Toward a Reverse Foundation Axiom:
Implicit Upward Unboundedness in the Von Neumann Ordinals

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Abstract

The standard Foundation Axiom of ZFC forbids infinite descending membership chains, ensuring that all sets are well-founded in a bottom-up sense. However, we argue that this condition is not sufficient to prevent problematic constructions. In particular, we show that the structure of the von Neumann ordinals permits internal constructions of unbounded membership height, and we propose a converse principle—a kind of "reverse foundation"—to rule out such behavior. Our critique focuses on ω_2, whose construction involves a transfinite accumulation of nested sets, each introducing a new local "top level" of membership. We introduce three core principles that make these issues apparent: Extensional Parsing, Length/Height Duality, and Local Top-Level Saturation.

1 Introduction

The von Neumann ordinals are defined recursively in ZFC as sets of all smaller ordinals:

$$\alpha = \{\beta \mid \beta < \alpha\}.$$

This elegant construction supports ordinal arithmetic, transfinite recursion, and the cumulative hierarchy. However, its structural implications are not always acknowledged.

We argue that although the Axiom of Foundation rules out infinite *descending* $\in$-chains, the theory admits transfinite *ascending* $\in$-chains implicitly. These arise in the form of unbounded brace-height, or internal membership structure, within large ordinals such as ω_2 [1].

2 The Case of ω_2

The ordinal ω_2 is defined as the set of all ordinals of cardinality less than $\aleph_2$:

$$\omega_2 = \{\alpha \mid \alpha < \omega_2\}.$$

In particular, it contains ω_1, $\omega_1 + 1$, $\omega_1 + 2$, and so on—a collection of ordinals that form a long sequence in the order topology. But under the von Neumann definition, each of these ordinals is a set of smaller ordinals, leading to deeply nested membership structures.

Example

The sequence

$$\omega + 1 \in \omega \cdot 2 + 1 \in \omega \cdot 3 + 1 \in \cdots$$

forms a countably infinite ascending $\in$-chain. While ZFC permits such a chain as an *external sequence*, it forbids sets containing them internally, as that would violate an implicit principle we will call Extensional Parsing below. Yet ω_2 contains sets with arbitrarily high brace structure, due to its role as the collection of all ordinals $< \omega_2$.

3 Three Guiding Principles

3.1 1. Extensional Parsing

A set is nothing more than its members. So, to "parse" the structure of a set means to read its top-level elements as a list of entities it directly contains. Every set, in this view, *is* its top level.

But every element of a set is itself a set. Thus, each member introduces its own top level, leading to recursive top-level structure nested arbitrarily deep.

3.2 2. Length/Height Duality

This principle can be phrased as:

> *"You want it long, you get it tall."*

To build a long sequence of ordinals—such as all those less than ω_2—one must accumulate nested sets. The ordinal ordering $<$ mirrors the $\in$ relation structurally, meaning a long order-type demands a tall membership tree. [2]

This duality means that internal membership height is the cost of ordinal length.

3.3 3. Local Top-Level Saturation

Every element of a set is a set, with its own members. That means every element introduces a new top level. So, any set with many elements also contains many new top levels. There is no global "top"; instead, the entire hierarchy is saturated with local summits.

As ChatGPT summarized:

> *"Once you admit that being a top level just is membership, and that every ordinal is a set of ordinals, you've guaranteed the possibility of internal upward unboundedness."*

4 Toward a Reverse Foundation Axiom

We propose that Foundation alone is insufficient to prevent pathological set structures. To remedy this, we introduce an informal converse:

> **Reverse Foundation (informal).** Every ascending $\in$-chain must terminate in finite time.

That is, for any set x and any sequence $x_0 \in x_1 \in x_2 \in \cdots$ with $x_0 \in x$, there exists n such that x_n is not a member of any set. Or more weakly, that the length of such chains is bounded below some fixed ordinal.

This principle would eliminate objects like ω_2 as legitimate sets, not because they are ill-founded in the classical sense, but because they embed unbounded internal tallness—structures with no final top level.

5 Conclusion

The standard ZFC axioms, particularly Foundation, fail to address the dangers of implicit upward membership growth. The von Neumann ordinals, though foundationally canonical, enable the construction of sets whose internal membership structure is unbounded in height. This critique follows directly from ZFC's own definitions and from natural semantic principles:

- **Extensional Parsing:** Membership *is* top-level containment.
- **Length/Height Duality:** Long ordinal chains imply tall brace nesting.
- **Local Top-Level Saturation:** Every level is a top level within some deeper structure.

To prevent the formation of objects whose legitimacy as sets is questionable due to infinite internal ascent, we advocate for consideration of a "Reverse Foundation" principle.

References

[1] David L. Selke. *A Pathology of the von Neumann Ordinals.* viXra preprint, 2024. `https://vixra.org/abs/2410.0123`

[2] David L. Selke and ChatGPT. *Brace Height and Ordinal Length: A Duality in the Von Neumann Hierarchy.* AI.viXra preprint, 2025. `https://ai.vixra.org/abs/2506.0112`