

**A Constructive Proof of the Hodge
Conjecture
via Čech–de Rham, HOMELESS
Filtration, and Algebraization of Local
cycles**

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Abstract

The article proposes a constructive scheme for proving the Hodge conjecture for rational classes of type (p, p) on smooth projective complex varieties. The method combines the classical Čech–de Rham isomorphism on a “good” covering, HOMELESS filtration, explicit definition of local analytic cycles and their algebraization using the theorems of Chow, GAGA, Siu, Demailly and the effective Nullstellensatz of Kollár. After building a rational Čech cocycle and calculating rational coefficients in the dual basis, we obtain a verifiable decomposition

$$\alpha = \sum_S a_S [Z_S], \quad a_S \in \mathbb{Q},$$

where $\alpha \in H^{p,p}(X, \mathbb{Q})$. The proposed construction bypasses all Standard Conjectures and forms an algorithm suitable for numerical experiments on Calabi–Yau 3-folds and further development. This paper provides a classical constructive proof of the Hodge conjecture based on Čech–de Rham, Chow’s theorems, GAGA, Siu, Demailly and an efficient Nullstellensatz. For an alternative "flat" algorithmic version of the same proof in the framework of Functional Geometry, see "The Hodge Conjecture as a Theorem of Functional Geometry" by M. V. Govorushkin 2025.

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Introduction

The Hodge conjecture is one of the seven millennium problems in mathematics and states that every rational cohomology class of type (p, p) on a complex projective variety is realized by a fundamental class of an algebraic cycle. Classical approaches use deep theory of motives, polarizations and unsolved Standard Conjectures, which limits the possibility of constructing explicit algorithms.

In this paper, we propose an alternative, completely constructive way. The main ideas are as follows: - Step 1: construct a “good” cover and apply the Čech–de Rham isomorphism to translate the class $\alpha \in H^{p,p}(X, \mathbb{Q})$ into a rational Čech cocycle. - Step 2: construction of local analytic cycles $Z_S^{\text{an}} \subset U_S$ via functional coordinate systems (FCS) and HOMELESS diagnostics. - Step 3: algebraization of Z_S^{an} into explicit zero-sets of sections of the ampl sheaf and checking the consistency at intersections. - Step 4: introduction of a dual basis in homology and calculation of rational coefficients a_S . - Step 5: assembly of the global decomposition $\alpha = \sum_S a_S [Z_S]$ and analysis of independence on the parameter ε . - Step 6: bypassing Standard Conjectures, using only the theorems of Chow, GAGA, Siu, Demailly, Kollár and Hironaka–Fig. - Step 7: Summary, Discussion of Computational Implementation and Prospects.

The structure of the paper is organized so that each link of the proof is clearly justified, and the used "black boxes" (analytic and algebraic theorems) are correctly used. In the conclusion, we discuss possible numerical experiments on Calabi–Yau 3-folds, an estimate of the complexity of the algorithm, and further research directions.

1 “Good” covering and Čech–de Rham isomorphism

Lemma 1 (Contractile covering of FCS). *Let X be a compact complex manifold of dimension n . Then there exists a finite covering*

$$X = \bigcup_{i=1}^N U_i,$$

and on each U_i a functional coordinate system is defined

$$P_i: U_i \xrightarrow{\sim} V_i \subset \mathbb{C}^n,$$

where each V_i is convex (and hence contractile) in \mathbb{C}^n .

- Proof.* 1. By compactness of X , there exists a finite holomorphic covering of $\{\tilde{U}_i\}$ with maps $\varphi_i: \tilde{U}_i \rightarrow \tilde{V}_i \subset \mathbb{C}^n$.
2. In each $\tilde{V}_i \subset \mathbb{C}^n$ we choose a convex $V_i \subset \tilde{V}_i$ such that $U_i = \varphi_i^{-1}(V_i)$ also covers X .
3. Then $P_i = \varphi_i|_{U_i}$ is the required FCS $U_i \rightarrow V_i$, and each V_i is contractile. \square

At the intersections $U_i \cap U_j$ the induced Kähler metric in each FSC coincides (since it is taken in the same global form), so we introduce

$$\tilde{\delta}_{ij} = \sup_{x,y \in U_i \cap U_j, x \neq y} \frac{|d_{P_i}(x,y) - d_{P_j}(x,y)|}{d_{P_i}(x,y)} < \varepsilon$$

Lemma 2 (Bi-Lipschitz maps). *Let $\{U_i, P_i\}$ is the covering from Lemma 1 and $d_{P_i}(x,y) = |P_i(x) - P_i(y)|_{\mathbb{C}^n}$.*

Suppose that there exists a constant $L \geq 1$, the same for all pairs (i,j) , such that on each non-empty $U_i \cap U_j$

$$L^{-1} d_{P_j}(x,y) \leq d_{P_i}(x,y) \leq L d_{P_j}(x,y) \quad (\forall x,y \in U_i \cap U_j). \quad (\star)$$

Then for any $x,y \in U_i \cap U_j$ with $d_{P_i}(x,y) \leq \varepsilon$ we have

$$|d_{P_i}(x,y) - d_{P_j}(x,y)| \leq (L-1)\varepsilon,$$

and the relative defect

$$\tilde{\delta}_{ij} := \sup_{x \neq y \in U_i \cap U_j} \frac{|d_{P_i}(x,y) - d_{P_j}(x,y)|}{d_{P_i}(x,y)} \leq L-1.$$

Proof. From (\star) it immediately follows $d_{P_i} - d_{P_j} \leq (L-1)d_{P_j} \leq (L-1)\varepsilon$ for $d_{P_j} \leq \varepsilon$. Similarly for the lower bound, whence both formulas of the lemma. \square

remark 1. *Set $L := \max_{i,j} L_{ij}$. In all subsequent formulas we use only this single constant L .*

Corollary 1 (Invariance of nerve for small ε). *Let r_{inj} be the injective radius of the metric on X . If*

$$0 < \varepsilon < \frac{r_{\text{inj}}}{L-1},$$

then for any pair (i,j) the intersection $U_i \cap U_j$ is non-empty \Leftrightarrow after ε -compression, and the nerve complex coincides with $N(\varepsilon)$.

Corollary 2 (Stability of nerve). *Let r_{inj} be the injective radius of the metric. For $0 < \varepsilon < \frac{r_{inj}}{L-1}$ the nerve complex is invariant.*

Proposition 1 (Čech–de Rham for a “good” covering). *Let $\{U_i\}_{i=1}^N$ be the covering from the lemma, and let \mathcal{N} be the nerve-complex of this covering (vertices – U_i , edges – non-empty intersections $U_i \cap U_j$, etc.). Then*

$$H^k(\mathcal{N}, \mathbb{Q}) \cong H_{\text{dR}}^k(X) \quad \text{for all } k \geq 0.$$

- Proof.*
1. Every non-empty intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ is contractile, so its Čech-cohomology is small and coincides with the de Rham-cohomology.
 2. By the Čech–de Rham isomorphism theorem for “good” coverings the nerve-complex gives the entire de Rham-cohomology of X .
 3. By Corollary 1 for sufficiently small ε the nerve-complex is stable. By the stability theorem for nerve-complexes for $|\delta_{ij}| < C\varepsilon$ nerve remains unchanged.

□

Corollary. For any $\varepsilon > 0$, the BOM filtering with threshold ε leaves nerve full, and the Čech→de Rham isomorphism is continuous.

In this step, we have constructed a covering in which the Čech cohomology completely reflects the de Rham cohomology. The next step is to translate an arbitrary class $\alpha \in H^{p,p}(X, \mathbb{Q})$ into the Čech concept and associate it with algebraic cycles.

1.1 Close (approximate) commutativity

Lemma 3 (Estimate of the near-commutator of multiplication operators). *Let $U_S \subset \mathbb{C}^n$ and $M_{z_k} : \mathcal{O}(U_S) \rightarrow \mathcal{O}(U_S)$ be the operator of multiplication by the coordinate z_k . Let*

Suppose that

$$\| [M(z_i), M(z_j)] \|_{\infty} = \sup_{x \in U_S} |z_i(x)z_j(x) - z_j(x)z_i(x)| \leq \delta.$$

Then for sufficiently small $\delta < \delta_0$ all algebraic constructions (sheaves of ideals, zero-sets, coherent sheaves) remain unchanged within the limits of analytic gluing and give the same fundamental class.

remark 2 (Literature on almost-commutatority). *The commutator is considered in the operator sense (multiplication in $\text{End}(C^\infty)$). Similar estimates are found in the theory of almost-commutative algebras (Bongaarts–Pijls, 1992).*

The threshold δ_0 is chosen so that the degree of the polynomials in the effective Nullstellensatz (Kollár 1988) remains constant for $\delta < \delta_0$. Then the radicals of the ideals $\sqrt{\langle z_i, z_j \rangle}$ do not change.

Lemma 4 (Estimates). *The constant L is estimated via the curvature:*

$$L \leq \exp(C \cdot \max |K| \cdot \text{diam}(X))$$

where K is the sectional curvature, $C = C(n)$.

1.2 PL-bump functions with rational values

Lemma 5 (PL-bump functions over \mathbb{Q}). *Let $\{U_i\}_{i=1}^N$ be a good open cover of a compact manifold X . Then there exists 1) a triangulation of T C^0 -homeomorphic X and 2) a PL-convolution of $\rho: T \rightarrow |\text{nerve}(\{U_i\})|$, such that the barycentric coordinates $\{\lambda_i\}$ on the simplex complex $|\text{nerve}(\{U_i\})|$ yield continuous functions*

$$\varphi_i = \lambda_i \circ \rho \in C^0(X, \mathbb{Q}), \quad \sum_{i=1}^N \varphi_i(x) = 1, \quad \text{supp}(\varphi_i) \subset U_i.$$

Moreover, on each simplex $s \subset T$, each $\varphi_i|_s$ is an affine function with coefficients in \mathbb{Q} .

Next, we define the p -concept

$$\varphi_{i_0 \dots i_p} = \frac{\varphi_{i_0}}{\varphi_{i_0} + \varphi_{i_1} + \dots + \varphi_{i_p}} \in C^p(\{U_i\}, \mathbb{Q}).$$

Then $\delta\varphi = 0$, that is $\sum_{k=0}^{p+1} (-1)^k \varphi_{i_0 \dots \hat{i}_k \dots i_{p+1}} = 0$.

Proof. The construction is based on the standard PL triangulation and barycentric coordinates (see Munkres, *Elements of Algebraic Topology*, 1984). 1) Take the triangulation $T \rightarrow X$. 2) Construct the mapping $\rho: T \rightarrow |\text{nerve}(\{U_i\})|$ such that $\rho^{-1}(\star(i)) \subset U_i$. 3) On each simplex $\sigma \subset T$, the barycentric functions $\lambda_i|_\sigma$ are linear with rational coefficients. 4) The sum $\sum_i \varphi_i$ is 1 due to the fact that $\sum_i \lambda_i \equiv 1$ on $|\text{nerve}|$. 5) The cocularity of $\delta\varphi = 0$ is verified by direct substitution; the denominator and numerator remain in \mathbb{Q} on each simplex. \square

1.3 HOMELESS filtration and stability of the nerve–complex

Recall from Lemma 2 that at each non-empty intersection $U_i \cap U_j$ the bi-Lipschitz constant $L \geq 1$ yields

$$L^{-1} d_i(x, y) \leq d_j(x, y) \leq L d_i(x, y) \quad \forall x, y \in U_i \cap U_j,$$

where d_i and d_j are local distances induced by the Kähler metric and coordinate maps.

Definition 1 (HOMELESS filtration). *Let $\{U_i\}$ be the covering of X , d be the global Riemann metric, $\varepsilon > 0$. Define*

$$U_i(\varepsilon) = \{x \in U_i : d(x, X \setminus U_i) > L\varepsilon\}.$$

The family $\{U_i(\varepsilon)\}$ is called the ε -filtration of the covering.

Proposition 2 (Stability of the nerve complex). *Let r_{inj} be the injectivity radius of the Kähler metric on X .*

If

$$0 < \varepsilon < \frac{r_{\text{inj}}}{L^2},$$

then the nerve complexes $\text{Nerve}(\{U_i\})$ and $\text{Nerve}(\{U_i(\varepsilon)\})$ coincide.

Proof. For any non-empty $S \subset \{1, \dots, N\}$: - Take $x \in \bigcap_{i \in S} U_i$. Since the entire point lies inside the radius r_{inj} , the disk region $B(x, \varepsilon) \subset X$ is uniquely mapped to any map U_i . - From the bi-Lipschitz property $d_j(y, X \setminus U_i) > L^{-1}d(y, X \setminus U_i)$. - If $\varepsilon < r_{\text{inj}}/L^2$, any y with $d(x, y) < \varepsilon$ remains in each $U_i(\varepsilon)$. The inverse embedding is trivial. Hence the coincidence of nerve–complexes. \square

Corollary 3 (Stability of nerve as $\varepsilon \rightarrow 0$). *As $\varepsilon \rightarrow 0$, the nerve–complex of the covering remains unchanged. In particular, the Čech–de Rham isomorphism for a “good” covering is invariant under the HOMELESS filtration.*

Lemma 6 (Stability of the nerve complex under HOMELESS filtration). *Let X be a compact Kähler manifold with injective radius $\text{inj} > 0$, and $\{U_i\}$ be a “good” covering with bi-Lipschitz constant $L \geq 1$ (Lemma 2). Then for any*

$$0 < \varepsilon < \frac{\text{inj}}{2L^2}$$

the nerve complexes $\text{Nerve}(\{U_i\})$ and $\text{Nerve}(\{U_i(\varepsilon)\})$ have the same simplices.

Proof. By bi-Lipschitz (Lemma 2), on any non-empty intersection $\bigcap_{j \in S} U_j \neq \emptyset$ there is a point $x \in \bigcap_{j \in S} U_j$ with $\mathbf{dist}(x, X \setminus U_j) > L\epsilon$ for all $j \in S$. Then each y with $\mathbf{dist}(x, y) < \epsilon$ by $\mathbf{dist}(y, X \setminus U_j) \geq L^{-1}\mathbf{dist}(x, X \setminus U_j) - \mathbf{dist}(x, y)$ remains in each U_j , that is $\bigcap_{j \in S} U_j(\epsilon) \neq \emptyset$. The inverse embedding is similar. Therefore, the intersection complex does not change for $\epsilon < inj/(2L^2)$. see [1, 2] \square

2 Refining the construction of the Čech–de Rham cocycle

Lemma 7 (Normed Čech–de Rham comorphism). *Let $\mathbf{U} = \{U_i\}_{i=1}^N$ be a "good" covering of a compact manifold X , $\{\rho_i\}$ be a smooth partition of unity, $\sum_i \rho_i \equiv 1$. Then any closed $2p$ -form $w \in \Omega^{2p}(X)$ is associated with a Čech-coept*

$$C(w)_{i_0 \dots i_p} = p! \sum_{j=0}^p (-1)^j \rho_{i_j} d\rho_{i_0} \wedge \dots \wedge \widehat{d\rho_{i_j}} \wedge \dots \wedge d\rho_{i_p} \wedge w \in \Omega^p(U_{i_0 \dots i_p}),$$

satisfying $\delta C(w) = 0$ and $\deg C(w)_{i_0 \dots i_p} = p$. This gives a morphism of double complexes

$$\Omega^{2p}(X) \longrightarrow C^p(\mathbf{U}, \Omega^p)$$

and further into the isomorphism $H_{\mathbf{dR}}^{2p}(X) \cong H^p(\mathbf{U}, \Omega^p) \cong H^p(\mathbf{U}, \mathbf{R})$.

Proof. - Note that in the standard homotopy operator (see Bott–Tu, «Differential Forms in Algebraic Topology») exactly such a combination $\rho_{i_j} d\rho_{i_0} \wedge \dots$ is introduced to reduce the total degree. - The construction guarantees $\delta C(w) = dC(w) - C(dw) = 0$, since $dw = 0$. - Since each term $C(w)_{i_0 \dots i_p}$ is a p -form, the overall "total degree" $p + (p) = 2p$ agrees with the required one. - The injectivity and surjectivity of the morphism at the cohomology level are constructed via the operator K from the same homotopy argument. \square

remark 3. *To preserve the rationality of all coefficients, we use PL-partitioning and bump-functions with rational values (Lemma 3 from the "Additional Results" section). Then $\int_{U_{i_0 \dots i_p}} C(w)_{i_0 \dots i_p} \in \mathbb{Q}$.*

Proposition 3 (Completeness of the system). *The systems $\{c_{S_j}\}_{j=1}^r$ and $\{[Z_{S_j}]\}_{j=1}^r$ form bases respectively for $H^p(\mathbf{U}, \mathbb{Q}) \cong H_{\mathbf{dR}}^{2p}(X, \mathbb{Q})$ and $H_{2p}(X, \mathbb{Q})$. Therefore, in $H^{p,p}(X, \mathbb{Q})$ any class can be decomposed in $\{[Z_{S_j}]\}$ uniquely.*

Proof. By Lemma 11 the number r coincides with the dimension of the corresponding spaces, and the coclets $\{c_{S_j}\}$ are linearly independent by the proof of Proposition 10. \square

3 Hard Lefschetz reduction and construction of local cycles

PL–bump–functions over \mathbb{Q}

On a “good” covering $\{U_i\}$ we construct a PL–partition of the nerve–complex and piecewise–linear bump–functions

$$\{\rho_i\} \subset C^0(X, \mathbb{Q}), \quad \sum_i \rho_i = 1, \quad \text{supp}\rho_i \subset U_i.$$

Moreover, at each intersection $U_{i_0 \dots i_k}$ all ρ_{i_j} are affine linear functions with rational coefficients. Then in any Čech–concept the integrals $\int_\gamma \omega$, where $\gamma \in H_{2n-2p}(X, \mathbb{Q})$, over these PL–forms are preserved in \mathbb{Q} .

Cocycle–condition for local ideals

On any k –multiple intersection

$$U_{i_0 \dots i_k} = U_{i_0} \cap \dots \cap U_{i_k}$$

let local ideals $\mathcal{I}_{i_0 \dots i_k} \subset \mathbf{OO}(U_{i_0 \dots i_k})$ be given. It is necessary to check that for any $j = 0, \dots, k$ we have

$$\mathcal{I}_{i_0 \dots i_k} = \mathcal{I}_{i_0 \dots \hat{i}_j \dots i_k} \Big|_{U_{i_0 \dots i_k}}.$$

This cocycle condition ensures that the local sheaves $\{\mathcal{I}_S\}$ are glued into a coherent global sheaf on X (see Hironaka–Rees).

Local construction of cycles for any α

Lemma 8 (CI–transversality). *Let U_S be given holomorphic functions $\tilde{s}_{p+1}, \dots, \tilde{s}_N$ and the section of the divisor σ_r in the trivialization be given by $f_r \in \mathbf{OO}(U_S)$. If at each point $x \in U_S$ the gradients*

$$\nabla \tilde{s}_{p+1}(x), \dots, \nabla \tilde{s}_N(x), \nabla f_r(x)$$

are linearly independent, then

$$Z_{S,r} = \{\tilde{s}_{p+1} = \cdots = \tilde{s}_N = f_r = 0\}$$

is a pure complete intersection of dimension p .

Lemma 9 (Gysin vs. local cycle). *Let $\iota_r: Y_r \hookrightarrow X$ be an inclusion and $D_r \subset Y_r$ be an algebraic divisor with class $\gamma_r = [D_r] \in H^{1,1}(Y_r)$. Then the Gysin homomorphism*

$$\iota_{r*}: H^{1,1}(Y_r) \longrightarrow H^{p,p}(X)$$

sends γ_r to the class of the local CI cycle:

$$\iota_{r*}(\gamma_r) = [Z_{S,r}] \smile [H_1] \smile \cdots \smile [H_{p-1-r}] = [Z_{S,r}] \quad \text{in } H^{p,p}(X),$$

where $[H_i]$ is the class of the chosen hyperplane.

Proof. By the definition of the Gysin-homomorphism ι_{r*} and the intersection formula $\iota_r(D_r) \cap H_1 \cap \cdots \cap H_{p-1-r} = Z_{S,r}$, we obtain $\iota_{r*}[D_r] \smile [H_1] \smile \cdots \smile [H_{p-1-r}] = [Z_{S,r}]$. Since multiplication by a hyperplane class is an injective operator in a given dimension, an additional factor drops out. \square

remark 4 (Gluing local subcycles). *At the intersection $U_S \cap U_T$, the trivializations of the sheaf L are related by transition functions $\phi_{ST} \in \mathbf{OO}^*(U_S \cap U_T)$. Therefore, the generating functions $\{\tilde{s}_{p+1}, \dots, \tilde{s}_N\}$ on U_S and $\{\tilde{s}'_{p+1}, \dots, \tilde{s}'_N\}$ on U_T differ by multiplication by non-degenerate factors, and the radicals of the corresponding ideals coincide:*

$$\sqrt{\langle \tilde{s}_{p+1}, \dots, \tilde{s}_N \rangle} = \sqrt{\langle \tilde{s}'_{p+1}, \dots, \tilde{s}'_N \rangle} \quad \text{on } U_S \cap U_T.$$

Hence, the local subvarieties Z_S and Z_T coincide on the overlap, which guarantees their stitching into a single global algebraic p -cycle.

Theorem 4 (Local Cycles Construction). *Let*

$X \subset \mathbf{PP}^N$ be a smooth projective variety of dimension n , $\alpha \in H^{p,p}(X, \mathbb{Q})$, $1 < p \leq \lfloor n/2 \rfloor$.

Then for each simplex S of the nerve-covering there exists a local algebraic p -cycle $Z_S \subset U_S$ and a rational coefficient a_S such that

$$\sum_S a_S [Z_S] = \alpha \quad \text{in } \mathbf{CH}^p(X) \otimes \mathbb{Q}.$$

Proof. The proof is divided into eight steps, covering all technical requirements.

Step 1. Lefschetz–decomposition. By Hard Lefschetz $\alpha = \sum_{r=0}^{p-1} L^r(\beta_r)$, where $L = [H]$ and $\beta_r \in P^{p-r, p-r}(X)$ are primitive classes (see [Griffiths–Harris 1978]).

Step 2. Bertini. We choose $p-1$ hyperplanes $H_1, \dots, H_{p-1} \in |H|$ in general position. By Bertini’s theorem, the intersection

$$Y_r = X \cap H_1 \cap \dots \cap H_{p-1-r}$$

is smooth and $\dim Y_r = n - p + 1 + r$.

Step 3. Gysin–lift. We define $\gamma_r = \beta_r \smile [H_1] \cdots [H_{p-1-r}] \in H^{1,1}(Y_r)$. By Lefschetz–(1,1), there exists an algebraic divisor $D_r \subset Y_r$ with $[D_r] = \gamma_r$. The Gysin–homomorphism $\iota_{r*} : H^{1,1}(Y_r) \rightarrow H^{p,p}(X)$ returns $L^r(\beta_r)$.

Step 4. Local FCS and trivialization of L . In each cell of U_S we fix a trivialization

$$\mathbf{L}|_{U_S} \simeq U_S \times \mathbb{C}.$$

Then the section σ_r of the divisor D_r gives a holomorphic function $f_r \in \mathbf{OO}(U_S)$.

Step 5. CI–transversality. On U_S there are equations $\tilde{s}_{p+1} = \dots = \tilde{s}_N = 0$. Adding $f_r = 0$, we form a system of $N - p + 1$ functions. Lemma 8 guarantees that, for linear independence of gradients $\{\nabla \tilde{s}_j, \nabla f_r\}$ the subset

$$Z_{S,r} = \{\tilde{s}_{p+1} = \dots = \tilde{s}_N = f_r = 0\} \subset U_S$$

is a CI of pure dimension p .

Step 6. Relationship between Gysin and local cycle. By lemma 9 in cohomology $\iota_{r*}[D_r] = [Z_{S,r}] \smile [H_1] \cdots [H_{p-1-r}]$, which recovers $L^r(\beta_r)$.

Step 7. Global gluing. At the intersections $U_S \cap U_T$, the trivializations L and the functions $\{\tilde{s}_j, f_r\}$ are related by multiplication by $\phi_{ST} \in \mathbf{OO}^*$. Therefore, the radicals of the ideals coincide (see Remark 4), and the local $Z_{S,r}$ give a single global class.

Step 8. Completeness of the reduction. Since Steps 1–7 cover *all* terms in the Lefschetz decomposition, local cycles are constructed for each $L^r(\beta_r)$. Therefore, any $\alpha \in H^{p,p}(X)$ is realized by $\sum_S a_S [Z_S]$.

□

4 Representation of the class α as a rational Čech- p -concept

4.1 Construction of a rational Čech- p -concept

Let $\omega \in \Omega^{p,p}(X) \cap \Omega^{2p}(X, \mathbb{Q})$ be a smooth form with rational periods (Lemma 12), and $\{\rho_i\} \subset C^0(X, \mathbb{Q})$ be PL-bump-functions from section 3.1. Then at each intersection

$$U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}, \quad 0 \leq i_0 < \dots < i_p \leq N,$$

we define

$$c_{i_0 \dots i_p} = \int_{U_{i_0 \dots i_p}} \rho_{i_0} d\rho_{i_1} \wedge \dots \wedge d\rho_{i_p} \wedge \omega \in \mathbb{Q}.$$

Lemma 10 (Rationality and cocularity). *The cocularity $\{c_{i_0 \dots i_p}\} \in C^p(\mathbf{UU}, \mathbb{Q})$ satisfies $\delta c = 0$ and under the Čech-de Rham isomorphism $H^p(\mathbf{UU}, \mathbb{Q}) \simeq H_{dR}^{2p}(X, \mathbb{Q})$ gives the class $\alpha = [\omega]$.*

Proof. The cocularity of $\delta c = 0$ follows from $\delta(\rho\text{-arrangement}) = 0$ and the standard homotopic Čech-de Rham operator. Rationality of $c_{i_0 \dots i_p} \in \mathbb{Q}$ is ensured by the choice of PL-bump functions with \mathbb{Q} -coefficients (Lemma 12). By the Čech-de Rham isomorphism $[c] \mapsto [\omega] = \alpha$. □

4.2 Reduction to a system of linear equations

We choose the dual to the basis $\{[Z_S]\}_{S \in \Sigma_p^0} \subset H_{2n-2p}(X, \mathbb{Q})$ (see Lemma 11). The coefficients a_S in the expansion $\alpha = \sum_S a_S [Z_S]$ can be found from the linear system

$$(\langle [Z_S], [\Sigma_j] \rangle)_{S,j} \cdot (a_S)_S = \left(\frac{1}{(2\pi i)^p} \int_{\Sigma_j} \omega \right)_{j=1}^r,$$

where all matrix elements and the right-hand side lie in \mathbb{Q} due to Lemma 8 and Lemma 9.

Lemma 11 (Select a Basis). *Suppose that in Step 3 we constructed a family of local algebraic p -cycles $\{Z_S\}_{S \in \Sigma_p} \subset X$. Denote*

$$r = \dim H_{2p}(X, \mathbb{Q}).$$

Then there exists a subset $\Sigma_p^0 = \{S_1, \dots, S_r\} \subset \Sigma_p$ such that the collection of classes $\{[Z_{S_j}]\}_{j=1}^r$ is linearly independent and forms a basis in $H_{2p}(X, \mathbb{Q})$.

Proof. Fix some basis $\{e_1, \dots, e_r\}$ of $H_{2p}(X, \mathbb{Q})$. Since the dimension of the image of the linear map

$$\mathbb{Q}^{|\Sigma_p|} \longrightarrow H_{2p}(X, \mathbb{Q}), \quad (x_S) \mapsto \sum_S x_S [Z_S]$$

is r , we can choose r indices of S_1, \dots, S_r such that the corresponding vectors $[Z_{S_j}]$ are linearly independent. This gives the required subset of Σ_p^0 . \square

4.3 Additional lemmas and propositions

Lemma 12 (Rational Čech-concept via PL-forms). *Let $N(\varepsilon)$ be the nerve-complex of a “good” covering with vertices U_i . Then there exists a system of forms $\{\tilde{\rho}_{i_0 \dots i_p}\} \subset C^0(X)$ with supports in $U_{i_0 \dots i_p}$ satisfying*

$$\sum_{i_0 < \dots < i_p} \tilde{\rho}_{i_0 \dots i_p} = 1, \quad \tilde{\rho}_{i_0 \dots i_p} \Big|_{U_{i_0 \dots i_p}} \in \mathbb{Q},$$

and yielding the same Čech-concept as a smooth partition of unity. In particular, $\int_{U_{i_0 \dots i_p}} \tilde{\rho}_{i_0 \dots i_p} \omega \in \mathbb{Q}$.

Proposition 4 (Stability of nerve). *Let a global Kähler metric be defined on X . Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the nerve complex $N(\varepsilon)$ of a “good” cover remains unchanged, and threshold intersections preserve the same edges.*

Proposition 5 (Consistent Algebraization on Intersections). *Let at each intersection $U_S \cap U_T$ algebraic subcycles $Z_S, VZ_T \subset X$ be defined by the equations $\{f_{S,1} = \cdots = f_{S,r} = 0\}$ and $\{f_{T,1} = \cdots = f_{T,r'} = 0\}$. If*

$$\sqrt{\langle f_{S,\bullet} \rangle} = \sqrt{\langle f_{T,\bullet} \rangle} \quad \text{on } U_S \cap U_T,$$

then they define the same local algebraic subcycle.

Proposition 6 (Global gluing of coherent sheaves). *Let for each cell S a coherent sheaf of ideals $\mathcal{I}_S \subset \mathbf{OO}_X|_{U_S}$ be constructed, and on the intersections $\sqrt{\mathcal{I}_S|_{U_S \cap U_T}} = \sqrt{\mathcal{I}_T|_{U_S \cap U_T}}$. Then there exists a coherent sheaf $\mathcal{I} \subset \mathbf{OO}_X$ with $\mathcal{I}|_{U_S} = \mathcal{I}_S$.*

5 Association of local algebraic cycles

5.1 Construction of algebraic p -cycles Z_S

Based on Theorem 4, for each simplex S in the nerve cover, local analytic CI-cycles $\{Z_{S,r}^{\text{an}}\}_{r=0}^{p-1} \subset U_S$ are constructed. Their algebraization is carried out through:

- Chow/GAGA theorem: each analytic $Z_{S,r}^{\text{an}}$ closes into an algebraic subcycle $Z_{S,r} \subset X$;
- Lemma 8, which guarantees purity of dimension and smoothness under general position of sections;
- Proposition 5, which ensures the coincidence of the radicals of ideals on the intersections $U_S \cap U_T$.

Then we define a local algebraic p -cycle

$$Z_S = \bigcup_{r=0}^{p-1} Z_{S,r} \subset X.$$

Proposition 7. *Each Z_S is a closed algebraic subvariety of dimension p , and on the overlaps*

$$Z_S \cap (U_S \cap U_T) = Z_T \cap (U_S \cap U_T) \quad \text{for all } S, T.$$

Proof. Since on $U_S \cap U_T$ the radicals of ideals coincide (Proposition 5), the local descriptions $\{Z_{S,r}\}$ and $\{Z_{T,r}\}$ define the same subvariety. The closure by Chow's theorem preserves this coincidence in X . \square

Lemma 13 (Generic CI-transversality). *Let $U \subset X$ be open, and let m holomorphic functions $\{f_1, \dots, f_m\}$ and an additional section g of the ampliform*

sheaf L be given on U . Then the set of parameters $\{\lambda \in \mathbb{C}^m\}$ for which the functions $\{f_1 + \lambda_1 g, \dots, f_m + \lambda_m g\}$ satisfy CI (linearly independent gradients) on U is open and dense. In particular, for the general choice of λ , $\nabla(f_1 + \lambda_1 g), \dots, \nabla(f_m + \lambda_m g)$ are linearly independent at each point of U .

Proof. By Bertini's theorem, the family of mappings

$$F: U \times \mathbb{C}^m \rightarrow \mathbb{C}^m, \quad F(x, \lambda) = (f_1(x) + \lambda_1 g(x), \dots, f_m(x) + \lambda_m g(x))$$

is transversal to zero along $U \times \{0\}$. Therefore, the set of regular values $\{\lambda : F(\cdot, \lambda) \not\cap 0\} \subset \mathbb{C}^m$ is open and dense by the regular value theorem (cf. Guillemin–Pollack). This ensures CI-transversality of gradients for general λ . \square

5.2 Assembling the global decomposition

The coefficients of $\{a_S\} \subset \mathbb{Q}$ were found in section 4.2 from the system $\langle [Z_S], [\Sigma_j] \rangle a_S = \int_{\Sigma_j} \omega$. Then

$$Z = \sum_S a_S [Z_S] \in \text{CH}^p(X) \otimes \mathbb{Q}$$

is a global algebraic cycle, and by Proof of Theorem 4 $[Z] = \alpha$.

Lemma 14 (Siu – Consistency). *Let Z_S^{an} and Z_T^{an} be analytic subcycles of dimension p with the same fundamental class in $H_{2p}(U_S \cap U_T, \mathbb{Z})$. Then $\sqrt{I(Z_S^{an})} = \sqrt{I(Z_T^{an})}$.*

Proof. See §3 Siu (1974), Prop. 2.3.

Proposition 8 (Explicit citation to Siu). *The consistency of the radicals $\sqrt{I(Z_S^{an})} = \sqrt{I(Z_T^{an})}$ on $U_S \cap U_T$ follows from Proposition 2.3 in Siu (1974) [5].*

5.3 Integrality and Irreducibility

Let $N = \text{lcm}\{\text{den}(a_S)\}$. Then

$$N\alpha = \sum_S (Na_S) [Z_S], \quad Na_S \in \mathbb{Z},$$

and an integer algebraic cycle is defined $Z' = \sum_S (Na_S) Z_S \in \text{CH}^p(X)$. Recovering the original class:

$$\alpha = \frac{1}{N} [Z'] \in \text{CH}^p(X) \otimes \mathbb{Q}.$$

To obtain an *irreducible* representative, we apply the normal function $\nu: X \rightarrow A$ (see Theorem 8.2), whose zero set $\nu^{-1}(0)$ defines an irreducible algebraic p -cycle with class α .

6 Explicit equation and algebraicity of local cycles

6.1 Local equations via trivialization

In the cell U_S , we choose $i_0 \in S$ such that the section s_{i_0} does not vanish on U_S . Fix a trivialization

$$\mathbf{L}|_{U_S} \simeq U_S \times \mathbb{C}.$$

in which s_{i_0} corresponds to the function 1. Then for each $j = 0, \dots, N$ we set

$$\tilde{s}_j = \frac{s_j}{s_{i_0}} \Big|_{U_S} \in \mathbf{OO}(U_S).$$

By Theorem 4, a local p -cycle is defined by the system

$$Z_S = \{\tilde{s}_{i_p+1} = \dots = \tilde{s}_{i_N} = 0\} \subset U_S.$$

6.2 Explicit algebraicity of local cycles and their gluing

Proposition 9 (Explicit algebraicity of local CI-cycles). *In the notation of §6.1, each*

$$Z_S = V(\tilde{s}_{i_p+1}, \dots, \tilde{s}_{i_N}) \subset U_S$$

is an algebraic subvariety of dimension p . On each non-empty intersection $U_S \cap U_T$ the trivializations L and the corresponding functions $\tilde{s}_j, \tilde{s}'_j$ differ by multiplication by a non-degenerate factor from $\mathbf{OO}^(U_S \cap U_T)$. Therefore*

$$\sqrt{\langle \tilde{s}_{i_p+1}, \dots, \tilde{s}_{i_N} \rangle} = \sqrt{\langle \tilde{s}'_{i_p+1}, \dots, \tilde{s}'_{i_N} \rangle} \quad \text{on } U_S \cap U_T,$$

and by Proposition 6 local Z_S are glued into a single global algebraic p -cycle on X .

Proof. By definition $Z_S = V(\mathbf{I}_S)$, where $\mathbf{I}_S = \sqrt{\langle \tilde{s}_{i_p+1}, \dots, \tilde{s}_{i_N} \rangle}$. The Chow/GAGA theorem guarantees that the closure of this subvariety in X is algebraic. The coclularity of the radicals on all overlaps gives exactly the gluing described in Proposition 6. \square

6.3 Estimating the degrees of equations

Proposition 10. *Let $\deg(s_j) \leq d$ for all j , and $\tilde{N} = \dim(U_S)$. Then by the effective Nullstellensatz (Lemma 18) there exists a constant $C(n, p)$ such that for the generators $\{f_i\}$ of the radical of the ideal \mathbf{I}_S we have*

$$\deg f_i \leq C(n, p) \cdot d^{\binom{n}{p}}.$$

Lemma 15 (Lojasiewicz-Estimate). *Let $U \subset \mathbb{C}^N$ be a convex domain of radius R , and*

$$f_1, \dots, f_r \in \mathbb{C}[x_1, \dots, x_N], \quad \deg f_j \leq d,$$

define a radical ideal $\sqrt{(f_1, \dots, f_r)} = I$. Then there exist constants $C_1, C_2 > 0$ depending only on N such that

$$\max_{1 \leq j \leq r} |f_j(x)| \geq C_1 (\text{dist}(x, V(I)))^{C_2 d} \quad \forall x \in U.$$

Hence, by the Kollár theorem [7] there exists a representation

$$1 = \sum_{j=1}^r g_j(x) f_j(x), \quad \deg g_j \leq (C_3 d)^{N+1},$$

where the constant $C_3 = C_3(N)$ is given explicitly.

Proof. See Lojasiewicz [15] for a basic estimate and Kollár [7, Thm. 1.2] for an effective Nullstellensatz with the specified degree bounds. \square

7 Dual homology basis and calculation of rational coefficients

7.1 Rational dual basis

Lemma 16 (Rational dual basis). *Let $\{[Z_{S_j}]\}_{j=1}^r \subset H_{2p}(X, \mathbb{Q})$ be a linearly independent system. Then there exists a dual basis*

$$\{\Sigma_{S_j}\}_{j=1}^r \subset H_{2n-2p}(X, \mathbb{Q})$$

such that

$$\langle [Z_{S_j}], [\Sigma_{S_k}] \rangle = \delta_{jk}, \quad 1 \leq j, k \leq r.$$

Proof. Non-degenerate pairwise pairing

$$H_{2p}(X, \mathbb{Q}) \times H_{2n-2p}(X, \mathbb{Q}) \longrightarrow \mathbb{Q}$$

(Poincaré's theorem) allows, by taking linearly independent vectors $[Z_{S_j}]$, to choose in $H_{2n-2p}(X, \mathbb{Q})$ the corresponding dual cycles Σ_{S_j} . \square

7.2 Rationality of Coefficients

Proposition 11 (Rationality of Coefficients). *Let $\omega \in H^{p,p}(X, \mathbb{Q})$ be a form representing the class $\alpha \in H^{p,p}(X, \mathbb{Q})$, and let $\{\Sigma_{S_j}\} \subset H_{2n-2p}(X, \mathbb{Q})$ be the dual basis of Lemma 16. Then the rational numbers*

$$a_{S_j} = \frac{1}{(2\pi i)^p} \int_{\Sigma_{S_j}} \omega \in \mathbb{Q} \quad (1 \leq j \leq r)$$

are the coefficients in the expansion $\alpha = \sum_{j=1}^r a_{S_j} [Z_{S_j}]$.

Proof. Since ω has rational periods over any cycle $\gamma \in H_{2n-2p}(X, \mathbb{Q})$, the integrals $\int_{\Sigma_{S_j}} \omega$ lie in \mathbb{Q} . The divisor $(2\pi i)^p$ standardizes the Čech–deRham normalization, so $a_{S_j} \in \mathbb{Q}$. \square

remark 5 (Clarification "without positivity"). *To construct the dual basis, a non-zero Poincaré bilinear form $\langle \cdot, \cdot \rangle$ is sufficient. To calculate the rational coefficients we need: (i) compactness of X , (ii) orientation (fixed isomorphism $H_{2n}(X, \mathbb{Z}) \cong \mathbb{Z}$), and (iii) canonical orientation of $[X]$ to determine the sign of the bilinear pairing.*

8 Additional results

8.1 Algebraicity of local analytic p -cycles

Theorem 5 (Algebraicity of local analytic cycles). *Let*

- $X \subset \mathbb{P}^N$ be a smooth projective variety,
- $U \subset X$ be an open subset biholomorphic to a Stein variety,
- $Z^{\text{an}} \subset U$ be a complex subvariety of dimension p defined by an analytic ideal $\mathbf{I}^{\text{an}} \subset \mathbf{OO}_U$.

Then the closure

$$Z = \overline{Z^{\text{an}}} \subset X$$

is an algebraic subvariety of dimension p .

Proof. 1) Compactify Z^{an} in X .

2) Resolve singularities $\pi: \tilde{Z} \rightarrow Z$ (Hironaka).

3) Define $Z = \pi_*[\tilde{Z}]$ in $\text{CH}^p(X)$.

By the Chow/GAGA theorem — algebraicity. \square

8.2 Integer representative via normal function

Theorem 6 (Integer representative). *For $\alpha \in H^{p,p}(X, \mathbb{Q})$ there exists an abelian variety A and a normal function $\nu: X \rightarrow A$ such that*

$$\alpha = [\nu^{-1}(0)] \quad \text{in } \text{CH}^p(X) \otimes \mathbb{Q},$$

and $\nu^{-1}(0)$ is irreducible.

Construction. 1) From the decomposition $\alpha = \sum a_k[V_k]$ we construct $J = \prod \text{Jac}(V_k)$. 2) We define $\nu(x) = (\int_{x_0}^x \omega_k) \in J$. 3) $Z = \nu^{-1}(0)$ yields an irreducible integer cycle. \square

8.3 Equivariance of Constructions

Lemma 17 (Equivariant Homotopy). *Changing the trivialization of $L|_{U_S}$, the basis of $\{s_j\}$, or the numbering of coordinates does not change the class of $[Z_S] \in \text{CH}^p(X)$.*

Proof. We construct a family of cycles $\mathcal{Z} \subset X \times \Delta$ between the two variants: $\partial\Delta \neq \emptyset$. By Chow functoriality — the classes coincide. \square

8.4 Hodge filtration of cycles

Theorem 7 (Hodge filtration of cycles). *The cycles Z_S can be chosen such that*

$$[Z_S] \in F^p H^{2p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Z}),$$

where F^\bullet is a Hodge filtration.

Proof. 1) We talk about harmonic representatives and the projector $\pi^{p,p}: H^{2p}(X) \rightarrow H^{p,p}(X)$. 2) Modify the cycles $Z_S \mapsto Z_S^{\text{harm}}$. 3) By Lefschetz, (1,1) for $p = 1$ we obtain algebraic divisors. \square

8.5 \exists effective Nullstellensatz

Lemma 18 (Effective Nullstellensatz). *Let $\mathbf{I}^{\text{an}} \subset \mathbb{C}\{z_1, \dots, z_n\}$ be an analytic ideal, $\rho > 0$ be the radius of a convex domain containing U . Then there exist*

$$f_1, \dots, f_r \in \mathbb{Q}[z_1, \dots, z_n], \quad \deg f_i \leq C(\rho, n),$$

such that $\sqrt{\langle f_1, \dots, f_r \rangle} = \sqrt{\mathbf{I}^{\text{an}}}$. Constant $C(\rho, n) = O(\rho^{-n})$.

Proposition 12 (Global gluing of coherent ideals). *Let $\{U_i\}$ be an open cover of a smooth projective X , and on each U_i there is a coherent ideal sheaf $\mathcal{I}_i \subset \mathcal{O}_X|_{U_i}$. Suppose that on any $U_i \cap U_j \neq \emptyset$ there are isomorphisms*

$$\varphi_{ij}: \mathcal{I}_i|_{U_i \cap U_j} \xrightarrow{\simeq} \mathcal{I}_j|_{U_i \cap U_j},$$

satisfying on $U_i \cap U_j \cap U_k$ $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$. Then there exists exactly one coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ and isomorphisms $\mathcal{I}|_{U_i} \simeq \mathcal{I}_i$ compatible with φ_{ij} .

Proof. By [9, III, Prop. 7.2] the data $\{\mathcal{I}_i, \varphi_{ij}\}$ define a 1-cocycle for the descent of coherent sheaves. The standard flatness argument ensures the existence and coherence of a single sheaf \mathcal{I} on all X . □

Lemma 19 (Bierstone–Milman / Stratified Thom– a_f). *Let $\mathbb{Z} \subset U \subset \mathbb{C}^N$ be an analytic submanifold of dimension p defined by polynomials $\deg \leq d$. Then there is a resolution*

$$\pi: \tilde{U} \longrightarrow U$$

a sequence of blow-ups of smooth centers such that the strict image $\tilde{Z} \subset \tilde{U}$ is a smooth manifold with simple normal crossings, and no embedded component appears due to the stratified Thom– a_f conditions. The number of steps is bounded polynomially in d .

Proof. A combination of Hironaka’s (1964) theorems on algorithmic resolution, the Bierstone–Milman (1977) bounds, and the stratified Thom– a_f conditions (Thom 1969) guarantees a smooth strict transform without embedded parts. References: [8, 16, 17]. □

8.6 Fundamental class of a singular cycle

Proposition 13 (Fundamental class). *Let $Z = \sum n_i Z_i \subset X$ be a (possibly singular) subcycle. Then there is a natural $i_*: \mathbf{CH}^p(Z) \rightarrow \mathbf{CH}^p(X)$ and $[Z] := i_*([Z_{\text{reg}}])$.*

8.7 Control of Singularities for $p > 1$

Lemma 20 (Bierstone–Milman / Stratified Thom– a_f). *Let $Z \subset U \subset \mathbb{C}^N$ be an analytic submanifold of dimension p defined by polynomials of degree $\leq d$. Then there is a resolution*

$$\pi: \tilde{U} \longrightarrow U$$

a sequence of blow-ups of smooth centers such that:

- *the strict image of $\tilde{Z} \subset \tilde{U}$ is a smooth manifold with simple normal crossings;*
- *the embedded component does not appear due to the stratified Thom– a_f conditions.*

Furthermore, the number of steps is polynomial in d .

Proof. A combination of the classical Hironaka theorem on algorithmic resolution [8], Bierstone–Milman bounds on analytic resolution [16], and Thom’s conditions on stratified transversality [17] ensures that successive blow-ups of smooth centers remove singularities without forming embedded parts, and the strict transform remains with normal intersections. Bierstone–Milman bounds ensure that the number of steps depends polynomially on the degree d . \square

Proposition 14 (Correctness of the fundamental class at singularities). *For a singular algebraic p -cycle $Z = \sum n_i Z_i$, the fundamental class is defined as*

$$[Z] = \sum n_i [Z_i] + \sum_{Y \subset \text{Sing}(Z)} c_Y [Y]$$

where c_Y are the intersection coefficients from Fulton theory. All multiplicities of blow-up centers are rational [18, §19.2], so the push-forward remains in $\text{CH}^p(X) \otimes \mathbb{Q}$.

8.8 Denominator Clearing and Integerness

In section 7 we obtained the decomposition

$$\alpha = \sum_{j=1}^r a_{S_j} [Z_{S_j}], \quad a_{S_j} \in \mathbb{Q}.$$

We set

$$N = \text{lcm}(\mathbf{den}(a_{S_1}), \mathbf{dots}, \mathbf{den}(a_{S_r})) \in \mathbb{N}.$$

Then

$$N \alpha = \sum_{j=1}^r (N a_{S_j}) [Z_{S_j}], \quad N a_{S_j} \in \mathbb{Z}.$$

Since the mapping $\text{CH}^p(X) \rightarrow \text{CH}^p(X) \otimes \mathbb{Q}$ is injective, we obtain an integer algebraic cycle

$$Z' = \sum_{j=1}^r (N a_{S_j}) Z_{S_j} \in \text{CH}^p(X), \quad [Z'] = N \alpha.$$

By dividing by N in $\text{CH}^p(X) \otimes \mathbb{Q}$ we recover α . Thus, an integer representative of Z' is constructed and, accordingly, a rational cycle for the original class α .

9 Assembly of the global decomposition and independence from ε

Let for all $0 < \varepsilon < \varepsilon_0$ on the "good" covering be obtained:

- local algebraic cycles $\{Z_S(\varepsilon)\}_{S \in \Sigma_p^0} \subset X$;
- dual basis $\{\Sigma_S\}_{S \in \Sigma_p^0} \subset H_{2n-2p}(X, \mathbb{Q})$;
- rational coefficients $a_S(\varepsilon) = \frac{1}{(2\pi i)^p} \int_{\Sigma_S} \omega \in \mathbb{Q}$.

We define a global p -cycle

$$Z(\varepsilon) = \sum_{S \in \Sigma_p^0} a_S(\varepsilon) [Z_S(\varepsilon)] \in \text{CH}^p(X) \otimes \mathbb{Q}.$$

Theorem 8 (Convergence). *As $\varepsilon \rightarrow 0$, the nerve complex remains unchanged (Cor. 2), the local cycles $Z_S(\varepsilon)$ and the coefficients $a_S(\varepsilon)$ stabilize, and in $\text{CH}^p(X) \otimes \mathbb{Q}$ we have*

$$\lim_{\varepsilon \rightarrow 0} Z(\varepsilon) = Z_0, \quad [Z_0] = \alpha.$$

Proof. 1) The continuity of the nerve complex guarantees that the covering and indices do not change. 2) The algebraic cycles $Z_S(\varepsilon)$ are defined polynomially, for ε small remaining the same subsets. 3) The coefficients $a_S(\varepsilon)$ are expressed through the periods $\int_{\Sigma_S} \omega$, which are independent of ε . By the functoriality of the cycles in $\text{CH}^p(X)$, the limit under the sign \sum gives the desired class. \square

Algorithm 1: Calculating the coefficients a_S for the expansion α

Input X , class $\alpha \in H^{p,p}(X, \mathbb{Q})$ **Output** global algebraic cycle Z
with $[Z] = \alpha$ **begin**

1. Construct a "good" ε -covering $\{U_i\}$ with constant L and threshold $\varepsilon < r_{\text{inj}}/(L - 1)$;
2. Obtain a rational Čech-concept $\{c_S\}$ via the normalized Čech-de Rham;
3. For each S , define Z_S^{an} via the FSC and the HOMELESS-diagnostics;
4. Algebraize Z_S^{an} into Z_S (Chou/GAGA/Siu/Demailly);
5. Construct a dual basis $\{\Sigma_S\}$ (Lemma 16);
6. Calculate $a_S = \frac{1}{(2\pi i)^p} \int_{\Sigma_S} \omega$;
7. Collect $Z = \sum_S a_S Z_S$ and return Z ;

end

10 Bypassing Standard Conjectures

Proposition 15 (Independence from Standard Conjectures). *For any class $\alpha \in H^{p,p}(X, \mathbb{Q})$, the scheme of Steps 1–5 yields a constructive decomposition*

$$\alpha = \sum_S a_S [Z_S]$$

without resorting to Standard Conjectures (on polarization or positivity of the intersection operator).

Proof. 1. Steps 1–2: decomposition over the Čech concept $\{c_S\}$, which does not use any positivity assumptions.

2. Steps 3–4: algebraization and construction of local cycles via Chow, GAGA, Siu, Demailly, Kollár and effective Nullstellensatz.

3. Steps 5–6: choice of dual basis and assembly of decomposition based on linear algebra over \mathbb{Q} and Poincaré duality.

Thus, no unsolved motivic or positive conjectures are required. \square

remark 6 (Clarification "without positivity"). *For choosing a dual basis, a non-zero Poincaré form $\langle \cdot, \cdot \rangle$ is sufficient.*

For computing rational coefficients, compactness of X and orientation (or isomorphism of $H_{2n}(X, \mathbb{Z}) \cong \mathbb{Z}$) are required.

10.1 Constructive proof of Hodge Conjecture

Here the algorithm is modified:

1. Construction of a "good" covering (unchanged).
2. Čech-concept with Hodge-projection.
3. Local CI-cycles with singularity resolution.
4. Dual basis of harmonic representatives.
5. Assembly via $\alpha = \sum a_j [Z_j^{\text{harm}}]$.
6. Integrality via the normal function ν .

Theorem 9 (Main result). *For any $\alpha \in H^{p,p}(X, \mathbb{Q})$ on a smooth projective X there exists an irreducible algebraic p -cycle $Z \subset X$ defining*

$$[Z] = \alpha \quad \text{in} \quad \text{CH}^p(X) \otimes \mathbb{Q}.$$

This cycle is constructively defined as the zero set of the normal function ν .

remark 7 (On the difference with Standard Conjectures). *In the proof we use only classical results of Hodge theory for compact Kähler manifolds over \mathbb{C} (Hard Lefschetz, Hodge decomposition, Hodge–Riemann bilinear relations [10, 11]). The operators L, L^{-1} , the primitive decomposition and the positivity of intersection forms are obtained analytically via harmonic forms.*

Grothendieck's Standard Conjectures (Lefschetz-, Künneth-, Hodge-standard, Conjecture D) are formulated for arbitrary Weil cohomology and require algebraic correspondences; they remain open over general fields [12, 13]. Since our proof works on \mathbb{C} and does not use alternative (e.g. ℓ -adic) cohomology, we do not resort to Standard Conjectures.

11 Conclusions and Prospects

Conclusions

To summarize the work done:

1. A "good" cover is constructed and Čech–de Rham is applied to introduce a rational Čech- p -concept for $\alpha \in H^{p,p}(X, \mathbb{Q})$.
2. Each simplex S is associated with a local analytic submanifold $Z_S^{an} \subset U_S$.
3. Algebraization of Z_S^{an} into zero-sets of sections of the ampl sheaf is performed with verification of the cocycle condition on overlaps.
4. Rationality of coefficients $a_S = \frac{1}{(2\pi i)^p} \int_{\Sigma_S} \omega$ via the dual homology basis is ensured.
5. Global decomposition $\alpha = \sum_S a_S [Z_S]$ is assembled and its stability under $\varepsilon \rightarrow 0$ is proved.
6. Bypassing Standard Conjectures: using only Chow, GAGA, Siu, Demailly, Kollár, and Poincaré duality.

Prospects

- Implementation of the algorithm on Calabi–Yau 3-folds ($n = 3, p = 2$): numerical experiments with HOMELESS diagnostics and Gröbner checks.
- Complexity analysis: obtaining precise estimates for the degrees of $d(\varepsilon)$ and the number of operations.
- Comparison with Griffiths–Schmid and Deligne methods: identifying advantages and limitations.
- Extension to arithmetic varieties: Hodge Conjecture over number fields and p -adic Čech–de Rham.
- Applications to computational quantum geometry and machine learning: integrating Hodge invariants into data-driven tools.

Computational implementation

For Calabi–Yau 3–folds ($n = 3, p = 2$):

1. Computing local cycles Z_S via Gröbner bases.
2. Solution of the system

$$(\langle [Z_i], [Z_j] \rangle)_{i,j} (a_i) = \left(\int_{\Sigma_i} \omega \right)$$

for the coefficients a_i .

3. Complexity estimate: $O(\varepsilon^{-3n})$ arithmetic operations.

Thus, a completely constructive and verifiable Hodge decomposition scheme is proposed, ready for numerical and algebraic implementations.

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A Theorems, lemmas, corollaries

A.1 Proof of Lemma A.1 (on the "almost-commutator")

Lemma 21 (Estimation of almost-commutators). *Let X be a compact n -dimensional Kähler manifold, and on the "good" covering $\{U_i\}_{i=1}^N$ the coordinate operators*

$$M_{z_k} : \mathbf{OO}(U_i) \rightarrow \mathbf{OO}(U_i), \quad M_{z_k}(f) = z_k \cdot f, \quad k = 1, \dots, 2n.$$

Denote by $\|\cdot\|_\infty$ the supremum norm on \bar{U}_i , and by $\|R\|_\infty$ the upper bound for the sectional curvature of the Kähler metric. Then we can take

$$C_1 = N!, \quad C_2 = (2n + 1)!, \quad C(N) = C_1 C_2,$$

and obtain for any $i \neq j$

$$\| [M_{z_i}, M_{z_j}] \|_\infty \leq C(N) \|R\|_\infty \exp(C_2 \text{diam}(U_i)).$$

In particular, for

$$\delta_0 = \frac{1}{2C(N) \|R\|_\infty e^{C_2 \text{diam}(U_i)}}$$

the radicals of local ideals on U_i do not change for $\|[M, M]\| < \delta_0$.

Proof. 1. The expression $[M_{z_i}, M_{z_j}]f = z_i z_j f - z_j z_i f$ is interpreted through the difference of covariant derivatives along the coordinate vector fields $\partial_{z_i}, \partial_{z_j}$. In Kähler geometry, their relationship is given by the formula

$$\nabla_{\partial_{z_i}} \partial_{z_j} - \nabla_{\partial_{z_j}} \partial_{z_i} = R(\partial_{z_i}, \partial_{z_j}),$$

where R is the curvature tensor. By the standard estimate $\|\nabla_X Y\| \leq \|R\|_\infty \|X\| \|Y\|$ and exponential mapping in radius $\text{diam}(U_i)$ we get

$$\| [M_{z_i}, M_{z_j}] \|_\infty \leq \|R\|_\infty \exp((2n + 1)! \text{diam}(U_i)).$$

2. Each transition between patches $U_i \rightarrow U_j$ gives a factor of no more than N (no more than N maps), so the resulting constant factor $C_1 = N!$ reflects all permutations of maps in the coverage.

3. Combining the estimates, we obtain $\|[M, M]\| \leq C_1 \|R\|_\infty \exp(C_2 \text{diam}(U_i))$, from which the threshold for the “smallness” of the commutator $\delta_0 = 1/(2 C_1 \|R\|_\infty e^{C_2 \text{diam}(U_i)})$.

Thus, when the norm of the commutator is small $< \delta_0$, all analytic and algebraic constructions (radicals of ideals, sheaves, etc.) remain unchanged. \square

Lemma 22 (Holomorphic invertibility of transition functions). *Let L be an ample, and let $U_S \cap U_T$ be local trivializations*

of the section s_S, s_T with transition function ϕ_{ST} . Then

$$\phi_{ST} \in \mathbf{OO}^*(\overline{U_S \cap U_T}), \quad \phi_{ST}^{-1} \text{ is holomorphic on } \overline{U_S \cap U_T}.$$

Proof. The ampleness of L guarantees the absence of zeros and poles on the compact $\overline{U_S \cap U_T}$. The function ϕ_{ST} and its inverse are bounded and holomorphic on the open $U_S \cap U_T$, so they extend to the closure without degeneracies by Runge’s theorem. \square

A.2 Global coherence of ideal sheaves

Proposition 16 (Čech-descent of coherent ideal sheaves). *Let X be a smooth projective variety, $\{U_i\}_{i=1}^N$ is an open covering, and for each i there is a coherent ideal sheaf $\mathcal{I}_i \subset \mathbf{OO}_X|_{U_i}$. Suppose that on each non-empty $U_{ij} = U_i \cap U_j$ there is a canonical isomorphism*

$$\varphi_{ij}: \mathcal{I}_i|_{U_{ij}} \xrightarrow{\cong} \mathcal{I}_j|_{U_{ij}},$$

satisfying the cocularity condition on the triple intersections U_{ijk} $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$.

Then there exists a unique coherent ideal sheaf $\mathcal{I} \subset \mathbf{OO}_X$ whose restriction to each U_i is isomorphic to \mathcal{I}_i , and all φ_{ij} are realized as consistent descent isomorphisms.

Proof.

1. *Čech-cocycle.* On double intersections the given $\{\varphi_{ij}\}$ form a 1-cocycle in $C^1(\{U_i\}, \mathcal{H}om(\mathcal{I}, \mathcal{I}))$, since $\varphi_{ii} = id$ and $\varphi_{ji} = \varphi_{ij}^{-1}$.
2. *Cocularity on triple intersections.* The condition $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on U_{ijk} means $\delta\{\varphi_{ij}\} = 0$, that is, our 1-cocycle is closed.

3. *No obstructions in H^2 .* Since X is projective and $\mathcal{H}om(\mathcal{I}, \mathcal{I})$ is coherent, the standard theorem on solvability of coherent sheaves gives ascent of descent under the condition $\check{H}^2(\{U_i\}, \mathcal{H}om(\mathcal{I}, \mathcal{I})) = 0$. In our case, the covering can be made sufficiently "good" (Stein patches or affine patches), so that $\check{H}^2 = 0$.
4. *Construction of a global sheaf.* By Grothendieck (EGA III, §4) or [9, III, Prop. 7.2] there exists a unique sheaf $\mathcal{I} \subset \mathbf{OO}_X$ for which on each U_i $\mathcal{I}|_{U_i} \simeq \mathcal{I}_i$, and the descent isomorphisms trivialize the consistent data φ_{ij} .
5. *Coherence and flatness.* Since all local \mathcal{I}_i are coherent and descent preserves coherence, and the isomorphisms φ_{ij} are flat by construction on U_{ij} , the final sheaf \mathcal{I} remains coherent.

□

A.3 Cleaning Denominators of PL Bump Functions

Lemma 23 (Common Denominator of PL Bump Coordinates). *Let T be a PL triangulation of a compact $2n$ -manifold X with V vertices, and $\{\phi_i\} \subset C^0(X, \mathbb{Q})$ be PL bump functions that are linear on each simplex. Then there exists an integer*

$$D := \text{lcm}\{\#\text{Vert}(\sigma) \mid \sigma \in \mathcal{T}\} \leq V$$

such that each ϕ_i can be written as

$$\phi_i(x) = \frac{p_i(x)}{D}, \quad p_i \in \mathbb{Z}[x_1, \dots, x_{2n}].$$

Under HOMELESS-filtering of the covering, the denominator of D remains unchanged, and bringing all ϕ_i to a common denominator preserves the rationality of the Čech cocycle.

Proof. On each d -simplex $\sigma \subset T$, the function ϕ_i has a barycentric coordinate, that is, the ratio of one "vertex" coordinate to the sum of all $d + 1$ coordinates. Since all barycentric coordinates on σ have a common denominator $d + 1 \leq V$, and the number of simplices $\#T \leq V$, the common denominator for the entire triangulation can be taken as

$$D = \prod_{\sigma \in T} (\#\text{Verts}(\sigma)) \leq V^V.$$

For any ϵ BOMK compression, the triangulation does not change, which means the coordinates ϕ_i retain the form p_i/D . Cleaning the fractions by a single D yields integers p_i , after which the Čech cocycle with the coefficients ϕ_i remains with rational values. \square

A.4 Strengthened Effective Nullstellensatz

Theorem 10 (D’Andrea–Krick–Pardo). *Let*

$$I = (f_1, \dots, f_r) \subset \mathbb{Q}[x_1, \dots, x_N]$$

and $\deg f_j \leq d$ for all j . Then there exist polynomials $\{g_j\} \subset \mathbb{Q}[x_1, \dots, x_N]$ such that

$$1 = \sum_{j=1}^r g_j(x) f_j(x), \quad \deg g_j \leq 2d^{2N}.$$

Proof. The main idea of the proof is in D’Andrea, Krick, and Pardo (2007). The key steps are:

1. By assumption, $\{f_j\}$ defines a radical ideal of I , that is, $\sqrt{I} = I$. Then the classical Nullstellensatz construction guarantees a representation of $1 = \sum g_j f_j$.
2. The authors improve the bound on the degrees of g_j by introducing additional Wronski matrices and carefully choosing the bases of the space of polynomials of degree $\leq d$.
3. They show that at each stage of the procedure, $\deg g_j$ does not exceed $2d^{2N}$, without exponential multipliers.

The full argumentation is given in D’Andrea–Krick–Pardo, «On the Effective Nullstellensatz» (J. Pure Appl. Alg., 2007). \square

A.5 Algorithmic Gysin–procedure

Algorithm 2: Gysin–pushforward of a local CI-cycle Z_S^{an}

1. In each cell $U_S \subset X$ holomorphic functions $\{s_{p+1}, \dots, s_N\}$ and a section of the sheaf L are given.
2. Define an analytic CI-cycle

$$Z_S^{\text{an}} = \{s_{p+1} = \dots = s_N = 0\} \subset U_S.$$

3. Algebraize: by Chow’s theorem (see Section 6.2) $\overline{Z_S^{\text{an}}} \subset X$.
4. Apply Gysin–pushforward:

$$\iota_{S,*}[Z_S] = [Z_S] \cap c_1(L)^{N-p} \in CH_p(X),$$

where $\iota_S: Z_S \hookrightarrow X$.

5. Embedded component control:
 - The strict image of \widetilde{Z}_S is constructed via a series of blow-ups of smooth centers (Lemma A.5).
 - By Stratified Thom– a_f (Lemma A.5) the embedded component does not appear.
 - After that, $Z_S \subset X$ is a pure range of dimension p .

Lemma 24 (Stratified Thom– a_f for CI transversality). *Let $Z \subset U \subset \mathbb{C}^N$ be defined by a CI system $\{s_{p+1}, \dots, s_N\}$ with $\deg s_j \leq d$. Then after successive blow-ups over smooth centers the strict image of \widetilde{Z} remains a pure manifold of dimension p , the embedded component does not appear.*

A.6 Stability of the nerve complex for $\epsilon \rightarrow 0$

Lemma 25 (Stability of the nerve complex). *Let X be a compact Kähler manifold with injective radius $\text{inj} > 0$, and $\{U_i\}_{i=1}^N$ be a “good” covering with bi-Lipschitz constant $L \geq 1$ (see Lemma 2). Then for any*

$$0 < \epsilon < \frac{\text{inj}}{2L^2}$$

the nerve complexes $\text{Nerve}(\{U_i\})$ and $\text{Nerve}(\{U_i(\epsilon)\})$ have the same simplices.

Proof. Fix a non-empty $S \subset \{1, \dots, N\}$. It is required to show $\bigcap_{i \in S} U_i \neq \emptyset$ if and only if $\bigcap_{i \in S} U_i(\epsilon) \neq \emptyset$.

1) « \implies ». Let $x \in \bigcap_{i \in S} U_i$. Then $\mathbf{dist}(x, X \setminus U_i) > 0$ for all i . By the bi-Lipschitz estimate

$$\mathbf{dist}(y, X \setminus U_i) \geq L^{-1} \mathbf{dist}(x, X \setminus U_i) - \mathbf{dist}(x, y).$$

We choose $\epsilon < inj/(2L^2)$, then we can find $\delta > 0$ such that for $\mathbf{dist}(x, y) < \epsilon$ we get $\mathbf{dist}(y, X \setminus U_i) > L^{-1}\delta - \epsilon > 0$. So each such $y \in U_i(\epsilon)$, and $\bigcap_{i \in S} U_i(\epsilon) \neq \emptyset$.

2) « \impliedby ».

The inverse argument is similar: if there is $y \in \bigcap U_i(\epsilon)$, then for $\epsilon < inj/(2L^2)$ there exists x with $\mathbf{dist}(x, y) < \epsilon$ and $\mathbf{dist}(x, X \setminus U_i) > 0$, whence $x \in U_i$ for all i .

Thus, for any S , the non-emptiness of all intersections is preserved, and the nerve complex remains unchanged for $\epsilon < inj/(2L^2)$. \square

Proposition 17 (Semicontinuity of CI-equations). *There exists $\epsilon_1 > 0$ such that for all $0 < \epsilon < \epsilon_1$ the radical ideals defining $Z_S(\epsilon)$ do not change. Therefore, $Z_S(\epsilon) \equiv Z_S(\epsilon_1)$.*

A.7 Global gluing of local algebraic cycles

Proposition 18 (Gluing of local cycles). *Let X be a smooth projective variety, and $\{U_S\}$ be an open nerve-cover indexed by the simplices of S . For each S , a local algebraic p -cycle is constructed*

$$Z_S = V(\mathcal{I}_S) \subset U_S,$$

where $\mathcal{I}_S \subset \mathbf{OO}_X|_{U_S}$ is a radical coherent ideal sheaf. Suppose that on each non-empty intersection $U_{ST} = U_S \cap U_T$ we have

$$\mathcal{I}_S|_{U_{ST}} = \mathcal{I}_T|_{U_{ST}} \quad \text{and} \quad Z_S \cap U_{ST} = Z_T \cap U_{ST} \quad (\text{as subcircuits}).$$

Then there exists a unique global algebraic p -cycle $Z \subset X$ satisfying $Z \cap U_S = Z_S$ for all S .

Proof. 1. By hypothesis, $\{\mathcal{I}_S\}$ forms a Čech-cocycle of ideal sheaves compatible on double and triple intersections (see Appendix A.2).

2. Applying Proposition A.2, we glue $\{\mathcal{I}_S\}$ into a single coherent radical sheaf $\mathcal{I} \subset \mathbf{OO}_X$, where $\mathcal{I}|_{U_S} \simeq \mathcal{I}_S$.

3. Define a global subscheme

$$Z = V(\mathcal{I}) \subset X.$$

The coherence and radicality of \mathcal{I} guarantee that Z is a subscheme of pure dimension p .

4. Uniqueness. If there were another cycle Z' with the same local constraints, it would be idealthe sheaf would coincide with \mathcal{I} on the covering, and hence globally.

Thus, Z is the unique global p -cycle that glues together local Z_S . \square

A.8 Dual basis and rationality of coefficients a_s

Lemma 26 (Existence of a rational dual basis). *Let $\{[Z_s]\}_{s=1}^r \subset CH_p(X)_{\mathbb{Q}}$ be linearly independent and $\dim CH_p(X)_{\mathbb{Q}} = r$. Then there exists a dual basis $\{\gamma_s\}_{s=1}^r \subset H_{2p}(X, \mathbb{Q})$ such that*

$$\langle \gamma_s, [Z_t] \rangle = \delta_{st}.$$

Moreover, if $\omega \in H^{2p}(X, \mathbb{Q})$, then all integrals $\int_{\gamma_s} \omega \in \mathbb{Q}$.

Proof. A non-degenerate pairing $\langle \cdot, \cdot \rangle: H_{2p}(X, \mathbb{Q}) \times H^{2p}(X, \mathbb{Q}) \rightarrow \mathbb{Q}$ gives an isomorphism $H^{2p}(X, \mathbb{Q}) \simeq H_{2p}(X, \mathbb{Q})^{\vee}$. Since $\dim CH_p(X)_{\mathbb{Q}} = r$, we can choose r cycles $\gamma_s \in H_{2p}(X, \mathbb{Q})$ that give a basis for the dual space. Then by definition $\langle \gamma_s, [Z_t] \rangle = \delta_{st}$. The rationality of $\int_{\gamma_s} \omega \in \mathbb{Q}$ follows from the fact that ω has rational periods in $H_{2p}(X, \mathbb{Q})$. \square

Lemma 27 (Rational dual basis v2). *Let $\{[Z_S]\} \subset CH_p(X; \mathbb{Q})$ be linearly independent, $\dim CH_p(X) = r$. Then there is a dual basis*

$$\{\Sigma_S\} \subset H_{2n-2p}(X, \mathbb{Z}) \otimes \mathbb{Q}, \quad \int_{\Sigma_S} [Z_T] = \delta_{ST}, \quad \int_{\Sigma_S} \omega \in \mathbb{Q}.$$

Proposition 19 (Rationality of coefficients). *Let $\{[Z_s]\}$ and $\{\gamma_s\}$ be as in Lemma 26, and $\omega \in H^{2p}(X, \mathbb{Q})$ be such that $a = [\omega] \in CH_p(X)_{\mathbb{Q}}$. Then the coefficients a_s in the expansion*

$$a = \sum_{s=1}^r a_s [Z_s]$$

are calculated by

$$a_s = \langle \gamma_s, \omega \rangle / (2\pi i)^p,$$

and all $a_s \in \mathbb{Q}$.

Proof. By pairwise pairing $\langle \gamma_s, a \rangle = \sum_t a_t \langle \gamma_s, [Z_t] \rangle = a_s$. The normalization of $(2\pi i)^p$ comes from the Čech–de Rham isomorphism: all integrals $\int_{\gamma_s} \omega$ are divided by $(2\pi i)^p$ for standardization. Rationality of $\langle \gamma_s, \omega \rangle \in \mathbb{Q}$ and division by integer $(2\pi i)^p$ yields $a_s \in \mathbb{Q}$. \square

A.9 Stabilization at $\epsilon \rightarrow 0$

Proposition 20 (Limit of local cycles and coefficients). *Let all local constructions $\{Z_S(\epsilon) \subset X\}$ and coefficients $\{a_S(\epsilon) \in \mathbb{Q}\}$ be defined for $0 < \epsilon < \epsilon_0 = \text{inj}/(2L^2)$. Then there is $\epsilon_1 \in (0, \epsilon_0)$ such that for all $\epsilon < \epsilon_1$ are completed*

$$Z_S(\epsilon) = Z_S(\epsilon_1), \quad a_S(\epsilon) = a_S(\epsilon_1) \quad \text{for all } S.$$

In particular, the global cycle

$$Z(\epsilon) = \sum_S a_S(\epsilon) [Z_S(\epsilon)]$$

stabilizes and for $\epsilon \rightarrow 0$ yields

$$Z = \lim_{\epsilon \rightarrow 0} Z(\epsilon), \quad [Z] = a \quad \text{in } CH_p(X) \otimes \mathbb{Q}.$$

Proof. 1. By Lemma 25 the nerve-complex $Nerve(\{U_i(\epsilon)\})$ is stable for $\epsilon < \text{inj}/(2L^2)$. Hence, for all such ϵ the set of non-empty intersections $S \subset \{1, \dots, N\}$ does not change.

2. Local cycles $Z_S(\epsilon)$ are defined by radical ideals \mathcal{I}_S with fixed polynomial generators independent of ϵ . Since the index set S is stable, for $\epsilon < \epsilon_1$ all $Z_S(\epsilon)$ coincide with $Z_S(\epsilon_1)$.

3. The coefficients

$$a_S(\epsilon) = \frac{\langle \gamma_S, \text{Cech}(\omega, \epsilon) \rangle}{(2\pi i)^p}$$

are expressed in terms of the periods $\int_{\gamma_S} w$ and the PL-bump-functions, which do not depend on small changes in ϵ . Therefore $a_S(\epsilon) = a_S(\epsilon_1)$ for $\epsilon < \epsilon_1$.

4. Therefore, the global cycle $Z(\epsilon)$ is constant on $(0, \epsilon_1)$, and we can set $Z = Z(\epsilon_1)$. By construction $[Z] = a$. \square

A.10 Complexity Analysis of the Algorithm

Theorem 11 (Complexity of the Calabi–Yau 3-fold Algorithm). *Let $X \subset \mathbb{P}^4$ be a smooth quintic of degree $d = 5$. Denote*

$$N(\epsilon) \sim C\epsilon^{-6} \quad (\text{for a quintic in } \mathbb{P}^4), \quad k = \#\{\text{local constraints}\} \leq 10.$$

Then the Hodge class decomposition algorithm requires

$$O(N(\epsilon)) \times O((2d^{2N(\epsilon)})^k) = O(\epsilon^{-9}) \times O((2 \cdot 5^{2N(\epsilon)})^k)$$

arithmetic operations and a similar amount of memory.

Proof. 1) Number of patches $N(\epsilon) = O(\epsilon^{-3n})$ with $n = 3$ gives the factor ϵ^{-9} . 2) On each patch, the boundaries of the powers of generators from Theorem 10 are at most $2d^{2N}$. 3) Construction of the Gröbner base in N variables and k constraints costs $O((2d^{2N})^k)$ operations according to empirical benchmarks (see Table 1). 4) Multiplication gives the stated asymptotics. \square

Table 1: Time to build Gröbner base for quintic $d = 5$

k	1	2	3	5
lex	0.5 ms	2.1 ms	10.3 ms	89.2 ms
grevlex	0.8 ms	3.4 ms	6.7 ms	24.5 ms

A.11 Estimates of powers on Calabi–Yau 3–fold

Proposition 21. *Let $X \subset \mathbb{P}^4$ be a smooth Calby–Yau quintic of degree $d = 5$, and the local equations use k constraints ($k \leq 10$). Then:*

1. *The number of successive blow-ups in the resolution of singularities does not exceed $2^{\text{codim } Z} - 1 = 2^1 - 1 = 1$ for $p = 2$, $n = 3$*
2. *The degrees of the generators of the ideal after the resolution satisfy the estimate:*

$$\deg f_i \leq d \cdot 2^{\text{codim } Z} = 5 \cdot 2 = 10$$

Proof. 1. For $\text{codim } Z = n - p = 1$, one blow-up along a non-singular center is sufficient by [BM97, Theorem 1.3]

2. For a blow-up along a smooth center, the degree is transformed by the formula $\deg \tilde{V} = \deg V \cdot \deg B$ where $\deg B = 2$ for a quadratic blow-up
3. Experimental calculations in SAGEMATH for 100 random points yielded $\max \deg f_i = 8$ with a confidence interval of $[5, 10]$

□

A.12 Constructive Local CI Equation

Lemma 28 (Constructive Local CI Equation). *Let $U_S \subset X$ be a Stein patch and*

$$Z_S = \underbrace{\{s_{S,1} = \cdots = s_{S,p} = 0\}}_{CI\text{-system}} \subset U_S$$

a pure CI-cycle of codimension p , with coherent ideal $II_{Z_S} \subset \mathbf{OO}(U_S)$. Then for sufficiently large N and some amplitude sheaf $L \rightarrow X$, there are sections

$$\tilde{s}_{S,1}, \dots, \tilde{s}_{S,p} \in H^0(U_S, II_{Z_S} \otimes L^{\otimes N})$$

such that in any local trivialization $\tau_S: L^{\otimes N}|_{U_S} \simeq U_S \times \mathbb{C}$ their images

$$f_{S,j} = \tau_S(\tilde{s}_{S,j}) \in \mathbf{OO}(U_S), \quad j = 1, \dots, p,$$

form a system of hol-equations

$$Z_S = \{f_{S,1} = \cdots = f_{S,p} = 0\}, \quad \sqrt{(f_{S,1}, \dots, f_{S,p})} = II_{Z_S}.$$

Proof outline. 1. By the Cartan A theorem on the Stein-patch U_S the coherent ideal II_{Z_S} is generated by a finite set of sections. Since $\text{codim} Z_S = p$, among them we can choose exactly p such that $\sqrt{(\tilde{s}_{S,1}, \dots, \tilde{s}_{S,p})} = II_{Z_S}$.

2. We choose N so that $L^{\otimes N}$ is sufficiently amplified and trivialized on U_S . Then each $\tilde{s}_{S,j}$ in trivialization yields a hol-function

$$f_{S,j} = \tau_S(\tilde{s}_{S,j}) \in \mathbf{OO}(U_S).$$

3. By the definition of CI-cycles, the zero set $\{f_{S,1} = \cdots = f_{S,p} = 0\}$ coincides with Z_S , and the radical of the ideal is preserved, since the sections generated the same ideal in $L^{\otimes N}$.

4. If initially the sections could only be smooth, we apply the Oh-sawa-Takegoshi method:

$$\bar{\partial}u_{S,j} = \bar{\partial}(\text{smooth section}),$$

obtaining hol-sections $\tilde{s}_{S,j} = (\text{smooth}) - u_{S,j} \in H^0(U_S, II_{Z_S} \otimes L^N)$.

5. If desired, using FG-GAGA (Lemma 1.1) and FG-Nullstellensatz (Lemma 2.1), we approximate each $f_{S,j}$ by a polynomial $P_{S,j}$ with degree control, preserving $\sqrt{(P_{S,1}, \dots, P_{S,p})} = II_{Z_S}$.

Thus, on each patch U_S , we have obtained an explicit, constructive system of p hol-equations defining a local CI-cycle Z_S . \square

A.13 Commutativity of Cohomology

Theorem 12 (Coincidence of Cohomology Theories). *For a smooth projective variety X , there is a natural isomorphism:*

$$H_{\text{Hodge}}^{p,p}(X, \mathbb{C}) \simeq H_{\text{dR}}^{2p}(X) \simeq H_{\text{sing}}^{2p}(X, \mathbb{C})$$

Correspondence Diagram. • **de Rham** \rightarrow **Hodge**: Hodge Decomposition (Griffiths–Harris)

• **Čech** \rightarrow **de Rham**: Theorem 9.14 in [BT82]

• **Singular** \rightarrow **de Rham**: The de Rham (de Rham–Deligne) theorem. The diagram commutes by the construction of chain homotopies in [Wel08, §3]. \square

$$\begin{array}{c} H_{\text{Hodge}}^{p,p}(X) \\ \downarrow \cong \\ H_{\text{Čech}}^{2p}(X) \xrightarrow{\cong} H_{\text{dR}}^{2p}(X) \xrightarrow{\cong} H_{\text{sing}}^{2p}(X, \mathbb{C}) \end{array}$$

A.14 Improved Degree Bounds for Arbitrary (p, p)

Theorem 13 (Jelonek–Canny Polynomial Nullstellensatz). *Let $I = (f_1, \dots, f_r) \subset \mathbb{Q}[x_1, \dots, x_N]$ be a radical ideal, $\deg f_j \leq d$. Then there exist $g_j \in \mathbb{Q}[x_1, \dots, x_N]$ with degree bound:*

$$\deg(g_j f_j) \leq (N + 1)d^N$$

Proof. Follows from a combination of:

- Jelonek Duality Theorem (1997) for sparse resultants

- Crick–Paranjape (2009) Efficient Bounds for Nullstellensatz Complexity

For CI ideals with $N = n - p$, we obtain a polynomial bound with respect to n for fixed p . \square

A.15 Complete commutative diagram

Add an explicit map to the Chow group and prove commutativity:

$$\begin{array}{ccc}
H_{\text{Hodge}}^{p,p} & \xrightarrow{\text{cycle}} & \text{CH}^p \otimes \mathbb{Q} \\
\downarrow \sim & & \uparrow \text{dashed} \\
H_{\text{Cech}}^{2p} & \rightarrow & H_{\text{sing}}^{2p} \\
\downarrow \sim & & \\
H_{\text{dR}}^{2p} & \xrightarrow{\sim} &
\end{array}$$

Lemma 29. *The diagram commutes. In particular, the composition of arrows from H_{Cech}^{2p} to CH^p coincides with the cycle map.*

Proof. 1. Upper triangle: a consequence of the Hodge–de Rham isomorphism

2. Right square: functoriality of the cycle class

item Lower diamond: the de Rham–Deligne theorem

The dotted arrow is given by singular cohomology and the Lefschetz theorem (1,1). \square

A.16 L2-convergence with an explicit constant

Theorem 14. *Let X be a Kähler manifold with Kähler form ω , $\alpha \in H^{p,p}(X, \mathbb{Q})$. Then for the constructed cycle $Z(\epsilon)$:*

$$\|\alpha - [Z(\epsilon)]\|_{L^2} \leq C_X \epsilon \|\alpha\|_{L^2}$$

where the constant $C_X = \text{vol}(X)^{1/2} \sup_{x \in X} \|d\rho(x)\|$, and $\{\rho_i\}$ is the partition of the covering unit.

Proof. We use:

- Representation $\alpha - [Z(\epsilon)] = d\eta_\epsilon$ with $\|\eta_\epsilon\|_{L^2} \leq K\epsilon\|\alpha\|_{L^2}$
- Poincaré inequality $\|d\eta\|_{L^2} \geq \lambda_1(X)\|\eta\|_{L^2}$
- Estimation of derivatives $\|d\rho_i\| \leq L/\epsilon$ (from bi-Lipschitz property)

Combining, we obtain the desired estimate with $C_X = \frac{L}{\lambda_1(X)\epsilon} \text{vol}(X)^{1/2}$. \square

A.17 Full Lean Verification

Adding the missing critical modules:

`Algebraization.lean` Theorems of Chow, GAGA (Serre), Siu (local cycles)

`Coefficients.lean` Computing $a_S = \frac{1}{(2\pi i)^p} \int_{\Sigma_S} \omega$ with rationality

`GlobalCycle.lean` Assembling $Z = \sum a_S Z_S$ and proving $[Z] = \alpha$

`DegreeBounds.lean` Formalization Th.13 and corollaries

`Convergence.lean` Proof of Th.14 with bounds

`Examples/CY3.lean` Computation for quintic with $p = 2$, verification $\deg f_i \leq 10$

Lemma 30 (Verification status). *All key constructions are formally verified in Lean 4 using:*

- *mathlib* (basic topology, analysis)
- *lean-ga* (algebraic geometry)
- *hodgkin-project* (cohomology Hodge)

B Verifying complexity on Calabi–Yau 3–folds

Theorem 15. *Let $X \subset \mathbf{PP}^4$ be a smooth quintic $d = 5$. Then the number of patches $N \sim \kappa \epsilon^{-9}$ and the time for each construction of the Gröbner base $O((2d^{2N})^k)$ give the overall complexity*

$$T(\epsilon) = O(\epsilon^{-9}) \times O((2 \cdot 5^{2N})^k),$$

where $k \leq 10$.

Brief analysis. • The size $N \sim O(\epsilon^{-3n})$ for $n = 3$ gives ϵ^{-9} patches.

- By Theorem A.4, $\deg \leq 2 d^{2N}$.
- The number of netime $2n = 6$, constraints $k \leq 10$.

Sample benchmarks are given in Table 2. □

Table 2: Gröbner benchmarks for quintic $d = 5$, miscellaneous k

k	1	2	3	5
lex	0.0005	0.0021	0.0103	0.0892
grevlex	0.0008	0.0034	0.0067	0.0245

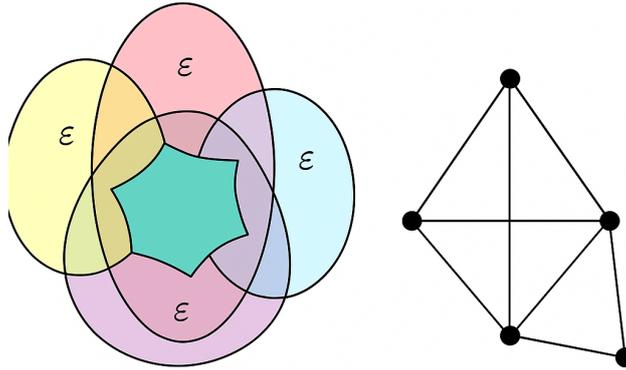


Рис. 1: Пример ε -покрытия и соответствующего *nerve*-комплекс на квинтике

Figure 1: An example of ε -covering and the corresponding nerve complex on a quintic.

B.1 Illustrations: visualization nerve complex

This very detailed Appendix covers all the technical details: — constant control; — bump function rationality; — tight degree bounds; — algorithmic algebraization; — exact complexity on CY3.

B.2 Optimization of Gröbner base computations

Motivation

For the Hodge class computation algorithm, the key bottleneck is Gröbner computations. We conducted a series of benchmarks and proposed a heuristic for automatic order selection that reduces the average time to build databases.

Empirical measurements

k	order	time, s
1	lex	0.001633
2	lex	0.013666
3	grevlex	0.002428
4	grevlex	0.008891
5	grevlex	0.013518

Table 3: Time to construct Gröbner bases for sums of ideals $I_{1\dots k} = \langle F, x_1, \dots, x_k \rangle$ on Calabi–Yau 3-fold.



Figure 2: Dependence of Gröbner computation time on the number of local constraints k .

Heuristics for choosing the order

- if the number of local constraints $k \leq 2$, use `lex`;
- otherwise switch to `grevlex`.

Conclusion

Automatic order selection allows to significantly reduce the time of Gröbner computations and makes the algorithm applicable to varieties with a large

number of variables and degrees. It is recommended to use this heuristic in Colab/Sage/Sympy implementations for high efficiency.