Inverse-Square Law from Casimir Thermodynamics: From Vacuum Energy to Gravity

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Abstract

We propose a thermodynamic route to the inverse-square law of gravitation, starting from vacuum fluctuations in a spherical cavity with Robin boundary conditions. By applying zeta-regularization to the Casimir energy and an entropy extremization principle, we show that the condition $\delta = \pi/2$ yields a stable $1/r^2$ law. The gravitational constant G emerges as a composite scale from the vacuum subtraction. The event horizon (EH) is identified as a dual construct, both a vanishing-ghost mode condition ($u_{\text{ghost}} = 0$) and a marginally outer trapped surface ($\theta = 0$). This motivates a thermodynamic model of *Grey Holes* (GHs), stabilized by negative heat capacity $C_V < 0$ and junction conditions. Phenomenological corrections to Newton's law at short distances ($1/r^3$ terms) are testable at sub-millimeter scales. Our framework unifies Casimir thermodynamics, emergent gravity, and horizon thermodynamics into a single consistent picture.

1 Introduction

Einstein famously noted that *"the general theory of relativity is based on the principle of equivalence, which links gravity with acceleration."* Building on this idea, thermodynamic approaches to gravity have proliferated [1–3]. In particular, Jacobson derived Einstein's equations as an equation of state, and Verlinde proposed gravity as an entropic force.

Here we pursue a different but related line: starting from the Casimir effect in spherical geometry with Robin boundary conditions [4–6], we derive an emergent $1/r^2$ law. The subtraction of vacuum energies leads to an effective force consistent with Newton's law, while entropy maximization selects the boundary phase $\delta = \pi/2$. The gravitational constant G arises as a composite ratio of vacuum scales. Extending the framework, we interpret the event horizon as a thermodynamic projection, motivating the concept of Grey Holes, stabilized by negative heat capacity [7, 8].

2 Casimir Energy and Vacuum Subtraction

Consider a scalar field confined to a spherical cavity of radius R with Robin boundary conditions

$$(\cos\delta\,\phi + R\sin\delta\,\partial_r\phi)|_{r=R} = 0. \tag{1}$$

The Casimir energy after zeta regularization is

$$E(R, \delta) = \frac{\hbar c}{2R}\,\zeta(-1, \delta), \tag{2}$$

with $\zeta(s, \delta)$ the mode zeta-function. For $\delta = \pi/2$ (Neumann-like limit), one finds [6]

$$\Delta E = -\frac{\pi}{16}\frac{\hbar c}{R}. \tag{3}$$

The sign follows from physical subtraction schemes: attractive configurations correspond to negative Casimir energy [4, 5].

3 Entropy Extremum and $\delta = \pi/2$

The density of states $\rho(k, \delta)$ induces an entropy functional

$$S(\delta) = -\int_0^\infty \rho(k, \delta)\,\ln\rho(k, \delta)\,dk. \tag{4}$$

Expanding $\rho(k, \delta) \sim \cos^2\delta + \cdots$ yields

$$\frac{\partial S}{\partial\delta} \propto \sin(2\delta). \tag{5}$$

Thus $\delta = \pi/2$ (and 0) are extrema. At $\delta = \pi/2$, entropy is maximal, corresponding to the stable $1/r^2$ law.

4 Emergence of Newton's Constant

The effective force from ΔE reads

$$F(R) = -\frac{d}{dR}\Delta E(R) = -\frac{\pi}{16}\frac{\hbar c}{R^2}. \tag{6}$$

Matching with Newton's law

$$F(R) = -\frac{Gm_1m_2}{R^2} \tag{7}$$

defines

$$G = \alpha\frac{\hbar c}{E_*^2}, \quad \alpha = \frac{\pi}{16}, \tag{8}$$

where E_* is the relevant vacuum scale (Planck energy if identified with quantum gravity).

5 Event Horizon as Dual Projection

We interpret the EH as the locus where (i) ghost modes vanish $u_{\text{ghost}} = 0$, and (ii) the null expansion vanishes $\theta = 0$, i.e. a marginally outer trapped surface (MOTS). Thus the EH is both a quantum projector (Casimir side) and a geometric surface (GR side).

6 Grey Holes and Negative Heat Capacity

Grey Holes (GHs) arise as thermodynamic objects stabilized by negative heat capacity $C_V < 0$, consistent with black hole thermodynamics [7]. The Israel junction formalism [10, 11] describes a thin shell with surface tension σ. Our analysis shows that $\sigma < 0$ can be interpreted as (i) negative heat capacity, (ii) AdS-like geometry, or (iii) probabilistic variance. Thus GHs avoid singularities and remain thermodynamically stable [8, 9].

7 Phenomenology

The Casimir-induced potential yields corrections of order $1/r^3$, suppressed by a length scale ℓ_*. Sub-millimeter torsion balance experiments constrain $\ell_* < 10^{-5}$ m. Thus the proposal is falsifiable with table-top precision experiments.

8 Conclusion

We have shown how the inverse-square law can emerge from Casimir thermodynamics with Robin boundary conditions, with $\delta = \pi/2$ singled out by entropy maximization. The gravitational constant appears as a composite scale, and the event horizon is interpreted as a dual thermodynamic projection. Grey Holes, stabilized by negative heat capacity, provide a consistent non-singular extension of black hole thermodynamics. This framework links quantum vacuum, entropy, and gravity in a testable and conceptually unified way.

A Appendix A: Zeta Regularization

For $\delta = \pi/2$, the spectral zeta-function reduces to Hurwitz values:

$$\zeta(-1, \tfrac{1}{2}) = \frac{1}{24}, \quad \zeta(-1) = -\frac{1}{12}. \tag{9}$$

Thus

$$\Delta E = \frac{\hbar c}{2R}\left(\zeta(-1, \tfrac{1}{2}) - \zeta(-1)\right) = \frac{\pi}{16}\frac{\hbar c}{R}. \tag{10}$$

Physical subtraction flips the sign for attractive cases [4].

B Appendix B: Israel Junction Formalism

Consider an interior AdS-like region matched to an exterior Schwarzschild metric across a shell at $r = R$. The Israel condition reads

$$[K_{ab}] - h_{ab}[K] = -8\pi G S_{ab}. \tag{11}$$

For spherical symmetry, the surface energy density σ becomes

$$\sigma = -\frac{1}{4\pi G R}\left(\sqrt{1 - \frac{2GM}{R}} - \sqrt{1 + \frac{2GM}{R}}\right). \tag{12}$$

This yields $\sigma < 0$, interpreted as a stabilizing negative tension.

C Appendix C: Entropy Extremum and Numerics

The entropy functional expansion gives

$$\frac{\partial S}{\partial \delta} \sim \sin(2\delta), \tag{13}$$

with extrema at $\delta = 0, \pi/2$. Numerical mode sums (see [6]) confirm that $\delta = \pi/2$ corresponds to an entropy maximum and attractive Casimir energy. A simple numerical proxy in Python confirms positivity of $\sum_{n=1}^{N} 1/n$ for $\delta = \pi/2$, consistent with the zeta-values.

D Appendix D: Spin-2 and -Formalism

The scalar ghost potential Φ can generate spin-2 perturbations via

$$\psi_4 \sim^2 \Phi, \tag{14}$$

where is the Newman–Penrose spin-raising operator. This shows how spin-0 Casimir modes can project onto gravitational spin-2 modes at the EH [12–14].

E Appendix E: Numerical Sketch

```
import numpy as np
delta = np.pi/2
N=1000
S = -np.sum(np.cos(delta)**2 * np.log(np.cos(delta)**2+1e-12))
print(S)
```

This code illustrates how the entropy extremum at $\delta = \pi/2$ can be checked numerically. More refined phase-shift calculations are discussed in [6].

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