The Habiro Purity Conjecture: Toward a Universal Cohomology of $\mathrm{Spec}(\mathbb{Z})$

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Abstract

We propose the *Habiro Purity Conjecture*, a structural statement asserting that Frobenius eigenvalues in Habiro cohomology satisfy a universal purity condition, mirroring Deligne's theorem for varieties over finite fields. Together with the determinantal formalism of Habiro cohomology, this conjecture would imply the Riemann Hypothesis for the Riemann zeta function and its generalizations.

The context is provided by two recent developments. Scholze's Habiro cohomology furnishes a cohomology theory interpolating between q-de Rham, prismatic, crystalline, and motivic theories, valid across all primes simultaneously and defined over the Habiro completion. In parallel, Garoufalidis–Scholze–Wheeler–Zagier introduced the Habiro ring of number fields, a universal coefficient ring encoding Frobenius compatibilities and regulators from algebraic K-theory.

We argue that these two constructions fit together as dual aspects of a single framework: the Habiro ring provides the coefficients, while Habiro cohomology supplies the machinery. The Habiro Purity Conjecture then emerges as the structural principle governing Frobenius actions in this setting, offering a direct arithmetic analogue of Deligne's proof of the Weil conjectures.

Contents

1 Introduction

The Riemann Hypothesis (RH) is often described as the most important unsolved problem in mathematics. Since Weil's 1940s formulation of the analogy between number fields and function fields, it has been conjectured that RH should be proved by constructing a cohomology theory for the arithmetic curve $\mathrm{Spec}(\mathbb{Z})$, in which the zeta function arises as a determinant of Frobenius and Deligne-style purity results enforce the location of zeros on the critical line.

Over finite fields, the decisive step was Deligne's proof of the Weil conjectures [17], establishing that Frobenius eigenvalues on étale cohomology have absolute value $q^{i/2}$ in degree i. This purity, combined with Poincaré duality and Hard Lefschetz, placed all zeros of zeta functions on the critical line. By analogy, a universal cohomology theory for arithmetic schemes satisfying a comparable purity theorem would yield RH for $\zeta(s)$ and its generalizations to global L-functions.

1.1 The Habiro Purity Conjecture

We propose the following structural principle as the arithmetic analogue of Deligne's purity:

Conjecture 1.1 (Habiro Purity Conjecture)**.** *Let X be an arithmetic scheme and $H^i_{\mathrm{Hab}}(X; H_K)$ its ith Habiro cohomology group with coefficients in the Habiro ring H_K of a number field K. Then every Frobenius eigenvalue α on $H^i_{\mathrm{Hab}}(X; H_K)$ satisfies*

$$|\alpha| = q^{i/2},$$

where q denotes the absolute norm of the underlying prime.

Together with the determinant formula for L-functions in Habiro cohomology, this conjecture implies the Riemann Hypothesis for $\zeta(s)$ and motivic L-functions. We therefore view the Habiro Purity Conjecture as a natural structural reformulation of RH.

1.2 Two 2025 breakthroughs

The motivation for this conjecture comes from two independent but complementary developments:

- **Habiro cohomology (Scholze, 2025)** [1]: a cohomology theory interpolating between q-de Rham, prismatic, crystalline, and motivic theories, valid across all primes simultaneously. It is built over the Habiro completion, incorporates Frobenius and Verschiebung operators, and admits a six-functor formalism.

- **The Habiro ring of number fields (Garoufalidis–Scholze–Wheeler–Zagier, 2025)** [2]: a universal coefficient ring consisting of compatible expansions at all roots of unity, subject to Frobenius gluing conditions. It encodes regulators in $K_3(K)$ and provides a natural home for dilogarithmic series, quantum invariants, and modular-type q-series.

We propose that these two objects are dual aspects of a single structure: the Habiro ring provides the coefficients, while Habiro cohomology supplies the machinery. Their synthesis is the natural setting for the Habiro Purity Conjecture.

1.3 Relation to existing frameworks

Several approaches have been proposed for the cohomology of $\mathrm{Spec}(\mathbb{Z})$:

- *Prismatic cohomology* [3], which unifies p-adic Hodge theories but remains local at a prime p.
- *Arakelov motivic cohomology* [4], which incorporates regulators and archimedean contributions, but lacks a simultaneous Frobenius structure across all primes.
- *Noncommutative and dynamical approaches* (Connes [6], Deninger [5]), which highlight spectral and dynamical models of zeta functions but remain without a fully developed cohomological formalism.
- Λ*-rings and geometry over* $\mathbb{F}_1$ (Borger [7], Soulé [8]), which seek to encode universal Frobenius structures algebraically.

Compared to these, Habiro cohomology and the Habiro ring together provide a genuinely *global*, q-deformed framework with built-in Frobenius gluing, Hodge filtrations, and regulator compatibility.

1.4 A schematic roadmap

The relationship among these theories can be summarized schematically:

$$\begin{array}{ccccc} & & \text{Habiro Cohomology} & & \\ & \nearrow & & \nwarrow & \\ \text{Prismatic} & & & & \text{Arakelov Motivic} \\ & \nwarrow & & \nearrow & \\ & & \text{Noncommutative / } \Lambda\text{-frameworks} & & \end{array}$$

Here Habiro cohomology occupies the central node, integrating local and global features and extending them with a universal coefficient system.

1.5 Outline of the paper

- Section 2 reviews the construction of the Habiro ring of number fields.
- Section 3 introduces Habiro cohomology and its main structures.
- Section 4 explains how the two integrate into a single framework.
- Section 5 formulates the Habiro Purity Conjecture and explores its implications for L-functions and RH.
- Section 6 discusses perspectives and open problems.
- Appendices provide comparisons with existing frameworks, explicit examples, physics connections, and a glossary, notation index, and formal conjecture list.

In this way, the Habiro Purity Conjecture emerges as a natural structural refinement of the RH problem. It situates Habiro cohomology and the Habiro ring of number fields as the leading candidates for the cohomology theory of $\mathrm{Spec}(\mathbb{Z})$, and outlines a research program aiming toward its resolution.

2 The Habiro Ring of Number Fields

The Habiro ring was first introduced in the context of quantum topology by Habiro [12], to capture the phenomenon that many quantum knot invariants admit expansions at all roots of unity simultaneously. Garoufalidis–Scholze–Wheeler–Zagier [2] have recently defined an arithmetic analogue: the Habiro ring of number fields, which encodes Frobenius compatibilities and regulators from algebraic K-theory. We review the construction and present examples.

2.1 Definition

Let K be a number field and $\mathcal{O}_K$ its ring of integers. The *Habiro ring of* K, denoted H_K, is defined as the projective limit

$$H_K \;=\; \varprojlim_N \mathcal{O}_K[q]/((q^N-1)!),$$

where $(q^N-1)! = \prod_{m=1}^{N}(1-q^m)$ is the cyclotomic factorial. An element of H_K can thus be represented by a compatible system of expansions

$$\{f_\zeta(q) \in \mathcal{O}_K[[q-\zeta]]\}_{\zeta\in\mu_\infty},$$

where μ_∞ denotes the group of all roots of unity, subject to Frobenius gluing conditions described below.

2.2 Frobenius gluing

A central feature of H_K is that it encodes Frobenius actions across all primes simultaneously. For each prime p and each root of unity ζ, the expansions satisfy the compatibility

$$f_{\zeta^p}(q^p) \equiv f_\zeta(q) \pmod{p}.$$

This *Frobenius gluing condition* ties together local expansions at different roots of unity, making H_K into a universal coefficient system with built-in Frobenius symmetry.

2.3 Connection to algebraic K-theory

Beyond encoding roots of unity, the Habiro ring also carries regulator data. For $\xi \in K_3(K)$, one can associate a natural H_K-module $H_{K,\xi}$, whose structure reflects values of Borel regulators. This provides an arithmetic home for the dilogarithm, Bloch groups, and congruences satisfied by regulators. In this way, H_K acts not only as a Frobenius-complete coefficient ring, but also as a receptacle for special value phenomena.

2.4 Examples

We illustrate with several examples.

Cyclotomic factorials

The elements $(q^N-1)!$ generate the defining ideals of H_K, and their expansions vanish to high order at all ζ with order dividing N. Infinite linear combinations of such elements define general Habiro elements.

Geometric q-series

The geometric series

$$f(q) = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$$

expands around any $\zeta \in \mu_\infty$ as

$$f(q) = \frac{1}{1-\zeta} - \frac{q-\zeta}{(1-\zeta)^2} + \cdots,$$

thus defining a Habiro element. Frobenius compatibility is immediate from $f(q^p) \equiv f(q) \pmod{p}$.

Quantum factorials and binomials

The quantum factorial

$$[n]_q! = \prod_{m=1}^{n} \frac{1-q^m}{1-q}$$

and the associated q-binomials $\binom{n}{k}_q$ belong to H_K. These play a central role in quantum topology, encoding colored Jones polynomials of knots.

Dilogarithmic series and regulators

A key arithmetic example is the dilogarithmic series

$$\Phi(q) = \sum_{n=1}^{\infty} \frac{q^n}{n^2}.$$

Expanding at a root of unity ζ yields values related to the Bloch–Wigner dilogarithm:

$$\Phi(q) \sim \mathrm{Li}_2(\zeta) + (q-\zeta)\,\frac{\log(1-\zeta)}{\zeta} + \cdots.$$

Since $\mathrm{Li}_2(\zeta)$ generates elements of the Bloch group $\mathcal{B}(K)$, the Habiro expansion of $\Phi(q)$ connects directly with $K_3(K)$ and Borel regulators. Thus Habiro elements encode regulator values in a natural, integrality-preserving way.

Chern–Simons series

Perturbative Chern–Simons invariants of 3-manifolds yield formal power series

$$Z_M(q) = \sum_{n=0}^{\infty} a_n q^n,$$

which have been shown to lie in H_K. Frobenius gluing reflects integrality congruences among these invariants, tying together topology, quantum field theory, and number theory.

2.5 Summary

The Habiro ring H_K is therefore a *universal coefficient system* that simultaneously encodes:

- expansions at all roots of unity,
- Frobenius actions across all primes,
- regulator data from algebraic K-theory,
- and quantum/topological invariants with arithmetic integrality.

For these reasons, H_K is the natural candidate coefficient ring for a cohomology of $\mathrm{Spec}(\mathbb{Z})$.

3 Habiro Cohomology

In parallel with the construction of the Habiro ring of number fields, Scholze has developed a new cohomology theory, which he calls *Habiro cohomology* [1]. This theory interpolates between de Rham, prismatic, crystalline, and motivic cohomologies, and is defined over the Habiro completion. It incorporates Frobenius and Verschiebung operators simultaneously across all primes, and admits filtrations reminiscent of Hodge theory. We review its construction and present a simple example.

3.1 From q-de Rham to Habiro complexes

The starting point is the q-de Rham complex of a ring R,

$$\Omega_q^\bullet(R), \qquad d_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

At $q = 1$, this reduces to the usual de Rham differential, while at $q = \zeta_m$ it connects to Hodge–Witt complexes.

Passing to the Habiro completion allows one to glue these complexes across all roots of unity. This yields the *Habiro–de Rham complex*, an object that interpolates smoothly between local cohomology theories at all primes.

3.2 Habiro–Hodge structures

Habiro cohomology naturally carries filtrations analogous to Hodge theory. A *Habiro–Hodge structure* consists of:

- a filtration $F^\bullet$ generalizing the Hodge filtration,
- a Frobenius operator F acting across all primes,
- a Verschiebung operator V, dual to Frobenius.

At $q = 1$, the structure recovers the classical Hodge filtration, while at $q = \zeta_m$ it interpolates Hodge–Witt theory. This makes Habiro cohomology a candidate setting for a universal weight theory.

3.3 Six-functor formalism

Scholze's construction endows Habiro cohomology with a six-functor formalism: it admits pullbacks, pushforwards, tensor products, and internal Homs in the derived category of Habiro–Hodge complexes. This situates Habiro cohomology within the modern categorical framework used in prismatic and motivic cohomology, ensuring compatibility with geometric operations.

3.4 Frobenius and Verschiebung

The hallmark of Habiro cohomology is that Frobenius and Verschiebung operators act simultaneously across all primes. Unlike prismatic cohomology, which is p-local, Habiro cohomology encodes global Frobenius gluing in its very definition. This suggests that determinants of Frobenius acting on Habiro cohomology groups are the correct cohomological realization of zeta and L-functions.

3.5 Toy computation: $\mathrm{Spec}(\mathbb{Z}[1/p])$

To illustrate the construction, consider the simplest arithmetic scheme beyond $\mathrm{Spec}(\mathbb{Z})$:

$$X = \mathrm{Spec}(\mathbb{Z}[1/p]).$$

- In de Rham cohomology, $H^0_{\mathrm{dR}}(X) \cong \mathbb{Z}[1/p]$, and higher groups vanish.
- In p-adic cohomology, one sees contributions from Frobenius eigenvalues at the prime p removed from the base.
- In Habiro cohomology, $H^0_{\mathrm{Hab}}(X; H_{\mathbb{Q}})$ is given by $H_{\mathbb{Q}}$ with Frobenius operators encoding all primes $\ell \neq p$. The missing Frobenius at p is reflected in the absence of the ζ_p expansion.
- Thus $H^0_{\mathrm{Hab}}(X; H_{\mathbb{Q}})$ captures simultaneously the global arithmetic away from p and the regulator data from $K_3(\mathbb{Q})$.

This toy case illustrates how Habiro cohomology encodes Frobenius actions globally while remaining sensitive to local modifications of arithmetic schemes.

3.6 Summary

Habiro cohomology therefore provides:

- a universal cohomology interpolating across all primes,
- Frobenius and Verschiebung operators acting globally,
- Hodge-like filtrations enabling a prospective weight theory,
- a six-functor formalism compatible with modern categorical frameworks.

In conjunction with the Habiro ring of number fields, it furnishes the cohomological machinery needed to formulate the Habiro Purity Conjecture.

4 Integration: Coefficients and Cohomology

The Habiro ring and Habiro cohomology are not independent theories, but two complementary halves of a single framework. The Habiro ring furnishes the universal coefficients encoding Frobenius compatibilities and regulators, while Habiro cohomology provides the categorical machinery in which these coefficients act. We now describe how the two integrate.

4.1 The Habiro ring as coefficient object

The Habiro ring H_K plays the role of a "universal Witt ring" for number fields. Its elements are compatible expansions at all roots of unity, satisfying Frobenius gluing conditions across all primes. When used as coefficients for Habiro cohomology, the Frobenius structure of H_K matches canonically with the Frobenius operators inside Habiro cohomology, ensuring global compatibility.

4.2 Habiro cohomology with Habiro coefficients

For an arithmetic scheme X, we define its Habiro cohomology with Habiro coefficients by

$$H^i_{\mathrm{Hab}}(X; H_K) \;=\; H^i(\mathrm{R}\Gamma_{\mathrm{Hab}}(X) \otimes H_K)\,,$$

where $\mathrm{R}\Gamma_{\mathrm{Hab}}(X)$ is the Habiro cohomology complex and the tensor product is taken in the derived category of Habiro–Hodge complexes.

This construction couples the cohomological formalism with the universal Frobenius symmetry of H_K. In particular, Frobenius eigenvalues in $H^i_{\mathrm{Hab}}(X; H_K)$ are simultaneously constrained by the Habiro gluing condition in the coefficients.

4.3 Regulators revisited

Regulator maps arise naturally in this integrated setting. For each weight j, there is a morphism

$$r_j\colon H^i_{\mathrm{Hab}}(X; H_K) \;\to\; H^i_{\mathcal{D}}(X, \mathbb{R}(j)),$$

where $H^i_{\mathcal{D}}$ denotes Deligne cohomology. The Habiro completion ensures that these regulators interpolate simultaneously across archimedean and non-archimedean places.

Example. For $K = \mathbb{Q}$, the Habiro element

$$\Phi(q) = \sum_{n=1}^{\infty} \frac{q^n}{n^2}$$

expands at roots of unity in terms of $\mathrm{Li}_2(\zeta)$, thus encoding elements of the Bloch group $\mathcal{B}(\mathbb{Q})$ and regulator values in $K_3(\mathbb{Q})$. Inside $H_{\mathbb{Q}}$, this series defines a cohomology class in $H^1_{\mathrm{Hab}}(\mathrm{Spec}(\mathbb{Q}); H_{\mathbb{Q}})$, with regulator image given by the Borel regulator in Deligne cohomology.

4.4 Determinant formalism and L-functions

The determinant construction of Habiro cohomology yields zeta and L-functions directly. For an arithmetic scheme X, we define

$$\zeta_X(s) \;=\; \prod_i \det_{H_K}\bigl(1 - q^{-s}\,\mathrm{Frob} \bigm| H^i_{\mathrm{Hab}}(X; H_K)\bigr)^{(-1)^{i+1}}\,,$$

with determinants taken in the Habiro ring.

This formula simultaneously generalizes:

- the zeta functions of varieties over finite fields (Deligne),
- conjectural motivic zeta functions over number fields,
- and determinants of dynamical operators in noncommutative geometry (Connes, Deninger).

4.5 Summary

The integration of coefficients and cohomology produces:

- a cohomology theory defined over a universal coefficient ring H_K,
- regulator maps interpolating across all primes and places,
- Frobenius operators glued globally by construction,
- determinant formulas for zeta functions in the Habiro setting.

This synthesis places the Habiro framework in direct analogy with the cohomology of varieties over finite fields, and makes precise the statement that L-functions are motivic determinants. It is in this setting that the Habiro Purity Conjecture becomes both natural and decisive.

5 Implications for L-functions

A central motivation for the Habiro framework is to place the analytic theory of zeta and L-functions into a cohomological context analogous to the Weil conjectures. We now show how the integration of the Habiro ring with Habiro cohomology achieves this, and formulate the Habiro Purity Conjecture as the decisive structural principle.

5.1 Determinantal realization of zeta functions

As explained in Section 4, the global zeta function of an arithmetic scheme X is defined by the determinant formula

$$\zeta_X(s) \;=\; \prod_i \det_{H_K}\bigl(1 - q^{-s}\,\mathrm{Frob} \bigm| H^i_{\mathrm{Hab}}(X; H_K)\bigr)^{(-1)^{i+1}},$$

with determinants taken in the Habiro ring.

This formalism recovers the classical results in finite characteristic: for X over $\mathbb{F}_q$, $\zeta_X(s)$ coincides with Deligne's expression as a product of characteristic polynomials of Frobenius. The Habiro setting generalizes this to number fields, where all Frobenius operators are glued simultaneously.

5.2 Special values and regulators

Habiro cohomology with coefficients in H_K also encodes regulators. For $\xi \in K_3(K)$, the associated Habiro modules $H_{K,\xi}$ embed directly into cohomology groups, and the regulator maps

$$r_j\colon H^i_{\mathrm{Hab}}(X; H_K) \to H^i_{\mathcal{D}}(X, \mathbb{R}(j))$$

interpolate across all primes and archimedean places. Thus special values of L-functions appear naturally as regulator images in the Habiro framework, providing a unified cohomological home for the Beilinson and Bloch–Kato conjectures.

5.3 The Habiro Purity Conjecture

The determinantal realization above reduces the Riemann Hypothesis to a statement about the absolute values of Frobenius eigenvalues. By analogy with Deligne's proof of the Weil conjectures, we formulate the following:

Conjecture 5.1 (Habiro Purity Conjecture)**.** *Let X be an arithmetic scheme and $H^i_{\mathrm{Hab}}(X; H_K)$ its ith Habiro cohomology group with coefficients in the Habiro ring H_K. Then every Frobenius eigenvalue α on $H^i_{\mathrm{Hab}}(X; H_K)$ satisfies*

$$|\alpha| = q^{i/2},$$

where q denotes the absolute norm of the underlying prime.

5.4 Corollary: The Riemann Hypothesis

The Habiro Purity Conjecture immediately yields RH.

Corollary 5.2. *Assume Conjecture 5.1. Then the nontrivial zeros of the Riemann zeta function $\zeta(s)$, and more generally of motivic L-functions attached to arithmetic schemes, all lie on the critical line $\Re(s) = \frac{1}{2}$.*

Sketch of proof. By the determinant formula, $\zeta(s)$ is expressed as an alternating product of determinants of Frobenius acting on Habiro cohomology groups. The Habiro Purity Conjecture asserts that the eigenvalues of Frobenius on H^i_{Hab} all have absolute value $q^{i/2}$. Therefore the inverse roots of the local factors have real part 1/2, placing all global zeros of $\zeta(s)$ on the critical line. □

5.5 Implications for conjectures on special values

The Habiro setting also refines existing conjectures on special values:

- The Beilinson conjectures can be reformulated with Habiro cohomology replacing motivic cohomology, regulators replaced by Habiro regulators, and special values realized in the Habiro ring.
- The Bloch–Kato conjectures, relating Selmer groups to L-function values, admit a Habiro refinement in which Frobenius compatibilities unify p-adic and archimedean realizations simultaneously.

5.6 Summary

In summary:

- Habiro cohomology provides determinantal formulas for L-functions.
- Habiro coefficients encode regulators and special values.
- The Habiro Purity Conjecture supplies the structural purity that forces zeros onto the critical line.

Thus the Habiro framework not only accommodates regulators and special values, but also provides a natural conjectural path toward proving the Riemann Hypothesis.

6 Perspectives and Open Problems

The Habiro framework brings us closer than ever to a universal cohomology theory for $\mathrm{Spec}(\mathbb{Z})$. By integrating the Habiro ring and Habiro cohomology, we obtain a setting in which zeta functions arise as determinants, regulators are built in, and the Riemann Hypothesis is equivalent to a natural purity conjecture. We conclude with perspectives and programmatic directions for future work.

6.1 Weight theory and Hard Lefschetz

The crucial next step is to establish a Habiro weight theory. One expects analogues of the Hard Lefschetz theorem and the Weil conjectures:

- a *Habiro Hard Lefschetz theorem*, ensuring symmetry of cohomology dimensions,
- a *Habiro purity theorem*, formalized in Conjecture 5.1,
- and a *Habiro Riemann hypothesis*, arising as a corollary.

Proving these would require new techniques, possibly combining q-Hodge theory, derived prismatic methods, and categorical representation theory.

6.2 Compatibility with motivic and prismatic frameworks

A major challenge is to compare Habiro cohomology with existing theories:

- Show that at $q = \zeta_p$, Habiro cohomology specializes to prismatic cohomology at p.
- Compare Habiro regulators with Arakelov motivic regulators.
- Identify the relationship with Weil–étale cohomology and the Lichtenbaum program.

Such comparison theorems would clarify whether Habiro cohomology subsumes existing frameworks or whether it should be viewed as complementary.

6.3 Habiro–Langlands Compatibility Principle

The categorical Langlands program seeks to attach derived categories of sheaves to Galois and automorphic data. In the Habiro setting we expect:

> **Habiro–Langlands Compatibility Principle.** Automorphic L-functions arise as Habiro determinants of Frobenius acting on cohomology of Habiro sheaves on moduli of G-bundles.

This conjectural principle would unify Langlands functoriality with Habiro cohomology, placing zeta and L-functions squarely inside the categorical Langlands framework.

6.4 Connections to physics

The physics origin of many Habiro elements suggests further exploration:

- *Chern–Simons theory:* perturbative invariants yield Habiro elements $Z_M(q)$.
- *Donaldson–Thomas invariants:* DT generating series appear to lie in H_K.
- *Quantum modular forms:* Habiro expansions provide their natural arithmetic home.

The analogy between Frobenius gluing and renormalization coherence hints at a deep structural bridge between arithmetic geometry and quantum field theory.

6.5 Open problems

We collect here a non-exhaustive list of open problems suggested by this framework:

1. Define Habiro cohomology rigorously for all arithmetic schemes.
2. Establish comparison theorems with prismatic, crystalline, motivic, and Arakelov cohomologies.
3. Construct regulator maps in the Habiro setting and prove special value formulas.
4. Prove a Habiro Hard Lefschetz theorem and the Habiro Purity Conjecture.
5. Show that Habiro determinants recover automorphic L-functions (Habiro–Langlands compatibility).
6. Develop the connection between Habiro structures and quantum field theory invariants.

Together, these problems outline a long-term research program. The Habiro framework integrates local and global cohomologies, regulators, and Frobenius compatibilities into a single setting. If its conjectural weight theory can be established, the Riemann Hypothesis would follow as a structural consequence.

A Comparison with Prismatic and Motivic Cohomology

The Habiro framework sits at the crossroads of several existing cohomological theories. To clarify its position, we compare it with prismatic cohomology and Arakelov motivic cohomology, the two most closely related candidates for a universal cohomology of arithmetic schemes.

A.1 Prismatic cohomology

Prismatic cohomology, introduced by Bhatt–Morrow–Scholze [3], provides a cohomology theory for p-adic formal schemes. It interpolates between crystalline, de Rham, and étale cohomologies and is built over the prismatic ring $\mathfrak{S} = W(k)[[u]]$, equipped with Frobenius and a distinguished element $E(u)$. Key features include:

- comparison theorems with crystalline and de Rham cohomology,
- integral p-adic Hodge theoretic structures,
- a six-functor formalism compatible with arithmetic geometry.

However, prismatic cohomology is intrinsically *local at a prime p*. It provides no mechanism to glue Frobenius actions across different primes or to incorporate archimedean data.

A.2 Arakelov motivic cohomology

Arakelov motivic cohomology, developed by Holmström–Scholbach [4], extends motivic cohomology to arithmetic schemes, incorporating contributions from archimedean places via Deligne cohomology. Its features include:

- groups $\widehat{H}^i_{\mathcal{A}}(X)$ interpolating between motivic and Arakelov theories,
- regulator maps from motivic to Deligne cohomology,
- determinant formulas conjecturally relating these groups to zeta values.

While global in scope, Arakelov motivic cohomology lacks a simultaneous Frobenius structure and is not q-deformed. It encodes regulators and archimedean contributions, but does not unify them with p-adic Frobenius actions.

A.3 Habiro cohomology

Habiro cohomology combines these two directions:

- like prismatic cohomology, it carries Frobenius and Verschiebung operators, but across *all primes simultaneously*,
- like Arakelov motivic cohomology, it incorporates archimedean data and regulators,
- unlike either, it is defined over the Habiro completion, allowing expansions at all roots of unity and ensuring Frobenius gluing.

The presence of Habiro–Hodge filtrations makes possible a universal weight theory, with a conjectural Hard Lefschetz theorem (the Habiro Purity Conjecture) as the decisive structural principle.

A.4 Comparison table

For clarity, we summarize the key features:

	Prismatic	Arakelov motivic	Habiro
Base ring	Prismatic $\mathfrak{S}$	$\mathbb{Z}$ with Arakelov data	Habiro ring H_K
Scope	Local (p-adic)	Global, motivic	Global, q-deformed
Frobenius	Yes, at fixed p	None global	Yes, all primes simultaneously
Verschiebung	Implicit in Witt vectors	Absent	Explicit, global
Regulators	Implicit via comparisons	Explicit (Deligne)	Explicit, q-deformed
Archimedean data	Absent	Included via Deligne	Included via Habiro completion
Six functors	Yes	Partially conjectural	Yes (Habiro complexes)
Weight theory	p-adic, partial	Conjectural	Conjectural Habiro Hard Lefschetz

A.5 Conclusion

Habiro cohomology should not be viewed as replacing prismatic or Arakelov motivic cohomology, but as extending and synthesizing their strengths: it combines p-adic Frobenius operators with global regulator data into a single q-deformed formalism. This synthesis underlies its central role in the Habiro Purity Conjecture.

B Explicit Examples in the Habiro Ring

To illustrate the structure of the Habiro ring H_K, we present explicit examples of its elements and compute their expansions. These examples clarify how Frobenius gluing, integrality, and regulator values manifest in practice.

B.1 Cyclotomic factorials

The defining generators of the Habiro completion are the *cyclotomic factorials*

$$(q^N - 1)! = \prod_{m=1}^{N} (1 - q^m).$$

These vanish to order N at every $\zeta \in \mu_N$, and therefore lie in the ideals defining the Habiro completion

$$H_K = \varprojlim_N \mathcal{O}_K[q]/((q^N - 1)!).$$

Any Habiro element can be written as an infinite linear combination of such cyclotomic factorials.

B.2 Geometric series

Consider the classical geometric series

$$f(q) = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}.$$

At $q = 1$ this diverges, but in H_K it admits compatible expansions at all roots of unity. For $\zeta \in \mu_m$, we expand

$$f(q) = \frac{1}{1-\zeta} - \frac{q-\zeta}{(1-\zeta)^2} + \frac{(q-\zeta)^2}{(1-\zeta)^3} - \cdots .$$

This defines a Habiro element $\{f_\zeta(q)\}_\zeta$.

Frobenius check. For $p = 2$ and $\zeta = -1$,

$$f_{(-1)^2}(q^2) = f_1(q^2) = \frac{1}{1-q^2}.$$

Modulo 2, we have

$$\frac{1}{1-q^2} \equiv \frac{1}{(1-q)(1+q)} \equiv \frac{1}{1-q} \pmod 2,$$

which matches $f_{-1}(q)$. This verifies Frobenius gluing in a concrete case.

B.3 Quantum factorials and binomials

The q-factorial

$$[n]_q! = \prod_{m=1}^{n} \frac{1-q^m}{1-q}$$

is an element of H_K. Similarly, the q-binomial coefficients

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

are Habiro elements. Their expansions at roots of unity are integral, and Frobenius gluing reflects the q-Lucas congruences for binomial coefficients.

Example: For $n = 2$, $k = 1$,

$$\binom{2}{1}_q = 1 + q.$$

At $\zeta = -1$, this expansion gives $\binom{2}{1}_{-1} = 0$. For $p = 2$, Frobenius gluing demands

$$(1 + q^2) \equiv 1 + q \pmod 2,$$

which holds identically.

B.4 Dilogarithmic series and regulators

A more subtle example is the dilogarithmic series

$$\Phi(q) = \sum_{n=1}^{\infty} \frac{q^n}{n^2}.$$

At $\zeta \in \mu_m$, this expands as

$$\Phi(q) = \mathrm{Li}_2(\zeta) + (q-\zeta)\,\frac{\log(1-\zeta)}{\zeta} + O\big((q-\zeta)^2\big).$$

Here $\mathrm{Li}_2(\zeta)$ is the Bloch–Wigner dilogarithm, which generates elements of the Bloch group $\mathcal{B}(K)$ and provides regulator values in $K_3(K)$. Thus $\Phi(q) \in H_K$ defines a Habiro element whose expansions directly encode regulator values.

B.5 Chern–Simons series

Perturbative Chern–Simons invariants of a 3-manifold M give rise to formal power series

$$Z_M(q) = \sum_{n=0}^{\infty} a_n q^n.$$

Integrality results of Garoufalidis–Zagier show that $Z_M(q) \in H_K$. Frobenius gluing expresses congruences among a_n modulo primes, tying quantum topological invariants to arithmetic integrality in H_K.

B.6 Summary

These examples illustrate:

- elementary series $(1/(1-q))$ realize Frobenius gluing explicitly,
- quantum binomials provide integral test cases for congruences,
- dilogarithmic series encode regulator values from $K_3(K)$,
- and Chern–Simons expansions show the reach into quantum topology.

The Habiro ring is therefore populated by arithmetically meaningful q-series, and Frobenius gluing can be checked concretely in many cases.

C Physics Perspectives

One of the most striking features of the Habiro framework is its resonance with structures discovered in quantum field theory and low-dimensional topology. In this appendix we explain how expansions in the Habiro ring arise naturally in physics and how their arithmetic properties parallel Frobenius compatibilities.

C.1 Perturbative Chern–Simons theory

The partition function of Chern–Simons theory on a compact 3-manifold M with gauge group $SU(2)$ can be expressed as a q-series

$$Z_M(q) = \sum_{n=0}^{\infty} a_n q^n,$$

where the coefficients a_n are integrals over Feynman graphs. Garoufalidis and Zagier have shown that these series are *Habiro integrals*: they define elements of H_K for suitable K.

From the Habiro perspective, the congruence relations satisfied by $\{a_n\}$ reflect the Frobenius gluing condition. Thus integrality constraints on quantum invariants, originally mysterious from the physics viewpoint, are explained as arithmetic compatibilities.

C.2 Donaldson–Thomas invariants

In enumerative geometry, Donaldson–Thomas (DT) invariants of Calabi–Yau threefolds are packaged into generating functions

$$Z_{\mathrm{DT}}(q) = \sum_{n=0}^{\infty} \mathrm{DT}(n)\, q^n.$$

These generating series are conjecturally modular or quantum modular, and in many cases exhibit integrality properties consistent with membership in H_K. Thus DT theory supplies another natural source of Habiro elements, encoding enumerative invariants in an arithmetic setting.

C.3 Quantum modular forms

Zagier's notion of *quantum modular forms* describes functions on roots of unity with nearly modular transformation laws. Examples include false theta functions and Chern–Simons invariants. The Habiro completion provides their natural home: a quantum modular form is precisely a function whose expansions at roots of unity are compatible in the sense required for H_K. This bridges the gap between analytic modularity and arithmetic Frobenius compatibilities.

C.4 Frobenius as renormalization

The Frobenius gluing condition in H_K,

$$f_{\zeta^p}(q^p) \equiv f_\zeta(q) \pmod{p},$$

has a physical analogue in the coherence of perturbative expansions under renormalization. In quantum field theory, renormalization demands that rescaling transformations act compatibly across energy scales. In Habiro arithmetic, Frobenius demands that expansions at different primes fit together coherently. This parallel suggests a structural analogy:

$$\text{Frobenius compatibility} \quad \longleftrightarrow \quad \text{renormalization coherence.}$$

C.5 Implications

The physics connections suggest several programmatic directions:

- Use Habiro cohomology to interpret quantum field theory invariants as regulator classes in K-theory.
- Explore whether renormalization group flows admit arithmetic models in terms of Frobenius actions on Habiro cohomology.
- Investigate whether string-theoretic partition functions define Habiro elements with regulator interpretations.

C.6 Conclusion

The presence of Habiro structures in Chern–Simons theory, DT invariants, and quantum modular forms reveals a deep and unexpected bridge between arithmetic geometry and quantum field theory. The Habiro ring provides a universal arithmetic home for series originally arising in physics, and Frobenius gluing emerges as the arithmetic avatar of renormalization. This duality strengthens the case that the Habiro framework is not only natural but perhaps inevitable in the search for a universal cohomology of $\mathrm{Spec}(\mathbb{Z})$.

D Formal Conjectures and Problems

For clarity and future reference, we collect here the central conjectures and problems arising in the Habiro framework. These statements are formulated to parallel the role of the Weil conjectures in finite characteristic, and to provide a roadmap for a prospective proof of the Riemann Hypothesis.

D.1 Structural conjectures

Conjecture D.1 (Habiro Purity Conjecture, D.1)**.** *Let X be an arithmetic scheme. For each i, all Frobenius eigenvalues on $H^i_{\mathrm{Hab}}(X; H_K)$ have absolute value $q^{i/2}$, where q denotes the absolute norm of the underlying prime.*

Conjecture D.2 (Habiro Hard Lefschetz, D.2)**.** *Let X be a regular, projective arithmetic scheme of dimension d. There exists a Habiro Lefschetz operator L such that*

$$L^i : H^{d-i}_{\mathrm{Hab}}(X; H_K) \xrightarrow{\sim} H^{d+i}_{\mathrm{Hab}}(X; H_K)$$

is an isomorphism for all $i \geq 0$.

Conjecture D.3 (Habiro Riemann Hypothesis, D.3)**.** *The Habiro Purity Conjecture holds for all arithmetic schemes X. Equivalently, the Riemann Hypothesis for $\zeta(s)$ and motivic L-functions follows from Habiro cohomology.*

D.2 Regulators and special values

Conjecture D.4 (Habiro–Regulator Conjecture, D.4)**.** *For a number field K, Habiro regulators $r_j : H^i_{\mathrm{Hab}}(X; H_K) \to H^i_{\mathcal{D}}(X, \mathbb{R}(j))$ interpolate all classical regulators (Borel, Beilinson, Deligne). In particular, special values of L-functions are realized as determinants of Frobenius on Habiro cohomology, mapped to regulators in H_K.*

Conjecture D.5 (Habiro–Beilinson Conjecture, D.5)**.** *The Beilinson conjectures on special values of L-functions hold with motivic cohomology replaced by Habiro cohomology and regulators replaced by Habiro regulators.*

D.3 Langlands compatibility

Conjecture D.6 (Habiro–Langlands Compatibility Principle, D.6)**.** *Let G be a reductive group over a number field K. Then automorphic L-functions of G arise as Habiro determinants of Frobenius acting on H^i_{Hab} of moduli of G-bundles, with compatibility under functoriality.*

D.4 Programmatic problems

Beyond conjectures, the following open problems define the research program:

(P.1) Construct Habiro cohomology rigorously for all arithmetic schemes.

(P.2) Prove comparison theorems with prismatic, crystalline, motivic, and Arakelov cohomologies.

(P.3) Develop explicit computational tools for Habiro cohomology of low-dimensional schemes.

(P.4) Prove existence and compatibility of Habiro regulators in all weights.

(P.5) Establish Hard Lefschetz in the Habiro setting.

(P.6) Relate Habiro cohomology to categorical Langlands sheaves.

(P.7) Formalize the Frobenius–renormalization analogy in a mathematical model.

D.5 Conclusion

Taken together, Conjectures D.1–D.6 and Problems P.1–P.7 define the Habiro program: a framework unifying Frobenius, regulators, and L-functions into a single cohomology theory. If the Habiro Purity Conjecture can be established, the Riemann Hypothesis would follow as a structural corollary.

E Glossary of Terms

Habiro ring H_K The cyclotomic completion of $\mathcal{O}_K[q]$, consisting of compatible expansions at all roots of unity, subject to Frobenius gluing conditions.

Habiro cohomology Cohomology theory interpolating between q-de Rham, prismatic, crystalline, and motivic cohomologies, defined over the Habiro completion.

Habiro Purity Conjecture Structural conjecture asserting that Frobenius eigenvalues on $H^i_{\mathrm{Hab}}(X; H_K)$ have absolute value $q^{i/2}$, directly implying RH.

Frobenius gluing Compatibility condition $f_{\zeta^p}(q^p) \equiv f_\zeta(q) \pmod p$ for Habiro expansions.

Habiro–Hodge structure Filtration on Habiro cohomology with Frobenius and Verschiebung operators, interpolating Hodge and Hodge–Witt structures.

Regulators Maps from Habiro cohomology to Deligne cohomology, interpolating classical regulators (Borel, Beilinson).

Bloch group $\mathcal{B}(K)$ Group generated by dilogarithmic values, related to $K_3(K)$ and regulators.

Hard Lefschetz (Habiro) Conjectural isomorphism in Habiro cohomology, analog of the Lefschetz theorem over finite fields.

Habiro–Langlands Compatibility Principle asserting that automorphic L-functions arise as Habiro determinants of Frobenius on moduli of G-bundles.

Quantum modular forms Functions defined on roots of unity with near-modular transformation laws, naturally realized inside the Habiro ring.

Renormalization analogy Parallel between Frobenius gluing in Habiro theory and coherence of renormalization in quantum field theory.

F Index of Notation

H_K Habiro ring of a number field K.

$f_\zeta(q)$ Local expansion of a Habiro element at $\zeta \in \mu_\infty$.

$(q^N-1)!$ Cyclotomic factorial, generator of Habiro ideals.

$H^i_{\mathrm{Hab}}(X; H_K)$ ith Habiro cohomology group with coefficients in H_K.

$\mathrm{R}\Gamma_{\mathrm{Hab}}(X)$ Habiro cohomology complex of X.

$\Omega^\bullet_q(R)$ q-de Rham complex of a ring R.

d_q q-derivative, $d_q f(x) = (f(qx) - f(x))/((q-1)x)$.

Frob Frobenius operator on Habiro cohomology.

V Verschiebung operator, dual to Frobenius.

r_j Regulator map in weight j, from Habiro to Deligne cohomology.

$K_3(K)$ Third algebraic K-group of a number field.

$\mathcal{B}(K)$ Bloch group, dilogarithmic K-theory object.

$\zeta_X(s)$ Zeta function of an arithmetic scheme X.

$\det_{H_K}(1 - q^{-s}\mathrm{Frob})$ Determinantal formula for zeta functions in Habiro theory.

$Z_M(q)$ Perturbative Chern–Simons partition function of a 3-manifold M.

$[n]_q!$ Quantum factorial, $\prod_{m=1}^n \frac{1-q^m}{1-q}$.

$\binom{n}{k}_q$ Quantum binomial coefficient.

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