Symbolic and p-adic Encodings of the Lonely Runner Conjecture: Structure and Suppression

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July 30, 2025

Abstract

We present symbolic and p-adic reformulations of the Lonely Runner Conjecture (LRC), introducing auxiliary constructions to analyze visibility dynamics. By encoding runners as symbolic sequences and shifted p-adic expansions, we define new criteria for loneliness and explore methods to suppress or structurally control its duration. This formulation allows for entropy-like analysis, symbolic compression, and potential finite verification heuristics.

Author's Note

This paper was written independently over the course of 10 days, motivated by deep personal study and conceptual exploration. It does not claim to resolve the Lonely Runner Conjecture, but rather offers symbolic and p-adic reformulations that may inspire further investigation. The author welcomes feedback and criticism.

Acknowledgment

This work was composed with the assistance of generative AI tools, specifically OpenAI's **Mathematics Research Assistant** and the custom model **Henri Poincaré**, used to structure exposition, clarify notation, and explore conceptual extensions. All mathematical content and core ideas are entirely the author's own. The use of these tools is part of an ongoing personal experiment in hybrid human–machine mathematical creativity.

1 Geometric and Symbolic Reformulation

Let $k+1$ runners move at constant speeds $v_0, v_1, \ldots, v_k \in \mathbb{Q}$ on the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Each runner starts at 0 and moves linearly modulo 1:

$$x_i(t) := v_i t \mod 1.$$

Definition (Loneliness)
Runner R_i is *lonely* at time t if:

$$\forall j \neq i, \quad \operatorname{dist}(x_i(t), x_j(t)) \geq \frac{1}{k+1}.$$

Equivalently, no other runner lies within the open interval:

$$V_i(t) := \left(x_i(t) - \frac{1}{k+1},\ x_i(t) + \frac{1}{k+1}\right) \mod 1.$$

2 Symbolic Encoding of Motion

Partition $\mathbb{T}^1$ into $k+1$ equal-length intervals:

$$I_j := \left[\frac{j}{k+1}, \frac{j+1}{k+1}\right), \quad j = 0, 1, \ldots, k.$$

Define a symbolic encoding map for each runner:

$$s_i(t) = j \quad \text{iff} \quad x_i(t) \in I_j.$$

This yields a symbolic sequence $s_i : \mathbb{N} \to \{0, 1, \ldots, k\}$ for each runner.

Shift System

Let σ denote the shift map:

$$\sigma^t s_i := (s_i(t), s_i(t+1), \ldots).$$

We analyze the shift orbits $\sigma^t s_i$ symbolically, interpreting loneliness as a form of cylinder avoidance.

Loneliness via Cylinder Avoidance

Fix runner R_0 as the origin. Let C_0 be the cylinder set (e.g., the arc labeled 0). Then R_0 is lonely at time t if:

$$\forall j \neq 0, \quad s_j(t) \neq s_0(t) = 0.$$

Equivalently, none of the other runners are in R_0's current visibility arc.

3 p-adic Encoding

Assume $v_i = \frac{a_i}{N}$ for some common denominator N (e.g., the LCM of all periods). Then:

$$x_i(t) = \frac{a_i t}{N} \mod 1.$$

Let $\mathbb{Z}_{k+1}$ be the ring of p-adic integers with base $k+1$. Each runner's symbolic sequence can be interpreted as a base-$(k+1)$ expansion:

$$s_i = (a_0^{(i)}, a_1^{(i)}, a_2^{(i)}, \ldots) \in \mathbb{Z}_{k+1}.$$

The p-adic distance between two runners s_i, s_j at depth n is:

$$d(s_i, s_j) = (k+1)^{-n} \iff \text{they agree on the first } n \text{ digits.}$$

p-adic Cylinder Avoidance Criterion

Runner R_0 is lonely at depth n if:

$$\forall j \neq 0, \quad s_j \notin [s_0]_n,$$

where $[s_0]_n$ denotes the symbolic cylinder of depth n rooted at s_0.

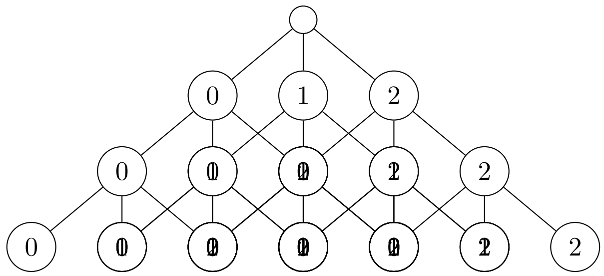

Figure 1: Ternary prefix tree illustrating symbolic sequences in base-3 up to depth 3. Cylinders of increasing depth isolate visibility patterns in p-adic encoding.

4 Loneliness Sequence and Density

Define the binary loneliness sequence for runner i:

$$\ell_i(t) = \begin{cases} 1 & \text{if } R_i \text{ is lonely at } t, \\ 0 & \text{otherwise.} \end{cases}$$

Density

Define the density of loneliness as:

$$\delta_i := \liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \ell_i(t).$$

This measures the asymptotic frequency of loneliness. For rational velocities, $\ell_i(t)$ is eventually periodic.

5 Loneliness Suppression via Auxiliary Runners

Introduce two auxiliary runners per original runner R_i, with speeds:

$$v_i^{\pm} := v_i \pm \varepsilon, \quad \text{for small } \varepsilon > 0.$$

These runners oscillate around R_i and enter its visibility arc with regularity.

Lemma. Let R_i have a visibility arc of length $2/(k+1)$. Then for any $\varepsilon > 0$, the maximum consecutive time steps during which R_i can remain lonely in the presence of $v_i \pm \varepsilon$ is bounded by:

$$T_{\max} \le \frac{1}{\varepsilon(k+1)}.$$

Proof. Let the position of runner R_i at time t be $x_i(t) = v_i t \mod 1$, and let two auxiliary runners move at velocities $v_i^+ = v_i + \varepsilon$ and $v_i^- = v_i - \varepsilon$, so their positions are $x_i^{\pm}(t) = (v_i \pm \varepsilon)t \mod 1$.

Define the visibility arc of R_i at time t as the open interval

$$V_i(t) := \left(x_i(t) - \frac{1}{k+1},\ x_i(t) + \frac{1}{k+1} \right) \mod 1.$$

R_i is lonely at time t if no other runner lies in $V_i(t)$.

Consider the relative position between R_i^+ and R_i:

$$x_i^+(t) - x_i(t) = \varepsilon t \mod 1.$$

This shows that R_i^+ advances around the circle relative to R_i at speed ε.

Since the arc $V_i(t)$ has length $2/(k+1)$, the auxiliary runner R_i^+ will enter this arc within time

$$T^+ \leq \frac{1}{\varepsilon(k+1)}.$$

An identical argument holds for R_i^-, which moves backwards relative to R_i at the same speed.

Hence, regardless of initial phase, at least one of the auxiliary runners must enter $V_i(t)$ within at most $T_{\max} \leq 1/\big(\varepsilon(k+1)\big)$ steps, interrupting loneliness. □

Lemma (Loneliness Bound via Symbolic Shift Sequences). Let $s_i : \mathbb{N} \to \{0, 1, \ldots, k\}$ be the symbolic encoding of runner R_i under partition of $\mathbb{T}^1$ into $k+1$ equal intervals. Then for any $\varepsilon > 0$, if we introduce auxiliary runners with velocities $v_i \pm \varepsilon$ having symbolic sequences $s_i^\pm$, the number of consecutive time steps during which R_i can remain symbolically lonely (i.e., $\forall j \neq i$, $s_j(t) \neq s_i(t)$) is bounded by

$$T_{\max} \leq \frac{1}{\varepsilon(k+1)}.$$

Proof. Each symbolic sequence $s_i(t)$ corresponds to a discrete encoding of position $x_i(t) = v_i t \mod 1$ via membership in intervals $I_j = [j/(k+1), (j+1)/(k+1))$. At each time t, runner R_i is symbolically lonely if no other sequence s_j satisfies $s_j(t) = s_i(t)$.

Now consider two auxiliary runners with velocities $v_i^\pm = v_i \pm \varepsilon$. Their symbolic sequences $s_i^\pm(t)$ differ from $s_i(t)$ because their underlying positions drift linearly over time at relative speed ε.

Since $x_i^\pm(t) - x_i(t) = \pm\varepsilon t \mod 1$, the auxiliary runners enter each arc I_j cyclically, advancing by one symbol approximately every $1/(\varepsilon(k+1))$ steps.

Therefore, for any fixed symbol $a = s_i(t)$, the sequence $s_i^\pm$ will match it within at most $T_{\max} \leq 1/(\varepsilon(k+1))$ time steps. Thus, s_i cannot remain the unique occupant of a symbol class for longer than this bound, i.e., loneliness is broken within $T_{\max}$ steps. □

Lemma (Loneliness Bound via $(k+1)$-adic Cylinder Avoidance.) Let $s_i \in \mathbb{Z}_{k+1}$ be the $(k+1)$-adic expansion of the symbolic sequence of runner R_i, with loneliness defined as exclusion of all other sequences from the cylinder $[s_i]_n$ (i.e., agreement on the first n digits). Then for any $\varepsilon > 0$, if auxiliary runners with velocities $v_i \pm \varepsilon$ are introduced, the maximum depth n for which s_i avoids intersection with $s_i^\pm$ satisfies:

$$n_{\max} \leq \log_{k+1}\left(\frac{1}{\varepsilon}\right), \quad \text{and hence} \quad T_{\max} \leq \frac{1}{\varepsilon(k+1)}.$$

Proof. The symbolic sequence s_i of a runner can be viewed as the $(k+1)$-adic expansion of a number in $\mathbb{Z}_{k+1}$:

$$s_i = (a_0^{(i)}, a_1^{(i)}, a_2^{(i)}, \ldots),$$

where $a_n^{(i)}$ corresponds to the symbolic state at time n. Two sequences agree up to depth n if they lie in the same cylinder set $[s_i]_n$, which corresponds to a ball of radius $(k+1)^{-n}$ in the p-adic metric.

Let $s_i^\pm$ denote the sequences of runners with speeds $v_i \pm \varepsilon$. Due to the linear drift in relative motion, these runners' symbolic expansions differ from s_i at a rate proportional to ε.

Since the relative angular distance between R_i and $R_i^{\pm}$ is $\pm\varepsilon t$, and the width of a cylinder of depth n is $(k+1)^{-n}$, the time required for $s_i^{\pm}$ to enter $[s_i]_n$ is at most

$$T_n \leq \frac{1}{\varepsilon(k+1)^n}.$$

Solving for n such that $T_n = 1$ gives:

$$(k+1)^n = \frac{1}{\varepsilon} \quad \Rightarrow \quad n = \log_{k+1}\left(\frac{1}{\varepsilon}\right).$$

Hence, $s_i^{\pm}$ must match s_i up to some depth n within $T_{\max} \leq 1/\big(\varepsilon(k+1)\big)$ steps, breaking the cylinder avoidance condition and thus ending p-adic loneliness. □

Refinement: Speed-Dependent Bound

Suppose we place auxiliary runners around R_i at speeds $v_i^{\pm} = v_i \pm \varepsilon$, with $\varepsilon > 0$ and $\varepsilon < v_i$. From the previous bound, we have:

$$T_{\max} \leq \frac{1}{\varepsilon(k+1)}.$$

If we now set $\varepsilon = v_i$, we obtain the general inequality:

$$T_{\max} \leq \frac{1}{v_i(k+1)},$$

showing that the maximum possible duration of loneliness for runner R_i is inversely proportional to their speed. Faster runners re-enter their own visibility arcs more frequently.

Corollary (Speed Bounds Loneliness Duration). Let R_i have speed $v_i > 0$. Then the maximum time interval during which R_i can remain lonely satisfies:

$$T_{\max} \leq \frac{1}{v_i(k+1)}.$$

Thus, slower runners may experience longer periods of loneliness, while faster runners naturally suppress it via self-return dynamics.

6 Compression via Substitution

Suppose $\ell_i(t)$ contains at most x consecutive 1s. Define a symbolic substitution:

$$\tau : 1^x \mapsto L, \quad 0 \mapsto 0.$$

This produces a compressed sequence $\tau(\ell_i)$ capturing blocks of loneliness and their frequency.

7 Worked Example: Three Runners

Let $k+1 = 3$ and choose rational velocities:

$$v_0 = 0, \quad v_1 = \frac{1}{4}, \quad v_2 = \frac{1}{2}.$$

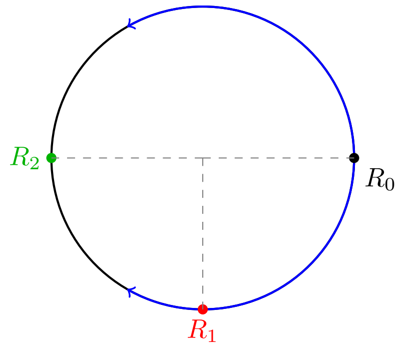

Figure 2: Runner positions at $t = 3$ with $v_0 = 0$, $v_1 = \frac{1}{4}$, $v_2 = \frac{1}{2}$. At this moment, R_0 is lonely since no other runner lies within its visibility arc (length 2/3).

The LCM of denominators is $N = 4$, so motion is periodic with period 4. We partition $\mathbb{T}^1$ into 3 arcs:

$$I_0 = \left[0, \tfrac{1}{3}\right), \quad I_1 = \left[\tfrac{1}{3}, \tfrac{2}{3}\right), \quad I_2 = \left[\tfrac{2}{3}, 1\right).$$

Symbolic Sequences

Compute $x_i(t) = v_i t \mod 1$ and assign $s_i(t)$ accordingly:

t	0	1	2	3
$x_0(t)$	0	0	0	0
$s_0(t)$	0	0	0	0
$x_1(t)$	0	0.25	0.5	0.75
$s_1(t)$	0	0	1	2
$x_2(t)$	0	0.5	0	0.5
$s_2(t)$	0	1	0	1

Loneliness Detection

At $t = 2$, $x_0 = 0$, while $x_2 = 0$ as well. So R_0 is not lonely. At $t = 3$, $x_1 = 0.75$, $x_2 = 0.5$, and $x_0 = 0$ — no one is near R_0.

Thus: $\ell_0 = (0, 0, 0, 1)$

8 Multiscale Visibility and Mandelbrot-Like Cascades for Irrational Speeds

We now investigate the Lonely Runner Conjecture in the case of *irrational* velocities using a symbolic, multiscale approach inspired by Mandelbrot cascades. Specifically, we model the visibility pattern of an irrational-speed runner as a cascade-like thinning of mass across nested visibility intervals.

Setup

Let two runners move on the unit circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$:

- R_0: fixed at the origin, $v_0 = 0$,
- R_1: moving with irrational speed $v_1 = \alpha \in (0,1) \setminus \mathbb{Q}$.

The position of R_1 at time $t \in \mathbb{N}$ is:

$$x_1(t) = \alpha t \mod 1.$$

We are interested in when R_1 is *close to the origin*—i.e., when $x_1(t) \in [-\varepsilon, \varepsilon] \mod 1$.

Multiscale Visibility Intervals

Since α is irrational, the orbit $\{x_1(t)\}_{t\in\mathbb{N}}$ is equidistributed in $\mathbb{T}^1$. However, we consider a cascade of nested visibility intervals defined at increasing resolution levels.

Let α have continued fraction convergents $\frac{p_n}{q_n} \approx \alpha$, with $\gcd(p_n, q_n) = 1$.

Each q_n serves as an approximate period and scale. Define:

$$\varepsilon_n := \frac{1}{q_n}, \quad I_n := [-\varepsilon_n, \varepsilon_n] \subset \mathbb{T}^1.$$

Define the depth-n visibility indicator function:

$$\mu_n(t) := \begin{cases} 1 & \text{if } x_1(t) \in I_n \mod 1, \\ 0 & \text{otherwise.} \end{cases}$$

This yields a nested visibility sequence:

$$\mu_0(t) \geq \mu_1(t) \geq \mu_2(t) \geq \dots$$

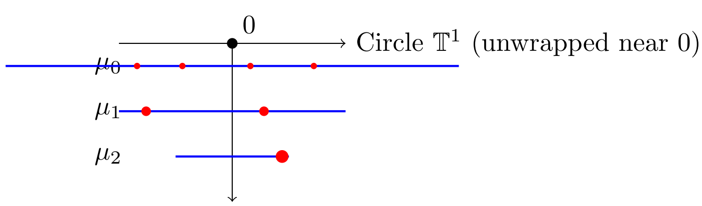

Figure 3: Nested visibility intervals $I_n = [-\frac{1}{q_n}, \frac{1}{q_n}]$ around 0, forming a Mandelbrot-style cascade. Red dots mark when $x_1(t) = \alpha t \mod 1$ enters each level. Density decreases with n.

Cascade Analogy

The structure $\{\mu_n(t)\}_n$ resembles a Mandelbrot cascade:

- At each scale n, visibility corresponds to an interval of length $2/q_n$,
- Frequency of visibility decays with n,
- Symbolic visibility sequence μ_n reflects a thinning measure, akin to multifractal energy distributions.

Define the scale-n visibility density as:

$$\rho_n := \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mu_n(t).$$

This quantifies how often R_1 falls within distance $1/q_n$ of the origin—providing a hierarchical view of visibility and potential loneliness.

Implications for LRC

While the classical LRC concerns exact loneliness (in arcs of length $1/(k+1)$), this cascade model:

- Captures the behavior of irrational runners across scales,
- Enables symbolic and density-based analysis,
- Suggests that even in irrational cases, structured scarcity of visibility exists through nested approximations.

In particular, the non-vanishing of ρ_n for infinitely many n provides a soft lower bound on recurrence frequency, and thus potential methods for proving visibility (or loneliness) over long time scales.

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