

On the Independence of the Generalized Goldbach Conjecture in Infinite Commutative Rings with Identity

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July 4, 2025

Abstract

We investigate the Generalized Goldbach Conjecture (GGC) in the context of Infinite Commutative Rings with Identity (ICRI). The conjecture states that every nonzero element in the even ideal, defined as the ideal generated by the sum of two units, can be expressed as the sum of two irreducible elements. We show that while GGC holds in the ring of integers \mathbb{Z} , it fails in the product ring $\mathbb{Z} \times \mathbb{Z}$, an infinite commutative ring with identity. This demonstrates that GGC is independent of the axioms of ICRI.

1 Introduction

The classical Goldbach conjecture asserts that every even integer greater than two is the sum of two primes. Generalizing this idea to algebraic structures, we consider whether elements in a suitable “even ideal” of a ring can be expressed as the sum of two irreducibles. In this paper, we formalize such a conjecture within the framework of infinite commutative rings with identity and examine its logical status. We provide a counterexample in $\mathbb{Z} \times \mathbb{Z}$, demonstrating that the conjecture is not derivable from the axioms of ICRI alone.

2 Definitions and Preliminaries

Let R be a commutative ring with identity.

- Let R^* denote the set of units in R .
- Define the even ideal $R_e = (R^* + R^*)$, the ideal generated by sums of two units.
- Let $\text{Irr}(R)$ denote the set of irreducible elements in R .

Generalized Goldbach Conjecture (GGC): For every nonzero $a \in R_e$, there exist irreducibles $x, y \in R$ such that $a = x + y$.

3 Failure of GGC in Product Ring $\mathbb{Z} \times \mathbb{Z}$

Let $Z_2 = \mathbb{Z} \times \mathbb{Z}$. This is a commutative ring with identity $(1, 1)$. The units of Z_2 are all pairs $(\pm 1, \pm 1)$, and the even ideal $Z_2^e = (Z_2^* + Z_2^*)$ consists of all pairs $(2a, 2b)$ where $a, b \in \mathbb{Z}$.

To identify irreducibles, define the norm map $N : Z_2 \rightarrow \mathbb{Z}$ by $N(a, b) = ab$. This homomorphism is multiplicative: $N((a, b)(c, d)) = N(a, b)N(c, d)$.

We define irreducibles in Z_2 as follows: $(a, b) \in Z_2$ is irreducible if and only if one component is a prime in \mathbb{Z} and the other is a unit, i.e., $(\pm p, \pm 1)$ or $(\pm 1, \pm p)$.

Counterexample: Let $z = (26, 26) \in Z_2^e$. Suppose $z = x + y$, where $x, y \in \text{Irr}(Z_2)$. Then the only possible irreducible pairs are of the forms $(a, \pm 1)$ and (c, d) . For $x + y = (26, 26)$, this implies:

$$\begin{aligned} a + c &= 26 \\ \pm 1 + d &= 26 \Rightarrow d = 25 \text{ or } 27 \end{aligned}$$

But $d = 25, 27 \notin \{\pm 1, \pm p\}$, so (c, d) is not irreducible. Thus, no such pair exists. Hence, $(26, 26) \in Z_2^e$ but is not the sum of two irreducibles.

4 Conclusion

Since GGC holds in \mathbb{Z} but fails in $\mathbb{Z} \times \mathbb{Z}$, and both are models of ICRI, we conclude that GGC is independent of the axioms of infinite commutative rings with identity. This result opens further exploration into the model-theoretic boundaries of number-theoretic conjectures in algebraic structures.

Acknowledgments

The author thanks the mathematical community for its open support of independent research and invites discussion and collaboration.