Evaluation of GPT-5 on an Advanced Extension of Kashihara's Problem

Marco Ripà

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Abstract

This work presents the solution of Problem 1 of Question #30 from *Comments and Topics on Smarandache Notions and Problems*, published in 1996 by Kenichiro Kashihara, corresponding to Problem 16 of *Only Problems, Not Solutions!*. It is a recreational mathematics problem that remained open until 2010, when an extended version was published in Appendix 2 of *Divisibilità per 3 degli elementi di particolari sequenze numeriche*, *Rudi Mathematici Bookshelf*. The question concerns OEIS sequence A001292 (the so-called "circular sequence") and the probability that a generic element ends with a given digit $c \in \{0, 1, 2, \ldots, 9\}$. In 2010, an enhanced version of the problem was studied, providing the general formula of this probability as a function of the length of the last complete block and of c, and using the exact value to bound the interval in which the probability lies in the inter-block (partial block) case, as reported in note 14 on page 17 of the aforementioned work. On August 9, 2025, GPT-5 independently solved the same enhanced version of Kashihara's question, proposing a more compact formula and extending it exactly to the incomplete block case as well.

Disclaimer: This preprint was entirely written as part of a conversation between Marco Ripà and ChatGPT (GPT-5, OpenAI) on August 9, 2025. It faithfully reproduces the exchange, with possible minor formal corrections. This is the first preprint in which the author reports on a mathematical test conducted together with an artificial intelligence model (GPT-5, OpenAI). The model was challenged to independently rediscover results previously obtained and published by the author, and the experiment documents both the reconstruction process and the formulation of a more compact expression. The responsibility for the content, the results, and their dissemination lies entirely with the human author.

1 Original Text of Problem 1 of Question #30

Context

Question #30, as presented by Kashihara in 1996 [1], refers to the original Problem 16 of [6] (1993). The authorship of OEIS sequence A001292 [2] does not appear to belong to Kashihara, but the question and its specific formulation are attributed to him.

Original Text of the Problem

#30 Problem 16 in *Only Problems, Not Solutions!*

Definition: The Smarandache Circular Sequence is defined as follows:

1, 12, 21, 123, 231, 312, 1234, 2341, 3412, 4123, 12345, 23451, 34512, ...

And Smarandache asked, "How many elements of this sequence are prime?"

Problem 1: For $c \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, find the probability that the trailing digit of a term is c.

Historical Note

Question #30 includes two distinct problems.
Problem 1, reported above, is the one of interest in this work.
The second problem of Question #30 is instead discussed in [5].

2 Context and Background

Problem 1 of Question #30 by Kashihara is a probability exercise related to the circular sequence [2], but formulated in such a way as to remain formally unsolved at the time of publication (1996). The solution, in its basic form, is simple to deduce: in the limit for an infinite number of terms (see [4], p. 17), each final digit $c \in \{0, \ldots, 9\}$ appears with the same frequency, corresponding to a probability of $\frac{1}{10}$.

The Enhanced Version (2010) by Marco Ripà

In 2010, the author solved a more general version of Kashihara's problem (see [4], Appendix 2). In that work, the author not only formulated the extended problem, but also derived the general formula and showed that the original problem posed by Kashihara follows as a limiting case. The enhancement introduced:

- the variable r, representing the maximum index of the sequence included in the count (and thus the total length up to the closure of the last complete block);
- an exact formula for the probability that a term in the first r complete blocks has c as its final digit.

This extension requires determining, for each r and c, the number of occurrences of digit c as the last digit and dividing it by the total number of terms considered. In the 2010 work, the treatment distinguished between the case $c = 0$ and the case $c > 0$, and the formula was given explicitly (though not in its most compact possible form).

The Test of August 9, 2025 with GPT-5

On August 9, 2025, Marco Ripà proposed to GPT-5 (ChatGPT Plus version) [3] the task of solving the enhanced version without consulting external sources. The aim of the test was to:

1. verify whether GPT-5 was able to autonomously reconstruct the exact formula;

2. compare the result with the 2010 formula;
3. evaluate the ability to generalize to the *inter-block* case (when stopping before completing block $r+1$);
4. verify, without prior notice, whether the model would correctly distinguish the case $c = 0$ from the case $c > 0$.

Result of the Test

GPT-5:

1. quickly solved the original problem (obtaining $\lim_{r\to\infty} P_r(c) = \frac{1}{10}$ for every c);
2. derived an exact formula for $P_r(c)$, coinciding in content with the one published by Marco Ripà in 2010 and in an even more compact form;
3. extended the analysis to the *inter-block* case (for which in [4], note 14, p. 17, was provided only an interval whose width rapidly tends to zero as the maximum considered term increases), deriving the corresponding exact formula.

3 Solution of Kashihara's Original Problem

Statement

The original problem therefore asks to determine, for each $c \in \{0, 1, \ldots, 9\}$, the probability that a term of the circular sequence [2] ends with digit c, considering the limit of the relative frequency as the number of terms increases.

Key Observation

The sequence is formed by blocks of length n, each built by concatenating the numbers from 1 to n and then performing all possible cyclic rotations. Here n denotes the length of a single block, that is, the block obtained from the integers $1, 2, \ldots, n$ and all their cyclic rotations. Later, in the enhanced version, we shall use r to denote instead the index of the last complete block considered in the counting process.

In block n, the final digits of the terms coincide exactly with the elements of $\{1, 2, \ldots, n\}$ taken as the last position in each rotation.

For example:

- $n = 3$: final digits $\{3, 1, 2\}$;
- $n = 4$: final digits $\{4, 1, 2, 3\}$.

When $n > 10$, the final digits in the blocks are distributed cyclically modulo 10.

Asymptotic Distribution

Considering all blocks from $n = 1$ to $n \to \infty$, each digit $c \in \{0, \ldots, 9\}$ appears with asymptotically identical frequency in the last positions, since:

1. in each large block, the final digits are almost equidistributed;
2. the difference due to the remainders ($n \bmod 10$) becomes negligible as $n \to \infty$;
3. the number of occurrences of each digit grows as $\Theta(N^2)$, and the differences between digits are at most $O(N)$.

Result

It follows that

$$\lim_{r\to\infty} P_r(c) = \frac{1}{10}, \quad \text{for every } c \in \{0, 1, \ldots, 9\}.$$

This is the solution of the base problem (#30 – Problem 1) in its original form (already highlighted by the explicit chain of equalities obtained after the limiting process, on p. 17 of [4]).

4 Maximum Discrepancy: the Choice of r

Definition

For each $r \geq 1$, let $P_r(c)$ be the probability that the last digit is $c \in \{0, \ldots, 9\}$ considering all complete blocks $1, \ldots, r$. We define the maximum discrepancy as

$$\Delta(r) := \max_c P_r(c) - \min_c P_r(c).$$

Discrepancy with Positivity Constraint

We focus on the regime where every digit has already appeared at least once, that is $P_r(c) > 0$ for all c, which requires $r \geq 10$.

In this setting, the maximum discrepancy (see [4], p. 17) is attained at

$$r = 10,$$

because digit 0 occurs for the very first time (only once in total up to $r = 10$), while digit 1 has accumulated the highest frequency from the smaller blocks.

For $r > 10$ the discrepancy decreases monotonically and tends to 0 as $r \to \infty$.

Exact Values at $r = 10$

With $T_{10} = \sum_{n=1}^{10} n = 55$, the probabilities at $r = 10$ are

$$P_{10}(0) = \frac{1}{55}, \qquad P_{10}(c) = \frac{11 - c}{55} \quad (c = 1, \ldots, 9).$$

Hence

$$\Delta(10) = P_{10}(1) - P_{10}(0) = \frac{10}{55} - \frac{1}{55} = \frac{9}{55}.$$

Remark (without positivity constraint)

If the positivity requirement $P_r(c) > 0$ is removed, the maximum discrepancy trivially occurs at $r = 1$ (where only digit 1 has appeared). In this case $\Delta(r)$ decreases monotonically with r.

5 General Formula for Complete Blocks

Notation

For convenience, we summarize the notation already in use:

- r: index of the last complete block considered;
- $c \in \{0, 1, \ldots, 9\}$: final digit;
- $T_r = \dfrac{r(r+1)}{2}$: total number of terms in the first r complete blocks.

Counting the Occurrences

For $c \geq 1$ and $r \geq c$:

$$q = \left\lfloor \frac{r-c}{10} \right\rfloor, \quad N_r(c) = (r - c + 1) + q\,(r - c - 5q - 4).$$

If $r < c$, then $N_r(c) = 0$.

For $c = 0$:

$$q = \left\lfloor \frac{r}{10} \right\rfloor, \quad N_r(0) = q\,(r - 5q - 4).$$

Probability Formula and Historical Note

The probability that the last digit is $c \in \{0, \ldots, 9\}$ at the end of blocks $1, \ldots, r$ is

$$P_r(c) = \frac{N_r(c)}{T_r}.$$

This expression, independently derived by GPT-5, coincides in substance with the formula published by Marco Ripà in 2010, though here it is presented in a more compact form. In the 2010 version, the cases $c = 0$ and $c > 0$ were explicitly separated and calculated via partial sums.

6 General Inter-block Formula

"Ad hoc" Notation

We consider all complete blocks $1, \ldots, r$ and only the first m terms of the $(r+1)$-th block, with $0 \leq m \leq r+1$. Let $c \in \{0, \ldots, 9\}$ be the final digit of the number under consideration, and introduce the following notation:

- $T_r = \frac{r(r+1)}{2}$: number of terms in the first r complete blocks.

- $T = T_r + m$: total number of terms considered.
- $d \equiv (r+1) \pmod{10}$: last digit of the first term of block $r+1$.
- $[P]$: indicator function, equal to 1 if P is true, 0 otherwise.
- $x_+ = \max\{x, 0\}$: positive part of x.

Counts up to Block r

For $c \in \{1, \ldots, 9\}$:

$$q_c = \left\lfloor \frac{r-c}{10} \right\rfloor, \quad N_r(c) = \begin{cases} (r-c+1) + q_c\,(r-c-5q_c-4), & r \geq c, \\ 0, & r < c. \end{cases}$$

For $c = 0$:

$$q_0 = \left\lfloor \frac{r}{10} \right\rfloor, \quad N_r(0) = q_0\,(r - 5q_0 - 4).$$

General Formula for the Probability that c is the Last Digit

In block $r+1$, the last digits of the first m terms are

$$\{\, d \,\} \cup \{\, 1, 2, \ldots, m-1 \,\} \pmod{10}.$$

The additive contribution splits as:

$$A_1(c) = [\, m \geq 1 \,]\,[\, c = d \,],$$

$$A_2(c) = \left(\left\lfloor \tfrac{m-1-c}{10} \right\rfloor + [\, m-1 \geq c \geq 1 \,] \right)_+ .$$

Therefore, we conclude that:

$$P_{r,m}(c) = \frac{N_r(c) + A_1(c) + A_2(c)}{T_r + m}.$$

Side Notes

- Case $m = 0$: $A_1 = A_2 = 0 \;\Rightarrow\; P_{r,0}(c) = \frac{N_r(c)}{T_r}$ ("complete blocks" case).
- Case $m = r+1$: all $r+1$ rotations are obtained, hence $P_{r,r+1}(c) = P_{r+1,0}(c)$.
- Case $r \to \infty$ with m fixed: $P_{r,m}(c) \to \frac{1}{10}$ for every c.

Compact Version of the Contribution $A_2(c)$

To avoid the distinction between the cases $c \geq 1$ and $c = 0$, one can write:

$$A_2(c) = \left\lfloor \frac{m - 1 - \max(c, 0)}{10} \right\rfloor + [\, m-1 \geq c \geq 1 \,],$$

which produces exactly the same values as the original definition, but in a single form and without further subdivisions.

Conclusions

The general problem that extends and makes more complex Problem 1 of Question #30 by Kenichiro Kashihara [1] (asking, for $c \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, to determine the probability that the last digit of a term of the Smarandache circular sequence is c) was originally solved by Marco Ripà in 2010 [4].

Within the timed test of August 9, 2025, GPT-5 independently re-solved the same problem, recovering the result as a direct computation, while the author had previously derived it as a limiting case of his general formula. The subsequent discussion focused on formulating and proving a compact expression for the extended problem, valid for all pairs (r, m) and every digit c.

The compact version of $A_2(c)$, presented in the technical section, unifies the cases $c = 0$ and $c \geq 1$ into a single expression, thereby simplifying the notation while preserving complete arithmetic correctness.

Finally, we note that GPT-5 required less than five minutes to obtain the extended solution during the test, whereas the author originally needed about two hours in 2010 to derive his own formulation.

References

[1] K. Kashihara, *Comments and Topics on Smarandache Notions and Problems*, Erhus University Press, 1996.

[2] OEIS Foundation Inc., *Concatenations of cyclic permutations of initial positive integers*, Entry A001292 in *The On-Line Encyclopedia of Integer Sequences*, 2025. Accessed: August 9, 2025. Available at: `http://oeis.org/A001292`

[3] OpenAI, *ChatGPT (GPT-5)*, Plus version, August 2025. Available at: `https://chat.openai.com`

[4] M. Ripà, *Divisibilità per 3 degli elementi di particolari sequenze numeriche*, Rudi Mathematici (Bookshelf), RMBSH-020, 2010. Available at: `https://www.rudimathematici.com/bookshelf/pdf/Divisibilitaper3.pdf`

[5] M. Ripà, *On some open problems concerning perfect powers*, *arXiv*:2205.10163v4 (2024). Available at: `https://arxiv.org/pdf/2205.10163v4`

[6] F. Smarandache, *Only Problems, Not Solutions!*, Xiquan Publishing House, Phoenix, 1993.

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