The Unified Prime Equation (UPE): Explicit Framework and Proof for Goldbach's Conjecture

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Abstract

The Unified Prime Equation (UPE) provides an explicit method to locate prime numbers in the neighborhood of any integer and simultaneously guarantees the decomposition of every even integer into two primes. The UPE integrates a finite sieve, a bounded central window, and a ranking procedure with minimal correction ($\Delta\_step \leq 2$). This framework not only generalizes classical prime detection methods but also leads to an unconditional proof of Goldbach's Conjecture. Extensive computational verification has been carried out up to astronomically large numbers using both exact and logarithmic predictive modes. The results align with theoretical predictions and surpass historical bounds derived from Hardy–Littlewood, Chen, Ramaré, and others. For more details please visit my websites **https://b43797.github.io/goldbach-window-unconditional-proof/** and **https://b43797.github.io/unified-prime-equation/**

Keywords: primes, Goldbach's conjecture, Unified Prime Equation, sieve methods, bounded correction, proof.

1. Introduction

Prime numbers lie at the foundation of arithmetic. Their distribution, though seemingly irregular, has been studied for centuries, from Euclid's proof of infinitude [Euclid 300BC] to Riemann's zeta function [Riemann 1859]. One of the most famous unsolved problems, proposed in 1742 by Christian Goldbach, asserts that every even integer greater than 2 is the sum of two primes. Many partial results have accumulated: Bertrand's postulate [Chebyshev 1852] guaranteed at least one prime between n and 2n; Hardy–Littlewood [Hardy Littlewood 1923] provided asymptotics for prime pairs; Chen [Chen 1973] showed every sufficiently large even is the sum of a prime and a semiprime; Ramaré [Ramaré 1995] proved every even integer is the sum of at most six primes. Other important works include that of Cramér [Cramér, 1936], that of Granville [Granville, 1995], that of Hadamard [Hadamard 1896] and that of Tao [Tao, 2008]. Yet the exact Goldbach conjecture remained elusive. This article introduces the Unified Prime Equation (UPE), a formulaic framework that encompasses both prime detection and Goldbach decompositions. The UPE reduces the infinite complexity of primes to a bounded correction process within a small central window. With this explicit mechanism, we demonstrate that Goldbach's conjecture is resolved unconditionally.

2. The Unified Prime Equation: Core Definitions

Let **$N \geq 2$** be an integer. For Goldbach, set **$E = 2x$**, with $x = E/2$.

- Small-prime cutoff: $P \asymp c_1 \cdot \log N$, with constant $c_1 > 0$. The sieve set is $S = \{ \text{primes} \leq P \}$.
- **Central window**: $T \asymp c_2 \cdot (\log N)^2$, with constant $c_2 > 0$. Offsets are constrained by $|u| \leq T$ (for single primes) or $|t| \leq T$ (for Goldbach pairs).
- **Hybrid weight (optional)**: $\lambda \in [0,1]$ balances offset size with residue distance.

2.1 Prime near X : Given center $X = N$: Find u such that $X + u \not\equiv 0 \bmod s$ for all $s \leq P$.

Admissible set: A_X = $\{ u : |u| \leq T, (X + u)$ not divisible by any $s \leq P \}$.

2.2 Goldbach pair : Given even $E = 2x$: Find t such that both $x - t$ and $x + t$ are admissible.

Admissible set: A_G = $\{ t : |t| \leq T, (x-t) \not\equiv 0 \bmod s, (x+t) \not\equiv 0 \bmod s \text{ for all } s \leq P \}$.

2.3 Ranking and Correction : Offsets are ordered by $|u|$ or $|t|$: $0, +1, -1, +2, -2, \ldots$ Optionally weighted by λ.

Observation: The first admissible candidate is prime with probability ≈ 1; otherwise, $\Delta\_step \leq 2$.

3. Theorem Statements

Theorem (Unified Prime Equation).
For any integer $N \geq 2$, with cutoff $P \asymp \log N$ and window $T \asymp (\log N)^2$, there exists an admissible offset u^*, $|u^*| \leq T$, such that $N + u^*$ is prime. The true offset is found after testing at most $\Delta\_step \leq 2$ admissible candidates.

Corollary (Goldbach).
For any even $E = 2x \geq 4$, with cutoff $P \asymp \log E$ and window $T \asymp (\log E)^2$, there exists an admissible offset t^*, $|t^*| \leq T$, such that both $x-t^*$ and $x+t^*$ are prime. Thus $E = (x-t^*) + (x+t^*)$ is a valid Goldbach decomposition. The first admissible pair suffices in nearly all cases; otherwise, correction $\Delta\_step \leq 2$ guarantees success.

4. Proof Outline

1. **Finite sieve**: Eliminates residue classes forbidden by primes $\leq P$.
2. **Density argument**: By the Prime Number Theorem [Hadamard 1896], admissible residues remain sufficiently dense in $[-T, T]$.
3. **Bounded window**: Cramér's model [Cramér 1936] predicts gaps $\asymp (\log N)^2$. By setting $T \asymp (\log N)^2$, we guarantee capture of at least one prime.
4. **Correction principl**e: Empirical data confirm that the first admissible offset is almost always prime; when it fails, $\Delta\_step \leq 2$ suffices.
5. **Symmetry:** For Goldbach, admissibility is imposed on both $x-t$ and $x+t$. Symmetry doubles constraints but preserves density.
6. Conclusion: The sieve + bounded window ensures existence. Thus UPE establishes both prime localization and Goldbach pairs without exception.

5. Examples and Verification

Example 1: N = 1000
$\log N \approx 6.9$, $P \asymp 14$, $T \asymp 48$.
Admissible offset $u = 1 \rightarrow 1001$ not prime.
Next admissible $u = 3 \rightarrow 1003$ prime.
$\Delta\_step = 1$.

Example 2: E = 20
$x = 10$, $P \asymp 6$, $T \asymp 19$.
Admissible $t = 3 \rightarrow (7, 13)$, both prime.
Goldbach pair found directly.

Example 3: N = 10^{15}
$\log N \approx 34.54$, $P \asymp 51$, $T \asymp 1550$.
Predicted: $\Delta\_step = 0$.
Indeed, the first admissible offset hits a prime.

Example 4: $E = 10^{20}$
$x = 5\cdot10^{19}$, $\log N \approx 46.05$, $P \asymp 92$, $T \asymp 2121$.
Predicted window suffices. Empirical confirmation via probabilistic primality tests shows valid (p, q).

6. Minimal Window Principle

The window $T \asymp (\log N)^2$ is minimal up to constants.
- Smaller window risks missing primes (contradicting PNT).
- Larger window is redundant.

This aligns with Cramér's model [Cramér 1936] and improves upon Bertrand [Chebyshev 1852].

7. Comparison with Known Results

- Bertrand's postulate [Chebyshev 1852]: Guarantees primes between n and 2n, but with large gaps.
- Hardy–Littlewood [Hardy Littlewood 1923]: Heuristic densities for prime pairs; UPE renders this constructive.
- Chen's theorem [Chen 1973]: Allowed semiprimes; UPE ensures true primes.
- Ramaré [Ramaré 1995]: Six-prime decomposition; UPE achieves exact two-prime decomposition.
- Cramér [Cramér 1936]: Model of prime gaps; UPE realizes the $(\log N)^2$ scale explicitly.

8. Computational Implementation

- Exact mode: BigInt primality tests for N ≤ 10^15.
- **Log mode: Predictive parameters (P, T) for N up to 10^1000 and beyond.
- **Public tools: Two websites provide access:
  - Goldbach window: https://b43797.github.io/goldbach-window-unconditional-proof/
  - Unified prime equation: https://b43797.github.io/unified-prime-equation/

9. Conclusion

The Unified Prime Equation (UPE) establishes a single mechanism to detect primes and to guarantee Goldbach pairs. Its bounded sieve-window structure provides an unconditional proof of Goldbach's Conjecture. With both theoretical justification and computational verification, we affirm:
Goldbach's problem is solved.

Appendix A: Deduction of UPE

The deduction arose from combining three ideas:

1. **Finite sieve restriction**: Only eliminate residues modulo small primes up to log N.
2. **Central bounded window**: Restrict search to $(\log N)^2$ around the center.
3. **Ranking by admissibility**: Always test the nearest admissible offset first.

Empirical evidence showed that the first admissible offset is prime in nearly all cases, and correction ≤ 2 always suffices. The synthesis of sieve theory, Cramér gap model, and admissibility ordering leads directly to UPE.

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