

# A Theorem on the Function $\Omega(n)$ and Mersenne Numbers

(Un teorema sulla funzione  $\Omega(n)$  e i numeri di Mersenne)

Giuseppe Fierro

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## Abstract

Dimostriamo che per ogni intero positivo  $n$ , vale la disuguaglianza  $\Omega(2^n - 1) \geq \Omega(n)$ , dove  $\Omega(n)$  denota il numero totale (con molteplicità) dei fattori primi di  $n$ . Questo risultato stabilisce un legame interessante tra la struttura moltiplicativa di  $n$  e quella del corrispondente numero di Mersenne  $2^n - 1$ .

**English version:** We prove that for every positive integer  $n$ , the inequality  $\Omega(2^n - 1) \geq \Omega(n)$  holds, where  $\Omega(n)$  denotes the total number of prime factors of  $n$ , counted with multiplicity. This result establishes an interesting connection between the multiplicative structure of  $n$  and that of the corresponding Mersenne number  $2^n - 1$ .

## 1 Introduction

Let  $\Omega(n)$  denote the total number of prime factors of a positive integer  $n$ , counted with multiplicity. In this short note, we prove that for every positive integer  $n$ , the inequality

$$\Omega(2^n - 1) \geq \Omega(n)$$

holds. The result emerged from computational observations and is demonstrated using a non-standard form of induction based on the value of  $\Omega(n)$ .

## 2 Proof of the Theorem

**Theorem 1.** *For every integer  $n > 0$ , the following inequality holds:*

$$\Omega(2^n - 1) \geq \Omega(n)$$

*Proof.* We proceed by induction on  $\Omega(n)$ .

**Base case:** If  $\Omega(n) = 1$ , then  $n$  is a prime number. In this case,  $2^n - 1$  is an odd number greater than 1, so it has at least one prime factor, and thus  $\Omega(2^n - 1) \geq 1 = \Omega(n)$ .

**Inductive step:** Assume the inequality holds for all integers  $n$  such that  $\Omega(n) = m$ . Let us consider  $n$  with  $\Omega(n) = m + 1$ . Write  $n = k \cdot p$  where  $p$  is a prime number and  $\Omega(k) = m$ .

We can write:

$$2^n - 1 = 2^{kp} - 1 = (2^k)^p - 1 = (2^k - 1) \cdot \sum_{i=0}^{p-1} 2^{ki}$$

Since  $\Omega$  is completely additive, we have:

$$\Omega(2^n - 1) = \Omega(2^k - 1) + \Omega\left(\sum_{i=0}^{p-1} 2^{ki}\right)$$

By the inductive hypothesis,  $\Omega(2^k - 1) \geq \Omega(k) = m$ , and the second term is at least 1. Therefore:

$$\Omega(2^n - 1) \geq m + 1 = \Omega(n)$$

□

### 3 Conclusion

The inequality  $\Omega(2^n - 1) \geq \Omega(n)$  provides a surprising link between the arithmetic structure of an integer  $n$  and the factorization of the associated Mersenne number  $2^n - 1$ . The inductive proof, based on the value of  $\Omega(n)$ , suggests possible generalizations and motivates further theoretical and computational investigation.

### References

- [1] G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 6th ed., 2008.
- [2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.