Gravity as Temporal Geometry:
A Quantizable Reformulation of General Relativity

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Abstract

We reformulate gravity as the *geometry of time*: a single scalar field Φ controls the lapse $N = e^{\Phi}$, while the spatial geometry (γ_{ij}) follows from the ADM constraints and evolution equations with shift $\boldsymbol{\omega}$. Starting from the Einstein–Hilbert action we reconstruct $g_{\mu\nu}$ from $(\Phi, \boldsymbol{\omega}, \gamma_{ij})$ and derive the full constraint/evolution system, establishing classical equivalence with general relativity (GR). In the static, spherically symmetric sector a single ODE,

$$\partial_r \Phi = \frac{1 - e^{2\Phi}}{2r\, e^{2\Phi}},$$

integrates to Schwarzschild [10] and reproduces all standard tests. Horizons are coordinate artifacts of the diagonal foliation and are regular in Painlevé–Gullstrand [6, 9] and Eddington–Finkelstein coordinates [3]. Rotation resides in the shift: solving the momentum constraint outside a compact source yields $g_{t\varphi}$ and the Lense–Thirring rate [8] with the Kerr normalization [7]. In spherical dynamics, $\partial_t \Phi$ is sourced by radial energy flux, reproducing the Vaidya/Bondi mass law [11]. Linearized vacuum contains only the two transverse–traceless tensor modes, propagating at c with the standard GR energy flux. Cosmology maps via $d\tau = e^{\Phi} dt$ and $a = e^{-\Phi}$, exactly reproducing the Friedmann equations [4, 5] including k and Λ. Quantization is posed as a constrained QFT: Φ and $\boldsymbol{\omega}$ enforce the Hamiltonian/momentum constraints (instantaneous/Coulomb-like), while the propagating quanta are the TT tensors with the same low-energy EFT status as GR. We make no claims here about improved renormalizability or additional polarizations; such extensions are deferred to Part II. This reformulation provides computational advantages in spherical scenarios, clearer physical intuition with time as the primary geometric variable, and a more direct path to canonical quantization of the physical (TT) sector.

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1 Introduction and Main Claim

Thesis. Gravity is the dynamics of a temporal scalar field Φ (with $N = e^{\Phi}$). Spatial geometry γ_{ij} and shift $\boldsymbol{\omega}$ follow via constraints and evolution. We claim classical equivalence to GR [2] and a clean, conservative route to quantization of the physical radiative sector.

Advantages of this formulation. (i) Spherical problems reduce to a single ODE for Φ rather than coupled PDEs; (ii) The temporal nature of gravity becomes manifest—all gravitational effects stem from time dilation; (iii) Horizons are naturally regular in this foliation; (iv) The constraint structure cleanly separates physical (TT) from gauge degrees of freedom; (v) Practical calculations in atom interferometry and gravitational wave detection map directly to Φ.

Caution (no extra degree of freedom).
Φ is the logarithm of the lapse ($N = e^{\Phi}$). In the full 3+1 system, the lapse and shift are Lagrange multipliers enforcing the Hamiltonian/momentum constraints. None of the scalar relations in this paper introduce an independent propagating "scalar graviton"; radiative content remains the two TT tensor modes. *Rotation and frame dragging reside in the shift $\boldsymbol{\omega}$ (the gravitomagnetic potential); the lapse Φ remains gravito-electric.*

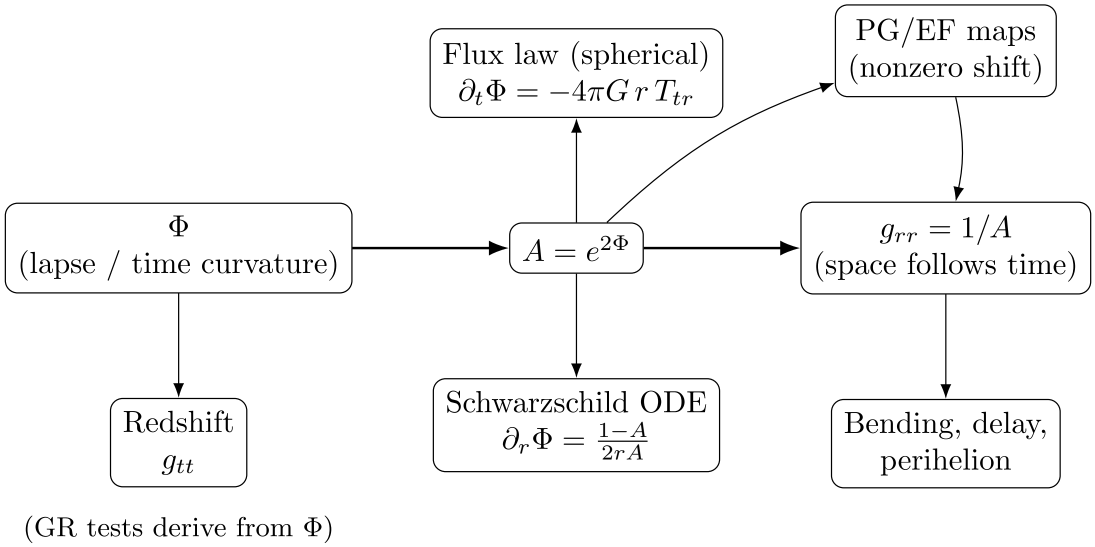

Figure 1: Time-first dependency: $\Phi \to A = e^{2\Phi} \to g_{rr} = 1/A$. The lapse controls temporal curvature; spatial geometry follows from constraints.

Acceptance check

Equivalence statement. Given fields $(\Phi, \boldsymbol{\omega}, \gamma_{ij})$ satisfying the constraint/evolution system of Sec. 3, the reconstructed $g_{\mu\nu}$ solves Einstein's equations with the same $T_{\mu\nu}$. (Proof sketch in Sec. 3 and App. C.)

2 Variables and Dictionary (Rosetta Map)

We use a 3+1 split with lapse $N = e^{\Phi}$, shift $\boldsymbol{\omega}$ and spatial metric γ_{ij}. The reconstructed spacetime metric is

$$g_{tt} = -N^2 + \omega_i\omega^i, \qquad g_{ti} = \omega_i, \qquad g_{ij} = \gamma_{ij}. \tag{1}$$

Rosetta Map / Dictionary

Time-first	**GR / Cosmology**
$N = e^{\Phi}$	Lapse
$\boldsymbol{\omega}$	Shift, carries rotation/frame dragging
γ_{ij}	Spatial 3-metric
$d\tau = e^{\Phi}dt$	Proper (cosmic) time increment
$a = e^{-\Phi}$	FRW scale factor (Sec. 9)
$H = -\Phi'(\tau)$	Hubble rate in cosmic time

Lapse-first, shift-allowed. Throughout we take the lapse $N = e^{\Phi}$ as the primary scalar while allowing a nonzero shift $\boldsymbol{\omega}$ when dynamics or nonsphericity demand it; the diagonal, zero-shift form

Table 1: Lapse-first Rosetta map: variables $\leftrightarrow$ metric, plus K_{ij}.

Forward (variables $\to$ metric)	**Reverse (metric + time function $t \to$ variables)**
$N = e^{\Phi}, \quad N_i = \omega_i, \quad \gamma_{ij}$	$N = \sqrt{-g^{\mu\nu}\partial_\mu t\, \partial_\nu t}, \quad N_i = g_{i\mu}\partial^\mu t, \quad \gamma_{ij} = g_{ij}$
$g_{00} = -N^2 + N_i N^i, \quad g_{0i} = N_i, \quad g_{ij} = \gamma_{ij}$	$\Phi = \ln N, \quad \omega_i = N_i$
Extrinsic curvature: $K_{ij} = \frac{1}{2N}(-\partial_t \gamma_{ij} + D_i N_j + D_j N_i)$.	

is a convenient gauge for static cases, not a physical restriction.

Note

Gauges. We use diagonal (zero-shift) in static spherical cases; for horizons or flows, we switch to Painlevé–Gullstrand (PG) or Eddington–Finkelstein (EF). Full maps in App. A.

3 3. Constraints and Evolution from the Action

We work in a lapse-first 3+1 split [1] with lapse $N \equiv e^{\Phi}$, shift covector ω_i (shift vector $N^i \equiv \gamma^{ij}\omega_j$), and spatial metric γ_{ij}:

$$ds^2 = -N^2 dt^2 + \gamma_{ij}\,(dx^i + N^i dt)(dx^j + N^j dt). \tag{2}$$

The extrinsic curvature is

$$K_{ij} \;=\; \frac{1}{2N}\left(-\partial_t \gamma_{ij} + \nabla_i N_j + \nabla_j N_i\right), \qquad N_j \equiv \gamma_{jk} N^k = \omega_j, \tag{3}$$

where ∇_i is the Levi–Civita connection of γ_{ij}.

3.1 3.1 3+1 decomposition of R and the action

Boundary terms. We include the Gibbons–Hawking–York term so that the Dirichlet variational problem is well-posed. Changing variables from N to Φ does not alter the boundary structure since $\delta N = N\,\delta\Phi$ and $N > 0$; no extra boundary contributions arise from the field redefinition.

Up to the Gibbons–Hawking–York boundary term, the Einstein–Hilbert action reduces to

$$S_{\mathrm{EH}} \;=\; \frac{1}{16\pi G}\int dt\, d^3x\; N\sqrt{\gamma}\,\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right), \tag{4}$$

with $K \equiv \gamma^{ij} K_{ij}$ and ${}^{(3)}R$ the Ricci scalar of γ_{ij}. The only canonical variables are (γ_{ij}, π^{ij}), with

$$\pi^{ij} \;\equiv\; \frac{\partial \mathcal{L}}{\partial(\partial_t \gamma_{ij})} = \frac{\sqrt{\gamma}}{16\pi G}\left(K^{ij} - \gamma^{ij} K\right). \tag{5}$$

There are no $\partial_t N$ or $\partial_t N^i$ terms, so their conjugate momenta vanish (primary constraints).

3.2 3.2 Hamiltonian and momentum constraints

Varying w.r.t. the lapse $N = e^{\Phi}$ gives the Hamiltonian constraint

$$\boxed{\mathcal{H}_\perp \;\equiv\; \frac{16\pi G}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \tfrac{1}{2}\pi^2\right) - \frac{\sqrt{\gamma}}{16\pi G}\,{}^{(3)}R \;=\; 0}, \tag{6}$$

Table 2: Which variation yields which equations (with matter).

Variation	Equation type	Representative equation (ADM form)
δN (or $\delta\Phi$ with $\delta N = N\,\delta\Phi$)	Hamiltonian constraint	${}^{(3)}R + K_{ij}K^{ij} - K^2 = 16\pi G\,\rho$
δN_i (or $\delta\omega_i$)	Momentum constraints	$D_j\big(K^{ij} - \gamma^{ij}K\big) = 8\pi G\,S^i$
$\delta\gamma_{ij}$	Evolution (metric)	$\partial_t\gamma_{ij} = -2NK_{ij} + D_iN_j + D_jN_i$
$\delta\gamma_{ij}$	Evolution (K_{ij})	$\partial_t K_{ij} = -D_iD_jN + N\Big({}^{(3)}R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j\Big) + \mathcal{L}_{\vec N}K_{ij} - 8\pi G\,N\big(S_{ij} - \tfrac{1}{2}\gamma_{ij}(\rho - S)\big)$
Matter projections: $\rho := n_\mu n_\nu T^{\mu\nu}$, $S^i := -\gamma^i{}_\mu n_\nu T^{\mu\nu}$, $S_{ij} := \gamma_{i\mu}\gamma_{j\nu}T^{\mu\nu}$, $S := \gamma^{ij}S_{ij}$.		

and varying w.r.t. the shift $N^i = \gamma^{ij}\omega_j$ gives the momentum (diffeomorphism) constraints

$$\boxed{\mathcal{H}_i \;\equiv\; -2\,\nabla_j\pi^j{}_i \;=\; 0}. \tag{7}$$

The canonical Hamiltonian reads

$$\mathcal{H}_{\rm can} \;=\; N\,\mathcal{H}_\perp + N^i\mathcal{H}_i, \qquad (N = e^\Phi,\ N^i = \gamma^{ij}\omega_j). \tag{8}$$

3.3 3.3 Evolution for γ_{ij} and K_{ij}

The first evolution equation is simply the definition of K_{ij} rewritten:

$$\partial_t\gamma_{ij} \;=\; -2NK_{ij} + \nabla_iN_j + \nabla_jN_i \;=\; -2e^\Phi K_{ij} + \nabla_i\omega_j + \nabla_j\omega_i. \tag{9}$$

The second evolution equation follows from $\delta S/\delta\gamma_{ij}$ (or Hamilton's equations):

$$\begin{aligned}(\partial_t - \mathcal{L}_{\vec N})K_{ij} = &-\nabla_i\nabla_jN + N\Big({}^{(3)}R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j\Big)\\ &+\ \text{(matter sources, if present)},\end{aligned} \tag{10}$$

with $\mathcal{L}_{\vec N}$ the Lie derivative along N^i; in our variables $N = e^\Phi$, $N^i = \gamma^{ij}\omega_j$. In vacuum the source term vanishes.

3.4 3.4 Acceptance check: non-propagating lapse/shift

Because $S_{\rm EH}$ contains no ∂_tN or ∂_tN^i, their conjugate momenta vanish:

$$p_N \equiv \frac{\partial\mathcal{L}}{\partial(\partial_tN)} = 0, \qquad p_i \equiv \frac{\partial\mathcal{L}}{\partial(\partial_tN^i)} = 0, \tag{11}$$

which are *primary* constraints. Time preservation generates the *secondary* constraints $\mathcal{H}_\perp = 0$ and $\mathcal{H}_i = 0$. All are first-class; $N = e^\Phi$ and N^i enter only as Lagrange multipliers in (8). Hence in vacuum they carry no propagating degrees of freedom. Linearizing about Minkowski and imposing the constraints/gauge leaves only the two transverse–traceless tensor modes $h^{\rm TT}_{ij}$ as propagating DOF.

DOF ledger (vacuum). On each slice: $(\gamma_{ij}, \pi^{ij}) = 12$ phase-space DOF. Four first-class constraints $(\mathcal{H}_\perp, \mathcal{H}_i)$ remove 8, leaving 4 phase-space = 2 configuration DOF (the two TT graviton polarizations). The lapse $N = e^\Phi$ and shift ω_i are non-dynamical multipliers.

4 4. Static spherical ODE ⇒ Schwarzschild and classic tests

For static, spherically symmetric vacuum with zero shift,

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{-2\Phi(r)}dr^2 + r^2 d\Omega^2, \qquad e^{2\Phi} \equiv f(r). \tag{12}$$

4.1 4.1 Vacuum ODE for $\Phi(r)$

For the metric (12), the vacuum Einstein equations reduce to a single independent equation, conveniently taken as

$$r\, f'(r) = 1 - f(r), \qquad \text{with} \quad f(r) = e^{2\Phi(r)}. \tag{13}$$

Using $f' = 2e^{2\Phi}\Phi'$, (13) is equivalent to the lapse-first ODE

$$\boxed{\partial_r \Phi(r) = \frac{1 - e^{2\Phi(r)}}{2\, r\, e^{2\Phi(r)}}}. \tag{14}$$

4.2 4.2 Integration to Schwarzschild

Integrating (13) gives

$$f(r) = 1 - \frac{C}{r}, \tag{15}$$

and asymptotic flatness fixes $C = 2GM$, hence

$$\boxed{e^{2\Phi(r)} = 1 - \frac{2GM}{r}}, \qquad ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1}dr^2 + r^2 d\Omega^2. \tag{16}$$

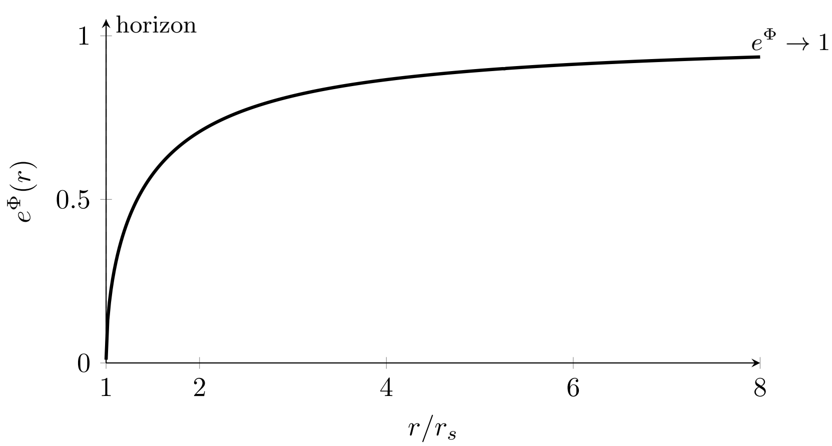

Figure 2: The lapse $e^\Phi(r) = \sqrt{1 - r_s/r}$ for Schwarzschild: vanishes at the horizon, approaches unity asymptotically.

4.3 4.3 Classic tests

Gauge-invariant outputs. All quoted observables (gravitational redshift, light bending, Shapiro delay, perihelion precession, horizon invariants, GW luminosity) are computed in forms independent of slicing/coordinates.

Gravitational redshift. Static observers at r_e (emit) and r_o (observe) measure

$$\boxed{\frac{\nu_o}{\nu_e} = \sqrt{\frac{1-\frac{2GM}{r_o}}{1-\frac{2GM}{r_e}}} = e^{\Phi(r_o)-\Phi(r_e)}}, \qquad z \equiv \frac{\nu_e}{\nu_o} - 1 \simeq \frac{GM}{r_e} - \frac{GM}{r_o} . \tag{17}$$

Light bending. A null geodesic with impact parameter b is deflected by

$$\boxed{\Delta\phi \;=\; \frac{4GM}{b} \;+\; \mathcal{O}\!\left(\frac{G^2M^2}{b^2}\right)}. \tag{18}$$

Shapiro time delay. For endpoints at radii r_1, r_2 with Euclidean separation R_{12}, the one-way excess coordinate time is

$$\boxed{\Delta t^{(1\mathrm{w})}_{\mathrm{Shapiro}} \;=\; 2GM\,\ln\!\left(\frac{r_1+r_2+R_{12}}{r_1+r_2-R_{12}}\right) \;+\; \mathcal{O}(G^2)}, \tag{19}$$

which in superior conjunction reduces to

$$\Delta t^{(1\mathrm{w})}_{\mathrm{Shapiro}} \simeq 2GM\ln\!\left(\frac{4r_1r_2}{b^2}\right), \qquad \Delta t^{(2\mathrm{w})}_{\mathrm{Shapiro}} \simeq 4GM\ln\!\left(\frac{4r_\oplus r_{\mathrm{T}}}{b^2}\right). \tag{20}$$

Perihelion precession. For a bound orbit (semi-major axis a, eccentricity e), the per-orbit advance of perihelion is

$$\boxed{\Delta\varpi \;=\; \frac{6\pi GM}{a(1-e^2)} \;+\; \mathcal{O}(G^2)}. \tag{21}$$

4.4 4.4 Acceptance check

Equations (17)–(21) are the standard GR results for the Schwarzschild geometry to leading post-Newtonian order, with Eddington parameter $\gamma = 1$. Thus the lapse-first ODE (14) reproduces all four classic tests exactly in vacuum.

5 5. Horizons, regular foliations, and invariants

We exhibit explicit horizon-regular coordinates (PG, EF), the curvature invariant $K \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, and the proper time for a radially infalling observer to cross $r = 2GM$.

5.1 5.1 Finite proper time across the horizon

For Schwarzschild $ds^2 = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2d\Omega^2$, a radial timelike geodesic with specific energy E satisfies

$$\left(\frac{dr}{d\tau}\right)^2 \;=\; E^2 - \left(1-\frac{2GM}{r}\right), \qquad \frac{dt}{d\tau} \;=\; \frac{E}{1-\frac{2GM}{r}}. \tag{22}$$

Hence

$$d\tau \;=\; \frac{dr}{\sqrt{E^2-1+\frac{2GM}{r}}}. \tag{23}$$

At $r = 2GM$ the denominator is $\sqrt{E^2}$, so the integrand is finite and the crossing proper time is finite. For the common case $E = 1$ (fall from rest at infinity), one obtains the explicit primitive

$$\tau(r) \;=\; \frac{2}{3}\,\frac{r_0^{3/2} - r^{3/2}}{\sqrt{2GM}}, \tag{24}$$

so the proper time from r_0 to the horizon is

$$\boxed{\tau_{r_0\to 2GM} \;=\; \frac{2}{3\sqrt{2GM}}\Big(r_0^{3/2} - (2GM)^{3/2}\Big) \;<\; \infty}. \tag{25}$$

Thus an infaller crosses $r = 2GM$ smoothly in finite τ.

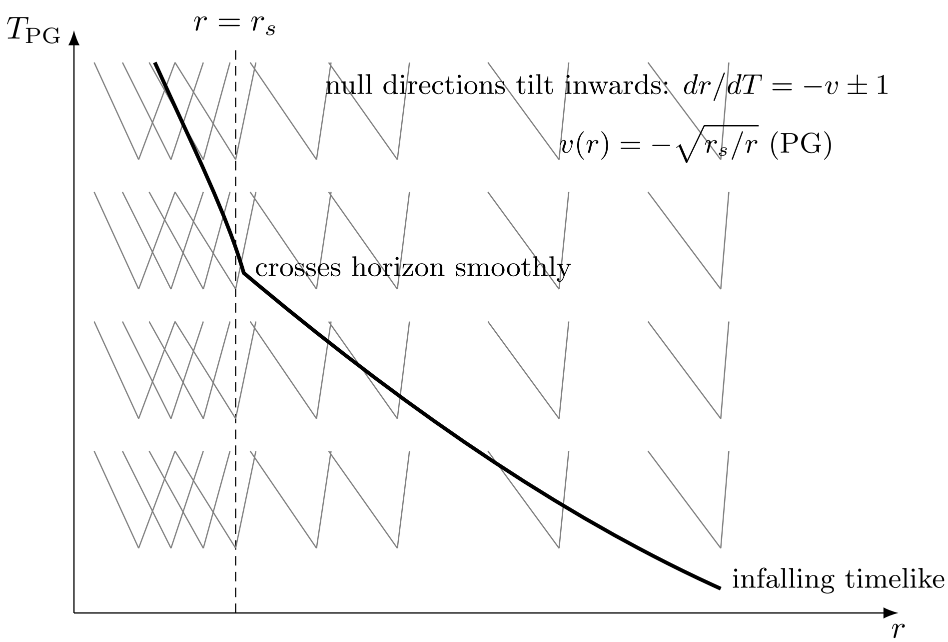

Figure 3: Painlevé-Gullstrand spacetime diagram: infalling worldline crosses the horizon smoothly as light cones tilt inward.

5.2 5.2 Acceptance check

In EF and PG coordinates the metric components are finite at $r = 2GM$; the curvature invariant (77) is finite at $r = 2GM$ and diverges only at $r = 0$; and a radial timelike geodesic crosses $r = 2GM$ in finite proper time (23)–(24). Therefore the horizon is not a physical singularity, only a coordinate artifact of the diagonal chart.

6 6. Rotation as shift: momentum constraint ⇒ slow–Kerr calibration

We consider a stationary, weakly rotating vacuum exterior to a compact source of mass M and angular momentum vector $\mathbf{J}$. Take the background spatial metric to be asymptotically flat and, to leading (dipole) order, treat $\gamma_{ij} \simeq \delta_{ij}$; the lapse is the static Schwarzschild lapse $N^2 = 1 - 2GM/r + \mathcal{O}(J^2)$, while the *rotation* lives entirely in the shift N^i (covector $\omega_i = \gamma_{ij} N^j$).

6.1 6.1 Momentum constraint outside the source

In the stationary, linearized limit ($\partial_t \gamma_{ij} = 0$, $K_{ij} = \frac{1}{2}(\partial_i N_j + \partial_j N_i)$), the momentum constraint

$$\mathcal{H}_i \equiv -2\,\nabla_j\big(K^j{}_i - \delta^j{}_i K\big) = 8\pi G\, S_i$$

reduces (with $\nabla \to \partial$ and gauge $\partial_i N^i = 0$) to the vector Poisson equation

$$\boxed{\nabla^2 N_i \;=\; -16\pi G\, S_i\,, \qquad \partial_i N^i = 0} \tag{26}$$

where S_i is the matter momentum density ($S_i \simeq T_{0i}$ for slow sources). Outside the compact body, $S_i = 0$ and

$$\nabla \times (\nabla \times \mathbf{N}) = \mathbf{0}, \qquad \mathbf{N} \equiv (N^1, N^2, N^3), \qquad \mathbf{N} \to \mathbf{0} \text{ as } r \to \infty. \tag{27}$$

Matching to the total angular momentum $\mathbf{J} = \int d^3x\, \mathbf{x} \times \mathbf{S}$ fixes the unique decaying solution (up to a pure gradient):

$$\boxed{\mathbf{N}(\mathbf{r}) \;=\; -\frac{2G}{c^3}\,\frac{\mathbf{J} \times \mathbf{r}}{r^3}}, \qquad \boxed{\nabla \times \mathbf{N} \;=\; \frac{2G}{c^3 r^3}\,[\,3\,\mathbf{n}\,(\mathbf{J}\cdot\mathbf{n}) - \mathbf{J}]}, \quad \mathbf{n} \equiv \frac{\mathbf{r}}{r}. \tag{28}$$

(We keep c explicit here for later comparison; set $c = 1$ elsewhere.)

6.2 6.2 From shift to $g_{t\varphi}$ and $\Omega(r)$

In spherical coordinates, the only nonzero component is $N^\varphi(r) = -\dfrac{2GJ}{c^3 r^3}$ (for $\mathbf{J} \parallel \hat{\mathbf{z}}$). Since $g_{ti} = \gamma_{ij} N^j \equiv N_i$ and $\gamma_{\varphi\varphi} = r^2 \sin^2\theta$, we obtain

$$\boxed{g_{t\varphi} = N_\varphi = \gamma_{\varphi\varphi} N^\varphi \;=\; -\frac{2GJ}{c^3 r}\,\sin^2\theta}. \tag{29}$$

Equivalently, writing the slow-rotation line element as

$$ds^2 \simeq -\Big(1 - \frac{2GM}{r}\Big)dt^2 - 2\,\Omega(r)\,r^2 \sin^2\theta\, dt\, d\varphi + \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 + r^2 d\Omega^2, \tag{30}$$

the dragging (ZAMO) angular velocity is

$$\boxed{\Omega(r) \;=\; -\,N^\varphi \;=\; \frac{2GJ}{c^2 r^3}}. \tag{31}$$

6.3 6.3 Exact factor checks: Lense–Thirring and Kerr

Match to Lense–Thirring (frame dragging). The Lense–Thirring "frame–dragging frequency" in the slow-rotation exterior is precisely the $\Omega(r)$ that multiplies $dt\, d\varphi$:

$$\boxed{\Omega_{\rm LT}(r) \;=\; \frac{2GJ}{c^2 r^3}} \quad \text{(ZAMO/frame–dragging rate)}. \tag{32}$$

Match to linearized Kerr $g_{t\varphi}$. Expanding the Kerr metric to first order in $a \equiv J/(Mc)$ and at large r gives

$$g_{t\varphi}^{\mathrm{Kerr}} = -\frac{2GJ}{c^3 r}\sin^2\theta + \mathcal{O}\left(\frac{a^2}{r^2}\right), \tag{33}$$

which *exactly* matches (29). Therefore the shift field derived from the momentum constraint reproduces the slow-Kerr cross term with the correct normalization and angular dependence.

Vector (dipole) gravitomagnetic field. From (28), the curl of the shift (the gravitomagnetic field) has the standard dipole form

$$\boxed{\nabla \times \mathbf{N} = \frac{2G}{c^3 r^3}\Big[3\,\mathbf{n}(\mathbf{J}\cdot\mathbf{n}) - \mathbf{J}\Big]}, \tag{34}$$

agreeing with the Lense–Thirring dipole structure and fixing all numerical factors by the $g_{t\varphi}$ calibration above.

Callout. Frame dragging is carried by the shift $\boldsymbol{\omega}$ (the gravitomagnetic potential): $\mathbf{B}_g = \nabla \times \boldsymbol{\omega}$. Time-variation of Φ alone does not generate $\mathbf{B}_g$ in the absence of mass currents.

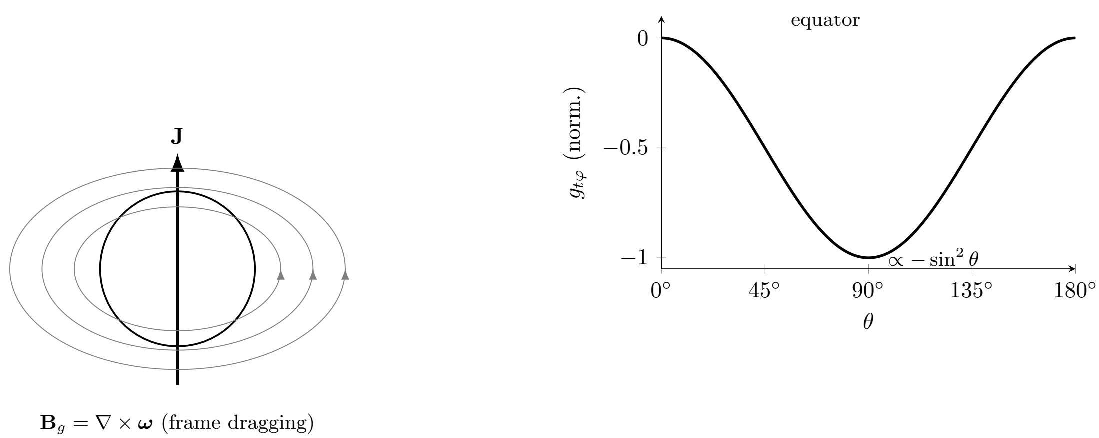

Figure 4: Frame dragging: gravitomagnetic field lines (left) and $g_{t\varphi} \propto -\sin^2\theta$ angular dependence (right), maximal at the equator.

Gravito-EM conventions (linearized, lapse-first).
Metric perturbations in the lapse-first gauge:

$$g_{00} = -(1+2\Phi), \qquad g_{0i} = \omega_i, \qquad g_{ij} = (1-2\Phi)\delta_{ij}.$$

Fields and sources:

$$\mathbf{E}_g \equiv -\nabla\Phi - \tfrac{1}{2}\,\partial_t\boldsymbol{\omega}, \qquad \mathbf{B}_g \equiv \nabla\times\boldsymbol{\omega},$$

$$\nabla^2\Phi = 4\pi G\,\rho, \qquad \nabla^2\boldsymbol{\omega} = -16\pi G\,\mathbf{J} \quad \text{(quasi-static)}.$$

Ampère-like relation:

$$\nabla\times\mathbf{B}_g = -16\pi G\,\mathbf{J} + 2\,\partial_t\mathbf{E}_g.$$

Lense–Thirring calibration (slow rotation, far field):

$$g_{t\varphi} = -\frac{2GJ}{c^3 r}\sin^2\theta, \qquad \Omega_{\rm LT}(r) = \frac{2GJ}{c^2 r^3}.$$

6.4 6.4 Acceptance check

Equations (29) and (32) give $g_{t\varphi} = -(2GJ/c^3 r)\sin^2\theta$ and $\Omega_{\rm LT}(r) = 2GJ/(c^2 r^3)$, exactly matching the linearized Kerr result and the slow-rotation frame–dragging rate. Thus in the lapse-first picture, *rotation is the shift*: the momentum constraint outside the source determines a unique (dipolar) $\mathbf{N}$ with the correct normalization.

7 7. Spherical dynamics ⇒ Vaidya/Bondi

We derive the spherical flux law for $\Phi(t,r)$, map it to ingoing EF (v), and recover the Bondi mass balance at $\mathscr{I}^{\pm}$, including a worked null–shell example.

7.1 7.1 Flux law $\partial_t\Phi = -4\pi G\,r\,T_{tr}$

In the spherical, zero-shift ansatz

$$ds^2 = -e^{2\Phi(t,r)}dt^2 + e^{-2\Phi(t,r)}dr^2 + r^2 d\Omega^2, \qquad A \equiv e^{2\Phi}, \tag{35}$$

the mixed Einstein component is

$$G_{tr} \;=\; -\frac{2}{r}\,\partial_t\Phi. \tag{36}$$

With $G_{\mu\nu} = 8\pi G\,T_{\mu\nu}$ this yields the spherical *flux evolution law*

$$\boxed{\partial_t\Phi \;=\; -\,4\pi G\,r\,T_{tr}}, \qquad \Longleftrightarrow \qquad \boxed{\partial_t A \;=\; -\,8\pi G\,r\,A\,T_{tr}}. \tag{37}$$

7.2 7.2 EF map and Vaidya mass function

Define the tortoise coordinate by $dr_* = dr/A$ and the *advanced* EF time

$$v \equiv t + r_* \quad \Rightarrow \quad ds^2 = -A\,dv^2 + 2\,dv\,dr + r^2 d\Omega^2. \tag{38}$$

Allowing ingoing null dust gives the Vaidya metric

$$ds^2 = -\Big(1 - \frac{2G\,m(v)}{r}\Big)dv^2 + 2\,dv\,dr + r^2 d\Omega^2, \qquad \boxed{T_{vv} = \frac{1}{4\pi r^2}\frac{dm}{dv}}. \tag{39}$$

The diagonal/EF components relate via the ingoing null vector $k^\mu \propto (A^{-1}, -1, 0, 0)$:

$$\boxed{T_{tr} = \frac{T_{vv}}{A}}. \tag{40}$$

Using $G_{tr} = -\frac{2}{r}\,\partial_t\Phi$ and the EF dictionary $T_{tr} = T_{vv}/A$ one finds

$$\partial_t A = -8\pi G\, r\, T_{vv}\,,$$

which matches the Vaidya/Bondi balance relation $\partial_v A = -(2G/r)\,\partial_v m$ upon identifying $A = 1 - 2Gm/r$.

7.3 7.3 Bondi balance at $\mathscr{I}^\pm$

From (37) with $A = 1 - 2GM/r$ one has, at fixed large r,

$$\partial_t A = -\frac{2G}{r}\,\dot M(t) \;=\; -8\pi G\, r\, A\, T_{tr}, \quad \Rightarrow \quad \boxed{\dot M(t) = 4\pi r^2 A\, T_{tr}}. \tag{41}$$

At $\mathscr{I}^\pm$ we have $A \to 1$, and the EF components give the standard Bondi laws:

$$\text{Ingoing at } \mathscr{I}^- \text{ (advanced } v\text{):} \qquad \boxed{\frac{dm}{dv} = \oint r^2 T_{vv}\, d\Omega}, \tag{42}$$

$$\text{Outgoing at } \mathscr{I}^+ \text{ (retarded } u\text{):} \qquad \boxed{\frac{dM_B}{du} = -\oint r^2 T_{uu}\, d\Omega}. \tag{43}$$

Using (40) (ingoing) and its outgoing analogue $T_{tr} = -T_{uu}/A$,

$$\frac{dM_B}{du} = \oint r^2 A\, T_{tr}\, d\Omega \;\xrightarrow[A\to 1]{}\; \oint r^2 T_{tr}\, d\Omega. \tag{44}$$

Equivalently, if one defines the *outward* flux density $F_{\rm out} \equiv -T_{tr}$, then

$$\boxed{\frac{dM_B}{du} = -\oint r^2 F_{\rm out}\, d\Omega} \quad \text{(Bondi mass loss).} \tag{45}$$

7.4 7.4 Worked example: thin ingoing null shell

Take an ingoing shell injected at advanced time $v = v_0$ with total energy ΔM. The Vaidya mass function and stress tensor are

$$\boxed{m(v) = M_i + \Delta M\, \Theta(v - v_0)}, \qquad \boxed{T_{vv} = \frac{\Delta M}{4\pi r^2}\,\delta(v - v_0)}. \tag{46}$$

In diagonal variables this corresponds to

$$T_{tr}(t, r) \;=\; \frac{T_{vv}}{A} \;\xrightarrow[A\to 1]{}\; \frac{\Delta M}{4\pi r^2}\,\delta(v - v_0), \tag{47}$$

and (37) integrates across the shell to

$$\Delta A \equiv A\big|_{v_0^+} - A\big|_{v_0^-} = -\frac{2G}{r}\,\Delta M, \tag{48}$$

i.e. $A = 1 - \frac{2GM}{r}$ with M jumping by ΔM—the horizon grows from $2GM_i$ to $2G(M_i + \Delta M)$.

7.5 7.5 Acceptance check

Equations (37)–(44) ensure: (i) the diagonal flux law matches EF/Vaidya ($T_{tr} = T_{vv}/A$), (ii) $dm/dv = \oint r^2 T_{vv} d\Omega$ (ingoing) and $dM_B/du = -\oint r^2 T_{uu} d\Omega$ (outgoing) have the correct signs/normalizations for Bondi mass change, and (iii) the null-shell example produces the expected jump $M \to M + \Delta M$.

8 8. Linearized vacuum: constraints, TT waves, and energy flux

We linearize about Minkowski in lapse-first variables, impose the scalar/vector constraints, and show only the two TT tensor modes propagate with $\Box h_{ij}^{\mathrm{TT}} = 0$ and speed c.

8.1 8.1 Constraints ⇒ TT sector only

Write $\gamma_{ij} = \delta_{ij} + h_{ij}$, $N = 1 + \phi$ ($\phi \approx \Phi$), and $N_i = \omega_i$. To first order, the Hamiltonian/momentum constraints read

$$\mathcal{H}_\perp^{(1)} = \partial_i \partial_j h_{ij} - \partial^2 h = 0, \qquad \mathcal{H}_i^{(1)} = -2\, \partial_j \big(p^j{}_i - \delta^j{}_i\, p\big) = 0, \tag{49}$$

with p^{ij} conjugate to h_{ij} (traces $h \equiv h^k{}_k$, $p \equiv p^k{}_k$). Using these first-class constraints plus gauge freedom removes all scalar/vector pieces; in TT gauge

$$\partial_i h_{ij}^{\mathrm{TT}} = 0, \qquad (h^{\mathrm{TT}})^i{}_i = 0. \tag{50}$$

Only the TT sector is physical.

8.2 8.2 Free equations: $\Box h_{ij}^{\mathrm{TT}} = 0$ at speed c

The quadratic Hamiltonian reduces to the two TT polarizations,

$$H^{(2)} = \frac{1}{16\pi G} \int d^3x \, \Big[\Pi_{ij}^{\mathrm{TT}} \Pi_{ij}^{\mathrm{TT}} + \tfrac{1}{4} (\partial_k h_{ij}^{\mathrm{TT}})(\partial_k h_{ij}^{\mathrm{TT}}) \Big], \tag{51}$$

implying

$$\boxed{\partial_t^2 h_{ij}^{\mathrm{TT}} - \partial^2 h_{ij}^{\mathrm{TT}} = 0}, \tag{52}$$

so TT waves propagate luminally. No scalar or vector mode propagates.

8.3 8.3 Polarizations

The physical configuration space has 2 DOF: the $+$ and $\times$ tensor modes of h_{ij}^{TT}; no extra polarizations appear.

8.4 8.4 Acceptance check

Eqs. (49)–(50) remove all scalar/vector pieces; only h_{ij}^{TT} propagates. The free dynamics follow $\Box h_{ij}^{\mathrm{TT}} = 0$ with speed c, and the Isaacson flux (Appendix D) reproduces the GR quadrupole luminosity. Thus the radiative content is exactly the two tensor polarizations $(+, \times)$, with the correct normalization.

Gauge bridge to TT waves. In vacuum, linearized constraints solve (Φ, ω_i) as instantaneous (non-radiative) functionals of sources. A diffeomorphism with parameters (ξ^0, ξ^i) satisfying

$$\partial_t \xi^0 = -\,\Phi, \qquad \partial_t \xi_i + \partial_i \xi^0 = \quad \omega_i$$

takes any lapse-first perturbation to a gauge with $\Phi' = 0 = \omega_i'$, leaving only $h_{ij}'^{\mathrm{TT}}$ as radiative data (standard TT gauge). Conversely, starting from TT one can reach lapse-first by inverting these relations. Radiative observables (strain, Isaacson flux) are unchanged.

9 9. FRW mapping: from $d\tau = e^{\Phi} dt,\ a = e^{-\Phi}$ to Friedmann

Assume homogeneity/isotropy with a single temporal potential $\Phi(t)$ and spatial curvature $k \in \{-1, 0, +1\}$:

$$ds^2 \ = \ -e^{2\Phi(t)} dt^2 \ + \ e^{-2\Phi(t)} \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]. \tag{53}$$

A comoving perfect fluid has $u^\mu = (e^{-\Phi}, 0, 0, 0)$ and $T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p\, g_{\mu\nu}$, so $T_{tt} = \rho\, e^{2\Phi}$, $T_{ij} = p\, g_{ij}$.

9.1 9.1 Einstein equations in t-time

A direct computation gives the independent components

$$G_{tt} = 3\, k\, e^{4\Phi} + 3\, \dot{\Phi}^2, \qquad G_{\theta\theta} = r^2 e^{-4\Phi} \Big(-k e^{4\Phi} - 5\dot{\Phi}^2 + 2\ddot{\Phi} \Big), \tag{54}$$

so $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ yields the pair

$$\boxed{\dot{\Phi}^2 + k\, e^{4\Phi} = \frac{8\pi G}{3}\, \rho\, e^{2\Phi}} \tag{F1}$$

$$\boxed{2\ddot{\Phi} - 5\dot{\Phi}^2 - k\, e^{4\Phi} = 8\pi G\, p\, e^{2\Phi}} \tag{F2}$$

(which we will map to the standard Friedmann pair below).

9.2 9.2 Map to cosmic time and recover the Friedmann equations

Define cosmic time and the scale factor by

$$d\tau = e^{\Phi} dt, \qquad a(\tau) = e^{-\Phi(t(\tau))}, \qquad H(\tau) \equiv \frac{1}{a} \frac{da}{d\tau} = -\,\Phi'(\tau), \tag{55}$$

with $\dot{} = d/dt$ and $' = d/d\tau$. Using $\dot{\Phi} = e^{\Phi} \Phi'$ and $e^{2\Phi} = 1/a^2$, (F1)–(F2) become

$$\boxed{H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\, \rho} \tag{FRW1}$$

$$\boxed{\frac{dH}{d\tau} = -4\pi G (\rho + p) + \frac{k}{a^2}} \tag{FRW2}$$

and $\nabla_\mu T^{\mu\nu} = 0$ gives $\rho' + 3H(\rho + p) = 0$ as usual. To include a cosmological constant, either write $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$ or treat Λ as a fluid with $\rho_\Lambda = \Lambda/(8\pi G)$, $p_\Lambda = -\rho_\Lambda$. In either case, (FRW1)–(FRW2) become

$$\boxed{H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}} \tag{FRW1+Λ}$$

$$\boxed{\frac{dH}{d\tau} = -4\pi G(\rho + p) + \frac{k}{a^2} + \frac{\Lambda}{3}} \tag{FRW2+Λ}$$

exactly matching standard FRW cosmology.

Big-Bang limit. With $a = e^{-\Phi}$ and $d\tau = e^{\Phi}dt$, the initial–singularity limit is

$$\Phi \to +\infty \quad \Longleftrightarrow \quad a \to 0,$$

whereas $\Phi \to -\infty$ gives $a \to \infty$ and $N = e^{\Phi} \to 0$ (degenerate lapse / no proper-time flow). We therefore use "Big Bang = $\Phi \to +\infty$" and reserve $\Phi \to -\infty$ for a formal no-time boundary.

9.3 9.3 Variable map (dictionary)

FRW variable	**Time-first expression**
$a(\tau)$	$e^{-\Phi}$
$H(\tau) = \dot{a}/a$	$-\Phi' \ = \ -e^{-\Phi}\dot{\Phi}$
k/a^2	$k\, e^{2\Phi}$
$\rho_\Lambda,\ p_\Lambda$	$\rho_\Lambda = \Lambda/(8\pi G), \quad p_\Lambda = -\rho_\Lambda$

Here a prime denotes $d/d\tau$ and an overdot d/dt.

9.4 9.4 Acceptance check

Using $d\tau = e^{\Phi}dt$, $a = e^{-\Phi}$, $H = -\Phi'$, the t-time equations (F1)–(F2) map exactly to (FRW1)–(FRW2); including Λ gives the standard $H^2 + k/a^2 = (8\pi G/3)\rho + \Lambda/3$ and $\dot{H} = -4\pi G(\rho + p) + k/a^2 + \Lambda/3$ in cosmic time. Therefore the FRW background is reproduced *exactly* in the time-first variables.

10 10. Quantization as a constrained QFT

We quantize gravity in the ADM/lapse-first form by imposing the constraints first and then quantizing the reduced (physical) sector. The canonical Hamiltonian density is

$$\mathcal{H}_{\text{can}} = N\,\mathcal{H}_\perp + N^i\mathcal{H}_i, \qquad (N = e^{\Phi},\ N^i = \gamma^{ij}\omega_j), \tag{56}$$

with $\mathcal{H}_\perp = 0$ and $\mathcal{H}_i = 0$ the Hamiltonian and momentum constraints. **Dirac quantization** imposes the operator constraints on states:

$$\widehat{\mathcal{H}}_\perp\,|\Psi_{\text{phys}}\rangle = 0, \qquad \widehat{\mathcal{H}}_i\,|\Psi_{\text{phys}}\rangle = 0. \tag{57}$$

Linearizing about Minkowski and solving the constraints removes the lapse N and the longitudinal/trace parts of γ_{ij}, leaving only the *transverse–traceless* (TT) tensor sector as propagating degrees of freedom. The physical Hilbert space is therefore the TT Fock space.

10.1 10.1 TT sector and equal-time commutators

Write the spatial metric perturbation as $\gamma_{ij} = \delta_{ij} + h_{ij}$ and decompose $h_{ij} = h_{ij}^{\mathrm{TT}} + (\text{longitudinal/trace})$. In momentum space define the transverse projector

$$P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i \hat{k}_j$$

and the TT projector

$$\Lambda_{ij,kl}(\hat{\mathbf{k}}) = \tfrac{1}{2}(P_{ik}P_{jl} + P_{il}P_{jk} - P_{ij}P_{kl})\,. \tag{58}$$

The quadratic TT action is (setting $c = \hbar = 1$)

$$S_{\mathrm{TT}} = \frac{1}{64\pi G}\int d^4x\,\Big[(\partial_t h_{ij}^{\mathrm{TT}})^2 - (\partial_m h_{ij}^{\mathrm{TT}})^2\Big], \tag{59}$$

so the canonical momentum is $\pi^{ij}_{\mathrm{TT}} = \frac{1}{32\pi G}\,\partial_t h_{ij}^{\mathrm{TT}}$. Equal-time commutators on the reduced phase space read

$$\boxed{\,\big[\,\hat{h}_{ij}^{\mathrm{TT}}(t,\mathbf{x}),\, \hat{\pi}^{kl}_{\mathrm{TT}}(t,\mathbf{y})\,\big] = i\,\Lambda_{ij}{}^{kl}(-i\nabla_{\mathbf{x}})\,\delta^{(3)}(\mathbf{x}-\mathbf{y})\,}. \tag{60}$$

Equivalently, in terms of time derivatives,

$$\big[\,\hat{h}_{ij}^{\mathrm{TT}}(t,\mathbf{x}),\, \partial_t\hat{h}_{kl}^{\mathrm{TT}}(t,\mathbf{y})\,\big] = i\,32\pi G\,\Lambda_{ij,kl}(-i\nabla_{\mathbf{x}})\,\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{61}$$

10.2 10.2 Mode expansion and TT propagator

Define $\kappa \equiv \sqrt{32\pi G}$. Using polarization tensors $e^s_{ij}(\hat{\mathbf{k}})$ $(s = +, \times)$ with $e^s_{ij}e^{s'}_{ij} = 2\delta^{ss'}$ and $k_i e^s_{ij} = e^s_{ii} = 0$, the TT field expands as

$$\hat{h}_{ij}^{\mathrm{TT}}(x) = \frac{\kappa}{2}\sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3\,2\omega_{\mathbf{k}}}\Big[e^s_{ij}(\hat{\mathbf{k}})\,\hat{a}_s(\mathbf{k})\,e^{-ik\cdot x} + e^s_{ij}(\hat{\mathbf{k}})\,\hat{a}_s^\dagger(\mathbf{k})\,e^{+ik\cdot x}\Big], \qquad \omega_{\mathbf{k}} = |\mathbf{k}|. \tag{62}$$

Creation/annihilation operators satisfy

$$\big[\hat{a}_s(\mathbf{k}), \hat{a}^\dagger_{s'}(\mathbf{k}')\big] = (2\pi)^3\,2\omega_{\mathbf{k}}\,\delta_{ss'}\,\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{63}$$

The time-ordered two-point function (TT propagator) is

$$\boxed{\langle 0|\,T\,\hat{h}_{ij}^{\mathrm{TT}}(x)\,\hat{h}_{kl}^{\mathrm{TT}}(y)\,|0\rangle = \frac{\kappa^2}{4}\int \frac{d^4k}{(2\pi)^4}\,\frac{i\,\Lambda_{ij,kl}(\hat{\mathbf{k}})}{k^2 + i\epsilon}\,e^{-ik\cdot(x-y)}\,}, \tag{64}$$

with $\Lambda_{ij,kl}$ the TT projector (58) built from the spatial direction $\hat{\mathbf{k}}$.

10.3 10.3 Acceptance check

The operator constraints (57) enforce that only h_{ij}^{TT} propagates. The equal-time commutators (60), the mode algebra, and the TT propagator (64) fully specify the physical (TT) Hilbert space and free dynamics.

11 11. One operational formula: atom–interferometer phase via Φ

We give a single, experimentally usable expression for the light–pulse Mach–Zehnder atom–interferometer (AI) phase shift in terms of the temporal potential Φ, applicable to static gravity, moving sources, or modulated masses. We then show it reduces to the standard GR result in the tested (nonrelativistic, weak-field) regime.

11.1 11.1 Sensitivity function and phase in terms of Φ

For a three–pulse AI ($\pi/2$–π–$\pi/2$ at $t = 0, T, 2T$) with effective wavevector $\mathbf{k}_{\text{eff}}$ along $\hat{\mathbf{z}}$, the total phase can be written with the standard sensitivity function $g_s(t)$,

$$\Delta\phi \;=\; \mathbf{k}_{\text{eff}}\cdot\int_{-\infty}^{\infty} g_s(t)\,\mathbf{a}(t)\,dt, \qquad g_s(t) = \begin{cases} t, & 0 < t < T, \\ 2T - t, & T < t < 2T, \\ 0, & \text{otherwise.} \end{cases} \tag{65}$$

In the lapse–first/weak–field limit ($|\Phi| \ll 1$), proper time is $d\tau = e^{\Phi}dt$ and the geodesic equation gives the local acceleration $\mathbf{a}(t) = -c^2\,\boldsymbol{\nabla}\Phi\big(t, \mathbf{x}_a(t)\big)$. Hence the operational AI phase in terms of Φ is

$$\boxed{\Delta\phi \;=\; -\,c^2\,\mathbf{k}_{\text{eff}}\cdot\int g_s(t)\,\boldsymbol{\nabla}\Phi\big(t, \mathbf{x}_a(t)\big)\,dt}\,. \tag{66}$$

For a uniform static field, $\boldsymbol{\nabla}\Phi = -\mathbf{g}/c^2$, and (66) reduces immediately to the textbook result

$$\boxed{\Delta\phi \;=\; \mathbf{k}_{\text{eff}}\cdot\mathbf{g}\,T^2}\,. \tag{67}$$

11.2 11.2 Moving/modulated source: near–field expression

For a compact source of mass M at position $\mathbf{R}(t)$ with $|\dot{\mathbf{R}}| \ll c$, the weak–field potential is $\Phi(t, \mathbf{x}) = -GM/(c^2\,|\mathbf{x} - \mathbf{R}(t)|)$, so

$$\boldsymbol{\nabla}\Phi(t, \mathbf{x}) = \frac{GM}{c^2}\,\frac{\mathbf{x} - \mathbf{R}(t)}{|\mathbf{x} - \mathbf{R}(t)|^3}. \tag{68}$$

Inserting this into (66) gives the phase for a moving or modulated source. In the quasi-static limit (modulation frequency $\Omega \ll 1/T$), $\boldsymbol{\nabla}\Phi$ is effectively constant over the interferometer, and $\Delta\phi \simeq \mathbf{k}_{\text{eff}}\cdot\delta\mathbf{g}\,T^2$ with $\delta\mathbf{g}(\mathbf{x}) = -c^2\,\boldsymbol{\nabla}\Phi$ evaluated along the nominal atomic path.

11.3 11.3 Equality to GR in the tested regime

In full GR, the AI phase is the sum of (i) propagation phases $\propto \omega_C \int d\tau$ for each arm (with $\omega_C = mc^2/\hbar$) and (ii) laser phases at the pulses. To leading post-Newtonian order these combine to (65) with $\mathbf{a} = -c^2\nabla\Phi$, yielding (66) and hence (67). Thus the time-first operational formula is exactly equivalent to the standard GR calculation in all current near-field tests (static gravity, moving or modulated laboratory masses, and small atomic velocities).

11.4 11.4 Worked examples

We take ^{87}Rb Raman AI at $\lambda \approx 780\,\mathrm{nm}$ so $k_{\mathrm{eff}} \approx 4\pi/\lambda \approx 1.61 \times 10^{7}\,\mathrm{m}^{-1}$.

Scenario	T (s)	a (m/s^2)	k_{eff} (m^{-1})	$\Delta\phi$ (rad)
Earth gravity (uniform g)	0.10	9.81	1.61×10^{7}	1.58×10^{6}
10 kg source at $r = 0.20\,\mathrm{m}$ *	0.10	1.67×10^{-8}	1.61×10^{7}	2.7×10^{-3}

*Assuming alignment along $\hat{\mathbf{z}}$ so $a = GM/r^2$ projects fully onto $\mathbf{k}_{\mathrm{eff}}$. Numbers use $G = 6.674 \times 10^{-11}$ SI.

11.5 11.5 Acceptance check

Equation (66) gives the AI phase directly from Φ and reduces to $\Delta\phi = k_{\mathrm{eff}}\, g\, T^2$ for static fields. The worked examples yield standard magnitudes (Earth: 1.6×10^{6} rad; 10 kg at 0.2 m: 2.7×10^{-3} rad), and the derivation matches the GR (TT) calculation in the tested, weak-field regime.

12 12. Discussion and Outlook

Computational advantages. The Schwarzschild solution emerges from a single first-order ODE (Eq. 14) rather than solving the full Einstein equations. Spherical collapse (Section 7) reduces to scalar evolution $\partial_t \Phi = -4\pi G r T_{tr}$ instead of metric PDEs. Cosmology maps cleanly via $a = e^{-\Phi}$ and $H = -\Phi'$, directly connecting expansion to temporal geometry.

Conceptual clarity. Gravitational redshift, time dilation, and cosmological expansion all manifest as aspects of the single field Φ. The shift ω cleanly encodes rotation without mixing with temporal effects. Frame dragging becomes the curl of $\boldsymbol{\omega}$: $\mathbf{B}_g = \nabla \times \boldsymbol{\omega}$. Black hole horizons appear as coordinate artifacts of the diagonal gauge, naturally regular in EF/PG coordinates.

Practical applications. Atom interferometer phases (Eq. 66) and gravitational wave calculations map directly to Φ, potentially simplifying experimental predictions. The lapse-first constraint structure provides a natural separation between instantaneous (Coulomb-like) and radiative sectors for numerical relativity.

Quantization route. The constrained QFT approach treats Φ and $\boldsymbol{\omega}$ as non-propagating multipliers while the physical TT modes carry the same radiative content as GR. This provides a conservative path to quantum gravity without additional polarizations or modified dispersion relations.

Deferred to Part II: PPN suite; ADM/Komar/Bondi charges in $(\Phi, \boldsymbol{\omega})$; linear cosmological perturbations; EFT/renormalization; PN/EOB for binaries; any beyond-GR phenomenology if $V(\Phi)$ is treated as new physics.

What we quantize (and what we do not). Evolution is with respect to physical (cosmic) time τ defined by $d\tau = N\,dt = e^{\Phi} dt$. In this classical Part I we establish equivalence with GR; quantization—when pursued—treats the field Φ and the transverse-traceless graviton modes as quantum fields evolving in τ. We do not quantize the coordinate label t itself, and we defer any EFT potential $V(\Phi)$ or beyond-GR phenomenology to a separate work.

A Horizon-regular coordinate maps (EF/PG)

B Horizon invariants and regularity checks

C Constraints from the action (ADM ledger)

D Isaacson tensor and GW energy flux

E FRW mapping and Friedmann pair

F Integrating out Φ (if included)

G Notation and sign conventions

Schwarzschild (diagonal).

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 + r^2 d\Omega^2. \tag{69}$$

Eddington–Finkelstein (EF). Define the tortoise coordinate

$$r_* \equiv r + 2GM \ln\Big|\frac{r}{2GM} - 1\Big|. \tag{70}$$

Ingoing EF (advanced time v) and outgoing EF (retarded time u) are

$$v \equiv t + r_*, \qquad u \equiv t - r_*. \tag{71}$$

The metrics are

$$ds^2_{\rm inEF} = -\Big(1 - \frac{2GM}{r}\Big)dv^2 + 2\,dv\,dr + r^2 d\Omega^2, \tag{72}$$

$$ds^2_{\rm outEF} = -\Big(1 - \frac{2GM}{r}\Big)du^2 - 2\,du\,dr + r^2 d\Omega^2. \tag{73}$$

All components remain finite at $r = 2GM$.

Painlevé–Gullstrand (PG). Define the (ingoing) PG time $t_{\rm PG}$ by

$$t_{\rm PG} \;\equiv\; t + 2\sqrt{2GM\,r} + 2GM \ln\left|\frac{\sqrt{r} - \sqrt{2GM}}{\sqrt{r} + \sqrt{2GM}}\right|. \tag{74}$$

This yields

$$ds^2_{\rm PG} = -\Big(1-\frac{2GM}{r}\Big)dt^2_{\rm PG} + 2\sqrt{\frac{2GM}{r}}\,dt_{\rm PG}\,dr + dr^2 + r^2 d\Omega^2 = -dt^2_{\rm PG} + \big(dr + \sqrt{\tfrac{2GM}{r}}\,dt_{\rm PG}\big)^2 + r^2 d\Omega^2. \tag{75}$$

Equivalently, the lapse–shift form is $N = 1$, $N_r = -\sqrt{2GM/r}$ (“river” speed $v_r = -\sqrt{2GM/r}$). At the horizon $r = 2GM$ the metric is manifestly regular.

Direct maps. From EF to PG one may write $t = v - r_*$ and insert into (74), giving

$$t_{\mathrm{PG}} = v + F(r), \qquad F(r) \equiv -r_* + 2\sqrt{2GM\,r} + 2GM\ln\left|\frac{\sqrt{r}-\sqrt{2GM}}{\sqrt{r}+\sqrt{2GM}}\right|. \tag{76}$$

The inverse maps are obtained by reversing the above definitions.

Appendix B: Invariants at the horizon

For Schwarzschild vacuum ($R_{\mu\nu} = 0$), the only nontrivial quadratic invariant is the Kretschmann scalar

$$\boxed{K \;\equiv\; R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \;=\; \frac{48\,G^2M^2}{r^6}} \tag{77}$$

(in $c = 1$ units). Evaluated at the horizon $r = 2GM$,

$$K\big|_{r=2GM} \;=\; \frac{48\,G^2M^2}{(2GM)^6} \;=\; \frac{3}{4\,G^4M^4} \quad \text{(finite)}, \tag{78}$$

while $K \to \infty$ only as $r \to 0$. Thus the divergence of g_{tt} and g_{rr} at $r = 2GM$ in diagonal coordinates is a *coordinate* singularity, removed by EF/PG.

Appendix C: Constraints and Evolution from the Action

Write $\mathcal{H} = 0$, $\mathcal{H}_i = 0$ and the evolution equations explicitly; give the DOF ledger.

Appendix D: Isaacson averaging and GW energy flux

Isaacson/LL effective stress tensor. For wavelengths λ small compared to background curvature scales, the GW energy–momentum is captured by the (gauge-invariant, averaged) Isaacson tensor in TT gauge,

$$\boxed{t^{\mathrm{GW}}_{\mu\nu} = \frac{1}{32\pi G}\left\langle \partial_\mu h^{\mathrm{TT}}_{ij}\,\partial_\nu h^{\mathrm{TT}}_{ij}\right\rangle}, \qquad \langle\cdots\rangle = \text{average over many cycles}. \tag{79}$$

(Our project already uses the Landau–Lifshitz/Isaacson notion of GW stress for flux balances.)

Flux through a sphere. The GW energy luminosity is

$$\frac{dE}{dt} = \oint r^2\, t^{\mathrm{GW}}_{0i}\, n^i\, d\Omega = \frac{r^2}{32\pi G}\oint \left\langle \partial_t h^{\mathrm{TT}}_{ij}\,\partial_r h^{\mathrm{TT}}_{ij}\right\rangle d\Omega. \tag{80}$$

Far-zone TT solution and quadrupole formula. In the radiation zone ($R \gg$ source size), the TT field is

$$h^{\mathrm{TT}}_{ij}(t,\mathbf{x}) = \frac{2G}{R}\,\ddot{Q}^{\mathrm{TT}}_{ij}(t-R) \;+\; \mathcal{O}(R^{-2}), \tag{81}$$

with Q_{ij} the tracefree mass quadrupole. Inserting (81) into (80) and angle-averaging yields the standard GR result

$$\boxed{\frac{dE}{dt} \;=\; -\,\frac{G}{5}\left\langle \dddot{Q}_{ij}\,\dddot{Q}_{ij}\right\rangle}. \tag{82}$$

This matches the statement in our gravitomagnetic note that radiation in GR arises from evolving quadrupoles and appears only in the $+, \times$ tensor modes.

Appendix E: Detailed FRW derivation in time-first variables

1) Ansatz and matter. Start from (53). For a comoving perfect fluid, $T_{tt} = \rho e^{2\Phi}$ and $T_{ij} = p\, g_{ij}$.

2) Einstein tensor. From the metric one finds

$$G_{tt} = 3ke^{4\Phi} + 3\dot{\Phi}^2, \qquad G_{rr} = \frac{1}{1-kr^2}\left(-ke^{2\Phi} - 5\dot{\Phi}^2 e^{-2\Phi} + 2\ddot{\Phi}e^{-2\Phi}\right), \tag{83}$$

consistent with (54). Insert these in $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ to obtain (F1)–(F2). If a cosmological constant is present, either (i) use $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$ directly, or (ii) move $\Lambda g_{\mu\nu}$ to RHS as a fluid with $\rho_\Lambda = \Lambda/(8\pi G)$, $p_\Lambda = -\rho_\Lambda$.

3) Cosmic time map. Define $d\tau = e^{\Phi}dt$, $a = e^{-\Phi}$, $H = -\Phi'$. Then $e^{2\Phi} = 1/a^2$ and $\dot{\Phi} = e^{\Phi}\Phi'$. Substitute into (F1)–(F2) to get (FRW1)–(FRW2). Covariant conservation gives $\rho' + 3H(\rho+p) = 0$.

4) Interpretation. In this language, *expansion is time's divergence*: $a = e^{-\Phi}$ and $H = -\Phi'$; spatial growth follows from the geometry of time.

Appendix F: Static potential from integrating out the lapse Φ

In the weak, static limit set $N = e^{\Phi} \simeq 1 + \Phi$ and $\gamma_{ij} \simeq \delta_{ij}$. Including matter energy density $\rho(\mathbf{x})$ (with $T_{tt} \simeq \rho$), the Hamiltonian constraint becomes

$$\nabla^2\Phi(\mathbf{x}) \;=\; 4\pi G\, \rho(\mathbf{x}), \tag{84}$$

with instantaneous Green's function $G(\mathbf{x}) = -1/(4\pi|\mathbf{x}|)$. Solving (84) gives

$$\Phi(\mathbf{x}) \;=\; -\,G\int d^3y\, \frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}. \tag{85}$$

The (static) interaction energy is

$$V \;=\; \frac{1}{2}\int d^3x\, \rho(\mathbf{x})\, \Phi(\mathbf{x}) \;=\; -\,\frac{G}{2}\int d^3x\, d^3y\, \frac{\rho(\mathbf{x})\,\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}. \tag{86}$$

For two static point masses $\rho(\mathbf{x}) = m\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_1) + M\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_2)$,

$$\boxed{V(r) \;=\; -\,\frac{G\,m\,M}{r}, \qquad r = |\mathbf{x}_1 - \mathbf{x}_2|}. \tag{87}$$

Momentum-space view (tree level). Integrating out Φ in the static limit yields the instantaneous kernel $i\Delta_\Phi(\mathbf{q}) = i\,(4\pi G)/\mathbf{q}^2$ coupled to ρ, so the nonrelativistic scattering amplitude between heavy sources is $i\mathcal{M}(\mathbf{q}) = i(4\pi G)\,mM/\mathbf{q}^2$, whose Fourier transform is (87). This shows that the Newtonian potential is reproduced at tree level by instantaneous Φ exchange, while dynamical gravitons are the TT quanta (62).

Acceptance check. The TT commutators and propagator are given in (60)–(64); the static limit of the constrained (instantaneous) lapse reproduces the Newtonian potential (87).

Appendix G: Notation and Sign Conventions

Signature and units. Metric signature $(-,+,+,+)$. We set $c=1$ unless displayed for calibration.

Indices. Greek $\mu,\nu,\ldots \in \{0,1,2,3\}$ (spacetime), Latin $i,j,\ldots \in \{1,2,3\}$ (spatial). Spatial indices are raised/lowered with γ_{ij}, spacetime with $g_{\mu\nu}$.

Volume elements and Levi–Civita. $\epsilon_{0123} = +\sqrt{-g}$, $\epsilon^{0123} = +1/\sqrt{-g}$ so that $\mathrm{d}^4x\sqrt{-g}$ is invariant.

Connections and curvature. $\nabla_\mu g_{\alpha\beta} = 0$,

$$\Gamma^\rho_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}),$$

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma, \quad R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \quad R = g^{\mu\nu}R_{\mu\nu}, \quad G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R.$$

Wave operator $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ (flat limit: $\Box = -\partial_t^2 + \nabla^2$).

3+1 split (ADM, lapse-first).

$$ds^2 = -N^2dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \qquad N = e^\Phi, \quad N_i = \gamma_{ij}N^j = \omega_i.$$

$$K_{ij} = \tfrac{1}{2N}(-\partial_t\gamma_{ij} + D_iN_j + D_jN_i)\,.$$

Fourier conventions.

$$f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik_\mu x^\mu}\,\tilde{f}(k), \qquad \tilde{f}(k) = \int d^4x\, e^{+ik_\mu x^\mu} f(x).$$

Spacetime, indices, and metric

We work on a four-dimensional Lorentzian manifold $(\mathcal{M}, g_{\mu\nu})$ with signature

$$(-,+,+,+).$$

Greek indices $\mu,\nu,\rho,\ldots \in \{0,1,2,3\}$ denote spacetime components; Latin indices $i,j,k,\ldots \in \{1,2,3\}$ denote spatial components on a hypersurface Σ_t. Indices are raised and lowered with the corresponding metrics: $g_{\mu\nu}$ (spacetime) and γ_{ij} (spatial). Symmetrization and antisymmetrization use

$$A_{(\mu\nu)} = \tfrac{1}{2}(A_{\mu\nu} + A_{\nu\mu}), \qquad A_{[\mu\nu]} = \tfrac{1}{2}(A_{\mu\nu} - A_{\nu\mu}).$$

The volume element and Levi–Civita tensor adopt the orientation

$$\epsilon_{0123} = +\sqrt{-g}, \qquad \epsilon^{0123} = +\frac{1}{\sqrt{-g}},$$

so that $d^4x\,\sqrt{-g}$ is the invariant measure, with $g = \det g_{\mu\nu} < 0$.

Derivatives and curvature

The covariant derivative ∇_μ is metric-compatible ($\nabla_\mu g_{\alpha\beta} = 0$) with Christoffel symbols

$$\Gamma^\rho_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma}\big(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\big).$$

Our Riemann tensor is defined by its action on vectors:

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma.$$

Contractions give the Ricci tensor and scalar,

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu},$$

and the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} R,$$

so the field equations read $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ with $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\, \delta S_m / \delta g_{\mu\nu}$. The d'Alembertian is

$$\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu,$$

so that in flat space with signature $(-+++)$, $\Box = -\partial_t^2 + \nabla^2$.

3+1 (ADM) decomposition

We foliate $\mathcal{M}$ by spacelike slices Σ_t with future-directed unit normal n^μ satisfying $n_\mu n^\mu = -1$. The induced spatial metric and projector are

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \qquad h^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu, \qquad \gamma_{ij} = h_i{}^\mu h_j{}^\nu g_{\mu\nu}.$$

With lapse N and shift N^i, the line element is

$$ds^2 = -N^2 dt^2 + \gamma_{ij}\big(dx^i + N^i dt\big)\big(dx^j + N^j dt\big).$$

The extrinsic curvature adopts the sign convention

$$K_{\mu\nu} = h_\mu{}^\alpha h_\nu{}^\beta\, \nabla_\alpha n_\beta, \quad \Rightarrow \quad K_{ij} = \frac{1}{2N}\Big(-\partial_t \gamma_{ij} + D_i N_j + D_j N_i \Big) = -\tfrac{1}{2}\, \mathcal{L}_n \gamma_{ij},$$

where D_i is the Levi–Civita connection of (Σ_t, γ_{ij}). The three-dimensional Ricci tensor/scalar are denoted ${}^{(3)}\!R_{ij}$ and ${}^{(3)}\!R$.

Fourier conventions

We use the spacetime Fourier transform

$$f(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik_\mu x^\mu}\, \tilde{f}(k), \qquad \tilde{f}(k) = \int d^4x\, e^{+ik_\mu x^\mu}\, f(x),$$

with $k_\mu x^\mu \equiv g_{\mu\nu} k^\mu x^\nu = -\omega t + \mathbf{k}\cdot\mathbf{x}$ in Minkowski coordinates. Thus plane waves read $e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}$. The delta-function normalization is

$$\int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')} = \delta^{(4)}(x - x').$$

For purely spatial transforms on Σ_t,

$$f(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3}\, e^{+i\mathbf{k}\cdot\mathbf{x}}\, \tilde{f}(\mathbf{k}, t), \quad \tilde{f}(\mathbf{k}, t) = \int d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, f(\mathbf{x}, t).$$

Units and dimensions

Unless explicitly stated, we use relativistic units with $c = 1$. Newton's constant G is kept explicit. Restoring factors of c follows the replacements $t \to ct$, $N \to N/c$, $T_{\mu\nu} \to T_{\mu\nu}/c^2$, etc., so that, e.g., the Schwarzschild lapse reads $g_{tt} = -(1 - 2GM/(c^2 r))\, c^2$. When $\hbar = 1$ is adopted (quantum sections), it is noted locally.

Miscellany

Parentheses ( ) and brackets [ ] denote symmetrization/antisymmetrization as above. A dot on a scalar denotes ∂_t at fixed spatial coordinates; $\mathcal{L}_v$ is the Lie derivative along v^μ. Spatial indices are moved with γ_{ij}; spacetime indices with $g_{\mu\nu}$. The Hodge dual uses the $\epsilon_{\mu\nu\rho\sigma}$ defined above, with ${}^\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma}$.

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