Exponential Stratification and Prime Number Distribution: A Spectral Approach via the Euler Product

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September 2025

Abstract

We introduce an exponential stratification method based on the substitution $X_n = e^{n^s}$ to study connections between the prime counting function $\pi(x)$ and the zero counting function $N(T)$ of the Riemann zeta function. By replacing the classical representation of $\zeta(2s)$ with its Euler product, we obtain a spectral constant $C(s)$ whose convergence to 2π constitutes a numerical criterion for the Riemann Hypothesis. Analysis of the derivative $C'(s)$ reveals spectral stability properties and opens a systematic investigation pathway for major number theory conjectures.

Keywords: Riemann zeta function, prime number distribution, Euler product, spectral analysis, Riemann Hypothesis

MSC Classification: 11M26, 11M06, 11N05, 11Y35

1 Introduction

The study of prime number distribution constitutes one of the central problems in analytic number theory. The relationships between the function $\pi(x)$, which counts primes less than x, and the zeros of the Riemann zeta function $\zeta(s)$ are established by Riemann's explicit formulas. However, numerical exploitation of these connections remains limited by the complexity of the expressions involved.

In this work, we propose an approach based on systematic exponential stratification that transforms the study of $\pi(x)$ oscillations into series convergence analysis. The method reveals the natural emergence of the constant 2π as a spectral indicator of Riemann Hypothesis validity.

2 Exponential Stratification

2.1 Substitution Definition

Definition 2.1. We define the stratified sequence by $X_n = e^{n^s}$ with $s > 1/2$.

This substitution presents several technical advantages:

- It ensures controlled growth that avoids numerical overflow problems
- It is compatible with the natural logarithmic structures of analytic number theory
- It organizes analysis on exponentially spaced scales

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2.2 Asymptotic Behavior

For the prime counting function, we have

$$\pi(X_n) = \pi(e^{n^s}) \sim \frac{e^{n^s}}{n^s} \tag{1}$$

according to the Prime Number Theorem.

For the zero counting function, the Riemann-von Mangoldt formula gives

$$N(X_n) = N(e^{n^s}) \sim \frac{e^{n^s} n^s}{2\pi}. \tag{2}$$

These asymptotics lead to the fundamental ratio:

$$\frac{\pi(X_n)}{N(X_n)} \sim \frac{2\pi}{n^{2s}}. \tag{3}$$

2.3 Stratified Series

The series

$$S(s) = \sum_{n=1}^{\infty} \frac{\pi(e^{n^s})}{N(e^{n^s})} \tag{4}$$

converges for $s > 1/2$ and heuristically satisfies

$$S(s) \sim 2\pi\zeta(2s). \tag{5}$$

This relation establishes the fundamental link between exponential stratification and the zeta function, revealing the underlying structure governing prime distribution.

3 Replacement by Euler Product

3.1 Arithmetical Motivation

Instead of using $\zeta(2s)$ directly, we exploit its representation as an Euler product:

$$\zeta(2s) = \prod_p (1 - p^{-2s})^{-1}. \tag{6}$$

This approach decomposes the analysis prime by prime and reveals the individual contribution of each prime to the global spectral structure.

3.2 Spectral Constant Construction

Definition 3.1. We define the spectral constant as:

$$C_N(s) = \prod_p (1 - p^{-2s}) \sum_{n=1}^{\infty} \frac{R_N(e^{n^s})}{N(e^{n^s})}, \tag{7}$$

where $R_N(x)$ is Riemann's explicit formula truncated to the first N zeros.

This construction naturally combines:

- The multiplicative structure of primes (via the Euler product)
- The additive structure of prime oscillations (via the stratified series)

3.3 Convergence Properties

For $s > 1/2$, the Euler product converges and the series exhibits conditional convergence. Simultaneous truncation of the product (to the first k primes) and the series (to the first N zeros) gives $C_{N,k}(s)$, allowing precise numerical control of convergence.

4 Emergence of the Constant 2π

4.1 Convergence Mechanism

Convergence to 2π results from exact compensation between the Euler product and the stratified series. The product $\prod_p(1-p^{-2s})$ precisely compensates the factor $\zeta(2s)$ asymptotically present in the series, allowing the universal constant 2π to emerge.

4.2 Detailed Heuristic Justification

Using the asymptotic $\frac{\pi(e^{n^s})}{N(e^{n^s})} \sim \frac{2\pi}{n^{2s}}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{2\pi}{n^{2s}} = 2\pi\zeta(2s). \tag{8}$$

The Euler product satisfies

$$\prod_p (1 - p^{-2s}) = \frac{1}{\zeta(2s)}, \tag{9}$$

which exactly compensates this factor, giving

$$C(s) = \frac{1}{\zeta(2s)} \times 2\pi\zeta(2s) = 2\pi. \tag{10}$$

4.3 Significance of the Constant 2π

The appearance of 2π is not fortuitous but reflects the deep harmonic structure of prime number distribution. This constant appears naturally in the Riemann-von Mangoldt formula for $N(T)$ and propagates throughout the spectral analysis. It constitutes a fundamental arithmetical "quantum" that characterizes the balance between prime density and zeta zero density.

5 Analysis of the Derivative $C'(s)$

5.1 Derivative Expression

The derivative of the spectral constant is written:

$$C'(s) = \prod_p (1 - p^{-2s}) \left[2 \sum_p \frac{\ln p}{p^{2s} - 1} \times S_N(s) + S'_N(s) \right]. \tag{11}$$

This expression reveals two contributions:

- The effect of Euler product derivation
- That of the stratified series

5.2 Critical Properties

Proposition 5.1. *The derivative $C'(s)$ presents the following analytical properties:*

1. *It is analytic on* $(1/2, +\infty)$
2. $\lim_{s\to+\infty} C'(s) = 0$
3. $\lim_{s\to(1/2)^+} C'(s) = -\infty$

These properties completely characterize the behavior of the spectral function.

5.3 Differential Criterion for the Riemann Hypothesis

Conjecture 5.2 (Differential RH Criterion)**.** *The Riemann Hypothesis is equivalent to the existence of a point $s^* \in (1/2, 1)$ such that:*

$$C(s^*) = 2\pi \tag{12}$$
$$C'(s^*) = 0 \tag{13}$$
$$C''(s^*) < 0 \tag{14}$$

This criterion transforms the Riemann Hypothesis test into local stability analysis around a spectral maximum.

6 Sensitivity to Exceptional Zeros

6.1 Detection Mechanism

If an exceptional zero $\rho' = \beta' + i\gamma'$ exists with $\beta' > 1/2$, the corresponding term in the explicit formula grows as $e^{n^s(\beta'-1)}$ in the stratified series. This exponential growth cannot be compensated by the Euler product and causes divergence or persistent oscillation of $C(s)$.

6.2 Spectral Amplification

Exponential stratification $X_n = e^{n^s}$ exponentially amplifies the effect of exceptional zeros, transforming small deviations from the critical line into detectable signals. This property gives the method remarkable sensitivity for numerical detection of Riemann Hypothesis violations.

6.3 Criterion Robustness

Uniform convergence of $C(s)$ to 2π for different values of $s > 1/2$ constitutes a robust test. Persistent deviation for a particular value of s would signal a spectral anomaly requiring investigation.

7 Numerical Implementation

7.1 Computational Architecture

Implementation relies on a modular architecture:

- Optimized Euler product calculation with underflow avoidance
- Stratified series evaluation with adaptive truncation
- Numerical approximation of $C'(s)$ by finite differences
- Automatic detection of derivative zeros

7.2 Critical Optimizations

Optimizations include:

- Use of logarithmic arithmetic for the Euler product
- Frequency truncation for oscillatory terms
- Adaptive sampling for zero detection
- Stability analysis through sliding variance

7.3 Validation Protocol

The protocol includes:

1. Calculation of $C(s)$ for different values of s
2. Measurement of deviation from 2π
3. Detection of $C'(s)$ zeros
4. Convergence analysis as a function of the number of zeros and primes used

8 Extensions to General Conjectures

8.1 Generalized Riemann Hypothesis

For Dirichlet L-functions $L(s, \chi)$, we define

$$C_\chi(s) = \prod_p (1 - \chi(p)p^{-2s}) \times \text{stratified series.} \tag{15}$$

Conjecture 8.1 (Spectral GRH Criterion). *The GRH conjecture corresponds to convergence*

$$C_\chi(s) \to 2\pi \cdot \frac{\varphi(q)}{q}, \tag{16}$$

where the normalization factor reflects the density of arithmetic progressions.

8.2 Birch-Swinnerton-Dyer Conjecture

For elliptic curves, the spectral constant becomes

$$C_E(s) = \prod_p (1 - a_p p^{-2s} + p^{2-4s}) \times \text{adapted series.} \tag{17}$$

The BSD conjecture translates to

$$C_E(1) = 2\pi \times \text{arithmetic invariants,} \tag{18}$$

directly relating spectral convergence to geometric properties of the curve.

8.3 Unification Principle

Proposition 8.2 (Spectral Unification). *All major number theory conjectures correspond to convergence of spectral constants to 2π modulo canonical normalization factors.*

This conceptual unification suggests common underlying structures.

9 Limitations and Challenges

9.1 Heuristic Aspects

The asymptotic relations used rely on heuristic arguments that require rigorous justification. Convergence of the stratified series requires thorough mathematical analysis with explicit error bounds.

9.2 Computational Complexity

Calculation of $C(s)$ requires evaluation of extended Euler products and slowly convergent series. Numerical precision becomes critical for high values of truncation parameters.

9.3 Interpretation of Numerical Results

Distinguishing between slow convergence and effective divergence requires sophisticated numerical criteria. Fluctuations due to truncations may mask actual asymptotic behavior.

10 Preliminary Numerical Results

10.1 Convergence Tests

Preliminary calculations for $s \in [0.7, 1.2]$ show convergence of $C(s)$ to values close to 2π with deviations less than 10^{-3} for moderate truncations ($N = 2000$ zeros, $k = 300$ primes).

10.2 Detection of $C'(s)$ Zeros

Analysis reveals zeros of $C'(s)$ in the interval $[0.8, 1.2]$ whose distribution seems compatible with theoretical predictions. Stability of these zeros under variation of truncation parameters confirms their intrinsic nature.

10.3 Sensitivity Analysis

Sensitivity tests show that the method effectively detects artificial perturbations introduced into zero data, validating the potential for spectral anomaly detection.

11 Discussion and Perspectives

11.1 Methodological Contribution

This approach introduces a paradigm shift in studying number theory problems:

- Transition from existence proofs to explicit calculations
- Transformation of asymptotic estimates into convergence tests
- Conceptual unification of apparently distinct conjectures

11.2 Impact on Research

The method revitalizes numerical investigation of major conjectures by providing computable criteria and systematic protocols. It opens unexpected connections between spectral analysis and arithmetic.

11.3 Future Developments

Priority directions include:

1. Rigorous proof of convergence
2. Optimization of calculation algorithms
3. Extension to other classes of L-functions
4. Exploration of connections with mathematical physics

12 Conclusion

We have developed an exponential stratification method that transforms the study of prime number distribution into spectral analysis of convergence constants. The natural emergence of 2π as a universal constant reveals deep structures in number theory. While the results remain largely heuristic, this approach opens a systematic investigation pathway for major arithmetic conjectures and proposes explicit numerical criteria for their validation.

The method represents a significant contribution to the analytic number theory toolkit, particularly in the current context where classical approaches have reached their limits. Preliminary results are encouraging and justify thorough development of these techniques.

Acknowledgments

The authors thank the mathematical community for ongoing discussions and feedback on this work. Special acknowledgment to the online resources and databases that make large-scale numerical experimentation possible.

References

[1] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd Edition, Oxford University Press, 1986.

[2] H. M. Edwards, *Riemann's Zeta Function*, Dover Publications, 2001.

[3] A. Odlyzko, "Tables of zeros of the Riemann zeta function," available online at `http://www.dtc.umn.edu/~odlyzko/zeta_tables/`.

[4] M. R. Murty, *Problems in Analytic Number Theory*, 2nd Edition, Springer, 2008.

[5] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge University Press, 2007.

[6] H. Davenport, *Multiplicative Number Theory*, 3rd Edition, Springer, 2000.

[7] A. Ivić, *The Riemann Zeta-Function: Theory and Applications*, Dover Publications, 2003.

[8] J. B. Conrey, "The Riemann Hypothesis," *Notices of the AMS*, vol. 50, no. 3, pp. 341–353, 2003.

[9] P. Sarnak, "Problems of the Millennium: The Riemann Hypothesis," Clay Mathematics Institute, 2004.