Thermodynamic Aspects of Timelike Thin-Shell Collapse in Classical General Relativity

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September 5, 2025

Abstract

Thermodynamic ideas in gravitating systems—linking geometry, horizon area/entropy, and the negative heat capacity of self-gravitating matter—are increasingly used to reason about black-hole physics and late–time dynamics. Motivated by this perspective, this study develops a conservative, fully classical treatment of late-time spherical collapse that relies only on standard GR tools while admitting a clear thermodynamic reading. A timelike thin shell joins a regular constant-curvature (de Sitter) interior to a Schwarzschild or Schwarzschild–de Sitter(SdS) exterior. After a brief formation stage, a post-transient zero-inflow regime is analyzed in which the exterior mass parameter (the ADM mass for $\Lambda_+ = 0$) is time–independent. Casting the Israel junction condition into an effective potential, the analysis yields: (i) a closed-form *sufficient* threshold for outward shell evolution that balances the interior SdS scale against exterior attraction; (ii) bounded scalar curvature invariants throughout the spacetime region covered by the construction; and (iii) a simple, falsifiable, mass-scaled redshift bound for near-shell quasi-trapped modes. No thermodynamic postulates enter the derivations; nonetheless, the late-time kinematics are consistent with standard coarse-graining intuitions (e.g., area scaling) and with the negative heat capacity of self-gravitating systems. Within the classical domain considered here, the premises that typically drive singularity formation and information-loss narratives are not realized, providing a classically consistent scenario for late-time collapse.

Keywords: timelike thin shell; junction geometry; static patch; classical collapse endpoint; Schwarzschild–de Sitter; curvature invariants; negative heat capacity; quasi-trapped modes.

1 Introduction

Thermodynamic aspects of gravitating systems—most notably the relations among geometry, horizon area/entropy, and temperature—have profoundly shaped modern views of strong gravity and black holes [1, 2, 3]. In parallel, it has long been appreciated that self-gravitating systems exhibit *negative heat capacity*, a hallmark of long-range attraction already visible in Newtonian theory [4, 5]. While such ideas are frequently discussed for stationary horizons, their concrete implications for *late-time gravitational collapse* can be assessed within strictly classical general relativity (GR), without recourse to quantum inputs.

Thin-shell methods provide a sharp classical framework: idealized interfaces can be modeled while enforcing exact geometric matching conditions across a hypersurface [6, 8]. In spherically symmetric vacuum exteriors, Birkhoff's theorem fixes the outside geometry to be Schwarzschild or Schwarzschild–de Sitter [7]. On this background, the Israel junction condition admits a mechanical, effective-potential formulation for timelike shells that is well suited to analyzing turning points, local tendencies, and regularity within static patches.

This paper adopts a minimalist setup in spherical symmetry. A *timelike* thin shell of areal radius $R(\tau)$ matches a regular constant-curvature (de Sitter) interior to a Schwarzschild or Schwarzschild–de Sitter exterior. After a brief formation stage, attention is restricted to a post-transient regime with negligible inflow so that the exterior mass parameter (ADM for $\Lambda_+ = 0$) is time-independent. Within this classical setting, the junction equation is cast into an effective-potential form and used to establish three conservative statements: a closed-form *sufficient* threshold for outward shell evolution, boundedness of scalar curvature invariants throughout the covered spacetime region, and a mass-scaled redshift bound that limits long-lived near-shell frequencies as measured at infinity. A separate discussion later in the paper relates these geometric findings to thermodynamic language (e.g., negative heat capacity) as interpretation rather than input.

Scope. The analysis remains strictly within classical GR, spherical symmetry, and timelike shells evolving in static patches; the null (Barrabès–Israel) limit is not employed [9]. Quantum-information aspects of black-hole evaporation are referenced for context only and lie outside the present scope; see [11, 12, 13] for seminal results and reviews.

Organization. Section 2 specifies the geometric setup and conservation laws and derives the effective-potential form of the junction condition. Section 3 establishes a sufficient outward-evolution threshold. Section 4 proves boundedness of curvature invariants in the covered domain. Section 5 derives a mass-scaled frequency bound for near-shell features. Section 6 discusses interpretation, observational handles, and extensions.

2 Geometric Setup and Methods

Exterior (vacuum, spherical symmetry). By Birkhoff's theorem, any spherically symmetric vacuum exterior with cosmological term is Schwarzschild or Schwarzschild–de Sitter:

$$ds_+^2 = -f_+(r)\,dt_+^2 + f_+^{-1}(r)\,dr^2 + r^2 d\Omega^2, \qquad f_+(r) = 1 - \frac{2M}{r} - \frac{\Lambda_+ r^2}{3}, \tag{1}$$

where M is the exterior mass parameter (ADM mass if $\Lambda_+ = 0$). We restrict to the static region $f_+(r) > 0$, consistent with timelike shell motion [7].

Interior (regular constant curvature). We model the interior as a constant-curvature (de Sitter) patch

$$ds_-^2 = -(1 - H^2 r^2)\,dt_-^2 + (1 - H^2 r^2)^{-1}\,dr^2 + r^2 d\Omega^2, \qquad H^2 = \Lambda_-/3 > 0, \tag{2}$$

so that interior curvature scalars are finite and set by Λ_- (e.g. $K_- = 8\Lambda_-^2/3$) [3].

Timelike thin shell and junction condition. The regions are matched across a spherical *timelike* shell at areal radius $R(\tau)$ with surface stress–energy $S^a{}_b = \mathrm{diag}(-\sigma, p, p)$. Israel's condition reads [6, 8]

$$\sqrt{f_+(R) + \dot{R}^2} - \sqrt{f_-(R) + \dot{R}^2} = \kappa(R), \qquad \kappa(R) = 4\pi\sigma(R)\,R. \tag{3}$$

Surface-energy conservation on the shell gives

$$\frac{d(\sigma A)}{d\tau} + p\,\frac{dA}{d\tau} = \Phi\,A, \qquad A = 4\pi R^2, \tag{4}$$

with Φ the net normal energy flux across the shell.

Post-transient zero-inflow regime. After a short formation stage we impose

$$T^r{}_t\big|_{\text{shell}} = 0 \quad \Longrightarrow \quad dM \simeq 0, \qquad R_S = 2M = \text{const}, \tag{5}$$

so the exterior mass parameter is time-independent and the dynamics focuses on $R(\tau)$ and Λ_-.

Domain and conventions. We work in static patches with $f_\pm(R) > 0$. Units $G = c = \hbar = k_B = 1$, signature $(-+++)$. Eliminating $\dot{R}$ from (3) yields the effective-potential form [8]

$$\dot{R}^2 + V(R) = 0, \qquad V(R) = f_-(R) - \frac{\big(f_+(R) - f_-(R) - \kappa(R)^2\big)^2}{4\,\kappa(R)^2}. \tag{6}$$

Turning points satisfy $V(R_*) = 0$ and local tendencies are set by $V'(R_*)$.

For sign control it is convenient to use a linear surface equation of state $p = w\sigma$ with $w > -1/2$ and $\Phi = 0$. Then (4) reduces to

$$\frac{d\sigma}{dR} = -\frac{2(\sigma + p)}{R}, \qquad \Longrightarrow \qquad \kappa'(R) = \frac{d}{dR}\big(4\pi\sigma R\big) = 4\pi\sigma(-1 - 2w) \; < \; 0. \tag{7}$$

3 A Sufficient Outward-Evolution Threshold

Balancing interior de Sitter acceleration ($\sim H^2 R$) against exterior attraction ($\sim M/R^2$) singles out

$$R_{\text{thr}} := \left(\frac{3M}{\Lambda_-}\right)^{1/3}. \tag{8}$$

Proposition 1 (Sufficient outward evolution at zero inflow)**.** *Assume $\Phi = 0$ (so that M is time-independent), finite surface stresses with $p = w\sigma$ and $w > -1/2$, and $f_\pm(R) > 0$ (static patches). If for some τ_0 the shell satisfies $R(\tau_0) \geq R_{\text{thr}}$, then $\dot{R}(\tau_0) \geq 0$ and the shell evolves outward on a finite interval beyond τ_0.*

Proof (sketch). Differentiate (6) and evaluate at a point with $H^2R^3 = M$ (for $\Lambda_+ = 0$). One has

$$f'_+(R) = \frac{2M}{R^2}, \qquad f'_-(R) = -2H^2R, \qquad \Delta f := f_+ - f_- = -\frac{2M}{R} + H^2R^2, \quad \Delta f' = \frac{2M}{R^2} + 2H^2R = 4H^2R > 0,$$

using $H^2R^3 = M$. Writing $V' = f'_- - \frac{\partial}{\partial R}\left[\frac{(\Delta f - \kappa^2)^2}{4\kappa^2}\right]$ and using $\kappa'(R) < 0$ from (7), one finds $V'(R_{\text{thr}}) < 0$ within the static domain $f_\pm > 0$. Hence $-V$ increases for $R > R_{\text{thr}}$ in a neighborhood of R_{thr}, implying outward evolution on a finite interval when $\dot{R}(\tau_0) \geq 0$. □

Remark 1 (Dimensionless form and static-patch consistency)**.** *Let $\chi = \Lambda_- M^2$. Then*

$$\frac{R_{\text{thr}}}{R_S} = \frac{1}{2}\left(\frac{3}{\chi}\right)^{1/3}.$$

Timelike consistency requires $R_{\text{thr}} > R_S$ and $HR_{\text{thr}} < 1$ (and, if $\Lambda_+ > 0$, R_{thr} below the cosmological horizon), i.e. $f_\pm(R) > 0$ throughout [7].

4 Boundedness of Curvature Scalars in the Covered Domain

Proposition 2 (Bounded curvature)**.** *Let $R(\tau) \geq R_{\min} > 0$, $f_\pm(R) > 0$, and $\sigma(\tau), p(\tau)$ finite. Then all scalar curvature invariants are finite everywhere in the spacetime covered by the construction.*

Proof (sketch). The interior is constant curvature (e.g. $K_- = 8\Lambda_-^2/3$). In the exterior, for $r \geq R(\tau) \geq R_{\min}$, the Schwarzschild/SdS Kretschmann scalar

$$K_+(r) = \frac{48M^2}{r^6} + \frac{8}{3}\Lambda_+^2$$

is bounded by $48M^2/R_{\min}^6 + 8\Lambda_+^2/3$. At the shell, the induced metric is continuous and curvature is distributional via Israel's condition; scalar invariants remain finite for finite σ, p [6, 8]. Since $R(\tau) > R_S$ and we work in static patches, no exterior trapping surface intersects the covered spacetime [7]. □

5 A Mass-Scaled Frequency Bound for Near-Shell Modes

A minimal near-shell storage criterion for quasi-trapped modes is $k_{\text{loc}}R \gtrsim \xi$ with $\xi = \mathcal{O}(1)$. Tolman redshift gives $\omega_\infty = \sqrt{f_+(R)}\,\omega_{\text{loc}}$ [3], hence for $f_c = \omega_\infty/(2\pi)$:

$$f_c\, R_S = \frac{\xi}{2\pi}\, \frac{\sqrt{1 - 2M/R - \Lambda_+ R^2/3}}{R/R_S}\,. \tag{9}$$

For a Schwarzschild exterior ($\Lambda_+ = 0$) the right-hand side is maximized at $R = \frac{3}{2}R_S$, giving

$$\boxed{f_c\, R_S \;\leq\; \frac{\xi}{3\sqrt{3}\,\pi}}\,. \tag{10}$$

A robust, persistent feature that violates (10) would falsify this classical near-shell storage picture under the assumptions stated above.

6 Discussion and Outlook

Classical content. Within classical GR and spherical symmetry, with a timelike thin shell evolving in static patches $f_\pm > 0$, a constant–curvature interior, and a post–transient zero–inflow regime ($dM \simeq 0$), the analysis establishes three points. First, a closed–form *sufficient* threshold for outward evolution, Eq. (8), arises from balancing interior de Sitter acceleration against exterior attraction. Second, scalar curvature invariants remain bounded throughout the spacetime region covered by the construction (Proposition 2), and no exterior trapping surface intersects that domain. Third, a mass–scaled redshift bound limits persistent near–shell frequencies as observed at infinity, Eq. (10). These statements follow directly from the junction geometry and standard conservation on the shell; no additional dynamical input beyond classical GR is required.

Thermodynamic interpretation (auxiliary). Thermodynamic language can provide intuition for the late–time regime while remaining distinct from the derivations. In a setting with fixed exterior mass parameter, the shell radius $R(\tau)$ serves as a macroscopic state variable. An area–scaling interpretation for coarse–grained entropy, $S \propto R^2$, then phrases outward motion as $dS \propto dR$. This picture is consistent with the negative heat capacity exhibited by self–gravitating systems [4, 5] and with Tolman redshift considerations near static configurations [3]. No fundamental statistical entropy is derived here; the thermodynamic reading is interpretive and not used in proofs.

Observational handle and falsifiability. The mass–scaled inequality in Eq. (10) furnishes a null test: given an independent estimate of M, any long–lived near–shell spectral feature measured at infinity should respect the bound. A robust, systematic violation would falsify the near–shell storage picture under the assumptions stated (timelike shell in static patches, negligible inflow, time–independent exterior mass). The bound also suggests a practical template for targeted searches formulated directly in the mass–scaled variable $f_c R_S$.

Scope and limitations. The statements are confined to classical GR, spherical symmetry, timelike shells restricted to static patches, finite surface stresses (e.g., linear $p = w\sigma$ with $w > -1/2$), and the absence of the null (Barrabès–Israel) limit [9]. The threshold in Eq. (8) is *sufficient* rather than necessary. Strong time dependence, rotation, nonspherical perturbations, and sustained inflow fall outside the present analysis.

Outlook. Natural extensions include: (i) axisymmetric generalizations with Kerr or Kerr–de Sitter exteriors; (ii) perturbative stability of the outward threshold against nonspherical fluctuations; (iii) controlled approaches to the null limit for rapidly evolving phases; (iv) nonzero inflow with evolving $M(t)$ and its impact on the effective potential and on the frequency bound; and (v) alternative regular interiors to quantify model dependence within the same junction framework. Each direction tests robustness of the classical statements developed here and connects them to observational strategies based on mass–scaled frequency constraints.

Quantum–information context (outside present scope). Discussions of black–hole evaporation and information in quantum settings lie beyond the classical framework adopted here; see [11, 12, 13] for seminal results and reviews.

Appendix A: Stationary Thin Shell and Tolman Potential

In a static geometry with Killing field ξ^a and lapse $N = \sqrt{-\xi^2}$, the Tolman relation gives $T\,N = \text{const}$ and $\theta := \ln T = -\ln N + \text{const}$ [3, 10]. In the Newtonian limit, $\nabla\theta$ reproduces the gravitational field. For a static thin shell, θ is continuous across the shell while its normal derivative jumps by a surface term obtained from a pillbox integration of $\nabla^2\theta = -(4\pi G/c^2)\rho$, reproducing the thin-sheet jump (potential continuous, normal field discontinuous). In full GR this coincides with the static/Newtonian limit of Israel's condition [6, 8] and introduces no new dynamics.

Appendix B: Proof of Proposition 1

Differentiate $V(R)$ in (6). At $H^2R^3 = M$ (for $\Lambda_+ = 0$),

$$f'_+ = \frac{2M}{R^2}, \quad f'_- = -2H^2R, \quad \Delta f = f_+ - f_- = -\frac{2M}{R} + H^2R^2, \quad \Delta f' = \frac{2M}{R^2} + 2H^2R = 4H^2R > 0.$$

Writing $V' = f'_- - \frac{\partial}{\partial R}\left[\frac{(\Delta f - \kappa^2)^2}{4\kappa^2}\right]$ and using $\kappa'(R) < 0$ from (7), one finds $V'(R_{\text{thr}}) < 0$ throughout the static domain $f_\pm > 0$. Therefore $-V$ increases for $R > R_{\text{thr}}$ in a neighborhood of R_{thr}, implying outward evolution on a finite interval once $\dot{R}(\tau_0) \geq 0$. Throughout this section, physically admissible shells with $\sigma > 0$ are assumed, so that for $w > -1/2$ one indeed has $\kappa'(R) < 0$.

Appendix C: Proof of Proposition 2

Interior: constant curvature implies finite invariants, e.g. $K_- = 8\Lambda_-^2/3$. Exterior: for Schwarzschild/SdS,

$$K_+(r) = \frac{48M^2}{r^6} + \frac{8}{3}\Lambda_+^2,$$

hence bounded for $r \geq R_{\min}$. Across the shell, the induced metric is continuous and distributional curvature is controlled by Israel's condition with finite σ, p [6, 8]. Since $R(\tau) > R_S$ and we work in static patches, no exterior trapping surface intersects the covered spacetime [7].

Appendix D: Derivation of the Frequency Bound

We assume units $G = c = \hbar = k_B = 1$. A minimal near–shell storage criterion for quasi–trapped modes is

$$k_{\rm loc}\, R \;\gtrsim\; \xi, \qquad \xi = \mathcal{O}(1). \tag{11}$$

With $\omega_{\rm loc} \simeq k_{\rm loc}$ and Tolman redshift $\omega_\infty = \sqrt{f_+(R)}\,\omega_{\rm loc}$, the observed (cyclic) frequency $f_c = \omega_\infty/(2\pi)$ satisfies

$$f_c \;=\; \frac{1}{2\pi}\,\sqrt{f_+(R)}\,\omega_{\rm loc} \;\gtrsim\; \frac{1}{2\pi}\,\sqrt{f_+(R)}\,\frac{\xi}{R}. \tag{12}$$

Multiplying by $R_S = 2M$ and writing the exterior lapse $f_+(R) = 1 - \dfrac{2M}{R} - \dfrac{\Lambda_+ R^2}{3}$, we obtain the general mass–scaled expression

$$f_c\, R_S \;\gtrsim\; \frac{\xi}{2\pi}\,\frac{\sqrt{1 - \dfrac{2M}{R} - \dfrac{\Lambda_+ R^2}{3}}}{R/R_S}. \tag{13}$$

Schwarzschild exterior ($\Lambda_+ = 0$). Let $x := R/M > 2$ so that $R/R_S = x/2$. Define

$$g(x) := \frac{\sqrt{1 - \dfrac{2}{x}}}{x/2} \;=\; \frac{2}{x}\sqrt{1 - \frac{2}{x}}, \qquad x > 2. \tag{14}$$

Then (13) reads $f_c\, R_S \gtrsim \dfrac{\xi}{2\pi}\, g(x)$. A straightforward maximization gives $g'(x) = 0 \Leftrightarrow x = 3$, and

$$g(3) = \frac{2}{3}\sqrt{1 - \frac{2}{3}} = \frac{2}{3\sqrt{3}}.$$

Therefore,

$$f_c\, R_S \;\lesssim\; \frac{\xi}{2\pi}\, g(3) \;=\; \frac{\xi}{3\sqrt{3}\,\pi}, \tag{15}$$

which is the bound (10) in the main text.

Including a cosmological term. For $\Lambda_+ > 0$, the same recipe applies on the static patch $f_+(R) > 0$. Since the extra $-\Lambda_+ R^2/3$ lowers $\sqrt{f_+(R)}$ at large R, the Schwarzschild value is an upper envelope; any $\Lambda_+ > 0$ tightens the bound.

Interpretation. Given an independent mass estimate M, any persistent late–time feature localized near the shell and measured at infinity must obey the mass–scaled inequality above. A robust violation would falsify the near–shell storage picture within the assumptions of the main text.

Author Contributions

Conceptualization, Methodology, Formal analysis, Writing—original draft, Writing—review & editing: A.Schubert.

Funding

This research received no external funding.

Data Availability

No new data were created or analyzed in this study.

Acknowledgments

The author thanks colleagues for discussions on thin shells and classical energy accounting.

Conflicts of Interest

The author declares no conflict of interest.

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