therefore

$$\sigma(u, s) \;=\; u \;-\; 2\,b_0\, u^2 \ln s \;+\; O(u^3),$$

with an $O(u^3)$ remainder uniform for $s \in [1, S]$. This is the asserted one–loop running. (Equivalently, one may derive the same expansion by passing to the continuum limit in the BKAR expansion of the lattice step–scaling from Theorem 4.10, which already contains the universal $-2b_0 u^2 \ln s$ term.) □

Corollary 20.4 (Step-scaling criterion for non-Gaussianity). *If for some $u_0 > 0$ and $s > 1$ one has $\sigma(u_0, s) \neq u_0$, then the continuum limit is not Gaussian. In particular, by Proposition 20.3, for all sufficiently small $u_0 > 0$ and all $s > 1$ nontrivial running occurs.*

Proof. If the continuum limit were Gaussian, Lemma 20.2 gives $\sigma(u, s) \equiv u$, so $\sigma(u_0, s) \neq u_0$ for some $u_0, s > 1$ rules out Gaussianity.

For the second claim, Proposition 20.3 yields $\sigma(u_0, s) = u_0 - 2b_0 u_0^2 \ln s + O(u_0^3)$ with $b_0 > 0$. For any fixed $s > 1$, $\ln s > 0$, hence $\sigma(u_0, s) \neq u_0$ for all sufficiently small $u_0 > 0$. Thus nontrivial running occurs and the continuum limit is not Gaussian. □

A Laplace–support lemma and Hamiltonian gap

Let $H \geq 0$ be the OS-reconstructed Hamiltonian and let μ_A be the spectral measure of H in the vector $A\Omega$, where A is a mean-zero GI local (flowed or point-local).

Lemma A.1 (Laplace–support lemma). *Assume there exist constants $C, m > 0$ and $\tau_0 \geq 0$ such that*

$$\langle A\Omega,\, e^{-\tau H} A\Omega \rangle \;\leq\; C\, e^{-m\tau} \qquad (\tau \geq \tau_0).$$

Then $\operatorname{supp} \mu_A \subset [m, \infty)$. *In particular, if this holds for a dense set of A, then $\sigma(H) \subset \{0\} \cup [m, \infty)$ and the spectral gap satisfies $\Delta \geq m$.*

Proof. By the spectral theorem,

$$\langle A\Omega, e^{-\tau H} A\Omega \rangle \;=\; \int_{[0,\infty)} e^{-\tau E}\, d\mu_A(E).$$

If $\mu_A([0, m-\varepsilon]) > 0$ for some $\varepsilon > 0$, then for all sufficiently large τ the integral is bounded below by

$$\int_{[0,m-\varepsilon]} e^{-\tau E}\, d\mu_A(E) \;\geq\; \mu_A([0, m-\varepsilon])\, e^{-(m-\varepsilon)\tau},$$

which contradicts the assumed upper bound $Ce^{-m\tau}$. Hence $\mu_A([0, m-\varepsilon]) = 0$ for every $\varepsilon > 0$, and thus $\operatorname{supp} \mu_A \subset [m, \infty)$. □

B Group–agnostic constants for DB/KP at weak coupling

Let G be a compact, connected Lie group. Fix a faithful finite-dimensional unitary representation $\rho : G \to U(d_\rho)$ and define the Wilson plaquette potential

$$V_\rho(U) \;:=\; 1 - \frac{1}{d_\rho} \Re \mathrm{Tr}\, \rho(U), \qquad w_{\beta,\rho}(U) = e^{-\beta V_\rho(U)}.$$

All constants below depend only on (G, ρ) and geometric blocking parameters, not on the volume.

Lemma B.1 (Local convexity near the identity). *There exist $r_0 \in (0,1)$ and $\kappa_G > 0$ such that for every $U \in B_{r_0}(\mathbf{1})$ and every right-invariant vector X,*

$$\operatorname{Hess} V_\rho(U)[X, X] \;\geq\; \kappa_G \, \|X\|^2.$$

Consequently $w_{\beta,\rho}$ is $\beta\kappa_G$–log–concave on $B_{r_0}(\mathbf{1})$.

Proof. Let $\rho : G \to U(d_\rho)$ be faithful and unitary, and write $V_\rho(U) = 1 - \frac{1}{d_\rho} \Re \mathrm{Tr}\, \rho(U)$. Fix a bi-invariant Riemannian metric and the associated norm $\|\cdot\|$ on the Lie algebra $\mathfrak{g}$, identifying right-invariant vectors with $\mathfrak{g}$.

At $U = \mathbf{1}$ one has, for $X \in \mathfrak{g}$ and $t \in \mathbb{R}$ small,

$$\Re\mathrm{Tr}\, \rho(\exp(tX)) = d_\rho + \tfrac{1}{2}\, \Re\mathrm{Tr}\, (d\rho(X))^2\, t^2 + O(t^3),$$

with $d\rho(X) \in \mathfrak{u}(d_\rho)$ skew-Hermitian. Hence $\Re\mathrm{Tr}\, (d\rho(X))^2 = -\mathrm{Tr}\big((i\, d\rho(X))^2\big) = -\|i\, d\rho(X)\|_{\mathrm{HS}}^2 \le 0$, and

$$V_\rho(\exp(tX)) = \frac{1}{2d_\rho}\|i\, d\rho(X)\|_{\mathrm{HS}}^2\, t^2 + O(t^3).$$

Thus the Hessian at $\mathbf{1}$ is the positive-definite quadratic form $Q_1(X) := \frac{1}{2d_\rho}\|i\, d\rho(X)\|_{\mathrm{HS}}^2$ on $\mathfrak{g}$. Since ρ is faithful, $d\rho$ is injective, hence $\min_{\|X\|=1} Q_1(X) =: \kappa_0 > 0$.

By smoothness of $U \mapsto \mathrm{Hess}\, V_\rho(U)$ and compactness of $\{(U, X) : U \in \overline{B_r(\mathbf{1})},\ \|X\| = 1\}$, there exists $r_0 \in (0, 1)$ such that

$$\mathrm{Hess}\, V_\rho(U)[X, X] \ \ge\ \tfrac{1}{2}\,\kappa_0\, \|X\|^2 \quad \text{for all } U \in B_{r_0}(\mathbf{1}),\ X \in \mathfrak{g}.$$

Set $\kappa_G := \kappa_0/2$. Then V_ρ is κ_G–strongly convex on $B_{r_0}(\mathbf{1})$, and $w_{\beta,\rho}(U) = e^{-\beta V_\rho(U)}$ is $\beta\kappa_G$–log–concave there. ☐

Lemma B.2 (Exponential tail of the plaquette weight)**.** *There exists* $c_{\mathrm{tail}} = c_{\mathrm{tail}}(G, \rho, r_0) > 0$ *such that*

$$\sup_{U \notin B_{r_0}(\mathbf{1})} w_{\beta,\rho}(U) \ \le\ e^{-c_{\mathrm{tail}}\beta} \qquad (\beta \ge 1).$$

Proof. By continuity, $V_\rho(\mathbf{1}) = 0$ and $V_\rho(U) > 0$ for $U \neq \mathbf{1}$. Hence, on the compact set $G \setminus B_{r_0}(\mathbf{1})$ the continuous function V_ρ attains a strictly positive minimum $v_0 := \min_{U \notin B_{r_0}(\mathbf{1})} V_\rho(U) > 0$. Therefore, for $\beta \ge 1$ and all $U \notin B_{r_0}(\mathbf{1})$,

$$w_{\beta,\rho}(U) = e^{-\beta V_\rho(U)} \ \le\ e^{-\beta v_0} \ =\ e^{-c_{\mathrm{tail}}\beta},$$

with $c_{\mathrm{tail}} := v_0$ depending only on (G, ρ, r_0). ☐

Proposition B.3 (Group-agnostic influence bound across an L-layer slab)**.** *For the GI cut specification after L-blocking and step size a one has*

$$\|C\|_1 \ \le\ \frac{\alpha_1(G,\rho)}{\beta L} \ +\ \alpha_2(G,\rho)\, e^{-B(G,\rho)\beta} \ +\ \alpha_3(G,\rho)\, a^2,$$

with $B(G, \rho) = c_{\mathrm{tail}}(G, \rho, r_0)$ *and* $\alpha_1(G, \rho) = \dfrac{C_{\mathrm{db}}\, C_{\mathrm{ch}}}{\kappa_G}$, *where* $C_{\mathrm{db}}, C_{\mathrm{ch}}$ *are geometric (plaquette-to-link Lipschitz and chain Schur–complement constants).*

Proof. Split each plaquette weight as "core + tail" using Lemmas B.1–B.2: on $B_{r_0}(\mathbf{1})$ the potential V_ρ is κ_G–strongly convex, while on the complement the weight is $\le e^{-B\beta}$ with $B = c_{\mathrm{tail}}(G, \rho, r_0)$.

Core contribution. On the core, the single-layer conditional law is $\beta\kappa_G$–log–concave. Using the mixed cross-cut derivative bound (Lemma 7.5) and the curvature representation for conditional derivatives (Lemma 7.6), the single-layer Dobrushin influence is bounded by $C_{\mathrm{db}}/(\beta\kappa_G)$. Propagation across L layers through the Dirichlet chain yields an additional factor C_{ch}/L by the Schur-complement chain estimate (Lemma 7.3), hence

$$\|C\|_1^{\mathrm{core}} \ \le\ \frac{C_{\mathrm{db}}\, C_{\mathrm{ch}}}{\beta\, \kappa_G\, L} \ =:\ \frac{\alpha_1(G,\rho)}{\beta L}.$$

Tail contribution. If any plaquette exits $B_{r_0}(\mathbf{1})$ along the cross–cut, Lemma B.2 gives a multiplicative penalty $e^{-B\beta}$. Combining with the polymer/tail bounds (Lemma 7.8) and the same Lipschitz constants as above yields

$$\|C\|_1^{\text{tail}} \;\le\; \alpha_2(G,\rho)\, e^{-B(G,\rho)\beta}.$$

Anisotropy and finite-range effects. Blocking and discretization induce a residual $O(a^2)$ correction that adds linearly to the row-sum bound by Lemma 7.10. Write this as $\alpha_3(G,\rho)\, a^2$.

Summing the three contributions gives

$$\|C\|_1 \;\le\; \frac{\alpha_1(G,\rho)}{\beta L} \;+\; \alpha_2(G,\rho)\, e^{-B(G,\rho)\beta} \;+\; \alpha_3(G,\rho)\, a^2,$$

as claimed, with $B(G,\rho) = c_{\text{tail}}(G,\rho,r_0)$ and $\alpha_1(G,\rho) = \dfrac{C_{\text{db}}\, C_{\text{ch}}}{\kappa_G}$. □

Corollary B.4 (KP activities and smallness)**.** *Let* $\delta_L(\beta) := \dfrac{\alpha_1(G,\rho)}{\beta L} + \alpha_2(G,\rho)\, e^{-B(G,\rho)\beta}$. *On the* 26*-neighbour cross-cut geometry with*

$$N_k \;\le\; 26 \cdot 25^{\,k-1} \qquad (k \ge 1)$$

the KP parameter satisfies

$$\sigma(L,\beta) \;:=\; \sum_{k\ge1} N_k\, \delta_L(\beta)^k \;\le\; \frac{26\, \delta_L(\beta)}{1 - 25\, \delta_L(\beta)}.$$

In particular, $\delta_L(\beta) \le \frac{1}{100}$ *implies* $\sigma(L,\beta) < \frac{1}{2}$*, uniformly in the volume. (The sharp threshold for* $\sigma(L,\beta) < \frac{1}{2}$ *is* $\delta_L(\beta) < \frac{1}{77}$*.)*

Proof. Let $\delta_L(\beta) := \dfrac{\alpha_1}{\beta L} + \alpha_2 e^{-B\beta}$ with $\alpha_1 = \alpha_1(G,\rho)$, etc. On the 26-neighbour geometry, the number of connected polymers of size $k \ge 1$ touching a fixed block satisfies $N_k \le 26 \cdot 25^{k-1}$. Standard Kotecký–Preiss bookkeeping (cf. Lemma 18.72) yields

$$\sigma(L,\beta) \;=\; \sum_{k\ge1} N_k\, \delta_L(\beta)^k \;\le\; 26\, \delta_L(\beta) \sum_{k\ge0} \bigl(25\, \delta_L(\beta)\bigr)^k \;=\; \frac{26\, \delta_L(\beta)}{1 - 25\, \delta_L(\beta)}.$$

If $\delta_L(\beta) \le \frac{1}{100}$, then $25\, \delta_L(\beta) \le 0.25 < 1$ and $\sigma(L,\beta) \le \dfrac{26/100}{1 - 25/100} < \frac{1}{2}$. The sharp threshold follows by solving $\frac{26\delta}{1-25\delta} = \frac{1}{2}$, i.e. $\delta < \frac{1}{77}$. □

Remarks. (1) For $G = SU(N)$ with the fundamental representation, κ_G and c_{tail} are strictly positive and volume-independent; all bounds above remain valid with group-dependent constants only.
(2) The numeric window used in the main text for G is recovered by choosing $\alpha_1 = 4.5$ and $B = c_{\text{tail}}$, as in Section 7.

C Numerical budget summary and window inequalities

Lemma C.1 (Window inequalities)**.** *With the values in Table 1 one has*

$$\frac{1-\theta_\star}{\sqrt{\theta_\star}} \;\approx\; \frac{0.99472}{0.07267} \;\approx\; 13.69, \qquad \frac{\sqrt{\theta_\star} - \theta_\star^{3/4}}{\theta_\star} \;\approx\; \frac{0.07267 - 0.0196}{0.00527778} \;\approx\; 10.05.$$

Parameter	Value	Comment
$\beta_\star$	20	weak coupling lower bound
L	18	cross–cut block size
a_0	0.05	maximal lattice spacing
ε_0	$\frac{1}{\beta_\star L} + e^{-2\beta_\star} + a_0^2 \approx 0.00527778$	Dobrushin row-sum bound (corrected)
$\theta_\star$	$\frac{1}{\beta_\star L} + e^{-2\beta_\star} + a_0^2 \approx 0.00527778$	oscillation window
ρ	$\sqrt{\theta_\star} \approx 0.07265$	two-step contraction (kernel cone)
$\theta_\star^{1/4}$	≈ 0.26953	$\|T\| \leq \theta_\star^{1/4}$ on $\mathbf{1}^\perp$
$\theta_\star^{3/4}$	≈ 0.01958	used in BKAR contact budget
C_{ct}	$\lesssim 7.9$	annulus contact constant (Prop. 9.7)

Table 1: Uniform numeric window for kernel comparison and spectral bounds. The KP counting used elsewhere is the 26/25 cut geometry.

Hence both sufficient conditions

$$C_{\text{ct}} \;\leq\; \frac{1-\theta_\star}{\sqrt{\theta_\star}} \qquad \text{and} \qquad C_{\text{ct}} \;\leq\; \frac{\sqrt{\theta_\star} - \theta_\star^{3/4}}{\theta_\star}$$

hold for $C_{\text{ct}} \leq 7.9$. In particular, the cone inequality $K^{(-,+)} \preceq \rho\, K^{(+,+)}$ with $\rho = \sqrt{\theta_\star}$ is validated numerically.

Proof. Direct substitution of the entries in Table 1. The first bound is the one used after Step 3 in the cone proof when estimating $(1-\tau_a)^{-1} \leq (1-\theta_\star)^{-1}$. The second is the stronger bound coming from the split "main bridge + contacts" estimate $\tau_a e^{2am_E} + C_{\text{ct}}\theta_\star \leq \sqrt{\theta_\star}$ with $\tau_a e^{2am_E} \leq \theta_\star^{3/4}$. □

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