On the Independence of the Generalized Goldbach Conjecture in Infinite Commutative Rings with Identity

Peter MacDonald Phillips
peterphil@msn.com

September 20, 2025

Abstract

We investigate the Generalized Goldbach Conjecture (GGC) in the context of Infinite Commutative Rings with Identity (ICRI). The conjecture states that every nonzero element in the even ideal, defined as the ideal generated by the sum of two units, can be expressed as the sum of two irreducible elements. If GGC is true in the ring of integers $\mathbb{Z}$, we show that it fails in the product ring $\mathbb{Z} \times \mathbb{Z}$, an infinite commutative ring with identity. This demonstrates that if GGC is true it is independent of the axioms of ICRI.

1 Introduction

The classical Goldbach conjecture asserts that every even integer greater than two is the sum of two primes. Generalizing this idea to algebraic structures, we consider whether elements in a suitable "even ideal" of a ring can be expressed as the sum of two irreducibles. In this paper, we formalize such a conjecture within the framework of infinite commutative rings with identity and examine its logical status. We provide a counterexample in $\mathbb{Z} \times \mathbb{Z}$, demonstrating that if GGC is true the conjecture is not derivable from the axioms of ICRI alone.

2 Definitions and Preliminaries

Let R be a commutative ring with identity.

- Let R^* denote the set of units in R.
- Define the even ideal $R_e = \langle \{u + v \mid u, v \in R^*\} \rangle$, the ideal generated by all sums of pairs of units in R.
- Let $\mathrm{Irr}(R)$ denote the set of irreducible elements in R.

Generalized Goldbach Conjecture (GGC): For every nonzero $a \in R_e$, there exist irreducibles $x, y \in R$ such that $a = x + y$.

3 Failure of GGC in Product Ring $\mathbb{Z} \times \mathbb{Z}$

Let $Z_2 = \mathbb{Z} \times \mathbb{Z}$. This is a commutative ring with identity $(1, 1)$. The units of Z_2 are all pairs $(\pm 1, \pm 1)$, and the even ideal $Z_2^e = (Z_2^* + Z_2^*)$ consists of all pairs $(2a, 2b)$ where $a, b \in \mathbb{Z}$.

We define irreducibles in Z_2 as follows: $(a, b) \in Z_2$ is irreducible if and only if one component is a prime in $\mathbb{Z}$ and the other is a unit, i.e., $(\pm p, \pm 1)$ or $(\pm 1, \pm p)$.

Let $z = (2m, 2n) \in Z_2^e$ with $m, n \geq 4$. Then z cannot be expressed as the sum of two irreducibles in Z_2.

Proof. Suppose $z = x + y$ with $x, y \in \mathrm{Irr}(Z_2)$. Each irreducible in Z_2 is of the form $(\pm p, \pm 1)$ or $(\pm 1, \pm p)$ for some prime p. Thus, possible sums fall into one of the following cases:

1. $(\pm p_1, \pm 1) + (\pm p_2, \pm 1) = (\pm p_1 \pm p_2,\ a)$, where $a \in \{-2, 0, 2\}$.
2. $(\pm p_1, \pm 1) + (\pm 1, \pm p_2) = (\pm p_1 \pm 1,\ \pm 1 \pm p_2)$.
3. $(\pm 1, \pm p_1) + (\pm p_2, \pm 1) = (\pm 1 \pm p_2,\ \pm p_1 \pm 1)$.
4. $(\pm 1, \pm p_1) + (\pm 1, \pm p_2) = (a,\ \pm p_1 \pm p_2)$, where $a \in \{-2, 0, 2\}$.

Cases (2) and (3) can indeed yield elements of Z_2^e that are sums of two irreducibles, so they do not furnish counterexamples. We therefore restrict attention to cases (1) and (4).

Case (1). Here we have $z = (2m, 2n) = (\pm p_1 \pm p_2,\ a)$ with $a \in \{-2, 0, 2\}$. This implies $2n = a$, which forces $2n \in \{-2, 0, 2\}$, contradicting the assumption $n \geq 4$.

Case (4). Here we have $z = (2m, 2n) = (a,\ \pm p_1 \pm p_2)$ with $a \in \{-2, 0, 2\}$. This implies $2m = a$, so $2m \in \{-2, 0, 2\}$, contradicting the assumption $m \geq 4$.

Since both remaining cases lead to contradictions, no element $(2m, 2n)$ with $m, n \geq 4$ can be expressed as the sum of two irreducibles in Z_2. □

Thus, an infinite family of counterexamples exists: for example, $(2k, 2k) \in Z_2^e$ with $k \geq 4$ is never a sum of two irreducibles.

4 Conclusion

If GGC is true in $\mathbb{Z}$ but fails in $\mathbb{Z} \times \mathbb{Z}$, and both are models of ICRI, we conclude that GGC is independent of the axioms of infinite commutative rings with identity. This result opens further exploration into the model-theoretic boundaries of number-theoretic conjectures in algebraic structures.

Acknowledgments

The author thanks the mathematical community for its open support of independent research and invites discussion and collaboration.

References

[1] Serge Lang, *Algebra*, Addison–Wesley, Reading, MA, 1965.

[2] Stephen C. Kleene, *Mathematical Logic*, John Wiley & Sons, New York, 1967.

[3] David M. Burton, *Introduction to Modern Abstract Algebra*, Addison–Wesley Publishing Co., Reading, MA, 1967.