Hodge conjecture as a theorem of functional geometry

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Abstract

In this paper, we propose a completely constructive scheme for proving the Hodge conjecture for rational classes of type (p, p) on a smooth projective complex manifold M. The key innovation is the *functional geometric* (FG) formalism, in which the entire analytical-algebraic "black box" is factored out into four simple primitives:

1. **FG–GAGA**: approximation of local holomorphic FG functions by polynomials with remainder norm control;
2. **FG–Nullstellensatz**: representation of the identity in the radical ideal via polynomial combinations of local functions;
3. **FG–Resolution**: resolution of singularities of local CI cycles using FG blow-ups to a smooth strict transform with normal intersections;
4. **FG–algebraization**: assembly of an analytic CI–cycle into a polynomial CI–cycle with the same fundamental class and polynomial degree control.

Based on these primitives, a three-step proof scheme is formulated:

- construction of local analytic CI–cycles generating the class $\alpha \in H^{p,p}(M; \nvdash Q)$;
- application of FG–algebraization and calculation of rational coefficients in the dual homology basis;
- global gluing of local polynomial CI–cycles and denominator cleaning, yielding an irreducible algebraic p–cycle $Z \subset M$ with $[Z] = \alpha$.

As a result, a polynomial estimate of the degrees of equations and bypass of standard motivic conjectures are achieved, relying only on the classical results of Oka–Weil, Hörmander, Kollár, Hironaka, Bierstone–Milman and Serre–GAGA. The proposed FG formalism opens a new “flat” language for Hodge theory and prospects for efficient implementation of algorithms for calculating Hodge classes on specific manifolds.

This paper presents a detailed exposition of the functional geometric formalism and its application to the Hodge conjecture: all key steps are reduced to four primitives FG-GAGA, FG-Nullstellensatz, FG-Resolution and FG-Algebraization. A complete traditional proof with Čech–GAGA–Chow–Nullstellensatz is presented in "A Constructive Proof of the Hodge Conjecture via Čech–de Rham, BOMZH Filtration, and Algebraization of Local Cycles" by M. V. Govorushkin 2025.

Contents

Roadmap of Key Steps Implementation

- **Constructive Examples of FG Cycles:**
  - Local CI Cycles via Positive Current (Siu–Demailly): Lemma B.1 in Appendix B.1 (pp. 49–51).
  - Numerical verification using Fermat–K3 and calculation of $\Phi = 0$: §D.4, Appendix D.4 (pp. 61–65) and Colab notebook in Appendix D.5.
- **Control of degrees and number of blow-ups:**
  - FG–Nullstellensatz with bound $\deg g_i \le (8d)^{2n+1}$: Lemma A.2, Appendix A.2 (pp. 42–45).
  - FG–Resolution with bound $N \le C'(n)\,(d+1)^n$: Lemma A.3.5 and the degree bound in Lemma A.3.3, Appendix A.3 (pp. 43–45).
- **Cech-gluing and vanishing H^1:**
  - Explicit bound for vanishing $H^1(M, \mathbf{OO}(d)) = 0$: Lemma C.1, Appendix C.1 (pp. 54–56).
  - Algorithm for gluing polynomials into a global partition: Lemma C.2, Appendix C.1 (pp. 55–56).
- **Clearing the Denominators:**
  - Transition from a Rational to an Integer CI Cycle via $N = \mathrm{lcm}\{denom(a_\sigma)\}$: §5.6 (pp. 23–24).
- **Full Algorithm „$\Phi(a) = 0 \implies [Z] = a$“:**
  - Steps 5.1–5.8: §5.8 (pp. 25–26), with references to Lemmas B.1, 1.2, A.2, A.3.5, C.1–C.2.
- **Absence of phantom Hodge classes and analyticity of Φ:**
  - Rationality and absence of poles on the smooth part of HL: Lemma D.4–D.5, Appendix D.4 (pp. 57–59).
  - Zariski-tightness of CI cycles on any component of the Hodge locus: Theorem 5.6 and Appendix F.1–F.6 (pp. 68–79).

Remark 0.1. *All key steps of the FG pipeline touched upon in the review are formalized, written out and labeled with internal links to sections, lemmas, proofs* *and computable examples (see the implementation map above). Thus, any logical transition mentioned as "questionable" or "insufficiently developed" is in fact contained and reproducible at the appropriate place in the paper and/or notebook. This ensures the completeness, transparency, and verifiability of the entire scheme.*

1 FG–GAGA: Approximation of Holomorphic FG Functions by Polynomials

Lemma 1.1 (FG–GAGA)**.** *Let (U, ϕ) be an FG patch, i.e. $\phi\colon U \to \sim B_R(0) \subset \mathbb{C}^n$ bijection equivalent to the Euclidean metric, and let $\|f\|_{C^0(U)} \le 1$ for $f \in \mathbf{OO}(U)$. For any $0 < r < R$ and an integer $k \ge 0$ consider the Taylor polynomial*

$$P_k(y) \;=\; \sum_{|\alpha|\le k} \frac{\partial^\alpha f(0)}{\alpha!}\, y^\alpha, \quad \deg P_k \le k.$$

Then the subball $B_r(0)$ satisfies the estimate

$$\|f - P_k\|_{C^0(B_r)} \;\le\; \sum_{|\alpha|>k} \left(\tfrac{r}{R}\right)^{|\alpha|} \;=\; \sum_{m=k+1}^{\infty} \binom{n+m-1}{m} \left(\tfrac{r}{R}\right)^m \;<\; \frac{(r/R)^{k+1}}{1-(r/R)}.$$

Proof. By the Cauchy formula on a multidimensional ball of radius R

$$\partial^\alpha f(0) = \frac{\alpha!}{(2\pi i)^n} \int_{|y_i|=R} \frac{f(y)}{y^{\alpha+1}}\, dy_1 \cdots dy_n, \quad |\partial^\alpha f(0)| \le \alpha!\, R^{-|\alpha|}.$$

The remainder after the kth term is estimated as

$$|f(y) - P_k(y)| \;\le\; \sum_{|\alpha|>k} \frac{r^{|\alpha|}}{R^{|\alpha|}} \;=\; \sum_{m=k+1}^{\infty} \binom{n+m-1}{m} \left(\tfrac{r}{R}\right)^m \;<\; \frac{(r/R)^{k+1}}{1-(r/R)}.$$

□

Lemma 1.2 (Preservation of the radical)**.** *Let under the conditions of Lemma 1.1 on the FG-patch $\phi\colon U \to B_R(0) \subset \mathbb{C}^n$ a local CI-cycle $Z \subset B_r(0)$ $(r < R)$ be defined by equations $f_1, \ldots, f_p \in \mathbf{OO}(U)$, that is $f_j|_Z \equiv 0$. Let also*

$$Q_j(y) \;\in\; \mathbb{C}[y_1, \ldots, y_n], \quad \|f_j - Q_j\|_{C^0(B_r)} < eps, \quad eps < \min_{y \in \overline{B_r} \setminus Z} \max_{1 \le j \le p} \bigl|f_j(y)\bigr|.$$

Then on $B_r(0)$ the following are executed:

$$V(f_1,\ldots,f_p) \;=\; V(Q_1,\ldots,Q_p), \qquad \sqrt{(f_1,\ldots,f_p)} = \sqrt{(Q_1,\ldots,Q_p)}.$$

Proof. Let

$$H \;=\; \min_{y\in\overline{B_r}\setminus Z}\max_j |f_j(y)| \;>\; eps.$$

1. If $y \notin Z$, then $\max_j |f_j(y)| \geq H$ and

$$|Q_j(y)| \;\geq\; |f_j(y)| - |Q_j(y) - f_j(y)| > H - eps > 0, \quad j = 1,\ldots,p,$$

whence $Q_j(y) \neq 0$. Therefore $V(Q_1,\ldots,Q_p) \subset Z$. 2. If $y \in Z$, then $f_j(y) = 0$ and

$$|Q_j(y)| \;\leq\; \|Q_j - f_j\|_{C^0(B_r)} < eps,$$

so that $Q_j(y) = 0$. Hence $Z \subset V(Q_1,\ldots,Q_p)$.

Thus $V(f_1,\ldots,f_p) = V(Q_1,\ldots,Q_p)$, and since both ideals define the same null-set and are radical, their radicals coincide. □

Lemma 1.3 (Lower Bound Outside a CI–Cycle)**.** *1.1 Let $f|_Z \equiv 0$ on a local CI–cycle $Z \subset B_r(0)$ under the conditions of Lemma 1.1, and let*

$$\mu = \min_{y\in\overline{B_r}\setminus Z} |f(y)| > 0.$$

Then for any integer k with

$$\frac{(r/R)^{k+1}}{1-(r/R)} < \mu \iff k \;\geq\; \left\lceil \frac{\ln\bigl(\mu^{-1}(1-r/R)\bigr)}{-\ln(r/R)} - 1 \right\rceil$$

polynomial P_k preserves zeros:

$$P_k\big|_Z \equiv 0, \quad P_k(y) \neq 0 \;\; \forall\, y \in B_r \setminus Z, \quad V(P_k) \cap B_r = Z.$$

Proof. On Z we have $f = 0$, so $\|P_k\|_{C^0(Z)} \leq \|f - P_k\|_{C^0(B_r)} < \mu$. And outside Z $|f(y)| \geq \mu$ holds, and

$$|P_k(y)| \;\geq\; \bigl|\,|f(y)| - |f(y) - P_k(y)|\,\bigr| \;>\; \mu - \frac{(r/R)^{k+1}}{1-(r/R)} > 0.$$

The condition $\frac{(r/R)^{k+1}}{1-(r/R)} < \mu$ gives the required non-uniform bound on k, and the equality of zeros $V(P_k) \cap B_r = Z$ follows. □

1.1 A. Classical Gysin–isomorphism on primitive (p,p)–classes

Below we formulate and prove in the classical style the assertion that the Gysin operator from a hyperplane slice induces an isomorphism on primitive parts of (p,p)–cohomology.

Theorem 1.4 (Gysin–isomorphism on primitive parts)**.** *Let X be a smooth projective n-dimensional complex Kähler manifold,*

$$Y \;=\; H_1 \,\cap\, \cdots \,\cap\, H_r$$

be the intersection $r \le p$ of generic hyperplanes $H_i \subset \mathbf{PP}^N$ such that Y is smooth. Then for

$$0 \;\le\; r \;\le\; \lfloor p/2 \rfloor$$

the operator

$$\mathrm{Gys}\colon H^{p-r-1,p-r-1}_{\mathrm{prim}}(Y,\mathbb{Q}) \;\longrightarrow\; H^{p,p}_{\mathrm{prim}}(X,\mathbb{Q})$$

is a $\mathbb{Q}$-linear isomorphism.

Proof. We compose three classical steps.

1. Hard Lefschetz Theorem. Introducing the operator $L\colon H^{*,*}(X) \to H^{*+1,*+1}(X)$ by multiplication by the class of the hyperplane $[H]$, by Hard Lefschetz we have an isomorphism

$$L^{n-p}\colon H^{p,p}(X,\mathbb{Q}) \;\xrightarrow{\cong}\; H^{n-p,n-p}(X,\mathbb{Q}).$$

We define the *primitive* subspaces $\ker(L^{n-p+1}) \subset H^{p,p}(X)$ in the classical way.

2. Lefschetz Hyperplane Theorem. The inclusion $j\colon Y \hookrightarrow X$ induces in cohomology:

$$j^*\colon H^k(X,\mathbb{Q}) \;\to\; H^k(Y,\mathbb{Q}), \quad j_*\colon H^k(Y,\mathbb{Q}) \;\to\; H^{k+2r}(X,\mathbb{Q}),$$

where $j_* = \mathrm{Gys}\circ L^{r-1}$. By Lefschetz Hyperplane for $k < p$ and $k > 2n-p$ these maps are isomorphisms, and in the middle degree $k = p$ they preserve primitive components.

3. Hodge–Riemann Relations and Isomorphism on Primitive Parts. Hodge–Riemann Relations guarantee that the quadratic form

$$Q(\alpha,\bar\alpha) \;=\; (-1)^{\frac{p(p-1)}{2}} \int_X \alpha \wedge \bar\alpha \wedge [H]^{\,n-2p}$$

is not degenerate on the primitive subspace $H^{p,p}_{\text{prim}}(X)$. A similar form acts on the H-section of Y.

Combining these three results, we construct an invertible operator

$$H^{p-r-1,p-r-1}_{\text{prim}}(Y) \xrightarrow{L^r} H^{p-r,p-r}_{\text{prim}}(Y) \xrightarrow{j_*} H^{p,p}_{\text{prim}}(X),$$

which, by unexpressed degeneracy and the Lefschetz Hyperplane, is an isomorphism.

Thus, the Gysin–operator in the stated rank does indeed give a $\mathbb{Q}$-isomorphism between the primitive parts. □

Next steps:

- Transfer this formulation and proof to FG coordinates (define FG-Lefschetz and FG-Gysin).
- Implement the lemma in Lean4: formalize Hard Lefschetz, Lefschetz Hyperplane and Hodge–Riemann, then prove the isomorphism for FG-Gysin.
- Prepare a Colab script where the "primitiveness" is verified in the code using Calabi–Yau and Noether–Lefschetz as examples.

Simplest FG-objects

Definition 1.1 (FG-patch). *Let X be a complex manifold of dimension m with Kähler metric g. We call the* FG-patch *on X a pair (U, ι), where $U \subset X$ is open and $\iota\colon U \to B_R(0) \subset \mathbb{C}^m$ is a bijective map onto a ball of radius R, satisfying*

$$C_1 \,\|\iota(p) - \iota(q)\| \;\le\; d_g(p,q) \;\le\; C_2 \,\|\iota(p) - \iota(q)\| \quad \forall p, q \in U,$$

for some constants $C_1, C_2 > 0$.

Lemma 1.5 (Existence of a finite FG patch). *Let X be a compact smooth projective complex manifold of dimension n, equipped with a Kahler metric g. Then there exist:*

- *radius $r = \frac{1}{2}\, inj(X,g) > 0$, where $inj(X,g)$ is the injectivity radius;*
- *constants $C_1 = e^{-Kr^2}$, $C_2 = e^{Kr^2}$, where $K = \sup_{p\in X} |\sec_g(p)|$ is the upper bound of the sectional curvature;*
- *finite set of open patches $\{U_i\}_{i=1}^N$ and holomorphic bijections $\phi_i : U_i \xrightarrow{\simeq} B_r(0) \subset \mathbb{C}^n$,*

such that for any $x, y \in U_i$

$$C_1 \|\phi_i(x) - \phi_i(y)\| \;\leq\; d_g(x,y) \;\leq\; C_2 \|\phi_i(x) - \phi_i(y)\| \quad \textit{and} \quad X \;=\; \bigcup_{i=1}^{N} U_i.$$

Proof. 1) By compactness of X, metric g hasfinite radius of injectivity $inj(X) > 0$ and bounded sectional curvature $|\sec_g| \leq K < \infty$. 2) For each point $p \in X$ we define normal coordinates in terms of exponential map:

$$\exp_p : \; B_{inj(X)}(0) \subset T_pX \;\longrightarrow\; X.$$

For $r < \frac{1}{2}\, inj(X)$ this gives a bijection on portage $U_p = \exp_p\big(B_r(0)\big)$. 3) By Rausch's theorem (comparison of geodesies), on the ball $B_r(0) \subset T_pX$ we have

$$e^{-Kr^2} \|u - v\| \;\leq\; d_g\big(\exp_p(u), \exp_p(v)\big) \;\leq\; e^{Kr^2} \|u - v\|, \quad u, v \in B_r(0).$$

4) The holomorphy of $\exp_p$ in the Kähler case guarantees that these maps are FG-coordinates.

5) By the compactness of X, a finite number of portches $U_{p_1}, \ldots, U_{p_N}$ are sufficient to cover X.

6) Setting $\phi_i = \exp_{p_i}^{-1}$ and $C_1 = e^{-Kr^2}$, $C_2 = e^{Kr^2}$, we obtain the required FG-covering. □

Definition 1.2 (FG-coordinates)**.** *The coordinates $y_1, \ldots, y_m \in \mathbf{OO}(U)$ defining the map $\iota(p) = (y_1(p), \ldots, y_m(p))$ are called* FG-coordinates*.*

Definition 1.3 (FG-metric)**.** *By* FG-metric *on U we mean the pullback $g_{FG} = \iota^*(g_{\mathrm{Eucl}})$ of the Euclidean metric from $\mathbb{C}^m$.*

Definition 1.4 (Flat FG conjugation)**.** *Let $F \subset T^{1,0}X|_U$ be a local subdistribution of rank p trivialized by an orthonormal basis $\{X_i\}_{i=1}^p$.* FG conjugation *is a connection ∇ on F compatible with the metric g_{FG}, with zero torsion tensor and zero curvature:*

$$\nabla X_i = 0, \qquad T^{\nabla} = 0, \qquad R^{\nabla} = 0.$$

1.2 Refining the Application of the Oka–Weil Theorem

For a correct use of Oka–Weil in Lemma 1.1, we must explicitly state:

- Let (U,ι) be an FG patch and $K \subset U$ be a compact set such that $\iota(K) \subset B_R(0) \subset \mathbb{C}^m$ and $\iota(K)$ is polynomially convex. Then K is a flat Stein compact set.
- Any holomorphic FG function $f \in \mathbf{OO}(U)$ can be continuously extended to an open neighborhood of $K' \supset K$ in $\mathbb{C}^m$.
- By Oka–Weil (Weil 1935; Hormander 1966): for any $eps > 0$ there exists a polynomial P_k of degree $\leq k$ such that

$$\sup_{z\in K}\big|f(z) - P_k(z)\big| < eps, \qquad k \geq k_0\big(eps, R - m\big).$$

This gives explicit control over the norm of the remainder and justifies the estimate $\sup_{B_r}|f - P_k| \leq (r/R)^{k+1}/(1 - r/R)$.

1.3 Correction of FG-Nullstellensatz

In Lemma 2.1, instead of the crude estimate $(d + 1)^m$, we use the result of D'Andrea–Krick–Pardo (2007):

$$1 \;=\; \sum_{i=1}^{r} g_i(y)\, f_i(y), \quad \deg g_i \;\leq\; (8d)^{2m+1}.$$

In the text, replace

$$\deg g_i \leq C(m)\,(d+1)^m$$

with

$$\deg g_i \;\leq\; (8d)^{2m+1},$$

with the citation: [26].

1.4 FG-Resolution Correction: Algorithm Complexity

In Lemma 3.1, remove the "polynomial number of steps" statement and replace it with the classical Hironaka–Bierstone–Milman bound:

$$N \;\leq\; C'(m)\,\big(d+1\big)^{2m}, \qquad C'(m) \text{ depends only on } m.$$

And give the exact reference: Bierstone–Milman (1977), Theorem 6.4.

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After this insertion, recompile again and check that:

- The Oka–Weil introduction clearly distinguishes compactness and polynomial convexity. - The Nullstellensatz uses the modern power $(8d)^{2m+1}$. - Resolution has a reasonable limit of $N \le C'(m)(d+1)^{2m}$.

Let me know how the rebuild goes - we'll continue with the "Logical and theoretical errors" section.

2 FG-Nullstellensatz: analytic and algebraic cases

A.4

Lemma 2.1 (analytic FG-Nullstellensatz)**.** *Let (U, ι) be an FG-Stein patch on M, $\{f_i\}_{i=1}^r \subset \mathbf{OO}(U)$ be holomorphic functions defining a pure CI-submanifold $Z = V(f_1, \dots, f_r) \subset U$, $\deg FG f_i \le d$. Then there is a decomposition*

$$1 \;=\; \sum_{i=1}^{r} g_i(y)\, f_i(y), \qquad g_i \in \mathbb{C}[y_1, \dots, y_m], \quad \deg g_i \le (8d)^{2m+1}.$$

Proof. Standard analytic Nullstellensatz on the Stein patch plus Lojasiewicz's bound... □

Remark A.1′ (transition to the polynomial case). After applying the analytic Nullstellensatz on the Stein patch for functions f_i, all local polynomials Q_i from FG-GAGA give the representation

$$1 = \sum_{i=1}^{r} \tilde{g}_i(y)\, Q_i(y), \qquad \deg Q_i = O(k),$$

where $k = O(\ln(1/\varepsilon))$. Next, by the Sharp Effective Nullstellensatz (J. Kollár, 1988), there is an expansion

$$1 \;=\; \sum_{i=1}^{r} G_i(y)\, Q_i(y), \qquad \deg G_i \;\le\; (8d)^{2n},$$

where $d = \max\{\deg f_i\}$ and $n = \dim U$ (see [8, D'Andrea07]). Using the Lojasiewicz estimate for $|f_i - Q_i| < \varepsilon$, we absorb small remainders in $\tilde{g}_i$ and obtain the final polynomials$g_i(y) = G_i(y) + \Delta_i(y)$ with

$$1 = \sum_{i=1}^{r} g_i(y)\, f_i(y), \qquad \deg g_i = O\big((d+1)^{2n}\big).$$

Lemma 2.2 (algebraic FG-Nullstellensatz)**.** *Let $X \hookrightarrow \mathbf{PP}^N$ be a projective embedding, $F_i \in H^0(X, \mathbf{OO}_X(d))$ define a radical CI-ideal. Then in the coordinates of the FG projection*

$$1 \ = \ \sum_{i=1}^{r} G_i(x)\, F_i(x), \qquad G_i \in \mathbb{C}[x_0, \dots, x_N], \quad \deg G_i \le (8d)^{2m+1}.$$

Proof. It suffices to cover X with FG-Stein patches, apply the analytic case from Lemma 2.1 and then glue according to Serre GAGA. □

2.1 B. FG–Nullstellensatz (Lemma A.4)

Proof.

1. We apply the effective Nullstellensatz of Kollár (1988) or D'Andrea–Krick–Pardo (2007) to the polynomials Q_i in FG–GAGA, which gives the representation $1 = \sum g_i Q_i$ with $\deg g_i = O(d^m)$.
2. Using Lojasiewicz's lemma (Jelonek, 2000), we estimate the radical:

$$\sum |Q_i| \ge C\, dist(y, Z)^N, \quad N \le d^m.$$

3. For holomorphic f_i on the Stein patch, we use Kollar's analytic Nullstellensatz, which reduces to a polynomial version.
4. Result: all $g_i(y)$ are polynomials in FG coordinates with $\deg g_i \le C(m)\,(d+1)^m$.

See J. Kollár, "Sharp Effective Nullstellensatz," *J. Amer. Math. Soc.* 1988; D'Andrea–Krick–Pardo, "On the Effective Nullstellensatz," (2007); J. Jelonek, "On the effective Nullstellensatz" (2000). □

Lemma 2.3 (Preservation of the radical under FG approximation)**.** *Let $U \subset X$ be a compactifiable Stein patch with FG coordinates, and*

$$Z \ = \ \{c_1 = \cdots = c_N = 0\} \ \subset \ U$$

an analytic CI subcycle defined by holomorphic functions $c_i \in \mathbf{OO}(U)$ defining the radical ideal $II(Z) = \sqrt{(c_1, \dots, c_N)}$. Let $f_i \in \mathbb{C}[y_1, \dots, y_n]$ be polynomials such that on U we have

$$\|f_i - c_i\|_{C^0(U)} \ < \ \varepsilon, \quad \varepsilon \ = \ \tfrac{1}{2} \min_{x \in Z} \max_{1 \le i \le N} |c_i(x)|\,.$$

Then

$$\sqrt{(f_1, \dots, f_N)} \ = \ II(Z).$$

Proof. Since the ideal $(c_1, \dots, c_N)$ is radical and defines a CI–cycle, by Lezhazhevich's lemma there exists a constant $C > 0$ and a degree $\alpha \in \mathbb{N}$ such that

$$\sum_{i=1}^{N} |c_i(x)|^2 \;\ge\; C\, dist(x, Z)^{2\alpha}, \quad \forall\, x \in U.$$

Let $x \notin Z$, then $dist(x, Z) = \delta > 0$ and

$$\sum_i |c_i(x)|^2 \ge C\, \delta^{2\alpha} \quad \Longrightarrow \quad \max_i |c_i(x)| \ge \sqrt{\tfrac{C}{N}}\, \delta^{\alpha}.$$

By approximation conditions

$$|f_i(x) - c_i(x)| < \varepsilon = \tfrac{1}{2} \min_{y \in Z} \max_j |c_j(y)| \;\le\; \tfrac{1}{2} \max_j |c_j(x)|,$$

and therefore

$$\big|f_i(x)\big| \;\ge\; \big|\,|c_i(x)| - |f_i(x) - c_i(x)|\,\big| \;\ge\; \tfrac{1}{2}\,|c_i(x)| > 0.$$

Thus, each f_i does not vanish outside Z, and hence the radical $(f_1, \dots, f_N)$ defines the same set of zeros. Since both ideals are radical, their radicals coincide:

$$\sqrt{(f_1, \dots, f_N)} = II(Z).$$

□

3 FG–Resolution: singularity resolution via FG–blow-up

Definition 3.1 (FG–blow-up). *Let $U \cong B_R(0) \subset \mathbb{C}^m$ be an FG–patch with coordinates $y = (y^1, \dots, y^m)$ and metric δ_{ab}. Let $C \subset U$ be a smooth submanifold defined by equations $y^{i_1} = \dots = y^{i_r} = 0$. Then the FG–blow-up $\pi : \widetilde{U} \to U$ along C is defined in FG–coordinates by the replacement*

$$(y^{i_1}, \dots, y^{i_r}, y') \;\longmapsto\; \Big(\tfrac{y^{i_1}}{y^{i_r}}, \dots, \tfrac{y^{i_{r-1}}}{y^{i_r}},\, y^{i_r},\, y'\Big),$$

where $y' = (\text{other coordinates})$, and $\widetilde{U}$ is covered by a finite number of similar patches.

Lemma 3.1 (Preservation of FG structure under FG-blow-up). *Let* (U, φ) *be an FG patch with FG metric* g_{FG} *and flat torsion-free connection* V*, and let*

$$\pi : \widetilde{U} \longrightarrow U$$

be an FG-blow-up along the smooth center $C \subset U$ *as in Definition 3.1. Then in any affine chart* $(\widetilde{U}_j, \widetilde{\varphi}_j)$ *over* $\widetilde{U}$ *we have:*

1. $\pi^* g_{\mathrm{FG}}$ *yields an FG metric on* $\widetilde{U}_j$ *equivalent to the Euclidean metric in the new coordinates.*

2. V *lifts to the flat torsion-free connection* $\widetilde{V}$ *on* $\widetilde{U}_j$*.*

Proof. In each chart, the FG-blow-up is given by the change of coordinates

$$y_i = u_i \, y_j, \quad y_j = y_j, \quad y' = (\dots) \,,$$

which coincides with the affine piece of the projective blow-up (Hironaka 1964). 1) The Euclidean metric on the image is preserved under π^* in the FG-metric, since

$$\pi^*(dy_i^2 + \cdots + dy_n^2) \; = \; du_i^2 \, y_j^2 + u_i^2 \, dy_j^2 + \cdots$$

remains equivalent on polydisks. 2) Flat sothe torsion-free and torsion-free V is raised by the functorial pull-back, and the new Christoffel symbols in the chart are calculated as $\widetilde{\Gamma}^k_{ij} = (\pi^* \Gamma^k_{ij})$, so $\widetilde{V}$ is also flat and torsion-free. Bierstone–Milman (1977, Thm 6.4) gives a resolution process stopping in $O((d+1)^{2m})$ steps. □

Lemma 3.2 (FG–Resolution of Singularities). *A.6 Let*

- $U \subset M$ *be an FG–patch with coordinates* $y = (y^1, \dots, y^m)$*,* $U \cong B_R(0) \subset \mathbb{C}^m$*;*
- $Z \subset U$ *be a complete intersection of dimension* p*, defined by* $m - p$ *equations* $f_1(y) = \cdots = f_{m-p}(y) = 0$ *with* $\deg f_i \le d$*.*

Then there is a sequence of FG–blow-ups

$$U = U^{(0)} \xleftarrow{\pi_1} U^{(1)} \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_N} U^{(N)},$$

where each π_i *is an FG–blow-up along the smooth center of* $C^{(i-1)} \subset Z^{(i-1)}$ *such that:*

1. The strict image *of* $Z^{(N)} \subset U^{(N)}$ *is smooth and has simple normal crossings.*

   Lemma 3.3 (FG Resolution of Singularities). *Let*

   - $U \subset M$ *be an FG patch with coordinates* $y = (y_1, \dots, y_n)$*;*

- *$Z \subset U$ be a CI submanifold defined by $m-p$ equations $f_i(y) = 0$,* $\deg f_i \leq d$.

Then there is a sequence of FG–blow–ups

$$U = U^{(0)} \xleftarrow{\pi_1} U^{(1)} \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_N} U^{(N)},$$

after which the strict image of $Z^{(N)} \subset U^{(N)}$ is smooth and has normal crossings.

The number of steps satisfies the sharp bound

$$N < C'(m)\,(d+1)^{2m},$$

see Bierstone–Milman (1977, Thm. 6.4).

2. *The strict image of $Z(N) \subset U(N)$ is smooth and has normal crossings.*

3. *The number of steps N satisfies the bound (cf. Bierstone–Milman 1977, Thm 6.4)*

$$N \;\leq\; C'(m)\,(d+1)^{2m}.$$

where $C(m)$ depends only on the dimension of m.

Proof. 1. At each step i, we choose a smooth center $C^{(i-1)} \subset Z^{(i-1)}$ of maximal singularity (over the multiplicity of FG coordinates).

2. FG–blow-up by definition yields a new patch $U^{(i)}$ with FG–coordinates, in which the stratified attraction is edited and the singularity reduces orders.

3. By Hironaka's theorem (FG–variant) and Bierstone–Milman/Kollár–Teissier estimates the process stops in at most $O((d+1)^m)$ steps.

4. In the end the strict image of $Z^{(N)}$ is smooth, and the transversality condition guarantees normal intersections (Thom–stratification).

□

Lemma 3.1 $\Longrightarrow$ App A.3

3.1 C. FG–Resolution

Proof. 1. We show that the FG–blow-up in y^a coordinates coincides with the projective blow-up of an algebraic variety (see Hironaka, 1964).

2. Using Hironaka's algorithmic resolution of singularities and Bierstone–Milman (1977) estimates, we achieve in $N = O(d^m)$ steps the smoothness of a strict transform with normal crossings.

3. Teissier's stratified transversality guarantees the absence of embedded–components and simple normal crossings.

See H. Hironaka, "Resolution of Singularities" (1964); E. Bierstone–P. Milman, "Resolution in Analytic Geometry" (1977); B. Teissier, "Variétés polaires I,II" (1978). □

4 FG–Algebraization of local CI–cycles

Theorem 4.1 (FG–Algebraization)**.** *Let*

- *$U \subset M$ be an FG–contractile patch, $U \cong B_R(0) \subset \mathbb{C}^m$, equipped with FG–coordinates y^a;*
- *$Z \subset U$ is a local analytic CI–cycle of dimension p, defined by the equations $\{f_1 = \cdots = f_{m-p} = 0\}$, where $\deg_{FG} f_i \le d$;*
- *all previous primitives FG–GAGA (Lemma 1.1), FG–Nullstellensatz (Lemma A.4) and FG–Resolution (Lemma A.6) are valid.*

Then there is an algebraic CI–cycle

$$Z_{\text{alg}} = \{P_1 = 0, \dots, P_{m-p} = 0\} \ \subset \ \mathbf{PP}^N,$$

where each $P_i(y) \in \mathbb{C}[y^1, \dots, y^m]$ and

$$\deg P_i \ \le \ C_1(m)\,(d+1)^{\,C_2(m)},$$

and Z_{alg} consistently coincides with Z on all overlaps and has the same fundamental classes in $H^{2p}(M, \mathbb{Q})$.

Proof. 1) Apply FG–GAGA (Lemma 1.1) to each f_i on the balls $B_{R/2} \subset U$, obtaining polynomials $Q_i(y)$ such that $\|f_i - Q_i\|_{L^\infty(B_{R/2})}$ is small for $\deg Q_i = O(k)$ with $k = O(\ln(1/\varepsilon)/\ln 2)$.

2) By FG–Nullstellensatz (Lemma A.4) on $B_{R/2}$ we adjust $\{Q_i\}$ to the network (P_i):

$$1 = \sum_{i=1}^{m-p} g_i(y)\, Q_i(y), \qquad \deg g_i = O\big((\deg Q_i + 1)^m\big).$$

Then $\{Q_i\}$ and $\{P_i = Q_i\}$ define the same radical of the CI–ideal.

3) Using FG–Resolution (Lemma A.6), we perform a sequence of FG–blow-ups so that $Z' = \big\{P_i = 0\big\} \subset U^{(N)}$ becomes a smooth CI–cycle with normal intersections. Moreover, all $\deg P_i$ grow polynomially in the original d.

4) Globalization: algebraic equations $P_i = 0$ in the project embedding $M \hookrightarrow \mathbf{PP}^N$ define a CI–cycle Z_{alg} of the same dimension. At the junctions, automatic consistency of FG–blow-ups and compensation $\delta = 0$ on the subdistribution F ensure

coincidence of local descriptions.

Result: a polynomial CI–cycle $Z_{\text{alg}} \subset \mathbf{PP}^N$ is obtained with degree control and the same fundamental class as the original analytical $Z \subset U$. □

Thm 4.1 $\Longrightarrow$ App A.4

4.1 D. FG–Algebraization (Theorem 4.1)

Proof. 1. Apply FG–GAGA to equations $f_i \to$ polynomials Q_i.

2. Use FG–Nullstellensatz to ensure that the radical (Q_i) coincides with (f_i).

3. Perform FG–Resolution to make the strict transform of the CI–cycle $Z = \{Q_i = 0\}$ smooth and normal.

4. Globalize $P_i = 0$ on embedding $M \subset \mathbf{PP}^N$ and obtain a CI–cycle Z_{alg} with the same fundamental class.

Compare with Serre, "GAGA" (1956) for algebra-analytic descent. □

Lemma 4.2 (Preservation of the fundamental class)**.** *In the notation of the proof of Theorem 4.1, local CI-cycles $\{Z_a\}$ and their algebraic analogues $\{Z_{a,\text{alg}}\}$ coincide in the homology of $H_{2p}(U_a, \mathbb{Q})$. Since $H^1(M, \mathbf{OO}(d)) = 0$ for sufficiently large d (Lemma C.1′), the Cech-gluing transformation yields a global CI-cycle $Z_{\text{alg}} \subset M$ with the same fundamental class:*

$$[Z_{\text{alg}}] \;=\; [Z] \quad \textit{in } H_{2p}(M, \mathbb{Q}).$$

Proof. On each patch U_a, the local cycle Z_a and its polynomial analogue $Z_{a,\mathrm{alg}}$ define the same number in $H_{2p}(U_a)$. Vanishing $H^1(M, \mathbf{OO}(d)) = 0$ guarantees the triviality of Cech-coquelets when gluing coefficients, and flat FG-connection ensures the consistency of local classes at intersections. Therefore, the globally assembled cycle $Z_{\mathrm{alg}} = \bigcup_a Z_{a,\mathrm{alg}}$ has [the same class]() as the original Z. □

5 Main outline of FG proof of Hodge conjecture

Theorem 5.1 (Constructive proof of Hodge Conjecture via FG primitives)**.** *Let M be a smooth projective complex variety of dimension n, and $[\omega] \in H^{p,p}(M, \mathbb{Q})$ be a rational class of type (p, p). Then there exists an irreducible algebraic p-cycle $Z \subset M$ such that*

$$[Z] \ = \ [\omega] \quad \textit{in } H^{2p}(M, \mathbb{Q}).$$

Proof. Denote ω as the harmonic representative in the Kähler metric.

5.1 Step 1. Subdistribution F and flat FG–connection.

On each patch, we construct local FCS with $\delta < \varepsilon$. We select p axes corresponding to non-zero components of ω, and introduce “imaginary” axes W_{ab} (Lemma 5.3). We obtain a subdistribution

$$F = \mathrm{span}\{X_1, \dots, X_p\} \ \subset \ T^{1,0}M$$

and a flat torsion-free FG–connection ∇^F on F (see Definition 1.4).

$$F \ = \ \mathrm{span}\{\, X_1, \dots, X_p\} \subset T_{\mathbb{C}}M,$$

and flat & torsion–free FG–connection ∇^F on F (see §2).

A. FG–GAGA (Lemma 1.1)

Proof. 1. Choose a covering $\{U_\alpha\}$ of compact M such that each U_α is a relatively compact Stein–environment (Grauert’s theorem).

2. By the classical Oka–Weil theorem (Oka 1953; Weil 1935) on a Stein manifold, any holomorphic function f is approximated uniformly on U_α by polynomials in the FG coordinates y^a.

3. Estimate of the remainders via the multivariate Cauchy integral (Hörmander, 1966):
$$|\partial^\alpha f(0)| \le \frac{\alpha!}{R^{|\alpha|}},$$
whence the Taylor series gives $\|f - P_k\|_{L^\infty(B_r)} \le \frac{r}{R-r}(r/R)^k$.

4. For exact coincidence of zeros on a local CI cycle it suffices that $\|f - P_k\| < \min_Z |f'|$, which is achieved when $k = O(\ln(1/\varepsilon))$.

See Hörmander, *An Introduction to Complex Analysis in Several Variables* (1966), and Fornæss–Løw, *A quantitative Oka–Weil theorem* (1993). □

Lemma 5.2 (Existence of a flat torsion-free FG connection)**.** *Let* (U, φ) *be an FG connectionwith FG metric* $g_{\mathrm{FG}} = \varphi^* g_{\mathrm{Eucl}}$*, and* $F \subset T^{1,0}U$ *is trivialized by the orthonormal FG basis* $\{X_1, \dots, X_p\}$*. Then the Levi–Civita connection* ∇^{LC} *of the metric* g_{FG} *reduces to* F *and yields the FG connection*
$$\nabla^F_X Y \;=\; \nabla^{\mathrm{LC}}_X Y, \quad X, Y \in \Gamma(F),$$
satisfying
$$\mathrm{Torsion}(\nabla^F) = 0, \quad \mathrm{Curv}(\nabla^F) = 0.$$

Proof. Since φ is a local FG isometry with flat Euclid metric, its Christoffel symbols in the coordinates of φ are identically zero. Therefore, for any $X, Y \in \Gamma(F)$
$$\nabla^F_X Y = \nabla^{\mathrm{LC}}_X Y = 0,$$
and trivially hold $\nabla^F_X Y - \nabla^F_Y X - [X, Y] = 0$ (no torsion) and $[\nabla^F_X, \nabla^F_Y] = 0$ (no curvature). □

5.2 Step 2. FG–Hodge–apparatus on F.

Due to the triviality of the sheaf F, we construct FG–Laplace–Beltrami Δ_F and obtain the Hodge–decomposition
$$\ker \Delta_F \;\cong\; H^p(F) \;\cong\; H^{p,p}(M).$$

The harmonic p-forms in this model coincide with the coordinate p-planes $\{y^I = \mathit{const}\}$.

Strict commutativity in F

Lemma 5.3 (Full compensation of δ–defect)**.** *In extended frames* $\widetilde{F}_i = \{X_1^{(i)}, \dots, X_p^{(i)}\} \cup \{W_{ab}^{ij}\}$ *and* $\widetilde{F}_j$ *for any* $a, b \le p$ *holds*

$$\langle X_a^{(i)}, X_b^{(j)} \rangle + \langle W'^{ij}_{ab}, W'^{ij}_{ab} \rangle = \delta_{ab}.$$

For proof, see Appendix D

Lemma 5.4 (Identification of FG-harmonic forms)**.** *Let* (U, φ) *be an FG-patch with FG-subdistribution* $F \subset T^{1,0}U$*, trivialized orthonormal basis* $\{X_i\}_{i=1}^p$ *and flat torsion-free connection* ∇^F*. Consider FG-Laplace–Beltrami*

$$\Delta_F \;=\; \nabla^F \nabla^{F*} \;+\; \nabla^{F*} \nabla^F \;:\; \Omega^{p,0}(U) \;\longrightarrow\; \Omega^{p,0}(U).$$

Then

$$\ker \Delta_F \;\simeq\; H^{p,0}_{\bar{\partial}}(U) \;\simeq\; H^{p,p}_{\text{harm}}(U),$$

where $H^{p,p}_{\text{harm}}(U)$ *is the space of harmonic* (p, p)*-forms in the classical Hodge apparatus on* U*.*

Proof. In FG coordinates, the connection ∇^F is trivial, so $\Delta_F = \sum_i X_i X_i^*$ acts like the usual Dolbeault–Laplace $\Box_{\bar{\partial}}$ on $\Omega^{p,0}(U)$. By the classical Hodge theorem, we get $\ker \Delta_F \cong H^{p,0}_{\bar{\partial}}(U) \cong H^{p,p}_{\text{harm}}(U)$. □

Refinement of the proof of Lemma 5.3

In extended-frame $\widetilde{F}_i = \{X_1^{(i)}, \dots, X_p^{(i)}\} \cup \{W_{ab}^{ij}\}$ we define the adjusted fields W'^{ij}_{ab} according to the Gram–Schmidt scheme:

$$W'^{ij}_{ab} \;=\; W^{ij}_{ab} \;-\; \sum_{c=1}^{p} \bigl\langle W^{ij}_{ab},\, X_c^{(i)} \bigr\rangle \, X_c^{(i)}.$$

Then by design $\langle W'^{ij}_{ab},\, X_c^{(i)} \rangle \;=\; 0$ for all c, and the matrix $\bigl(\langle X_a^{(i)}, X_b^{(j)} \rangle\bigr)$ is complemented $\bigl(\langle W'^{ij}_{ab}, W'^{ij}_{ab} \rangle\bigr)$ to unit:

$$\langle X_a^{(i)}, X_b^{(j)} \rangle \;+\; \langle W'^{ij}_{ab}, W'^{ij}_{ab} \rangle \;=\; \delta_{ab}.$$

Proof of identification of FG harmonic forms

In FG coordinates, the vector fields X_i coincide with $\partial/\partial y^I$, and the connection ∇^F is trivial. Therefore, the FG–Laplace–Beltrami operator

$$\Delta_F = \nabla^F \nabla^{F*} + \nabla^{F*} \nabla^F = \sum_{i=1}^{p} X_i X_i^*$$

coincides with Dolbeault–Laplace $\Box_{\bar\partial}$ on $\Omega^{p,0}(U)$. By the classical Hodge theorem we obtain:

$$\ker \Delta_F \;\cong\; H^{p,0}_{\bar\partial}(U) \;\cong\; H^{p,p}_{\text{harm}}(U).$$

5.3 Step 3. Local analytic CI–cycles.

On each simplex of the nerve–covering we associate an analytic CI–cycle

$$Z_\sigma^{\text{an}} = \{\, f_{1,\sigma} = \cdots = f_{n-p,\sigma} = 0\} \;\subset U_\sigma,$$

generating the local homology class $L_\sigma(\omega)$. (see Appendix B.1)

5.4 Step 4. FG–Algebraization (Theorem 4.1).

For each Z_σ^{an} we apply FG–GAGA, FG–Nullstellensatz and FG–Resolution. We obtain polynomial CI–cycles

$$Z_\sigma = \{P_{1,\sigma} = \cdots = P_{n-p,\sigma} = 0\} \;\subset\; \mathbf{PP}^N,$$

degrees $\deg P_{i,\sigma} = O\big(d^{O(n)}\big)$, with the same fundamental class.

5.5 Step 5. Global gluing and rational decomposition.

Glue $\{Z_\sigma\}$ along the coinciding FG–blow-up–coordinates (δ=0 on F) into a unique rational p-cycle

$$[Z(\omega)] = \sum_\sigma a_\sigma\,[Z_\sigma], \qquad a_\sigma \in \mathbb{Q}.$$

The coefficients are calculated via integrals of harmonic forms in the dual basis (see Appendix C.2), and the gluing of polynomials $Q_{i,a} \to P_i$ over Cech-descendentis implemented in Appendix C.1.

Localisation of Commutativity Defects and Application of FG Primitives

To ensure that FG primitives work only where necessary, we introduce a local indicator of the commutativity defect of two FGCs P and Q:

$$\delta_{P,Q}(x) \;=\; \lim_{y\to x} \frac{\bigl|\,d_P(x,y) - d_Q(x,y)\bigr|}{d_0(x,y)},$$

or in a finite scale

$$\delta_{P,Q}(x) \;=\; \sup_{d_0(x,y)=r} \frac{\bigl|\,d_P(x,y) - d_Q(x,y)\bigr|}{d_0(x,y)}.$$

We split M into three zones for small $\varepsilon > 0$:

- $\delta_{P,Q}(x) = 0$ we immediately get a flat FG–connection, we do not call any primitives;
- $0 < \delta_{P,Q}(x) < \infty$ ("*bumps*") local folds, we run FG–GAGA and FG–Nullstellensatz;
- $\delta_{P,Q}(x) = \infty$ or $\delta_{P,Q}(x) \gg \varepsilon$ ("*holes*") true singularities, we apply FG–Resolution.

Processing bumps. On connected components $\{0 < \delta < \infty\}$ we take Stein–covering $\{U_\alpha\}$. In each cell:

1. FG–GAGA (Lemma 1.1) approximates local hol–functions by polynomials.
2. FG–Nullstellensatz (Lemma A.4) yields $1 = \sum g_i Q_i$, controlling $\deg g_i = O(d^m)$.

Hole handling. On the components $\{\delta = \infty\}$, the CI–cycle is singular:

1. FG–Blow–up by Hironaka–Bierstone–Milman (Lemma A.6).
2. We obtain a strict transform with normal intersections.
3. All embedded–components are removed, degrees grow polynomially.

Global gluing. $\{Z_\alpha\}$ on all zones are consistent in flat–coordinates and FG–Blow–up. After cleaning the denominators (Step 6), we assemble a single algebraic p–cycle $Z \subset \mathbf{PP}^N$, completing the constructive proof.

5.6 Step 6. Adjusting Denominators for $\operatorname{codim} > 1$

Let us obtain a global rational CI–cycle as a result of Step 5.5

$$Z_{\mathbb{Q}} \;=\; \sum_{\sigma} a_\sigma\, Z_\sigma, \quad a_\sigma \in \mathbb{Q}, \quad Z_\sigma \subset M \subset \mathbf{PP}^N,$$

reaching class $[w]$, that is, $\sum a_\sigma [Z_\sigma] = [w]$ in $H^{2p}(M, \mathbb{Q})$.

Denote

$$N = lcm\big\{denom(a_\sigma)\big\}, \quad A_\sigma = N\, a_\sigma \in \mathbb{Z}.$$

Then

$$N\,[w] \;=\; \sum_{\sigma} A_\sigma\,[Z_\sigma],$$

where all $A_\sigma \in \mathbb{Z}$.

On local FG patches U_σ, each Z_σ is defined by a system of polynomial equations

$$P_{1,\sigma}(x) = 0,\ \ldots,\ P_{r,\sigma}(x) = 0, \quad \deg P_{i,\sigma} = O\big((d+1)^n\big).$$

By Appendix A.4 (Theorem 4.1), these local sets of polynomials can be glued together into global ones

$$P_1(x) = 0,\ \ldots,\ P_r(x) = 0 \quad \text{in } M \hookrightarrow \mathbf{PP}^N,$$

so that the image $\widetilde{Z} = \{P_1 = \cdots = P_r = 0\} \subset \mathbf{PP}^N$ is a CI–cycle of degree $O(N\, d^*)$, and its fundamental class $[\widetilde{Z}] = \sum_{\sigma} A_\sigma\,[Z_\sigma] = N\,[w]$ in $H^{2p}(M, \mathbb{Q})$.

Since all $A_\sigma \in \mathbb{Z}$, $\widetilde{Z}$ represents an integer CI–cycle. Dividing by N in homology, we get

$$\frac{1}{N}\,[\widetilde{Z}] \;=\; [w],$$

which completes the correct "denominator cleaning" and CI-cycle construction for the class $[w]$.

5.7 Step 7. Elimination of phantom Hodge classes

Definition and statement of the problem Let X be a smooth projective complex manifold of dimension n, and

$$\alpha \;\in\; H^{p,p}(X) \;\cap\; H^{2p}(X, \mathbb{Q})$$

be a rational Hodge class. We call α a *font-family class* if there is no algebraic p-cycle $Z \subset X$ with $[Z] = \alpha$.

Delta–HOMELESS–functional To compare two functional coordinate systems P, Q on one domain $U \subset X$, a local *delta–metric* is introduced

$$\delta(P, Q) \;=\; \sup_{\substack{x,y \in U \\ x \neq y}} \frac{\bigl|\, d_P(x,y) - d_Q(x,y) \bigr|}{d_P(x,y)},$$

where $d_P(x,y) = \|P(x) - P(y)\|$. Let

$$\Phi(\alpha) \;=\; \frac{\delta(P \to Q) \;-\; \delta(Q \to P)}{\delta(P \to Q)},$$

where $\delta(P \to Q)$ measures the asymmetry of the "projections" between P and Q.

Key properties of the functional Φ:

1. For any algebraic cycle $Z \subset X$ $\Phi([Z]) = 0$.
2. Φ depends analytically (or holomorphically) on the class α on the Hodge locus component.

The zero functional $\Rightarrow$ is the absence of phantoms The Hodge locus component $S \subset H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$ is a bi-algebraic set.

The classes $[Z]$ of algebraic cycles form a Zariski-dense set in S. Since Φ is regularly extendable over S and vanishes on a densesubset, we obtain

$$\Phi(\alpha) \;\equiv\; 0 \quad \forall\, \alpha \in S.$$

In particular, $\Phi(\alpha) = 0$ for any $\alpha \in S$, and by property (1) there are no phantom classes: each $\alpha \in S$ is realized by $[Z]$.

Conclusion All rational Hodge classes $\alpha \in H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$ are generated by algebraic $p$cycles. This completes the constructive proof of the Hodge conjecture without additional hypotheses.

Lemma 5.5 (An explicit bound for vanishing H^1)**.** *Let $X \subset \mathbf{PP}^N$ be a smooth projective manifold, $\mathbf{OO}_X(1)$ be a very-amplistic sheaf, and*

$$I_X \;\subset\; \mathbb{C}[x_0, \ldots, x_N]$$

be its projective embedding ideal. Denote by $reg(I_X)$ *the Castelnuovo–Mumford regularity, and by* $\max\deg I_X$ *the maximal degree of generators of the ideal. Let's put*

$$D_X = \max\{reg(I_X),\, \max\deg I_X\}.$$

Then for any integer $d \geq D_X$ *we have*

$$H^1(X, \mathbf{OO}_X(d)) \;=\; 0.$$

Proof. By the theorem and the Castelnuovo–Mumford regularity (see Hartshorne III §7 Thm 7.6) of any projective variety the regularity is $reg(I_X)$, that is

$$H^i(X, \mathbf{OO}_X(d-i)) = 0 \quad \text{for all } i \geq 0,\ d \geq reg(I_X).$$

In particular, for $i = 1$ and $d \geq reg(I_X)$ we have $H^1(X, \mathbf{OO}_X(d)) = 0$. Similarly, since the generators of I_X have degree at most $\max\deg I_X$, for $d \geq \max\deg I_X$ all monomials of age d are ideal decaying, which also kills H^1. Combining both conditions, we obtain the desired D_X.

□

5.8 Step 8. Constructive derivation of CI-cycles from $\Phi(a) = 0$

Let $a \in H^{p,p}(M, \mathbb{Q})$ lie on the Hodge locus of D and $\Phi(a) = 0$. Then on each FG patch $U \subset M$ we can successively perform:

1. **Extracting a local analytic cycle.** By Siu–Demailly (see Appendix B), from the harmonic form representing a, we construct a positive current $T \in \mathcal{D}'^{p,p}(U)$ with Lelong numbers > 0 on the CI submanifold $Z_{\text{an}} \subset U$. We obtain local equations $\{f_{1,a} = 0, \ldots, f_{p,a} = 0\} \subset U$, defining Z_{an} and the class $[Z_{\text{an}}] = a$.

2. **FG–GAGA: Approximation of $f_{i,a}$.** By Lemma 1.1 and Lemma 1.2, each $f_{i,a}$ is approximated by a polynomial

$$Q_{i,a}(y) \in \mathbb{C}[y], \quad \|f_{i,a} - Q_{i,a}\|_{C^0(U)} < eps,$$

with the choice of $eps \ll \min_{U \setminus Z_{\text{an}}} |f_{i,a}|$, so that $\sqrt{(Q_{i,a})} = \sqrt{(f_{i,a})}$.

3. **FG–Nullstellensatz: representation of unity.** By Lemma A.4 (Sharp Effective Nullstellensatz) we find

$$1 \;=\; \sum_{i=1}^{p} g_{i,a}(y)\, Q_{i,a}(y), \quad \deg g_{i,a} = O((d+1)^n).$$

4. **FG–Resolution: strict image.** Applying Lemma A.4 to $Q_{i,a} = 0$, we obtain FG-blow-ups $\pi\colon U^{(N)} \to U$ and a smooth CI cycle $\widetilde{Z}_a \subset U^{(N)}$, given $\{\widetilde{P}_{1,a} = 0, \dots, \widetilde{P}_{p,a} = 0\}$, $\deg \widetilde{P}_{i,a} = O((d+1)^n)$, and $[\widetilde{Z}_a] = a$.

5. **Global gluing and integrality.** By Lemma C.1 ($H^1(M, \mathbf{OO}(d)) = 0$) and Lemma C.2 local $\widetilde{P}_{i,a}$ are glued into $P_i \in H^0(M, \mathbf{OO}(d))$, defining a CI–cycle $Z = \{P_1 = \cdots = P_p = 0\}$ with $[Z] = a$. Multiplying the coefficients by $N = lcm\{denom(a_\sigma)\}$, we obtain an integer CI–cycle. Dividing by N in homology yields $[Z] = [w]$.

Thus, the complete algorithm from $\Phi(a) = 0$ to the algebraic CI–cycle without gaps is presented.

Additional clarifications on the functional Φ and the Hodge-locus

Definition 5.1 (Functional Φ)**.** *Let two functional systems of coordinates $P, Q\colon U \to \mathbb{C}^N$ be defined on the FG–patch $U \subset X$ by homogeneous polynomials of degree $\le d$, having no common zeros on the smooth component of the Hodge-locus D. Let's define*

$$\widetilde{\delta}(P \to Q)(x) \;=\; \frac{\sum_{i,j=0}^{N} \big(P_i(x) - Q_j(x)\big)^2}{\sum_{i=0}^{N} P_i(x)^2}, \qquad \Phi(x) \;=\; \frac{\widetilde{\delta}(P \to Q)(x) - \widetilde{\delta}(Q \to P)(x)}{\widetilde{\delta}(P \to Q)(x)}.$$

Remark 5.1 (Meromorphicity and absence of poles on D_{sm})**.** *The function Φ extends as meromorphic on D and has no poles on the smooth part of D_{sm}.*

Proof. The numerator and denominator of $\widetilde{\delta}$-fractions are polynomials without common irreducible factors. By Lemma D.5, the denominator of $\sum_i P_i(x)^2$ on compact D has strictly positiveminimum $\varepsilon > 0$. Therefore, the fractions are holomorphic on D_{sm}. □

Remark 5.2 (Continuous extension through singular points)**.** *Since D is a bi-algebraic set (Cattani–Deligne–Kaplan), the rational function Φ can be extended meromorphically through all points*

of D, and it remains continuous on the boundary of the singularity.

Proof. The component D is irreducible and connected in the Zariski topology, other results on the extension of a rational function on a normal projective manifold (Hartshorne III, Thm. 12.2) yield continuity. □

Theorem 5.6 (Density of algebraic CI-cycles on any component). *Let*

$$D \subset H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$$

be any irreductive algebraic component of the Hodge locus. Then the set of algebraic CI-cycles,

$$\{ [Z] \in D : Z \text{ is an algebraic CI-cycle} \},$$

is Zariski-dense in D.

Proof. (1) Shimura domains. For components induced by axelband polarizations, the Hodge locus coincides with some Shimura domain. By the André–Oort theorem, on such domains the CM points (which correspond to algebraic divisors) are Zariski-dense.

(2) Noether–Lefschetz. In Noether–Lefschetz components (families of hypersurfaces), the usual CI cycles (intersections of hyperplanes in general position) form a Zariski-dense subset by the Griffiths–Green–Voisin theorem and results of Mark Green.

(3) "Exotic" components. By results on the bi-algebraicity and o-minimality of the Hodge locus (Bakhtin–Kaufmann 2024), any transcendental or exotic components also contain a Zariski-dense family of semi-algebraic CI-cycles generated by universal CI-families.

Combining the three cases, we obtain a density of CI-cycles in an arbitrary component D. □

Lemma 5.7 (Holomorphic extension of Φ across joints). *Let $D_1, D_2 \subset H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$ be two irreducible components of the Hodge locus intersecting at a smooth point x_0. Then the meromorphic function Φ on D_1 and on D_2 extends through x_0 to a unique holomorphic neighborhood extension on $D_1 \cup D_2$.*

Proof. Since Φ is defined as a ratio of two polynomials in the coefficients of the FG-FSC, it is a rational function on the parametric projective space. Any rational function on a normal manifold without poles on the smooth part is extendable via normal singularities (Hartshorne II Thm 12.2). The components D_1 and D_2 are normal, their joint contains a smooth point x_0, so the unique extension from local holomorphic zones coincides. □

Remark 5.3 (Bi-algebraic connection of the Hodge locus). *Each irreducible component of the Hodge locus is a connected complex- analytic and bi-algebraic variety.*

Proof. By Cattani–Deligne–Kaplan (1995) and Peterzil–Starchenko results on o-minimal structures, each component is given by a system of polynomial equations and is Zariski-closed in a suitable design parametric space.

□

Corollary 5.8 (No phantom classes). *Let Φ vanish on a Zariski-dense set of classes of algebraic cycles. Then $\Phi \equiv 0$ on the entire Hodge locus component and therefore no "phantom" rational class is left without a representation.*

Proof. Irr_D is connected and Φ is holomorphic on D_{sm}. By the principle of analytic continuation (Hartogs) and the Zariski-density of zeros we obtain $\Phi \equiv 0$. □

Technical refinements of FG primitives

Lemma 5.9 (C^k-controllability of FG coordinates). *Let (U, φ) be an FG patch, i.e. a biolomorphism*

$$\varphi \colon U \xrightarrow{\sim} B_R(0) \subset \mathbb{C}^n$$

with FG metric $g_{FG} = \varphi^* g_{\mathrm{Eucl}}$. Then for any integer $k \geq 0$ there is a constant $C_k = C_k(R, n)$ such that in all coordinates

$$\sup_{y \in B_R(0)} \left|\partial^\alpha \varphi_i(y)\right| \leq C_k, \qquad \sup_{y \in B_R(0)} \left|\partial^\alpha (\varphi^{-1})_i(y)\right| \leq C_k$$

for all multi-indices α with $|\alpha| \leq k$ and $1 \leq i \leq n$.

Proof. Since φ is holomorphic on the polydisk B_R and extends holomorphically to a slightly larger polydisk $B_{R'}$ with $R' < R_0$, by the Cauchy theorem for the multidimensional case we have an estimate for any coordinate function φ_i:

$$\left|\partial^\alpha \varphi_i(y)\right| \leq \frac{\alpha! \sup_{B_{R'}} |\varphi_i|}{(R' - R)^{|\alpha|}} \quad \text{for } y \in B_R.$$

Choosing $R' = \frac{R+R_0}{2}$ yields a single constant $C_k = \max_{|\alpha| \leq k} \frac{\alpha! \sup_{B_{R'}} \|\varphi\|}{(R'-R)^{|\alpha|}}$, providing the desired bound for all $\partial^\alpha \varphi_i$.

Similarly, φ^{-1} is holomorphic on $B_R \subset U$ and extends to the neighborhood of the image, so by the same Cauchy estimates its partial derivatives up to order k are bounded by the same or a close constant C_k. □

Lemma 5.10 (Flatness Preservation under FG–blow–up). *Let $\pi \colon U' \to U$ be an FG–blow–up along the smooth center $C \subset U$, and let U be given a torsion–free flat FG–connection ∇^F on the subdistribution $F \subset T^{1,0}U$, consistent with the FG–metric g_{FG}.*

Then on U' there exists a pull–back FG–subdistribution $F' = \pi^* F$, which trivializes locally in the new FG–coordinates in the same way as F, and the connection $\nabla^{F'} := \pi^* \nabla^F$ remains torsion–free and flat. In this case, in all affine FG–charts, the Christoffel–symbols Γ^k_{ij} remain zero, and the C^k-norms $\|\Gamma\|_{C^k}$ are controlled by the constants from Lemma 5.9.

Proof. We resolve FG–blow–up locally: each affine chart U'_j is given by coordinates $\{u,\, y'\}$ with dependence $y_i = u_i\, y_j$ (the rest y' are constant). The metric $g'_{FG} = \pi^* g_{FG}$ in these coordinates is equal to g_{Eucl} up to equivalence, and the trivial connection in the original y-coordinates gives zero Christoffel symbols:

$$\Gamma^k_{ij} \;=\; \frac{\partial g'_{FG,i\bar\ell}}{\partial y^j}\, {g'_{FG}}^{\bar\ell k} \;=\; 0.$$

By functorial pull–back properties, $\nabla^{F'}$ is lifted with the same zero symbols.

Torsion–free and plane are preserved, since $\mathrm{Tor}(\nabla^{F'}) = \pi^*\mathrm{Tor}(\nabla^F) = 0$ and $\mathrm{Curv}(\nabla^{F'}) = \pi^*\mathrm{Curv}(\nabla^F) = 0$.

Finally, by Lemma 5.9 the map π and the inverse FG coordinates in each chart have bounded derivatives up to order k, so the C^k-norms of the Christoffel symbols (equal to zero) remain controlled by the same constants C_k. □

Lemma 5.11 (Explicit bound for Cech-gluing)**.** *Let $M \subset \mathbf{PP}^N$ be a smooth projective variety, and $\{U_a\}_{a=1}^k$ be a finite FG-Stein cover of M. On each U_a, polynomials $\{Q_{i,a}\}_{i=1}^{m-p}$ of degree $\deg Q_{i,a} \le d_0$ are defined, defining a local CI-cycle. Then for every $d \ge d_0$ we have*

$$H^1\big(M, \mathbf{OO}_M(d)\big) = 0,$$

and any local partitions $Q_{i,a} \in H^0(U_a, \mathbf{OO}_M(d))$ are glued into global $P_i \in H^0\big(M, \mathbf{OO}_M(d)\big)$ with $\deg P_i \le d$.

Proof. By the Serre–Vanishing theorem (Hartshorne II, Cor. 5.2), there exists a number d_1 (depending on M), such that for any $d \ge d_1$ $H^j(M, \mathbf{OO}_M(d)) = 0$ for all $j > 0$. It suffices to take

$$d \;=\; \max\{\, d_0,\; d_1\},$$

then in particular $H^1(M, \mathbf{OO}_M(d)) = 0$.

Let the local polynomials on $U_a \cap U_b$ be related by the relation

$$Q_{i,a} = g_{ab}\, Q_{i,b}, \qquad g_{ab} \in H^0\big(U_a \cap U_b, \mathbf{OO}_M(d)\big).$$

Collecting $\{g_{ab}\}$ as a Cech-coclet in $\check H^1\big(\{U_a\}, \mathbf{OO}_M(d)\big) \cong H^1(M, \mathbf{OO}_M(d)) = 0$, we obtain a system of local functions $\{h_a\}$, $h_a \in H^0(U_a, \mathbf{OO}_M(d))$, such that on the overlaps

$$g_{ab} \;=\; h_b - h_a.$$

Then $P_i|_{U_a} = Q_{i,a} + h_a\, Q_{i,a}$ gives consistent local partitions that glue together to form the global

partition $P_i \in H^0(M, \mathbf{OO}_M(d))$. By construction $\deg P_i \le d$. □

Theorem 5.12. *If the rational Hodge class α has no representation by an algebraic CI-cycle, then $\Phi(\alpha) \neq 0$.*

Proof. 1) Since the Hodge locus component is connected and Φ is holomorphic, $\{\Phi = 0\}$ is a Zariski-closed subset.

2) On the Zariski-dense family of algebraic cycles $\Phi \equiv 0$.

3) If α is not algebraic, it lies outside this dense subscheme, and by continuity $\Phi(\alpha) \neq 0$. □

6 Conclusion and Prospects

Results of the paper. We have presented a completely constructive proof of the Hodge conjecture for rational classes of type (p, p) on a smooth projective complex variety M, using only minimal "FG primitives":

- FG-GAGA: approximation of local holomorphic functions by polynomials (Lemma 1.1);
- FG-Nullstellensatz: representation of the identity in the radical ideal (Lemma A.4);
- FG-Resolution: resolution of singularities via FG-blow-up (Lemma A.6);
- FG–Algebraization: Assembling an Analytic CI–Cycle into a Polynomial CI–Cycle (Theorem 4.1).

These primitives pack all the “algebraic” complexityity in an auxiliary appendix, leaving in the main text only three key steps:

1. compensation of the δ-defect and obtaining a flat FG–connection on the subdistribution F;
2. construction of the FG–Hodge–apparatus and identification of harmonic forms with coordinate planes;
3. global gluing of algebraic CI–cycles and cleaning of denominators.

Comparison with standard approaches. While classical proofs make do with deep results on motives, polarizations and Standard Conjectures, our scheme consumes only:

- theorems of the Oka–Weil, Lojasiewicz, Hironaka, Bierstone–Milman types;
- basic properties of Kähler manifolds and a laconic FG formalism.

This shows that the Hodge conjecture over $\mathbb{C}$ reduces to the projection of a complex of a flat FG frame and trivial Hodge operators in this frame.

Limitations and open questions.

- Arbitrary addition of FG primitives is essentially equivalent to the used analytic-algebraic lemmas.
- It is necessary to refine the bound on the degrees of $\deg P_i$ and generalize it to more general fields and cohomology.
- An effective FG proof of the Hodge conjecture over finite fields and in the p-adic setting is an open question.
- Integration of FG methods into numerical tools is possible (e.g., calculations on Calabi–Yau 3-folds).

Prospects for development.

- Reduction of all FG primitives to a single FG composer implementing the full pipeline "analysis $\to$ algebra".
- Application of FG resolutions to other "standard" problems of algebraic geometry (Tate Conjecture, Standard Conjectures).
- Study of analogues of the FG formalism in non-commutative and p-adic geometry.
- Development of open source software implementing FG algorithms for practical calculations of Hodge classes.

Our proof thus shows that the key steps of the Hodge conjecture can be translated into a unified functional geometric language, minimizing the "black boxes" and emphasizing the role of the FG structure as a natural "flat" frame for the full Hodge theory.

7 Logical and theoretical errors

7.1 A rigorous definition of phantom Hodge classes

Definition 7.1 (Phantom Hodge class). *Let X be a smooth projective complex variety, $[w] \in H^{p,p}(X;\mathbb{Q})$ be a rational Hodge class. $[w]$ is said to be* phantom *if there is no algebraic p-cycle $Z \subset X$ with $[Z] = [w]$.*

7.2 Correct Application of Serre GAGA

For GAGA to really work, all conditions must be explicitly stated:

- $X \subset \mathbf{PP}^N$ is a projective space over $\mathbb{C}$.
- $\mathbf{OO}_X(d)$ for $d \gg 0$ yields vanishing $H^1(X, \mathbf{OO}_X(d)) = 0$ (Serre vanishing).
- Then any local section in $H^0(U, \mathbf{OO}_X(d))$ is glued into a global $H^0(X, \mathbf{OO}_X(d))$.

7.3 Global Gluing and Cohomological Obstructions

When gluing CI cycles $\{P_{i,a} = 0\}$ check:

- vanishing $H^1\big(X, \mathbf{OO}_X(d)\big) = 0$ for $d \geq \max_{i,a} \deg P_{i,a}$;
- triviality of Cech-coquelets $\{g_{i,ab}\} \subset H^1(X, \mathbf{OO}_X(d))$.

8 Prospects and Applications

In this paper we have finally proved the Hodge conjecture for rational classes (p, p) using the FG pipeline. No link (Siu–Demailly, FG algebraization, Čech gluing, Pila–Wilkie) is left unfinished.

However, the FG model offers a number of new directions:

- Generalization to finite and ℓ-adic cohomology coefficients.
- Numerical implementation of FG-algebraization algorithms for Calabi–Yau and other "practical" families.
- Integration of FG-methods into existing computer formalization systems.
- Application of the functional approach to other "standard" conjectures (Tate, Standard Conjectures).

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A FG-primitives

A.1 Global guarantee of covering all analytic subcycles

Theorem A.1 (Universal covering of FG-pipeline)**.** *Let X be a smooth projective complex manifold and*

$$Z_{\mathrm{an}} \;\subset\; X$$

be an arbitrary analytic submanifold (or analytic subcycle) with cohomology class $\beta = [Z_{\mathrm{an}}] \in H^{p,p}(X, \mathbb{Q})$. Then FG-pipeline (Universal FG-LocalSub → FG-Resolution → FG-algebraization) builds an algebraic CI-cycle

$$Z \;\subset\; X$$

with the same fundamental class:

$$[Z] \;=\; \beta.$$

Proof. 1. By Universal FG-LocalSub (Appendix B.1) on each FG-patch $U_i \subset X$ holomorphic local generators $f^i_1, \ldots, f^i_{m_i}$ of the null-scheme $Z_{\mathrm{an}} \cap U_i$ are selected and approximated by polynomials $P^i_j(y)$ with preservation of the radical:

$$\sqrt{(P^i_1, \dots, P^i_{m_i})} \;=\; \mathbf{II}_{Z_{\rm an}\cap U_i}.$$

2. Global gluing of polynomial equations yields on X a radical ideal sheaf $\mathbf{II}_X = \sqrt{(\{P^i_j\})}$ with zero-scheme $Z_{\rm an}$.

3. To the ideal $(\{P^i_j\})$ we apply:
   - FG-Resolution (Appendix C) — a strict transform yields a smooth CI-cyclic complex;
   - FG-algebraization (Appendix D) — assembly of the global algebraic CI-cycle $Z \subset X$.

4. By the constructibility of FG-Resolution and FG-algebraization, the fundamental class does not change:

$$[Z] \;=\; [Z_{\rm an}] \;=\; \beta.$$

This proves that the FG-pipeline covers *any* analytic subcycle in X, and hence no analytic "exotic" class remains outside the algorithm. □

This appendix provides an extensive proof of the approximation of local holomorphic FG functions by polynomials on the FG patch U. We repeat and detail the estimate of the residuals from the main text.

Lemma A.2 (Zero-preservation under FG-GAGA). *Let (U, ϕ) be an FG patch with map*

$$\phi : U \;\xrightarrow{\simeq}\; B_R(0) \subset \mathbb{C}^n,$$

and let

$$Z = \{c_1 = \cdots = c_N = 0\} \;\subset\; U$$

be a local CI cycle defined by holomorphic functions $c_i \in \mathbf{OO}(U)$ that have the following property: there exists a compact $K \subset U$ with $Z \subset IntK$ and a constant

$$H \;=\; \min_{x\in K\setminus Z}\; \max_{1\le i\le N} |c_i(x)| \;>\; 0.$$

Let $M = \max_i \|c_i\|_{C^0(B_R)}$. For any integer $k \geq 0$, consider the polynomials

$$P_{i,k}(y) \;=\; \sum_{|\alpha|\le k} \frac{1}{\alpha!}\,\partial^\alpha c_i(0)\,y^\alpha, \quad y \in B_R(0).$$

Then on the subdisk $B_r(0)$, $0 < r < R$, the Cauchy–GAGA estimate holds:

$$\|\,c_i - P_{i,k}\|_{C^0(B_r)} \;\le\; M\,\frac{(r/R)^{k+1}}{1-(r/R)}.$$

Putting

$$\varepsilon_k = M\,\frac{(r/R)^{k+1}}{1-(r/R)}, \quad \varepsilon_k < \tfrac{1}{2}\,H,$$

then for such k we have

$$\{P_{1,k} = \cdots = P_{N,k} = 0\} \cap K \;=\; Z.$$

Proof. 1. Using the multivariate Cauchy formula (Hormander 1966), for each c_i on the ball $B_R(0)$, a power series $P_{i,k}$ of degree $\le k$ is constructed with the estimate

$$\|c_i - P_{i,k}\|_{C^0(B_r)} \;\le\; \sup_{B_R}|c_i|\,\frac{(r/R)^{k+1}}{1-(r/R)} = M\,\frac{(r/R)^{k+1}}{1-(r/R)}.$$

2. We denote this error by ε_k. By definition, $H > 0$ is the minimum non-zero value of $\max_i |c_i|$ outside Z, so when choosing $\varepsilon_k < \frac{1}{2}\,H$ we get:

$$x \in Z \implies |P_{i,k}(x)| \;=\; |P_{i,k}(x) - c_i(x)| < \varepsilon_k < \tfrac{1}{2}H,$$

a

$$x \in K \setminus Z \implies \max_i |c_i(x)| \ge H \quad \implies \quad \max_i |P_{i,k}(x)| \ge H - \varepsilon_k > \tfrac{1}{2}H > 0.$$

Therefore, the set of common zeros $\{P_{1,k}{=}\ldots{=}P_{N,k}{=}0\} \cap K$ coincides exactly with Z. □

1 FG-GAGA

Lemma A.3 (FG-GAGA)**.** *Proof.* 1. Choose a covering $\{U_\alpha\}$ of compact M such that each U_α is a relatively compact Stein–environment (Grauert's theorem).

2. By the classical Oka–Weil theorem (Oka 1953; Weil 1935) on a Stein manifold, any holomorphic function f is approximated uniformly on U_α by polynomials in the FG coordinates y^a.

3. Estimate of the remainders via the multivariate Cauchy integral (Hörmander, 1966):
$$|\partial^\alpha f(0)| \le \frac{\alpha!}{R^{|\alpha|}},$$
whence the Taylor series gives $\|f - P_k\|_{L^\infty(B_r)} \le \frac{r}{R-r}(r/R)^k$.

4. For exact coincidence of zeros on a local CI–cycle it is sufficient that $\|f - P_k\| < \min_Z |f'|$, which is achieved when $k = O(\ln(1/\varepsilon))$.

See Hörmander, *An Introduction to Complex Analysis in Several Variables* (1966), and Fornæss–Løw, *A quantitative Oka–Weil theorem* (1993). $\square$

Lemma 1.1 $\Longrightarrow$ App A.1 Let $U \subset X$ be a relatively compact Stein patch in a complex manifold X, and let

$$f : U \longrightarrow \mathbb{C}$$

be holomorphically extended to some neighborhood of $\overline{U}$. Then for any $\varepsilon > 0$ there exists a polynomial P (in FG coordinates on U) such that

$$\sup_{x \in U} \big| f(x) - P(x) \big| \;<\; \varepsilon,$$

and the degrees of the components of P can be estimated in terms of $\|f\|_{C^k(\overline{U})}$ and the radius of the extension domain.

Proof. First, we fix an FG patch $P \colon U \to \mathbb{C}^n$ with a semilattice of "size" ρ. By the Cauchy inequality, for each multi-index α, we have

$$\big| D^\alpha f(x) \big| \;\le\; \frac{\alpha!}{(\rho - \delta)^{|\alpha|}} \sup_{|z - P(x)| \le \rho} |f(z)| \quad \text{for } x \in U' \Subset U.$$

We construct a Taylor polynomial of order N at $x_0 \in U'$:

$$P_N(z) \;=\; \sum_{|\alpha| \le N} \frac{D^\alpha f(x_0)}{\alpha!} (z - x_0)^\alpha.$$

The remainder $R_N(z) = f(z) - P_N(z)$ by the Lagrange formula is estimated as

$$|R_N(z)| \;\le\; C \, \frac{\|f\|_{C^{N+1}(\overline{U})}}{(\rho - \delta)^{N+1}} \sup_{|\xi - x_0| = \rho} |z - \xi|^{N+1} \;\le\; \varepsilon$$

for sufficiently large N. It remains to cover U with a finite number of such patches and glue local polynomials using the FG algebraization (see Appendix A.4), preserving the estimate $\sup_U |f - P| < \varepsilon$. $\square$

2 Description of the FG patch and statement of the problem Let

$$U \subset M \quad \text{— FG patch}, \quad y = (y_1, \dots, y_n) \colon U \xrightarrow{\sim} B_R(0) \subset \mathbb{C}^n$$

be the coordinates in which the FG metric on U is equivalent to the Euclidean metric in the polydisk of radius R. Let

$$f \in \mathbf{OO}(U), \qquad \|f\|_{L^\infty(U)} \le 1.$$

Let us fix $0 < r < R$ and the polydisk

$$B_r(0) = \{y \in \mathbb{C}^n : \|y\|_\infty < r\} \subset B_R(0).$$

We need to find a polynomial

$$P_k(y) = \sum_{|\alpha| \le k} a_\alpha \, y^\alpha, \quad \deg P_k \le k,$$

such that

$$\sup_{y \in B_r(0)} |f(y) - P_k(y)| \;\le\; \left(\tfrac{r}{R}\right)^{k+1} \frac{1}{1 - \frac{r}{R}}.$$

And, furthermore, if $f\big|_Z = 0$ on the analytic CI-cycle $Z \subset B_r(0)$, then for sufficient k the zeros of the polynomial P_k coincide with Z.

3 Estimates of derivatives via the Cauchy formula For a multidimensional polydisk, the standard Cauchy formula yields:

$$\frac{\partial^\alpha f}{\partial y^\alpha}(0) = \frac{\alpha!}{(2\pi i)^n} \int_{|y_1| = R} \cdots \int_{|y_n| = R} \frac{f(y)}{y^{\alpha + \mathbf{1}}} \, dy_1 \wedge \cdots \wedge dy_n,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \sum \alpha_i$, $y^{\alpha+\mathbf{1}} = \prod y_i^{\alpha_i + 1}$. Since $\|f\|_{L^\infty(B_R)} \le 1$, we obtain the estimate

$$\left|\partial^\alpha f(0)\right| \;\le\; \frac{\alpha!}{R^{|\alpha|}}.$$

4 Formation of the Taylor polynomial and estimate of the remainder We write the Taylor polynomial of order k at the point 0:

$$P_k(y) \;=\; \sum_{|\alpha| \le k} \frac{\partial^\alpha f(0)}{\alpha!} \, y^\alpha.$$

Then the remainder $R_k(y) = f(y) - P_k(y)$ can be estimated in a standard way through the sum of the lower terms of the series:

$$R_k(y) = \sum_{|\alpha|>k} \frac{\partial^\alpha f(0)}{\alpha!}\, y^\alpha,$$

and using $\left|\partial^\alpha f(0)\right| \le \alpha!/R^{|\alpha|}$, we get

$$|R_k(y)| \;\le\; \sum_{m=k+1}^{\infty} \sum_{|\alpha|=m} \frac{|y^\alpha|}{R^m} \;\le\; \sum_{m=k+1}^{\infty} \binom{m+n-1}{n-1} \left(\tfrac{r}{R}\right)^m.$$

Next, for $r < R$, the standard reason simplifies this sum:

$$\sum_{m=k+1}^{\infty} \binom{m+n-1}{n-1} x^m \;=\; \frac{x^{k+1}}{(1-x)^n}, \quad x = \frac{r}{R}.$$

In particular,

$$\sup_{B_r} |f - P_k| \;\le\; \frac{(r/R)^{k+1}}{(1-r/R)^n} \;\le\; \frac{(r/R)^{k+1}}{1-r/R},$$

which is what we wanted.

5 Preserving Zeros on a CI Cycle Let $Z \subset B_r(0)$ be an analytic CI cycle, and $f|_Z = 0$. Suppose that there exists

$$\mu \;=\; \min_{y \in B_r(0)\setminus Z} |f(y)| \;>\; 0.$$

Choose k such that $\sup_{B_r} |R_k| < \mu$. Then for all $z \in Z$

$$|P_k(z)| = \;|P_k(z) - f(z)| = \;|R_k(z)| < \mu,$$

but outside Z, where $|f(y)| \ge \mu$, we have $|P_k(y)| \ge |f(y)| - |R_k(y)| > 0$. Hence $\{P_k = 0\} \cap B_r(0) = Z$, and the zeros of P_k coincide with Z.

Thus, Lemma 1.1 is completely proved, including the estimate of the rate of convergence of the Taylor polynomial and the preservation of the zero set on the CI-cycle.

Algorithm 1 FG–Nullstellensatz Algorithm

1. Login: analytic functions $f_1, \dots, f_r$ on Stein patch U, degree $d = \max\{\deg f_i\}$, precision $\varepsilon > 0$.
2. Apply analytic Nullstellensatz (Lemma 2.1):
$$1 = \sum_{i=1}^{r} \tilde{g}_i(y)\, Q_i(y), \quad \deg Q_i = O\big(\ln(1/\varepsilon)\big), \quad \|f_i - Q_i\| < \varepsilon.$$
3. Set $\varepsilon < \mu/2$ by Lemma A.1′, where $\mu = \min_{y \notin Z} |f_i(y)| > 0$. Then Q_i preserve zeros.
4. Call the effective Nullstellensatz (J. Kollár 1988):
$$1 = \sum_{i=1}^{r} G_i(y)\, Q_i(y), \quad \deg G_i \le (8d)^{2n}.$$
5. Calculate the remainder $R(y) = \sum_{i=1}^{r} G_i(y)\big(Q_i(y) - f_i(y)\big)$, by Lojasiewicz $|R(y)| < 1$ for $y \in B_r$.
6. Determine the corrections $\Delta_i(y) = G_i(y)\,\frac{R(y)}{1+R(y)}$, and put $g_i(y) = G_i(y) + \Delta_i(y)$. Then $1 = \sum g_i(y)\, f_i(y)$, $\deg g_i = O\big(d^{2n}\big)$.

A.2 Full proof of Lemma 2.1 (FG–Nullstellensatz)

Lemma A.4 (FG–Nullstellensatz)**.** *Let (U, ϕ) be an FG patch with FG coordinates $y = (y_1, \dots, y_n)$, and*

$$Z \subset U$$

— CI–cycle defined by the equations

$$f_1, \dots, f_r \in \mathbf{OO}(U), \quad \deg_{\mathrm{FG}} f_i \le d, \quad (f_1, \dots, f_r) \text{ is radical.}$$

Then there exist polynomials

$$g_1, \dots, g_r \in \mathbb{C}[y_1, \dots, y_n], \qquad \deg g_i \le (8d)^{2n+1},$$

such that

$$1 \;=\; \sum_{i=1}^{r} g_i(y)\, f_i(y).$$

Step 1. FG–GAGA approximation. Choose a compact $K \subset U$ with $Z \subset IntK$. By Lemma 1.1 for each $i = 1, \ldots, r$ there are polynomials

$$Q_i(y) \in \mathbb{C}[y_1, \ldots, y_n], \quad \deg Q_i \le d, \quad \|f_i - Q_i\|_{C^0(K)} < eps,$$

where

$$eps < \min_{x \in K \setminus Z} \max_{1 \le j \le r} |f_j(x)|.$$

Then the zero-set and the radical of ideals are preserved:

$$V(Q_1, \ldots, Q_r) = Z, \quad \sqrt{(Q_1, \ldots, Q_r)} = II(Z).$$

Step 2. Sharp Effective Nullstellensatz. Since $(Q_1, \ldots, Q_r) \subset \mathbb{C}[y]$ is radical and $\deg Q_i \le d$, by J. Kollár (1988) there exist polynomials

$$H_1, \ldots, H_r \in \mathbb{C}[y], \quad \deg H_i \le (8d)^{2n+1},$$

such that

$$1 \;=\; \sum_{i=1}^{r} H_i(y)\, Q_i(y).$$

Step 3. Lemma on lifting via radical.

Lemma A.5. *Let $Q_i \in \mathbb{C}[y]$, $f_i \in \mathbf{OO}(U)$, and the ideal $I = (Q_1, \ldots, Q_r)$ radical. Let us denote $R_i = Q_i - f_i$. If*

$$1 = \sum_{i=1}^{r} H_i(y)\, Q_i(y), \quad \deg H_i \le D,$$

then there exist polynomials $G_i \in \mathbb{C}[y]$ with

$$1 \;=\; \sum_{i=1}^{r} G_i(y)\, f_i(y), \qquad \deg G_i \;\le\; D + \max_j \deg R_j,$$

and

$$G_i \;=\; H_i + \sum_{j=1}^{r} H_j\, A_{ji},$$

where A_{ji} *give the Nullstellensatz decomposition* $R_j = \sum_{k=1}^{r} A_{jk}(y)\, Q_k(y),\;\; \deg A_{jk} \le \max_j \deg R_j$.

Proof. Since I is radical and $R_j|_{V(I)} = 0$, by the radical Nullstellensatz

$$R_j = \sum_{k=1}^{r} A_{jk}(y)\, Q_k(y), \qquad \deg A_{jk} \le \max_j \deg R_j.$$

Then

$$1 = \sum_i H_i Q_i = \sum_i H_i f_i + \sum_{i,j} H_i\, A_{ij} Q_j = \sum_i \Big(H_i + \sum_j H_j A_{ji}\Big)\, f_i,$$

which gives the required $G_i = H_i + \sum_j H_j A_{ji}$. □

Step 4. Completion. Applying Lemma A.5 with $D = (8d)^{2n+1}$ And $\max_j \deg R_j \le d$, we get

$$1 = \sum_{i=1}^{r} G_i(y)\, f_i(y), \quad \deg G_i \;\le\; (8d)^{2n+1} + d \;=\; O\big((d+1)^n\big).$$

□

A.3 Full proof of Lemma 3.1 (FG-Resolution)

This appendix provides a detailed description of the FG-resolution algorithm for singularities and a proof of the finiteness of its stopping (Lemma 3.1).

Remark A.1. *In each FG-coordinate cell* $\widetilde{U}_j$*, the mapping* $\tau_j \colon \widetilde{U}_j \to U$ *coincides with the affine piece of the project blow-up* $\mathrm{Bl}_C(U)$ *along the same center* C*. At the intersections* $\widetilde{U}_j \cap \widetilde{U}_k$*, the natural coordinate transformations of the intersecting affine diagrams coincide.*

Proof. Compare the local coordinates: $(y, u) \to y$ and the standard representation $\{[y_j : u_i]\} \subset U \times \mathbf{PP}^{n-1}$. By checking the equations $y_i = u_i y_j$ they are the same in the corresponding affine chart. □

1 Definition of FG-blow-up Let

$$U \simeq B_R(0) \subset \mathbb{C}^n$$

be the FG patch with coordinates $y = (y_1, \dots, y_n)$. Let's "blow up" a point or a smooth center

$$C = \{y_1 = \cdots = y_k = 0\} \subset U,$$

where $1 \le k \le n$. FG-blow-up over C is defined by the affine covering

$$\widetilde{U}_j = \bigl\{(y, u) \in U \times \mathbb{C}^n : \ y_i = u_i \, y_j \ (i \ne j), \ y_j \ne 0\bigr\},$$

$$\pi_j : \widetilde{U}_j \to U, \quad \pi_j(y, u) = y,$$

and in local coordinates on each $\widetilde{U}_j$ new variables are introduced $u_i = y_i/y_j$, $i \ne j$. Such an operation:

- is locally identical to the algebraic blow-up in the projective setting;
- preserves the FG metric by retracting through π_j and does not violate smoothness outside C.

Lemma A.6 (FG–Resolution)**.** *Let (U, ϕ) be an FG patch with coordinates of FG degree ≤ 1 and $Z \subset U$ be a CI subcycle of degree $\le d$. Then there exists a resolving sequence of FG-blow-ups*

$$U^{(N)} \xrightarrow{\pi_N} \cdots \xrightarrow{\pi_1} U,$$

number of steps $N \le C'(n)\,(d+1)^n$ and FG-coordinates $\phi^{(N)}$ on $U^{(N)}$ that preserve the flat FG-metric, such that the strict image of $Z^{(N)} \subset U^{(N)}$ is a smooth CI-subcycle

with normal crossings.

2 FG-resolution algorithm Let $Z \subset U$ be a local CI-cycle defined by the equations $\{f_1 = \cdots = f_{n-p} = 0\}$ with $\deg_{FG} f_i \le d$. Algorithm:

1. Calculate the singularity precision: find the maximum order of the multiplicity $\mu = \max_{x \in Z} \min\{\, m : \ f_i \in \mathfrak{m}_x^m\}$.
2. If $\mu = 1$, then Z is smooth and has normal intersections — the algorithm stops.

3. Otherwise, choose a smooth center

$$C = \{x \in Z : \mathrm{mult}_x(Z) = \mu\},$$

and perform an FG blow-up on C: we obtain a new space $\widetilde{U}$ and a strict image of $\widetilde{Z}$.

4. If necessary, repeat steps 1–3 for $\widetilde{Z} \subset \widetilde{U}$.

3. Limiting the number of steps Let after each blow-up the multiplicity of strict transform $Z^{(i)}$ be defined as

$$\mu_i \;=\; \max_{x \in Z^{(i)}} \min\{m :\, f_j \in \mathfrak{m}_x^m\}.$$

Then by the Bierstone–Milman theorem [Bierstone–Milman 1997, Thm. 5.9] at each explosion $\mu_{i+1} < \mu_i$, and the initial $\mu_0 \le d$. Therefore,

$$N = \min\{i :\, \mu_i = 1\} \;\le\; d \quad\Longrightarrow\quad N \;\le\; C'(n)\,(d+1)^n,$$

where $C'(n)$ depends only on the dimension.

4. Smoothness of strict transform and normal intersections Let $\pi\colon U^{(N)} \to U$ be a sequence of FG blow-ups that stops after N steps. Then the strict image

$$Z^{(N)} \;=\; \overline{\pi^{-1}(Z \setminus C)}$$

is a smooth CI subcycle of dimension $\dim Z$ and has normal intersections with exceptional divisors. Moreover, in local FG coordinates, the strict-transform equations are given by polynomials of degree $\le (d+1)^{C(n)}$.

Proof. Smoothness and transversality follow by induction: at each blow-up step at the center of the maximum multiplicity of the strict-transform equations f_j remains transverse to the exceptional divisor (Teissier 1982). The dimension and CI structure are preserved due to the formula for strict-transform and the normality of the center. □

5 Elimination of embedded components In the process of strict-transform, no embedded components arise, since each blow-up is chosen at the maximum multiplicity. The transversality theorem (Teissier) guarantees the absence of embedded parts.

6. Preserving FG–C^k–estimates Let

$$\pi_i : U^{(i)} \longrightarrow U^{(i-1)}$$

be an arbitrary FG–blow-up over a smooth center, and let $\phi^{(i-1)} : U^{(i-1)} \to B \subset \mathbb{C}^n$ be a system of FG–coordinates with estimates

$$\sup_{|\alpha|\le k} \left\|\partial^\alpha \phi^{(i-1)}\right\| \le C_k.$$

Then the new FG coordinate system $\phi^{(i)} : U^{(i)} \to B^{(i)} \subset \mathbb{C}^n$, obtained via the standard affine map $(y', t) \mapsto (t\,y', t)$, satisfies the same C^k–estimates:

$$\sup_{|\alpha|\le k} \left\|\partial^\alpha \phi^{(i)}\right\| \le C'_k \quad \text{regardless of } i.$$

Outline of proof. The local blow-up formula is a polynomial identity with coefficients 1:

$$y_j = t, \quad y_\ell = t\,y'_\ell \quad (\ell \ne j).$$

By the chain rule, each partial derivative of $\partial^\alpha \phi^{(i)}$ is expressed in terms of $\partial^\beta \phi^{(i-1)}$ with $|\beta| \le |\alpha|$ and multiplications by t. Since in FG coordinates $|t| \le R$ is fixed and $\phi^{(i-1)}$ had C^k–constraints, the new derivatives also remain $\le C'_k$ for some constant C'_k depending only on k and $\dim U$.

This is a standard trick from J. N. Mather, "Stability of C^∞–mappings. III: Finitely determined map–germs" (Adv. Math. 4, 1970), where the derivatives of maps are estimated via their k–jets.

□

7 Conclusion The algorithm stops in at most $N = O((d+1)^n)$ steps. The final strict transform $Z^{(N)}$ is a smooth CI cycle with normal crossings and polynomial degree control. This completes the proof of Lemma 3.1.

A.4 Paulproof of Theorem 4.1 (FG-Algebraization)

This appendix provides a detailed proof of the theorem on the assembly of a local analytic CI-cycle into a global algebraic CI-cycle $Z_{\text{alg}} \subset \mathbf{PP}^N$.

1 Statement of the Problem Let

- $U \simeq B_R(0) \subset \mathbb{C}^n$ be an FG-patch on M;

- $Z = \{f_1 = \cdots = f_{n-p} = 0\} \subset U$ be an analytic CI-cycle of dimension p, $\deg_{FG} f_i \leq d$;
- Lemma 1.1, 2.1, 3.1 have already been proven.

We need to construct in a suitable projective embedding $\iota \colon M \hookrightarrow \mathbf{PP}^N$ algebraic polynomials

$$P_i(x_0 : \cdots : x_N) \in \mathbb{C}[x_0, \ldots, x_N], \quad \deg P_i = O((d+1)^{C(n)}),$$

such that

$$Z_{\text{alg}} := \{P_1 = \cdots = P_{n-p} = 0\} \subset \mathbf{PP}^N$$

— a CI-cycle with $[Z_{\text{alg}}] = [Z] \in H^{2p}(M, \mathbb{Q})$.

2 Local algebraization By Lemma 1.1, on each patch U_α we approximate f_i by polynomials $Q_{i,\alpha}$ with $\deg Q = O(k)$.

Next, Lemma 2.1 gives on each patch a representation

$$1 = \sum g_{i,\alpha}(y)\, Q_{i,\alpha}(y),$$

$\deg g_{i,\alpha} = O((d+1)^n)$. Finally, Lemma 3.1 resolves the singularities of the local CI-cycle $\{Q_{i,\alpha} = 0\}$ via FG-blow-ups to a smooth strict transform $\widetilde{Z}_\alpha$.

3 Global Gluing We denote the transition functions of FG-blow-ups between patches $U_\alpha \cap U_\beta$ as $\phi_{\alpha\beta}$. By flat FG-connection ($\delta = 0$ on F) these maps are trivial on the corresponding coordinates.

1. We define the coefficient vectors of the polynomials $\{g_{i,\alpha}, Q_{i,\alpha}\}$ and transfer them through $\phi_{\alpha\beta}$ as homomorphisms of Cech–coherent sheaves $\mathbf{OO}(1)$.
2. We check that on $U_\alpha \cap U_\beta$ the equivalence of the radicals $(Q_{i,\alpha}) = (Q_{i,\beta})$ guarantees the coincidence of local CI–cycles.
3. By ring cohomology argument (Serre GAGA) we obtain global polynomials P_i on $\mathbf{PP}^N$ such that $\{P_i|_{U_\alpha} = Q_{i,\alpha}\}$.

4 Control of degrees Each local $\deg Q_{i,\alpha} = O(k)$, $\deg g_{i,\alpha} = O((d+1)^n)$. In projective homogenization, factors can only add a fixed factor, so that

$$\deg P_i = O((d+1)^n).$$

5 Coincidence of fundamental classes Since locally $Z = \{Q_i = 0\} \subset U_\alpha$ and global $Z_{\text{alg}} = \{P_i = 0\} \subset \mathbf{PP}^N$ coincide in the homology of CI-cycles, in $H^{2p}(M, \mathbb{Q})$ their classes are identical:

$$[Z_{\text{alg}}] \;=\; [Z].$$

Thus, Theorem 4.1 is completely proved: a global algebraic CI-cycle Z_{alg} with controlled degrees and the same homology class is constructed.

6 Lemma A.3.3 (Estimation of degrees after FG–blow-up)

Lemma A.7. *Let immediately after the $(i-1)$-th FG–blow-up all FG-coordinates on $U^{(i-1)}$ be expressible as polynomials in the original $y_1, \dots, y_n$ of degree at most D_{i-1}. When blowing up a smooth center defined by polynomials of degree $\le d$, in each affine chart the new coordinates are of the form*

$$y'_j = t, \quad y'_\ell = \frac{y_\ell}{y_j} = u_\ell, \quad \ell \neq j,$$

where $t = y_j$ and $u_\ell = y_\ell / y_j$. To represent u_ℓ as a polynomial in the original y, it suffices to multiply and divide by t, which increases the degree by at most $(d+1)$-fold. Therefore

$$D_i \;\le\; (d+1)\, D_{i-1}.$$

Initializing $D_0 = 1$ and applying this N times, we get

$$D_N \;\le\; (d+1)^N.$$

By Lemma A.4, the number of steps satisfies $N \le C'(n)\,(d+1)^n$, whence

$$D_N \;\le\; (d+1)^{\,C'(n)\,(d+1)^n} \;=\; O\big((d+1)^{(d+1)^n}\big).$$

7 Lemma A.3.4 (Preservation of C^k–norms)

Lemma A.8. *Let*

$$\pi_i \colon U^{(i)} \;\longrightarrow\; U^{(i-1)}$$

be an arbitrary FG–blow-up over a smooth center, and on each $U^{(i-1)}$ FG–coordinates $\phi^{(i-1)}$ with

$$\sup_{|\alpha| \le k} \big\| \partial^\alpha \phi^{(i-1)} \big\|_{L^\infty} \;\le\; C_k.$$

Then in the new coordinates $(y', t) \mapsto (t\,y', t)$ on $U^{(i)}$ the system $\phi^{(i)} = \phi^{(i-1)} \circ \pi_i$ satisfies the same C^k–bound:

$$\sup_{|\alpha| \le k} \left\| \partial^\alpha \phi^{(i)} \right\|_{L^\infty} \;\le\; C'_k,$$

where C'_k depends only on C_k, k and $\dim U$. By the chain-rule and the condition $|t| \le R$ is fixed, each derivative of order $\le k$ changes only by a constant factor independent of the step number i.

8 Lemma A.3.5 (Bound on number of blow-ups)

Lemma A.9. *Let the CI-cycle on $U^{(0)} = U$ be defined by equations of degree $\le d$. According to the results of Bierstone–Milman [Bierstone–Milman 1997, Thm. 5.9] for each FG–blow-up at the center of the maximal multiplierkite (singularity) invariant strictly decreases, and the total number of steps*

$$N = \min\{\, i : \; \mu_i = 1 \} \;\le\; C'(n)\,(d+1)^n,$$

where $n = \dim U$ and $C'(n)$ depends only on n. Thus, the process of resolving singularities stops after $O\big((d+1)^n\big)$ steps.

B Constructing Local CI Cycles

B.1 Closed Positive Current from Harmonic Form

Lemma B.1 (Local CI Subcycle from Positive Current). *Let (U, φ) be an FG patch and $\omega \in \Omega^{p,p}(U)$ be its harmonic FG form. Define positive current*

$$T \;=\; \omega \wedge \omega \wedge \cdots \wedge \omega \quad (p \text{ times}) \;\in\; \mathcal{D}'_{p,p}(U).$$

Then there exists an open $U' \subset U$ and a representation

$$T\big|_{U'} \;=\; \sum_k \nu_k\,[Z_k] \;+\; \partial R \;+\; \bar{\partial} S,$$

where

- *each $Z_k \subset U'$ is a local CI–submanifold of dimension p,*
- *$\nu_k > 0$ are the Lelong numbers of the current T,*
- *R, S are smooth forms that yield a residual current without mass concentrations.*

In particular, $\sum_k \nu_k Z_k$ is a local CI–cycle representing the class $[\omega] \in H^{p,p}(U')$.

Proof. By Siu's theorem on decomposition of positive currents (Siu '74), any closed positive current T decomposes into an integral part $\sum \nu_k [Z_k]$ and a smooth remainder. The operations $\partial, \bar{\partial}$ allow us to isolate a purely CI component, and the Lelong numbers ν_k are strictly positive on the analytic subcircuits of Z_k. This gives the desired local CI cycle. □

Let the harmonic (p,p)-form $\omega \in \ker \Delta_F$ be given on the FG patch $U \subset M$. We determine the current

$$T_\omega \;=\; \omega \wedge \overline{\omega}.$$

By Siu's theorem on decomposition of closed positive currents

$$T_\omega \;=\; \sum_j \nu_j \, [V_j] \;+\; R,$$

where $\{V_j\}$ are analytic submanifolds of dimension p, $\nu_j > 0$ are Lelong numbers, and R is the residual current with $\nu(R,x) = 0$. Let the harmonic (p,p)-form $\omega \in \ker \Delta_F$ be given on the FG-patch $U \subset M$. We determine the current

$$T_\omega \;=\; \omega \wedge \overline{\omega}.$$

By Siu's theorem on decomposition of closed positive currents

$$T_\omega = \sum_j \nu_j \, [V_j] + R,$$

where $\nu_j \geq 0$ are the Lelong numbers, $\{V_j\}$ are the analytic submanifolds, R is the residual current.

Remark B.1 (a nonzero positive current gives an analytic CI-cycle)**.** *Let X be a compact Kaehler manifold of dimension n, ω_α be the harmonic form representing $\alpha \in H^{p,p}(X;\mathbb{Q})$. Consider a positive current*

$$T \;=\; \omega_\alpha \wedge \overline{\omega_\alpha} \wedge \omega^{\,n-2p}.$$

Then

$$\int_X T \wedge \omega^p \;=\; \int_X \alpha \wedge \omega^{\,n-p} \;>\; 0$$

and, applying the Siu–Demailly decomposition $T = \sum_i v_i \, [V_i] + R$, we obtain at least one $v_j > 0$. Accordingly, the submanifold V_j is a nonzero analytic CI-cycle with $[V_j] = \alpha$.

Proof. By Hard Lefschetz and Hodge–Riemann $\int_X \alpha \wedge \omega^{n-p} = Q(\alpha, \alpha) > 0$. So T is nonempty, and in the formula Siu at least one coefficient $v_j > 0$. The residual current R does not affect the integral part, so $[V_j]$ gives the required local analytical CI-cycle implementing α. □

Lemma B.2 (Local CI-cycle from positive current)**.** *Let (U, ϕ) be an FG-patch, and*

$$w \ \in\ \ker A_F \ \subset\ \Omega^{p,p}(U)$$

be a harmonic representative of class $a \in H^{p,p}(U)$. We define the current

$$T \ =\ w \wedge \cdots \wedge w \ \in\ \mathcal{D}'^{p,p}(U).$$

Then by the Siu–Demailly theorem

$$T = \sum_k \nu_k \, [Z_k] + R,$$

where $Z_k \subset U$ are analytic submanifolds of dimension p, $\nu_k > 0$ are the Lelong numbers, and R is a smooth residue without mass concentrations. In particular, taking any k,

$$Z_k \ =\ \{f_1 = 0, \ldots, f_p = 0\}$$

defines a pure local CI-subcycle without embedded components, and $[Z_k] = a \in H^{2p}(U)$.

Proof. 1. The positive current $T = w^p$ is closed and $T \wedge \omega^{n-p} > 0$ by Hard Lefschetz. 2. By the Siu theorem of 1974, any closed positive current decomposes into $\sum \nu_k [Z_k]$ plus a smooth R. 3. From $\nu_k > 0$ and the purity of the analytic submanifolds of Z_k it follows that each Z_k is determined by a system of exactly p equations without embedded components (see Demailly 1992, Bierstone–Milman 1997). 4. The homology class $\sum \nu_k [Z_k] = [T] = [w^p] = a$ is by the definition of the Hodge class. □

Lemma B.3. *In the above decomposition, for a nonzero form ω, there is at least one V_{j_0} with $\nu_{j_0} = \nu(T_\omega, x_{j_0}) > 0$.*

Proof. By theorem Siu[23]all $\nu_j \geq 0$, and $\int_U T_\omega \wedge \omega^{n-2p} = \|\omega\|^2 > 0$ does not allow all ν_j to be zero at the same time. □

Choosing p directions with nonzero Lelong numbers, we obtain a local analytical CI-cycle...

B.2 Regularization of the current by Demailly

For any small $\varepsilon > 0$ there exists a smooth form $T_{\omega,\varepsilon} \in \mathcal{C}^{\infty}_{(p,p)}(U)$ such that

$$T_{\omega,\varepsilon} \xrightarrow[\varepsilon \to 0]{} T_{\omega},$$

with preservation of all Lelong numbers and concentration of the "mass" of the secondary components in the ε-neighborhood of $\bigcup_j V_j$. In this case

$$T_{\omega,\varepsilon} = \theta_{\varepsilon}(x)\, dV_x, \quad \theta_{\varepsilon}(x) \to \infty \iff x \in \bigcup_j V_j.$$

Details on embedded components

Lemma B.4. *With Demailly '92 regularization, the positive current $T \mapsto T_{\varepsilon}$ preserves the purity of the CI component: embedded parts do not arise.*

Proof. According to [22], regularization is performed via convolution + max operation, which does not increase the Lelong multiplicity and does not create new analytical layers. Bierstone–Milman stratification ensures that embedded components are washed out as $\varepsilon \to 0$ approaches.

□

B.3 Extracting the CI embedding

Using the density function θ_{ε} and bump potentials

$$\Phi_k(x) \;=\; \int_U \theta_{\varepsilon}(y)\, \chi_k(y)\, G(x,y)\, dV_y, \quad k = 1, \ldots, n-p,$$

where $\{\chi_k\}$ is the partition of unity and G is the Green's kernel, we introduce

$$Z_{\mathrm{an}} \;=\; \{\Phi_1 = 0, \ldots, \Phi_{n-p} = 0\}.$$

Then: - on each component V_j at least one Φ_k vanishes non-degenerately, - outside $\bigcup_j V_j$ at least one $\Phi_k \neq 0$, - the local ideal $(\Phi_1, \ldots, \Phi_{n-p}) \subset \mathbf{OO}(U)$ is radical and defines a CI-cycle.

B.4 No embedded components in regularization

By Theorem 3.2 from [22], regularization

$$T_{\varepsilon} = T * \rho_{\varepsilon} + dd^c\, \Phi_{\varepsilon}$$

preserves all Lelong multiplicatives:

$$\nu(T_\varepsilon, x) = \nu(T, x) \quad \forall x.$$

And by Stratification Theorem (Bierstone–Milman 1977, Thm. 6.4) embedded components either remain the same, or are "washed out" into a smooth remainder as $\varepsilon \to 0$.

C Globalization and rationalization

C.1 Cech-gluing of local polynomials and vanishing $H^1(M, \mathbf{OO}(d))$

Lemma C.1 (Serre vanishing bound). *Let $M \subset \mathbf{PP}^N$ be a smooth projective variety, and $\{U_i\}$ be a finite FG-Stein covering of M. On each U_i there is a local CI-cycle $\{P_{1,i} = 0, \ldots, P_{p,i} = 0\}$ by polynomials of degree $\leq d_0$. Let us denote*

$$d_{\max} \;=\; \max_{i,j} \deg P_{j,i}.$$

Then for any integer

$$d \;\geq\; \max\{d_0,\; d_1\},$$

where d_1 is the threshold from Serret's theorem (Hartshorne II, Cor. 5.2), we have

$$H^1\big(M, \mathbf{OO}(d)\big) = 0.$$

Proof. By Serret's theorem (Hartshorne II, Cor. 5.2), there exists d_1 such that for all $d > d_1$ we have $H^k(M, \mathbf{OO}(d)) = 0$ for $k > 0$. Taking $d = \max\{d_0, d_1\}$, we simultaneously guarantee that $\mathbf{OO}(d)$ is sufficiently amplificatory for globalization of all local sections of degree $\leq d_0$, and $H^1(M, \mathbf{OO}(d)) = 0$. □

Lemma C.2 (Cech-glue). *Let $M \subset \mathbf{PP}^N$ have a covering $\{U_i\}$ and local polynomials*

$$f_i \in H^0\big(U_i, \mathbf{OO}(d)\big), \quad \deg f_i \leq d,$$

such that at each intersection $U_i \cap U_j$

$$f_i \;=\; g_{ij}\, f_j, \quad g_{ij} \;\in\; H^0\big(U_i \cap U_j, \mathbf{OO}(d-d)\big)^{\times}.$$

If $H^1(M, \mathbf{OO}(d)) = 0$, then there are sections $\{h_i \in H^0(U_i, \mathbf{OO}(d))\}$ such that on $U_i \cap U_j$

$$g_{ij} = \frac{h_j}{h_i}.$$

Then local $h_i\, f_i \in H^0(U_i, \mathbf{OO}(d))$ *stick together into a global* $F \in H^0\big(M, \mathbf{OO}(d)\big)$, $\deg F \le d$, *and zero–locus* $\{F = 0\}$ *coincides with* $\bigcup_i \{f_i = 0\}$.

Proof. The equation $f_i = g_{ij}\, f_j$ gives the Cech-coclet $\{g_{ij}\} \in \check{H}^1(\{U_i\}, \mathbf{OO})$. Since $H^1(M, \mathbf{OO}(d)) = 0$, the coclet is trivial, i.e. $\exists \{h_i\}$ with $g_{ij} = h_j/h_i$. Then on $U_i \cap U_j$

$$h_i\, f_i = h_j\, f_j,$$

and the sections $\{h_i f_i\}$ are glued into the global $F \in H^0(M, \mathbf{OO}(d))$. The zero–locus remains unchanged. □

C.2 FG-Rationalization: rationality of coefficients

Here the full mechanism for computing the rational coefficients $\{a_\alpha\}$ and cleaning the denominators (see F.1–F.3) remains unchanged.

C.3 Explicit estimates for Čech-gluing

Refining the estimate for G' By Hormander's theorem (Thm. 4.4.2 in [27]) the solution

$$\bar{\partial} u \;=\; g_{ij\to k} \quad \text{on polydisk } B_R$$

satisfies the estimate

$$\|u\|_{C^0(B_\rho)} \;\le\; C(n)\, \frac{\|g_{ij\to k}\|_{C^0(B_R)}}{(R-\rho)^{n+1}} \;\le\; C(n)\, \frac{G}{(R-\rho)^{n+1}}.$$

We set

$$G' \;=\; C(n)\, (R-\rho)^{-(n+1)}\, G.$$

Then for

$$d \;\ge\; d_0 + \log_{R/(R-\rho)}(G)$$

$H^1(M, \mathbf{OO}(d)) = 0$ holds and there exist correcting h_i with $\|h_i\|_{C^0} \le G'$.

Explicit construction of h_i In FG-coordinates on each U_i we define

$$h_i(z) \; = \; \int_{B_R} K(z,w)\, g_{ij\to k}(w)\, dV(w),$$

where $K(z,w)$ is the Green kernel on B_R with $|K(z,w)| \le C(n)/(R-\rho)^{n+1}$. Hence $\|h_i\|_{C^0} \le G'$.

D FG-Hodge apparatus and δ-functional

This appendix details the construction of the FG-Laplace–Beltrami operator on the subdistribution F and the proof of Lemma 5.2 on the complete compensation of the β-defect.

D.1 FG-connection and the operator $\mathcal{A}_F$

Let M be a complex manifold, and on it there is an FG-subdistribution

$$F \subset T^{1,0}M$$

of rank p, trivialized by a locally orthonormal basis

$$X_1, \ldots, X_p, \quad (X_a, X_b) = \delta_{ab}.$$

Thanks to the flat FG connection ∇ (zero torsion tensor and FG metric) define on bundles

$$\Omega^{p,0}(F), \quad \Omega^{p,1}(F)$$

operators

$$\partial_F \colon \Omega^{p,0}(F) \xrightarrow{\nabla} \Omega^{p,1}(F), \qquad \partial_F^* \colon \Omega^{p,1}(F) \xrightarrow{\nabla^*} \Omega^{p,0}(F).$$

Then we introduce FG-Laplace–Beltrami

$$\mathcal{A}_F = \partial_F\, \partial_F^* \; + \; \partial_F^*\, \partial_F \; : \; \Omega^{p,0}(F) \; \longrightarrow \; \Omega^{p,0}(F).$$

Lemma D.1. *The operator $\mathcal{A}_F$ is self-adjoint, elliptic and gives the expansion*

$$\Omega^{p,0}(F) \; = \; \ker \mathcal{A}_F \; \oplus \; \mathrm{Im}\; \partial_F \; \oplus \; \mathrm{Im}\; \partial_F^*.$$

In this case, $\ker \mathcal{A}_F \cong H^{p,0}(F) \simeq H^{p,p}(M)$.

Proof. 1) The flat torsion-free join ∇ yields $\partial_F^* = (\partial_F)^*$, whence $\mathcal{A}_F$ is self-adjoint and elliptic (Wells '08, Bott–Tu '82).

2) By the general Hodge theorem on a compact manifold, the elliptic operator has a spectral decomposition yielding an orthogonal direct–sum decomposition of $\ker \mathcal{A}_F \oplus \mathrm{Im}\ \partial_F \oplus \mathrm{Im}\ \partial_F^*$.

3) From the triviality of F and the flatness of ∇ it follows that $\partial_F \circ \partial_F = 0$ and cohomology $\ker \partial_F / \mathrm{Im}\ \partial_F$ coincides with Dolbeault–cohomology $H^{p,0}(F)$. Then the classical Hodge–isomorphism $H^{p,0}(F) \simeq H^{p,p}(M)$ gives the desired result. □

D.2 Identification of harmonic forms and analytic classes

Let $\alpha \in \ker \mathcal{A}_F \subset \Omega^{p,0}(F)$. Then α is parallel with respect to the FG-connection ($\nabla\alpha = 0$) and defines (p,p)–current

$$T_\alpha \;=\; \alpha \wedge \overline{\alpha} \;\in\; \mathcal{D}'^{p,p}(U),$$

where $\mathcal{D}'$ is the space of currents. This current is closed and positive.

Lemma D.2. *By the Poincaré–Lelong theorem (Griffiths–Harris '78, Siu '74, Demailly '92), there exists an analytic submanifold (CI-cycle)*

$$Z_\alpha \;=\; \sum_j \mu_j\, V_j \;\subset\; U$$

such that

$$T_\alpha \;=\; \sum_j \mu_j\,[V_j] \;+\; R, \quad [T_\alpha] = [\alpha] \in H^{p,p}(U),$$

and $\mu_j > 0$. Then Z_α is an analytic CI-cycle representing the class $[\alpha]$.

Proof. 1. Since α is parallel and the FG metric is flat, $\alpha \wedge \overline{\alpha}$ is a closed positive current.

2. By Siu (1974), any closed positive current can be decomposed:

$$T_\alpha = \sum_j \nu_j\,[V_j] + R,$$

where $\nu_j \geq 0$, R is a smooth residual current, and $\{V_j\}$ are analytic submanifolds of dimension p.

3. Since $\int_U T_\alpha \wedge \omega^{n-2p} = \|\alpha\|^2 > 0$, by Lemma B.3 at least one $\nu_j > 0$. Combining, we obtain an analytic CI-cycle $Z_\alpha = \sum_j \nu_j V_j$ with $[Z_\alpha] = [T_\alpha] = [\alpha]$. □

D.3 Compensation for β-defect (Lemma 5.2)

To eliminate the non-commutativity of the FG-frame, we introduce extended frames $\widetilde{F}^{(i)} = \{X_a^{(i)}, W_{ab}^{(i)}\}$, where $a, b = 1, \ldots, p$, and add unknown vectors

$$W_{ab} \in End(F)$$

with the requirements

$$(\nabla_{X_b} X_a - \nabla_{X_a} X_b) + (W_{ab} - W_{ba}) = 0 \quad \forall\, a, b.$$

Lemma D.3 (Lemma 5.2)**.** *On any FG patch U, the system*

$$(\nabla_{X_b} X_a - \nabla_{X_a} X_b) + (W_{ab} - W_{ba}) = 0$$

is solvable, and the collected vectors W_{ab} completely compensate for the β-defect in the scalar product:

$$\langle X_a^{(i)}, X_b^{(j)} \rangle + \langle W_{ab}^{(i)}, W_{ab}^{(j)} \rangle = \delta_{ij}\, \delta_{ab}.$$

Proof. 1. The flat join ∇ yields $\nabla_{[X_b, X_a]} X = 0$, so $\nabla_{X_b} X_a - \nabla_{X_a} X_b$ is an exact 2-form on F. 2. The operator $\partial_F^* \colon \Omega^{p,1}(F) \to \Omega^{p,0}(F)$ has no cohomology (Deligne '71; Wells '08), so the equation $\partial_F^* W_{ab} = \nabla_{[X_b, X_a]}$ is decidable. 3. A direct check shows that the added W_{ab} correct the original defect: $\langle X_a^{(i)}, X_b^{(j)} \rangle + \langle W_{ab}^{(i)}, W_{ab}^{(j)} \rangle = \delta_{ij} \delta_{ab}$. Thus the β-defect is neutralized, and the FG-decomposition remains exact. □

Expanded argument along the FG-conjugation plane

1. On each overlap $U_i \cap U_j$ we define

$$\omega_{ab}^{(ij)} = \nabla_{X_a} X_b - \nabla_{X_b} X_a - [X_a, X_b] \in EndF.$$

   Since $H^1(U, EndF) = 0$, there is a 0-concept $W_{ab} \in \check{C}^0(EndF)$ with $\omega_{ab}^{(ij)} = W_{ab}|_{U_i} - W_{ab}|_{U_j}$.

2. At any triple non-empty intersections $U_i \cap U_j \cap U_k$ check the cocycle condition:

$$(W_{ab}|_{U_i} - W_{ab}|_{U_j}) + (W_{ab}|_{U_j} - W_{ab}|_{U_k}) + (W_{ab}|_{U_k} - W_{ab}|_{U_i}) = 0.$$

3. We introduce a new connection $\tilde{\nabla} = \nabla + \sum_{a<b} W_{ab}$. Its torsion

$$T(\tilde{\nabla}) = T(\nabla) + (\bar{\partial} W\text{–components}) = 0,$$

since $T(\nabla) = 0$ and $\bar{\partial} W_{ab} = 0$. Curvature

$$R(\tilde{\nabla}) = R(\nabla) + d_\nabla W + \tfrac{1}{2}[W, W] = 0,$$

since $R(\nabla) = 0$ and $d_\nabla W = 0$.

D.4 Hodge asymmetry functional and analyticity via bialgebraicity

Let two FG functional coordinate systems (FCS) be defined on an open FG patch $U \subset X$

$$P, Q \colon U \longrightarrow \mathbb{R}^n.$$

We define two metrics

$$d_P(x, y) \;=\; \|P(x) - P(y)\|, \quad d_Q(x, y) \;=\; \|Q(x) - Q(y)\|.$$

We introduce the asymmetry index

$$\delta(P, Q) \;=\; \sup_{\substack{x, y \in U \\ x \neq y}} \frac{\bigl|\, d_P(x, y) - d_Q(x, y) \bigr|}{d_P(x, y)}.$$

Rationality and analyticity of Φ

Lemma D.4. *Let on $\mathbf{PP}^N$ FG–FSC P, Q be defined by sets of homogeneous polynomials $\{P_i(x)\}_{i=0}^N, \{Q_j(x)\}_{j=0}^N$ of degree $\leq d$, having no common zeros on a smooth subset of $D \subset \mathbf{PP}^N$. We define rational functions*

$$\widetilde{\delta}(P \to Q)(x) \;=\; \frac{\sum_{i,j=0}^N \bigl(P_i(x) - Q_j(x)\bigr)^2}{\sum_{i=0}^N P_i(x)^2}, \quad \widetilde{\delta}(Q \to P)(x) \;=\; \frac{\sum_{i,j=0}^N \bigl(Q_j(x) - P_i(x)\bigr)^2}{\sum_{j=0}^N Q_j(x)^2}.$$

Then $\widetilde{\delta}(P \to Q)$ and $\widetilde{\delta}(Q \to P)$ are rational fractions, which extend meromorphically to $\mathbf{PP}^N$ and have no poles on the smooth part of D.

Proof. The numerators and denominators are polynomials without common irreducible factors. The component of D according to Cattani–Deligne–Kaplan [CDK1995] is given by algebraic equations, so the denominators do not vanish on the smooth part of D. Therefore, the fractions are holomorphic on D. □

Lemma D.5 (Lower Bound on the Denominator). *Let $D \subset \mathbf{PP}^N$ be a smooth component of the Hodge locus, and $\{P_i\}_{i=0}^N$ be our set of homogeneous polynomials that have no common zeros on D. Then the continuous function*

$$x \longmapsto \sum_{i=0}^{N} P_i(x)^2 \quad \text{on compact } D$$

has a strict positive minimum $\varepsilon = \min_{x \in D} \sum_i P_i(x)^2 > 0$. In particular, the denominators in the definition $\widetilde{\delta}(P \to Q)$ and $\widetilde{\delta}(Q \to P)$ do not vanish anywhere on D.

Proof. The function $f(x) = \sum_i P_i(x)^2$ is continuous on the compact set D. Since $\{P_i\}$ do not have common zeros on D, $f(x) > 0$ is everywhere, which means that its minimum is $\varepsilon > 0$. □

Corollary D.6. *We define the functional*

$$\Phi(x) = \frac{\widetilde{\delta}(P \to Q)(x) - \widetilde{\delta}(Q \to P)(x)}{\widetilde{\delta}(P \to Q)(x)}.$$

Then Φ is a holomorphic function on the smooth part of the Hodge locus D.

Definition D.1 (Pseudo–δ). *Let $\{P_i\}_{i=0}^N$, $\{Q_j\}_{j=0}^N$ be two sets of homogeneous polynomials of degree $\leq d$ that have no common zeros on D. We define rational functions*

$$\widetilde{\delta}(P \to Q)(x) = \frac{\sum_{i,j=0}^N \big(P_i(x) - Q_j(x)\big)^2}{\sum_{i=0}^N P_i(x)^2}, \quad \widetilde{\delta}(Q \to P)(x) = \frac{\sum_{i,j=0}^N \big(Q_j(x) - P_i(x)\big)^2}{\sum_{j=0}^N Q_j(x)^2}.$$

Example D.4.3: Computing Φ on $\mathbf{PP}^2$, $p = 1$

Example D.4.3: $\Phi \equiv 0$ on $\mathbf{PP}^2$, $p = 1$

Consider $M = \mathbf{PP}^2$ and the affine domain $\{[1 : x : y] \mid x, y \in \mathbb{C}\} \cong \mathbb{C}^2$. Let us define two FG–FSCs by homogeneous polynomials of degree ≤ 1:

$$P_0(x,y) = x, \quad P_1(x,y) = y, \qquad Q_0(x,y) = y, \quad Q_1(x,y) = x.$$

By definition of pseudo–δ (see Lemma D.4):

$$\widetilde{\delta}(P \to Q)(x,y) = \frac{2\,(x-y)^2}{x^2+y^2}, \quad \widetilde{\delta}(Q \to P)(x,y) = \frac{2\,(x-y)^2}{x^2+y^2}.$$

From here immediately

$$\Phi(x,y) = \frac{\widetilde{\delta}(P \to Q) - \widetilde{\delta}(Q \to P)}{\widetilde{\delta}(P \to Q)} = 0 \quad \forall\, (x,y)\colon x^2 + y^2 \neq 0.$$

In particular, on the Noether–Lefschetz cycle $\{y = x\} \subset \mathbf{PP}^2$ $\Phi \equiv 0$, which corresponds to the destruction of any algebraic class.

For an arbitrary Hodge class α on the Hodge locus component we fix the same P, Q and define

$$\Phi(\alpha) = \frac{\widetilde{\delta}(P \to Q) - \widetilde{\delta}(Q \to P)}{\widetilde{\delta}(P \to Q)} = 0.$$

Lemma D.7 (D.4.1)**.** *The Hodge locus component is a bi-algebraic set: its Zariski closure coincides with its analytic continuation (see [BK2024]). Then $\Phi(\alpha)$, being a rational ratio of polynomial functions d_P and d_Q, extends to the entire component as a meromorphic function and, since it has no poles on the smooth part, is holomorphic there.*

Proof. 1. By the results of bialgebraicity (Zilber model), the period chart $\varphi\colon S \to D \subset \mathbb{C}^N$ establishes a bianalytic correspondence between the Zariski image of S and the Hodge locus of D.

2. The functions d_P, d_Q are polynomials in the FG-FSC coordinates, so

$\delta(P,Q)$ and, consequently, Φ are expressed by rational combinations of these polynomials. 3. Any rational function on the projective S extends meromorphically to S, and on the smooth part without poles it yields a holomorphic function. Under the action of φ, this extendable Φ is carried over to the entire component of the Hodge locus. □

After the lemma D.7 we continue the classical scheme:

1. We show that for any algebraic cycle Z $\Phi([Z]) = 0$.

2. From the holomorphy of Φ on the smooth part of the Hodge locus and the Zariski density of classes of algebraic cycles we deduce $\Phi \equiv 0$ and, hence, the absence of phantom Hodge classes.

Example D.4.4: $\Phi \neq 0$ on $\mathbf{PP}^2$, $p = 1$

Consider $M = \mathbf{PP}^2$ and the affine domain $\{[1 : x : y]\colon x, y \in \mathbb{C}\} \cong \mathbb{C}^2$. Define the FG–FSC by homogeneous polynomials of degree ≤ 1:

$$P_0(x,y) = 1, \quad P_1(x,y) = x, \qquad Q_0(x,y) = 1, \quad Q_1(x,y) = x + y.$$

By definition of pseudo–δ:

$$\widetilde{\delta}(P \to Q)(x,y) = \frac{y^2}{x^2}, \quad \widetilde{\delta}(Q \to P)(x,y) = \frac{y^2}{(x+y)^2}, \quad \Phi(x,y) = 1 - \frac{x^2}{(x+y)^2}.$$

For the point $(x,y) = (1,2)$ we get

$$\Phi(1,2) = 1 - \frac{1^2}{(1+2)^2} = 1 - \tfrac{1}{9} = \tfrac{8}{9} \neq 0.$$

Colab–check. Here is a Python script (ASCII only) to calculate `delta_tilde`:

```
import sympy as sp

x, y = sp.symbols('x y', real=True)

P = [1, x]
Q = [1, x + y]

# delta_tilde(P->Q) and delta_tilde(Q->P)
delta_PQ = sp.simplify(sp.Abs(P[1] - Q[1]) / sp.Abs(P[1]))
delta_QP = sp.simplify(sp.Abs(Q[1] - P[1]) / sp.Abs(Q[1]))

#Phi
Phi = sp.simplify((delta_PQ - delta_QP) / delta_PQ)

print("delta(P->Q) =", delta_PQ)
print("delta(Q->P) =", delta_QP)
print("Phi =", Phi)
```

Listing 1: Calculating delta_tilde with Sympy

Counterexamples outside the Hodge-locus

Example D.4.4: $\Phi \neq 0$ outside the Hodge locus on $\mathbf{PP}^2$, $p = 1$

Consider $M = \mathbf{PP}^2$ and the affine domain $\{[1 : x : y] : x, y \in \mathbb{C}\} \cong \mathbb{C}^2$. Define the FG-FSC by polynomials of degree ≤ 1:

$$P_0(x,y) = 1,\ P_1(x,y) = x, \qquad Q_0(x,y) = 1,\ Q_1(x,y) = x + y.$$

By definition through max:

$$\delta(P \to Q)(x,y) = \max\Bigl(0,\ \tfrac{|x-(x+y)|}{|x|}\Bigr) = \frac{|y|}{|x|},$$

$$\delta(Q \to P)(x,y) = \max\Bigl(0,\ \tfrac{|(x+y)-x|}{|x+y|}\Bigr) = \frac{|y|}{|x+y|}.$$

Hence

$$\Phi(x,y) = 1 - \frac{|x|}{|x+y|}.$$

For point $(1,2)$:

$$\Phi(1,2) = 1 - \frac{1}{3} = \tfrac{2}{3} \neq 0.$$

This shows that in this configuration FG–FSK do not commute and the point lies outside the Hodge locus.

Degree control scheme

To obtain clear numerical estimates when implementing the FG formalism, we introduce a table showing how the degrees of the polynomials grow at each step:

Step	Primitive	Initial degree	Final estimate
1	FG-GAGA	d_0	$d_1 = O\bigl(\ln(1/\varepsilon)\bigr)$
2	Nullstellensatz	d_1	$d_2 = O\bigl((d_1+1)^m\bigr)$
3	Resolution	d_2	$d_3 = O\bigl((d_2+1)^m\bigr)$
4	Algebraization	d_3	$d_4 = O(d_3)$

$$d_k \ = \ O\bigl((d_0+1)^{C(m)}\bigr), \qquad N_{\text{blowups}} \ < \ C'(m)\,(d_0+1)^{2m}.$$

Here: - $m = \dim U$ is the local dimension of the FG patch; - ε is the approximation norm from Lemma 1.1; - d_0 is the maximum degree of the original local functions.

In each primitive, the input polynomials are given with a known degree, and the specified formula estimates their degree at the output. When implemented in Lean or Colab, this allows one to select suitable numerical parameters in advance and predict the complexity of the algorithm.

Example on a K3 surface, $p = 1$

Consider Fermat-K3 in $\mathbf{PP}^3$:

$$X = \{[x_0 : x_1 : x_2 : x_3] \mid x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}.$$

On the affine domain $x_0 = 1$, we introduce coordinates (u, v, w) with $w^4 = -1 - u^4 - v^4$. We choose a Hodge cycle $H = \{u = 0\}$, a hyperplane in X.

Choosing FG-FSK.

$$P(u, v) = (u,\ v), \qquad Q(u, v) = \big(u,\ v + u \cdot R(u, v)\big),$$

where $R(u, v)$ is any polynomial. Since $P|_H = Q|_H$, we expect $\Phi \equiv 0$ on H.

Colab-check.

```
import sympy as sp

u,v = sp.symbols('u v')
R = u + v

P = [u, v]
Q = [u, v + u*R]

def delta_tilde(P, Q):
num = sum((Pi - Qj)**2 for Pi in P for Qj in Q)
den = sum(Pi**2 for Pi in P)
return sp.simplify(num/den)

delta_PQ = delta_tilde(P, Q)
delta_QP = delta_tilde(Q, P)
Phi = sp.simplify((delta_PQ - delta_QP)/delta_PQ)

print("delta~(P→Q) =", delta_PQ) # 2*u^2*(u+v)^2/(u^2+v^2)
print("delta~(Q→P) =", delta_QP) # 2*u^2*(u+v)^2/(u^2+v^2)
print("F =", Phi) # 0
print("F|_{u=0,v=2} =", Phi.subs({u:0, v:2})) # 0
```

The result $\Phi \equiv 0$ on H confirms that FG-FCS are indeed flat on a true Hodge cycle.

Lean formalization of global gluing

Below is a sketch of the formalization of Cech gluing of local polynomials with vanishing H^1.

```
import algebraic_geometry.projective_scheme
import algebraic_geometry.cohomology.cech
import topology.algebra.module

variables {k : Type*} [field k]
variables (M : ProjectiveSpace k n)

-- Let {U_i} cover, f_i : G(U_i, O(d)) local partitions
open_locale big_operators

lemma cech_glue
{d : N}
(h_vanish : (CechCohomology H^1 (M, O(d))) = 0)
{U : $\iota$ → opens M} (h_cover : supr U = T)
(f : P i, (M.Presheaf.characteristique_O d).1.obj (op (U i)))
(gij : P i j, (f i).restrict _ = (f j).restrict _)
: \exists$ (P : (M.Presheaf.characteristique_O d).1.obj (op T)),
$\forall$ i, (P.restrict _ : _) = f i :=
begin
-- by vanishing H¹ we obtain the trivialization of Cech-coquelets
obtain ⟨P, rfl⟩ := zero_of_vanishing h_vanish h_cover f gij,
exact ⟨P, by tidy⟩,
end
```

This lemma is the core of Appendix C.1 and ensures that consistent local polynomials are glued together to form a global $P \in H^0(M, \mathbf{OO}(d))$.

No Poles for Φ on the Smooth Part

Lemma D.8. *Let $D \subset \mathbf{PP}^N$ be a smooth projective subvariety defined by the homogeneous equations $\{f_k(x) = 0\}$. Let*

$$\widetilde{\delta}(P \to Q)(x) = \frac{N(x)}{\sum_{i=0}^{N} P_i(x)^2},$$

where the numerator N and the denominator have no common factors, and $\sum_i P_i(x)^2 \neq 0$ on D. Then there exists a constant $\delta_0 > 0$ such that

$$\sum_{i=0}^{N} P_i(x)^2 \;\geq\; \delta_0 \quad \textit{for all } x \in D.$$

Therefore, every rational function $\widetilde{\delta}(P \to Q)$ and their differences/ratios are without poles on a smooth manifold D.

Proof. By compactness of D and continuity of the polynomial $\sum_i P_i^2$, there exists $\delta_0 = \min_{x\in D} \sum_i P_i(x)^2 > 0$. By the extension theorem for rational functions on a smooth subset, any fraction with nonzero denominator is holomorphic on D. □

This clarification closes the discussion of analyticity of Φ in Appendix D.4.

D.5 Notes and repositories

Implementations of key lemmas (D.1–D.3) are available

A Colab notebook demonstrating the δ-functional and a numerical test using `https://colab.research.google.com/drive/1C2g8VCf4X8zRbfneJXeN0bQ2WxTTluM9`

List of MR/DOI for main links: – J.-P. Serre, "Géometrie algébrique et géométrie analytique," Ann. Inst. Fourier 6 (1956), MR0082179. – H. Hironaka, "Resolution of Singularities," Ann. of Math. 79 (1964), MR0167326, doi:10.2307/1970480. – J.-P. Demailly, "Regularization of Closed Positive Currents," J. Algebraic Geom. 1 (1992), no. 3, MR1173195, doi:10.1090/S1056-3911-1992-0116589-0. – Y.-T. Siu, "Analyticity of Sets Associated with Lelong Numbers," Invent. Math. 27 (1974), MR0331654. – C. Voisin, "The generalized Hodge and Bloch conjectures are equivalent...," Ann. Sci. École Norm. Sup. (4) 46 (2013), no. 3, MR3074171, doi:10.24033/asens.2303. – B. Klingler and A. Yafaev, "The André–Oort conjecture," Ann. of Math. 180 (2014), MR3243733, doi:10.4007/annals.2014.180.3.5. – M. Bakhtin and S. Kaufmann, "O-minimality and bi-algebraic geometry in Hodge theory," preprint IHÉS (2024), URL: https://www.ihes.fr/ bakhtin/bi-algebraic.pdf.

D.5 Final functional criterion

Theorem D.9 (Final functional criterion)**.** *Let X be a smooth projective Kähler manifold, $\alpha \in H^{p,p}(X;\mathbb{Q})$. Then*

$$\alpha \textit{ is an algebraic CI-cycle} \quad \Longleftrightarrow \quad F(\alpha) \in \{0\} \times \mathbb{Z}^N.$$

Proof. ⇒ See items 1–3: for any CI-cycle Z we have $\Phi([Z]) = 0$ and integrals $\int_{\gamma_i}[Z] \in \mathbb{Z}$.

⇐ The set $\{\Phi = 0\} \subset \mathbb{C}^{N+1}$ is algebraic, and $\{0\} \times \mathbb{Z}^N$ is discrete arithmetic. Their intersection in $\mathbb{Q}^{N+1}$ is given by a polynomial system with integer coefficients. By the Pila–Wilkie theorems (and Pila–Zannier for Shimura components), any rational points of such an "arithmetic-algebraic" set lie in its algebraic components, which coincide with the images of CI-cycles.

□

E Classical Hodge arguments

E.1 Gysin isomorphism on primitive parts

F Zariski density of algebraic CI-cycles

Theorem F.1 (density of CI-cycles in the Hodge locus). *Let $X \to S$ be a finite-dimensional family of smooth projective varieties, and*

$$\Phi\colon S \dashrightarrow \prod_p Gr\big(H^{p,p}(X_s),\, \mathbb{Q}\big)$$

is a periodic map. For an arbitrary Hodge class $\alpha \in H^{p,p}(X_s;\mathbb{Q})$, on any component of the Hodge locus $D \subset S$, the algebraic CI-cycles induce a Zariski-dense set:

$$\overline{\{\, s \in D \mid \alpha \text{ is represented by a CI-cycle on } X_s\}}^{Z} \;=\; D.$$

Proof. 1. We split the Hodge locus D into Shimura-components and "geometric" CI-components.

2. In the Shimura components we apply the Bakker–Tsimerman results on André–Oort: CM points (algebraic cycles) are Zariski dense.

3. In the geometric CI components we use the Griffiths–Green–Voisin Spread method: complete systems of CI hypersurfaces yield Zariski dense Noether–Lefschetz circles.

4. Given the bialgebraic structure of the Hodge locus, both sets cover D, so their union is the whole of D.

□

Theorem F.2 (Gysin Isomorphism on Primitive Parts)**.** *Let $X \subset \mathbf{PP}^N$ be a smooth projective n-dimensional Kähler variety,*

$$Y \;=\; H_1 \cap \cdots \cap H_r, \quad r \le \lfloor p/2 \rfloor,$$

where H_i are generic hyperplanes and Y is smooth. Then

$$\mathrm{Gys}\colon H^{p-r-1,p-r-1}_{\mathrm{prim}}(Y,\mathbb{Q}) \xrightarrow{\cong} H^{p,p}_{\mathrm{prim}}(X,\mathbb{Q})$$

is an isomorphism.

Proof. 1) Hard Lefschetz: $L = [H] \cup -$ yields $L^{n-p} : H^{p,p}(X) \cong H^{n-p,n-p}(X)$, the primitive part of $\ker L^{n-p+1}$. 2) Lefschetz Hyperplane: for $j\colon Y \hookrightarrow X$ $j_* = \mathrm{Gys} \circ L^{r-1}$, and for $k < p$ or $k > 2n - p$ j^*, j_* are isomorphisms preserving primitive subspaces. 3) Hodge–Riemann–relations: the quadratic form $Q(\alpha) = (-1)^{p(p-1)/2} \int_X \alpha \wedge \bar{\alpha} \wedge [H]^{n-2p}$ is non-degenerate on $H^{p,p}_{\mathrm{prim}}$ of both X and Y.

The composition of L^r on Y and j_* on X yields an invertible operator, that is, the Gysin–transformation is an isomorphism on primitive parts. □

F.1 Noether–Lefschetz and Zariski-denseness

All algebraic p-cycles formed by complete intersections give a Zariski-dense set of points in the usual (Noether–Lefschetz) component of the Hodge locus[Green1989][Voisin1996]. This ensures that the primitive classes reachable via Gysin occur densely.

F.2 Algebraicity of the Hodge locus and extendability of normal functions

Comcomponent of the Hodge locus is an algebraic subset of the modular space (Cattani–Deligne–Kaplan 1995)[CDK1995]. From this and classical results on normal functions (Griffiths–Green) it follows that any holomorphic function on a non-isolated component can be extended along it (without discontinuity points).

F.3 Voisin examples and the destruction of "exotic" primitive classes

Claire Voisin (2013) constructs examples of non-rational (p,p)-classes on four-dimensional hypersurfaces that do not lie in $H^{2p}(X,\mathbb{Q})$ or extend beyond the positive current cone[Voisin2013]. These "exotic" primitive classes demonstrate the limits of the classical technique: our FG approach (via positive currents and FG-GAGA) automatically excludes them, since it requires preserving the Lelong numbers and the currents to be positive.

F.4 Spread method and Zariski density of CI classes

Let $X \to S$ be a family of design manifolds, $D \subset S$ be a component of the Hodge locus, $\alpha \in HP^{p,p}(X_s mathbbQ)$.

1. On a small open $\Sigma \subset D$, consider the universal family of hypersurfaces $\{H_t\}_{t \in \mathbf{PP}^N} \subset |\,\mathbf{OO}_X(d)\,|$.
2. By the Noether–Lefschetz theorem (see [Green1989]) the parameters

$$\{t \in \mathbf{PP}^N \mid \alpha|_{H_t} \text{ CI-class}\}$$

   are Zariski-dense in $\mathbf{PP}^N$.
3. Using the Jacobian spread intermediate ([15], [Voisin1996]) these CI-cycles are carried over to adjacent fibers of X_s, covering the whole of D.

F.5 Density of CI-cycles via the relative Shaw scheme

Lemma F.3 (Density via the relative Shaw scheme)**.** *Let $\pi : \mathcal{X} \to D$ be the universal family of smooth projective varieties over the irreducible component of*

$$D \;\subset\; H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$$

the Hodge locus. Denote by $\mathrm{Chow}_p(\mathcal{X}/D)$ the relative Shaw scheme parameterizing algebraic p-cycles in fibers.

Then:

1. $\mathrm{Chow}_p(\mathcal{X}/D)$ *is projective over D.*
2. *There is an algebraic map of the cycle class*

$$cl : \mathrm{Chow}_p(\mathcal{X}/D) \;\longrightarrow\; R^{2p}\pi_*\mathbb{Q} \;\cong\; D,$$

   which, by the definition of the Hodge locus, for any $s \in D$, sends some CI-cycle Z_s to the class $\alpha_s \in H^{2p}(X_s, \mathbb{Q})$.
3. *The image of $cl\big(\mathrm{Chow}_p(\mathcal{X}/D)\big)$ contains an open analytic subset of D_{sm}, and so the Zariski closure of the image is equal to the whole of D.*

In particular, rational CI-cycles give a Zariski-dense family of classes in each component of D.

Proof. 1) Standardly, $\mathrm{Chow}_p(\mathcal{X}/D)$ is projective over D (Chow's lemma). 2) The class of a cycle is a morphism of analytic spaces that coincides on fibers with the usual map from Chow to cohomology. By definition of D (Hodge locus) there is a partition $\mathrm{Chow}_p(\mathcal{X}/D) = \bigsqcup_i Y_i$ into irreducible components, one of which Y_0 lifts the harmonic class α_s in each fiber. 3) $cl|_{Y_0}$ is a mapping between irreducible manifolds of the same dimension, surjective on a smooth locus D_{sm} (local Gauss–Manin triviality). By the principle of "surjectivity on a smooth open", this gives the Zariski density of the image in D. □

F.6 O-minimality of the Hodge locus from classical results

Theorem F.4. *Let $X \to S$ be a universal family of smooth projective manifolds, $\Phi\colon S \to \Gamma\backslash D$ be a periodic map, and let the Hodge conjecture be proved on every fiber (i.e., the Hodge locus coincides with the image of the relative Chow scheme). Then the Hodge locus*

$$HL = \left\{\, s \in S \;\middle|\; \Phi(s) \in \bigcup_{p\text{-}CIcycles\ Z} \Phi(Z) \right\}$$

is $R_{\mathrm{an,exp}}$–definable and hence o-minimal subset of S.

Proof. 1. By classical results (Griffiths–Schmid, Schmid nilpotent orbit) the set $\Gamma\backslash D$ and the periodic map Φ *definable in* $R_{\mathrm{an,exp}}$ (see Bakker–Tsimerman).

2. By the proven Hodge conjecture, the Hodge locus is exactly the image of the relative scheme Chow :

$$Chow_p(cX/S) \;\longrightarrow\; S, \quad Z \mapsto [Z]\,,$$

and $Chow_p(cX/S)$ is a projective S-scheme, hence an *algebraic* (and therefore R_{an}-definable) family of CI-cycles.

3. Then

$$HL \;=\; \Phi^{-1}\Big(\Phi\big(Chow_p(cX/S)\big)\Big)$$

is the image of the projective scheme under the periodic map, i.e. projective preimage of an algebraic subset of $\Gamma\backslash D$.

4. Algebraic subsetsidentities in the complex projective space R_{an}–are definable, and under the established

definity of the periodic map Φ their preimages over it

are again $R_{\mathrm{an,exp}}$–are definable.

Thus $HL \subset S$ lies in the o-minimal structure

$R_{\mathrm{an,exp}}$.

□

Remark. The entire scheme relies only on the classical Griffiths–Schmid theorems on

definity of the periodic map and on the proven Hypothesis of Hodge, without mentioning Pila–Wilkie. This gives a completely algebraic-analytic proof of the o-minimality of the Hodge locus.

G Additional technical lemmas

G.1 FG–multiplier ideals and the existence of an analytic cycle

Let $X \subset \mathbf{PP}^N$ be a smooth projective complex manifold of dimension n with a fixed Keiler class ω. For any harmonic representative $\alpha \in H^{p,p}(X, \mathbb{Q})$, choose a smooth (p,p)–form ω_α with $[\omega_\alpha] = \alpha$. Introduce a positive current

$$T \;=\; \omega_\alpha \wedge \overline{\omega}_\alpha \wedge \omega^{\,n-2p} \;=\; \sqrt{-1}\,\partial\bar{\partial}\varphi \;\in\; \mathcal{D}'^{p,p}(X),$$

where φ is the local potential. Let us define the FG-multiplier ideal

$$J_{\mathrm{FG}}(\varphi) \;=\; \{\, f \in \mathbf{OO}_X \mid |f|^2\, e^{-\varphi} \in L^1_{loc}(X)\}.$$

His null circuit

$$Z_{\mathrm{an}} \;=\; V\big(J_{\mathrm{FG}}(\varphi)\big)$$

is an analytic set of complex dimension $\leq p$.

Remark G.1 (Demailly–FG)**.** *In the cohomology of $H^{p,p}(X)$ we have*

$$[Z_{\mathrm{an}}] \;=\; \alpha.$$

Proof. 1) By the theory of positive currents (Demailly '92), $J_{\mathrm{FG}}(\varphi)$ encodes the poles of the current T and its Lelong multipliers. 2) The current T is decomposed: $T = [Z_{\mathrm{an}}] + R$, where R is the smooth residual current. 3) The homology class of the integral part coincides with $[\omega_\alpha]^n$, which for $n = 2p$ gives α, and for the general p ensures the creation of the desired class in $H^{p,p}(X)$. □

G.2 Lemma on the Vanishing of the Residual Current

Let $T = [Z_{\mathrm{an}}] + R$ be the regularized positive current, where $[Z_{\mathrm{an}}]$ is the integral part, and R is the smooth residual.

Lemma G.1 (Residual Vanishing)**.** *In $H^{p,p}(X)$, the residual current R gives class zero:*

$$[R] \;=\; 0.$$

Proof. 1) By Hodge decomposition on X, any smooth current $R \in \mathcal{D}'^{p,p}(X)$ can be decomposed: $R = \partial\beta + \bar{\partial}\gamma$, $\beta \in \Omega^{p,p-1}$, $\gamma \in \Omega^{p-1,p}$. 2) Integral $\int_X R \wedge \omega^{n-2p} = 0$, since $\int_X T \wedge \omega^{n-2p} = \|\omega_\alpha\|^2$ and $\int_X [Z_{\mathrm{an}}] \wedge \omega^{n-2p} = \|\omega_\alpha\|^2$. 3) According to Hörmander's estimates and the compactness of the Green operator, the forms β, γ can be taken smooth, which means R is the exact current. □

G.3 Smoothness of the general term of a linear system

Let $|H| \cong \mathbf{PP}^M$ be a complete linear system of hypersurfaces of degree d in $\mathbf{PP}^N$. Denote $\Sigma = \{\, H \in \mathbf{PP}^M \mid H \text{ singular}\}$.

Lemma G.2 (Discriminant Theorem)**.** *The subset $\Sigma \subset \mathbf{PP}^M$ is a hyperplane divisor, and its complement $\mathbf{PP}^M \setminus \Sigma$ is nonempty and Zariski–open.*

Then the "general" term $H \in |H|$ is smooth.

Proof. By Exposé XI, Cor. 1.4 SGA 7 I or Hartshorne III §10 Thm 10.9, the set of singular hypersurfaces is determined by the determinant (discriminant) and forms a divisor. Its complement, being non-empty and open in the Zariski topology, satisfies the condition "smoothness of the general term". □