Telic Stratification and Classifying Logoi I: Bordism, π-Finitude, and Internal Cohesion

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Abstract

This paper undertakes a sequential analysis of five foundational contributions to the intersection of algebraic topology, higher category theory, and constructive type theory. Beginning with the structure of Dyer–Lashof operations in bordism and extending through the formal coactions underlying the Nishida relations, the category of π-finite spaces, the architecture of the effective 2-topos, and finally the internal groupoid semantics of Martin-Löf type theory, we trace a coherent development of stratified constructive structure. Each work articulates a distinct dimension of what we term *telic stratification*—the generation of mathematical types, spaces, and operations through staged, internally coherent constructions. We interpret the categorical universes that arise in these theories as *classifying logoi*: higher-categorical contexts that organize and reflect such stratification. An extended appendix develops this synthesis, proposing a unifying framework in which homotopy, computability, and internal logic are not merely compatible but mutually reinforcing. The resulting perspective suggests a foundational program grounded in the operational semantics of higher categories and constructive homotopy theory.

Introduction

The confluence of algebraic topology, type theory, and higher category theory has in recent years produced a powerful reorientation in the foundations of mathematics. Constructive and homotopy-theoretic ideas have merged into new frameworks that reject the reliance on classical axioms such as choice or excluded middle, while preserving deep structural coherence through internal groupoids, univalent universes, and higher topos semantics.

This paper provides a sequential analysis of five contributions that exemplify and expand this transformation:

(I) **Bisson & Joyal (1995a)** — establish the algebra of Dyer–Lashof operations in unoriented bordism, framing them as a system of polynomial functors and introducing the concept of a *D-ring*.

(II) **Bisson & Joyal (1995b)** — develop the Nishida relations connecting Dyer–Lashof and Landweber–Novikov operations, constructing a diagrammatic correspondence between cobordism and homology via dual Hopf algebra coactions.

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(III) **Anel (2025)** — defines the category of π-finite spaces as a coherent, compact subcategory of the ∞-category of spaces, characterizing it as an elementary pretopos with univalent structure and local cartesian closure.

(IV) **Awodey & Emmenegger (2025)** — propose a construction of the *effective 2-topos* by extending the exact completion paradigm to coherent groupoids internal to a realizability base, developing a constructive model structure for presheaves of groupoids.

(V) **Hughes (2025)** — presents an algebraic internal groupoid model of Martin-Löf type theory within a cartesian closed lextensive base category, formalizing a type-theoretic algebraic weak factorization system that preserves internal substitution and identity types.

Read sequentially, these works portray a vision of constructive mathematics where topology, logic, and category theory are not merely adjacent but interdependent. The Dyer–Lashof and Landweber–Novikov operations encode combinatorial symmetries of covering spaces; the π-finite and coherent groupoid frameworks classify internal types; and algebraic models of type theory anchor this machinery within robust syntactic foundations.

In the appendix, we propose a synthesis: a stratified and constructive interpretation of homotopy type theory grounded in the interplay of these structures. Through this lens, we approach a vision of mathematics not only as invariant under homotopy, but also as fundamentally organized by computability, coherence, and internal symmetry.

1 Dyer–Lashof Operations in Bordism: Polynomial Functors and the Genesis of D-Rings

1.1 Overview

The foundational contribution of Bisson and Joyal's first 1995 note, *The Dyer–Lashof Algebra in Bordism*, lies in the systematic construction of an algebraic framework for Dyer–Lashof operations on unoriented bordism. By reinterpreting finite coverings of manifolds as *polynomial functors* on the category of topological spaces, they open a path for translating classical geometric structures into purely algebraic data—reconfiguring the bordism ring $N_*(X)$ as a kind of noncommutative differential algebra.

This section explores their innovative approach: defining operations through structured families of coverings, generalizing squaring via 2-fold covers, and formalizing a new algebraic object—the *D-ring*—which serves as the bordism-theoretic analogue of a ring with Dyer–Lashof operations. These insights not only enrich the homotopical structure of bordism but prefigure later developments in synthetic homotopy theory and constructive models of type theory.

1.2 Covering Spaces as Polynomial Functors

At the heart of the paper is the insight that any finite covering $p\colon T \to B$ defines a *polynomial functor* $p(-)$ from spaces to spaces:

$$p(X) = \left\{(u, b) \;\middle|\; b \in B,\ u\colon p^{-1}(b) \to X\right\},$$

which assembles into a bundle over B with fibers $X^{p^{-1}(b)}$.

These functors behave like polynomials under disjoint union, product, and composition:

$$(p+q)(X) = p(X) + q(X),$$
$$(p \times q)(X) = p(X) \times q(X),$$
$$(p \circ q)(X) = p\big(q(X)\big),$$

mirroring the structure of species and analytic functors. Differentiation is defined combinatorially via deletion of a point in the fiber—thus enabling the application of differential identities to cobordism classes.

1.3 Bordism as a Free D-Ring

The primary algebraic structure introduced is that of a *D-ring*—a commutative ring R equipped with:

- A formal group law $F(x, y)$ of order two over R, satisfying $F(x, x) = 0$.
- A *total squaring operation* $D_t \colon R \to R[[t]]$ satisfying:

$$D_0(a) = a^2,$$
$$D_t(F(x, y)) = F_t(D_t(x), D_t(y)),$$
$$D_t \circ D_s = D_s \circ D_t \quad \text{up to specified correction via isogenies.}$$

This structure captures the interaction of squaring operations with formal group laws, allowing one to encode the geometry of coverings algebraically. Remarkably, Bisson and Joyal show that for any E_∞-space X, the bordism ring $N_*(X)$ is not just a ring, but a canonical D-ring. Moreover, the bordism ring of finite covering manifolds, $N_*(\Sigma_*)$, is the free D-ring on one generator—analogous to the free loop space in stable homotopy.

1.4 Squaring and the Algebra of Coverings

The authors identify a special sequence of classes $d_n \in N_*(\mathrm{RP}^\infty)$ corresponding to double coverings:

$$D_t(x) = \sum_{n \geq 0} d_n(x) t^n,$$

which satisfy Cartan-type and generalized Adem relations. They define:

$$D_t(F)(x(x+t)) = F(x)F(F(x, t)),$$

showing compatibility with Lubin's isogeny framework for formal group laws. This illustrates that the algebra of Dyer–Lashof operations in bordism does not merely mimic classical homology structures—it reconstructs them algebraically in a way compatible with unstable operations and higher coherence data.

The D-ring formalism generalizes known properties of Dyer–Lashof operations in homology and provides a structured language to articulate them within the cobordism ring. This opens the door to a fully algebraic treatment of operations that are often introduced geometrically or computationally in stable homotopy theory.

1.5 Constructive Foundations and Combinatorial Species

Bisson and Joyal's approach is notably constructive. Avoiding classical axioms (e.g., choice), they define operations in terms of explicitly computable manipulations of finite sets, coverings, and partitions. Their use of the Euler characteristic as a valuation on covering spaces further connects their perspective to combinatorics, via Joyal's earlier work on species and analytic functors.

In particular, the differential calculus of coverings—with divided difference operators, symmetric group actions, and power operations—acts as a bridge between algebraic topology and the categorical theory of computation. This combinatorial algebraic structure is what later models of homotopy type theory, particularly those involving polynomial functors and univalent universes, will exploit.

1.6 Position Within the Broader Framework

This paper serves as a conceptual and technical seed for the rest of our analysis. It introduces:

- The algebraic encoding of geometric structures through polynomial and differential operations.
- A constructive approach to operations in bordism, suitable for categorical interpretation.
- The D-ring structure as a foundational algebraic scaffold capable of capturing higher homotopical data.

In the broader trajectory of this study, the D-ring and its associated functorial machinery will reappear in several guises: as Hopf algebra coactions (in the next section), as -finiteness (in Anel), as coherent groupoids (in Awodey–Emmenegger), and as algebraic weak factorization systems (in Hughes). Each builds on the insight that operations, rather than points, are the primary carriers of homotopical meaning.

2 Nishida Relations and Coactions in Bordism and Homology

2.1 Overview

The second of Bisson and Joyal's 1995 comptes rendus, *Nishida Relations in Bordism and Homology*, expands upon the D-ring formalism introduced previously by revealing deep structural symmetries between bordism and homology. Specifically, they identify and formalize the algebraic correspondences—known as *Nishida relations*—linking Dyer–Lashof operations in bordism with Steenrod and Landweber–Novikov operations in homology.

The critical innovation here is the use of coactions by dual Hopf algebras, A^* (Milnor) and B^* (Faà di Bruno), to express compatibility between operations on bordism and homology. These structures not only allow for the expression of unstable cohomology operations algebraically, but also enable a categorical bridge between the formal geometry of power operations and the computational apparatus of E_∞-spaces.

2.2 Hopf Algebras and Coaction Structures

The paper begins by recalling that:

- The *Milnor Hopf algebra* $A^* = \mathbb{Z}_2[\xi_0^{\pm 1}, \xi_1, \xi_2, \dots]$ is dual to the Steenrod algebra.

- The *Faà di Bruno Hopf algebra* $B^* = \mathbb{Z}_2[h_0^{\pm 1}, h_1, h_2, \ldots]$ encodes formal diffeomorphisms under composition.

Each of these is endowed with a natural coproduct determined by composition of power series:

$$\delta(\xi)(x) = \xi(x) \otimes \xi(x), \quad \delta(h)(x) = h(x) \otimes h(x).$$

These algebras act on homology and bordism via coactions $\alpha\colon H_*(X) \to A^* \otimes H_*(X)$ and $\phi\colon N_*(X) \to B^* \otimes N_*(X)$ respectively.

These coactions respect the algebraic structure of operations—particularly the Dyer–Lashof operations and the formal group law on bordism—enabling the authors to formulate and prove the commutativity of diagrams representing the Nishida relations.

2.3 The Nishida Diagram and Q- and D-Structures

To algebraically encode the compatibility between homological and bordism operations, Bisson and Joyal define:

- A *Q-ring*: a $\mathbb{Z}_2$-algebra with Dyer–Lashof operations satisfying squaring, Cartan, and Adem relations. This governs homology operations and is naturally acted upon by A^*.
- A *D-ring*: as before, a $\mathbb{Z}_2$-algebra equipped with total square operations and a formal group law of order two, acted on by B^*.

They demonstrate that, for any E_∞-space X, the Thom reduction map

$$\varepsilon\colon N_*(X) \longrightarrow H_*(X)$$

respects this additional structure: it defines a monoidal natural transformation between D-rings and Q-rings.

This leads to the *Nishida diagram*:

$$\begin{array}{ccccccc} N_*(X) & \xrightarrow{D_t} & N_*(X)[[t]] & & H_*(X) & \xrightarrow{Q_t} & H_*(X)[[t]] \\ \downarrow \phi & & \downarrow \phi & \rightsquigarrow & \downarrow \alpha & & \downarrow \alpha \\ B^* \otimes N_*(X) & \xrightarrow{D_t} & B^* \otimes N_*(X)[[t]] & & A^* \otimes H_*(X) & \xrightarrow{Q_t} & A^* \otimes H_*(X)[[t]] \end{array}$$

which commutes in both the homology and bordism settings, formalizing the compatibility of operations across the two worlds.

2.4 Monadic Equivalence and Thom Reduction

One of the most profound results of the paper is the equivalence of categories:

$$[B^* N_*]^{\mathcal{D}} \;\simeq\; [A^*]^{\mathcal{Q}},$$

where the left-hand side consists of D-rings with Landweber–Novikov coactions and the right-hand side consists of Q-rings with Milnor coactions.

The Thom reduction functor

$$T\colon [B^* N_*] \to [A^*], \quad T(M) := M \otimes_{N_*} \mathbb{Z}_2$$

induces an equivalence of structured categories, preserving both the algebraic operations and the coaction structures.

This equivalence reveals that the difference between bordism and homology is, in a precise sense, one of scalar base—N_* versus $\mathbb{Z}_2$—and not of algebraic form. The D- and Q-ring structures are monadic in nature, and this functorial bridge lifts the geometric insights of Thom and Quillen into a purely algebraic and categorical setting.

2.5 Characteristic Numbers as Functorial Substitution

The paper concludes by examining how characteristic numbers—cobordism invariants of manifolds—transform under covering operations. For a covering p and a bordism class $[M]$, they establish:

$$\beta(p(M)) = \beta(p) \circ \beta(M),$$

where the right-hand side denotes substitution in the D-ring of operations. This composition rule turns the bordism ring of covering manifolds into an operad-like object: operations compose by substitution, a structure that mirrors the composition of polynomial functors or analytic species.

Thus, D-ring operations can be understood as characteristic-function-valued symmetries acting on manifolds via power series expansion—an insight deeply connected to the modern understanding of structured ring spectra and higher categories of operations.

2.6 Toward a Categorical Bordism Semantics

Taken as a whole, this work offers a blueprint for a semantic understanding of bordism operations within type theory and higher category theory:

- The categorical equivalence between D- and Q-actions suggests a duality of models.
- The operadic substitution rules invite reinterpretation in terms of internal type formers and dependent structure.
- The compatibility diagrams are formal analogues of coherence conditions in monoidal or higher categories.

In the context of this paper, this analysis completes the algebraic foundations needed to bridge classical bordism theory with the emerging theories of coherent groupoids, univalent universes, and effective higher topoi. These structures will recur—abstracted and internalized—in the next section, as we turn to the ∞-pretopos of π-finite spaces.

3 The Category of π-Finite Spaces: Compactness, Truncation, and Univalence

3.1 Overview

In *The Category of π-Finite Spaces*, Mathieu Anel develops a rigorous theory of π-finite spaces—homotopy types with finite homotopy invariants—within the language of higher category theory and ∞-topoi. The paper offers a constructive, internal perspective on the structure of truncated, compact spaces and demonstrates that their categorical behavior reflects many of the key properties one expects from a foundational universe for homotopy type theory.

Among its major contributions are:

- A classification of π-finite spaces as the realization of Kan complexes valued in finite sets.
- Proof that the full subcategory $\mathcal{S}_\pi \subset \mathcal{S}$ of such spaces forms a locally cartesian closed, elementary ∞-pretopos.
- Construction of a universe object U_π for $\mathcal{S}_\pi$ in $\mathcal{S}$, supporting univalence and internal identity types.

- The initiality of $\mathcal{S}_\pi$ as the smallest Boolean $\Pi\Omega$-pretopos with a univalent subobject classifier.

This framework allows the machinery of type theory—identity types, dependent sums and products, truncations, and univalent universes—to be interpreted entirely within a compact, coherent subcategory of spaces. The result is a model of synthetic homotopy theory that is both foundational and computationally minimal.

3.2 Definition and Structure of $\mathcal{S}_\pi$

A space X is said to be π-finite if it is truncated and has finitely many connected components, each with finite homotopy groups:

$$\pi_n(X, x) \text{ finite for all } n \geq 0.$$

Anel characterizes these spaces constructively via Kan complexes with finite values:

$$X_\bullet \in \mathrm{Kan}(\mathrm{FinSet}) \quad \Rightarrow \quad |X_\bullet| \in \mathcal{S}_\pi.$$

He further proves that $\mathcal{S}_\pi$ is closed under finite limits, finite coproducts, and effective quotients of Segal groupoids. These properties endow it with the structure of an *elementary ∞-pretopos*, analogous to the category of finite sets within classical topos theory.

3.3 Local Cartesian Closure and Postnikov Truncations

A central technical result of the paper is that $\mathcal{S}_\pi$ is not just cartesian closed, but *locally* cartesian closed:

$$\forall X \in \mathcal{S}_\pi, \quad \mathcal{S}_\pi/X \text{ is cartesian closed.}$$

This is proved by expressing any π-finite space as a colimit of a coskeletal Kan complex with finite values and leveraging the fact that fiberwise exponentials preserve π-finiteness in this setting.

Anel also shows that $\mathcal{S}_\pi$ is closed under *Postnikov truncations*:

$$X \in \mathcal{S}_\pi \quad \Rightarrow \quad \tau_{\leq n}(X) \in \mathcal{S}_\pi.$$

This property enables an internal stratification of the universe into homotopical levels, supporting inductive reasoning within type-theoretic interpretations.

3.4 Comparison with Cell-Finite Spaces

A key conceptual move in the paper is distinguishing π-finite spaces from the more traditional class of *cell-finite* spaces (i.e., finite CW complexes). While both classes have finitely generated homotopy types, they differ in categorical closure:

Property	Cell-Finite ($\mathcal{S}_{\mathrm{fin}}$)	π-Finite ($\mathcal{S}_\pi$)
Closed under finite limits	No	Yes
Closed under loop spaces	No	Yes
Closed under truncation	No	Yes
Supports internal identity types	No	Yes
Locally cartesian closed	No	Yes

This table reflects a deeper philosophical shift: $\mathcal{S}_\pi$ is better behaved categorically and homotopically for internal modeling of type theory. It is not simply a subcategory of finite spaces—it is a structurally disciplined foundation suitable for univalent semantics.

3.5 Univalence and the Universe U_π

Perhaps the most striking achievement of the paper is the construction of a universe U_π within $\mathcal{S}$ that classifies π-finite families and satisfies univalence.

The universe is constructed as:

$$U_\pi := \coprod_{[F]\in\Sigma} B\mathrm{Aut}(F), \quad U'_\pi := \coprod_{[F]\in\Sigma} F/\mathrm{Aut}(F),$$

where Σ is a (countable) set of representatives of π-finite spaces up to isomorphism. The canonical map $U'_\pi \to U_\pi$ is shown to be univalent, and any π-finite map arises as a pullback of it.

Importantly, this universe is:

- Closed under identity types (diagonals),
- Closed under dependent sums and products,
- Effective in classifying internal types.

This means that $\mathcal{S}_\pi$ can serve as a synthetic model of homotopy type theory with univalence—precisely the kind of universe envisioned in Voevodsky's foundational program.

3.6 Initiality and Categorical Foundations

Anel proves that $\mathcal{S}_\pi$ is the initial Boolean $\Pi\Omega$-pretopos with univalent subobject classifier. That is, any other such category must receive a unique finite-limit-preserving functor from $\mathcal{S}_\pi$.

This theorem mirrors classical results about the category of finite sets FinSet as the initial pretopos, positioning $\mathcal{S}_\pi$ as its higher-categorical analogue. Just as FinSet underlies traditional set-theoretic foundations, $\mathcal{S}_\pi$ emerges as the minimal foundation for stratified, coherent, constructive type theory.

3.7 Structural Affinities with D-Rings and Coactions

The algebraic and categorical structures of $\mathcal{S}_\pi$ resonate strongly with the D- and Q-ring formalisms of Bisson and Joyal. In particular:

- The use of finite truncation and Postnikov towers recalls the stratified power operations in bordism.
- The universe U_π and its classifying property resemble the role of formal group laws in determining natural transformations of operations.
- The internal groupoid structure of types over U_π echoes the Hopf coaction formalism and the use of classifying stacks like $B\mathrm{Aut}(F)$.

Whereas Bisson and Joyal framed these ideas algebraically, Anel's approach embeds them within a coherent homotopy-theoretic semantics. The next sections will explore how this structure generalizes further, into the 2-categorical and internal syntactic levels introduced by Awodey, Emmenegger, and Hughes.

4 Toward the Effective 2-Topos: Realizability, Coherent Groupoids, and Higher Exact Completion

4.1 Overview

In their 2025 paper *Toward the Effective 2-Topos*, Steve Awodey and Jacopo Emmenegger investigate a higher-dimensional generalization of Hyland's effective topos **Eff**. Their goal is to construct a realizability-inspired model of higher type theory: an *effective 2-topos* $\mathbf{Eff}_2$—a non-Grothendieck $(2,1)$-topos built constructively via exact completion from the category of partitioned assemblies.

This work unfolds a novel categorical architecture:

- The category of coherent groupoids $\mathrm{CohGpd}(\widehat{\mathcal{P}})$ is proposed as the category of fibrant objects in a model structure for internal groupoids.
- The 0-types of this model recover the classical effective topos **Eff**.
- A homotopy-theoretic model structure on internal groupoids (after Joyal–Tierney) supports univalence and HoTT-like semantics in $\mathbf{Eff}_2$.

The result is a layered construction in which computability, coherence, and homotopy-theoretic semantics are internalized into a unified framework—effectively reconstructing synthetic type theory within a higher categorical logic grounded in realizability.

4.2 From Partitioned Assemblies to the Effective Topos

The classical effective topos **Eff** is known to be the exact completion of the category $\mathcal{P}$ of partitioned assemblies. Awodey and Emmenegger review this construction via the factorization:

$$\mathcal{P} \to \mathrm{Asm} \to \mathbf{Eff} \to \widehat{\mathcal{P}},$$

where:

- Asm is the regular completion of $\mathcal{P}$ (assemblies),
- **Eff** is the exact completion of Asm,
- $\widehat{\mathcal{P}}$ denotes presheaves on $\mathcal{P}$.

They introduce a spectrum of intermediate subcategories:

$$\mathcal{P} \subset \mathrm{IndProj} \subset \mathrm{Asm} \subset \mathrm{Coh} \subset \mathbf{Eff} \subset \widehat{\mathcal{P}},$$

classifying representables, projectives, compact and coherent objects. These categories provide the scaffolding upon which the higher structure of $\mathbf{Eff}_2$ will be built.

4.3 Internal Groupoids and Coherence Conditions

The effective 2-topos is constructed via a category of internal groupoids in $\widehat{\mathcal{P}}$:

$$\mathrm{Gpd}(\widehat{\mathcal{P}}) \simeq [\mathcal{P}^{op}, \mathrm{Gpd}].$$

Within this framework, Awodey and Emmenegger define *coherent groupoids* as those satisfying:

1. Pseudo-compactness: there exists a covering from a compact discrete groupoid.
2. Coherent diagonal: pullbacks of the diagonal along compact maps remain compact.
3. Coherent second diagonal: iterated pullbacks remain compact.

These conditions guarantee that the 0-types (objects equivalent to discrete groupoids) in the resulting category correspond precisely to coherent presheaves—recovering **Eff** as a subcategory of $\mathbf{Eff}_2$.

4.4 Homotopical Structure and Fibrant Groupoids

To support a model of homotopy type theory, the authors equip $\mathrm{Gpd}(\widehat{\mathcal{P}})$ with a model structure whose:

- Cofibrations are objectwise injections on objects,
- Weak equivalences are objectwise equivalences of groupoids,
- Fibrant objects are stacks (i.e., presheaves satisfying descent).

This model structure, inherited from Joyal–Tierney and Shulman, allows them to characterize fibrant replacements and homotopy pullbacks. In particular, the diagonal $\Delta : G \to G \times G$ is a homotopy monomorphism when G is fibrant, capturing identity types internally.

This formalism yields a robust notion of internal ∞-groupoid semantics over a base realizability category—mirroring the internal type-theoretic models used in univalent foundations.

4.5 Effective 2-Topos and Stratified Semantics

The resulting category $\mathrm{CohGpd}(\widehat{\mathcal{P}})$ serves as the category of 0- and 1-types in the effective 2-topos. Its homotopy theory supports:

- A Quillen model structure suitable for modeling HoTT,
- Homotopy coherent diagrams of compact groupoids,
- Interpretation of identity types, higher paths, and truncations,
- Internal univalence for compact groupoids.

Importantly, the internal logic is *constructive*: the effective 2-topos does not assume choice, excluded middle, or other classical axioms. The development is compatible with realizability models, enabling computational interpretations of type theory.

4.6 Parallelism with Anel and Bisson–Joyal

The structural affinities with previous sections are striking:

- **Anel:** The coherent groupoids of $\mathbf{Eff}_2$ mirror the compact truncations of $\mathcal{S}_\pi$. Both support stratified universes with internal homotopy semantics.
- **Bisson–Joyal:** The effective groupoids play the role of geometric realizations of D-ring operations; their substitution rules, diagonal coherence, and pseudo-compactness echo the coaction diagrams and operadic composition in bordism.

In effect, Awodey and Emmenegger realize the homotopical potential implicit in bordism and -finitude: they lift these to internal higher categorical structure grounded in computation and logic.

4.7 Toward Algebraic Models of Type Theory

This construction lays essential groundwork for the final stage of our analysis. The effective 2-topos is not merely a semantic structure: it is an algebraic setting for type theory with higher identity types, univalent universes, and realizability. As we move to Hughes' work on internal algebraic groupoids, we will see these ideas further internalized—giving syntactic form to the semantic structure developed here.

5 Algebraic Internal Groupoids and Type Theory: A Constructive Foundation via Category Theory

5.1 Overview

Calum Hughes' 2025 paper, *The Algebraic Internal Groupoid Model of Martin-Löf Type Theory*, presents a refined algebraic semantics for intensional type theory grounded in internal category theory. Building on earlier categorical semantics of identity types and coherence conditions, Hughes internalizes the construction of algebraic weak factorization systems and establishes a model of Martin-Löf Type Theory (MLTT) inside a lextensive, cartesian closed category.

Unlike prior realizability or topological models, this approach emphasizes:

- **Internalization:** all groupoids, homs, and factorizations are internal to a base category $\mathcal{E}$.
- **Algebraic structure:** substitution, identity types, and path structures are encoded via functorial, algebraic operations.
- **Coherence:** the model is strictly functorial and satisfies necessary coherence laws for interpreting dependent types.

The result is a highly tractable, constructive model of MLTT with identity types, homotopy structure, and categorical semantics that align closely with the frameworks of π-finiteness, effective 2-topoi, and homotopical bordism.

5.2 Internal Categories and Lextensive Structure

Hughes begins by defining the category $\mathrm{Cat}(\mathcal{E})$ of small internal categories in a lextensive cartesian closed category $\mathcal{E}$. The assumptions on $\mathcal{E}$ ensure:

- finite limits and colimits,
- disjoint coproducts,
- exponential objects,
- a well-behaved internal logic (i.e., $\mathcal{E}$ is a type-theoretic base).

This setup allows $\mathrm{Cat}(\mathcal{E})$ to carry enriched categorical structure: it has internal homs, factorization systems, and supports constructions like slice categories, fibrations, and pullbacks—all internally defined.

5.3 Algebraic Weak Factorization Systems

Central to Hughes' model is the construction of a *type-theoretic algebraic weak factorization system* (AWFS) on $\mathrm{Gpd}(\mathcal{E})$, the subcategory of internal groupoids. This AWFS supports:

- a class of cofibrations (representing constructors or type formers),
- a class of fibrations (representing dependent projections),
- functorial factorizations that interpret identity types.

By using a variation of Garner's small object argument adapted to internal categories, Hughes shows that one can construct an algebraic model where every map factors as:

$$A \xrightarrow{c} I(A) \xrightarrow{f} B$$

with c in a generating cofibration class and f in a class of algebraically defined fibrations—mirroring the inductive structure of identity types in type theory.

5.4 Identity Types and Internal Homotopy Structure

A standout feature of the model is its treatment of identity types. Hughes defines:

- an internal interval object I,
- path objects via functorial factorization over I,
- identity contexts as homotopically defined internal groupoids.

These definitions mirror those in homotopy type theory and validate key rules:

$$\begin{aligned} \text{Refl:} \quad & \mathrm{id}_A \colon A \to \mathrm{Path}_A \text{ is a section,} \\ \text{J-rule:} \quad & \text{holds via dependent elimination over path objects.} \end{aligned}$$

By interpreting the identity type $x =_A y$ as an internal hom in a groupoid object, the model preserves both extensional and intensional features—allowing for both judgmental and propositional equality.

5.5 Functorial Substitution and Contextuality

Unlike many higher categorical models of type theory, which suffer from coherence or substitution issues, Hughes' model is strictly functorial. The key reasons:

- All structures—types, contexts, morphisms—are internal and algebraic.
- The substitution structure is defined functorially in terms of pullbacks in $\mathcal{E}$.
- The entire semantics is framed within a fibrational framework over the base category.

This strict functoriality ensures that dependent type formers (e.g., Σ-types, Π-types, Id-types) respect substitution up to definitional equality—not just up to homotopy.

5.6 Enriched Hom and Truncated Nerve

To articulate the enrichment of internal categories, Hughes constructs a truncated internal nerve and an internal Yoneda structure. This allows:

- Internal homs: $[\mathcal{C}, \mathcal{D}]_{\mathcal{E}}$ as enriched objects,
- Classification of natural transformations as internal spans,
- Modeling of homotopy coherent diagrams internally.

These constructions reinforce the model's ability to interpret the full syntax of type theory, including higher inductive types and path spaces—closing the gap between syntax and semantics.

5.7 Relation to Previous Sections

Hughes' algebraic model represents a culmination of themes developed throughout this paper:

- It internalizes the π-finite coherence of Anel and the effective groupoids of Awodey–Emmenegger.
- It realizes the operadic and coaction structures of Bisson–Joyal within the logic of type formers and identity types.
- It embeds all of this within a constructive, functorial setting that avoids reliance on global universes or meta-theoretical assumptions.

This final link bridges the purely categorical (2-topos), the homotopical (π-finite spaces), and the algebraic (bordism/D-rings) into a single model-theoretic system. Hughes' framework provides the infrastructure needed for a stratified and syntactic model of homotopy type theory based on internal groupoid semantics.

Appendix: Toward Telic Stratification and Classifying Logoi

A.1 Prelude: From Operations to Ontologies

Each of the five works examined in this paper—spanning bordism operations, π-finiteness, effective groupoids, and internal type theory—presents a distinct but convergent view of mathematical structure. What unites them is a transition from sets and types as static containers of values to types and spaces as dynamically stratified operations or transformations. In this appendix, we propose a synthetic unification of these perspectives under the twin concepts of:

- **Telic Stratification:** the view that mathematical structures are layered according to their internal constructive capacity, expressed in terms of coherent operations, truncations, and dependent type formation.
- **Classifying Logoi:** higher categorical universes that classify the possible stages or layers of such structures, equipped with internal groupoids, fibrational models, and univalent semantics.

Our goal is to map the key constructions in each work onto this unifying dialectic.

A.2 Bordism and Operadic Telicity

In Bisson and Joyal's theory of Dyer–Lashof operations and Nishida relations, we find the first manifestation of telic layering: bordism classes are not merely sets of manifolds but are organized by sequences of operations—indexed by covering structures, formal group laws, and algebraic substitutions.

The D-ring and Q-ring structures constitute a *telic grammar*: operations act on types (manifolds, homology classes), and their interaction is mediated by homotopically meaningful coactions. The resulting structure is stratified by degree, dimension, and substitution depth—much like the homotopy levels of types in HoTT.

A.3 π-Finiteness and Coherent Classification

In Anel's construction of $\mathcal{S}_\pi$, we see the rise of a logos: a minimal, coherent category that classifies truncated, compact homotopy types. This ∞-pretopos is stratified by truncation level (e.g., 0-types, 1-types, etc.), forming an explicit stage structure for internal type theory.

The univalent universe U_π acts as a *classifying logos* for -finite types. Each type is internally represented by a classifying map into U_π, and internal substitution corresponds to pullback. This tight correspondence between internal logic and homotopy structure makes $\mathcal{S}_\pi$ a prototypical logoi space.

A.4 The Effective 2-Topos as a Telically Enriched Logos

Awodey and Emmenegger's $\mathbf{Eff}_2$ refines the idea of stratified classification. Here, types are not just -finite—they are enriched via realizability and computability. The coherent groupoids internal to $\widehat{\mathcal{P}}$ are layers of computational structure, ordered by coherence and fibrancy.

Each such groupoid defines a stratum in the logos of effective higher types. The fibrant replacement corresponds to a form of *telic completion*: ensuring that identity, substitution, and homotopy coherence hold at each level. The full effective 2-topos then functions as a classifying logos for constructive homotopy types with computational content.

A.5 Algebraic Internal Groupoids and Syntactic Telicity

Hughes' model completes the picture by internalizing these structures into the syntax of type theory. The AWFS (algebraic weak factorization system) and functorial identity types provide a stratified internal semantics: types are not primitive but derived via operations of substitution, path formation, and factorization.

His use of internal groupoids and Yoneda enrichment connects directly to the categorical universes in Anel and Awodey–Emmenegger, but does so from within a purely algebraic framework. This makes the model a syntactic instantiation of a classifying logos: one where the operations that define typehood—identity, substitution, context extension—are themselves stratified and constructed from within.

A.6 Diagram of Stratified Interpretation

We summarize the interrelations as a stratified diagram:

$$\begin{array}{rcl}\textbf{Level 0: Algebraic Operations} & \longrightarrow & \text{D-Rings, Q-Rings (Bisson–Joyal)} \\ \textbf{Level 1: Homotopy Spaces} & \longrightarrow & \pi\text{-Finite Types } (\mathcal{S}_\pi) \\ \textbf{Level 2: Internal Groupoids} & \longrightarrow & \text{Coherent Groupoids } (\mathbf{Eff}_2) \\ \textbf{Level 3: Type-Theoretic Syntax} & \longrightarrow & \text{Internal AWFS Models (Hughes)}\end{array}$$

Each level stratifies the previous one: operations induce homotopy types, which form groupoids, which define type-theoretic contexts. This is the essence of telic stratification: types emerge from operations in staged, internally coherent layers.

A.7 Classifying Logoi and Stratified Universes

The term *classifying logos* here generalizes the notion of a universe object. In each case:

- $\mathcal{S}_\pi$ classifies π-finite types via U_π,
- $\mathbf{Eff}_2$ classifies effective groupoids via fibrant replacements,
- Hughes' model classifies type formers via internal groupoid structure,
- Bisson–Joyal's coaction diagrams classify operations via Hopf algebras.

In all cases, these structures serve as internal universes—object classifiers, operation classifiers, or groupoid classifiers. They are not external sets of types but internal scaffolds through which types and operations are generated and stratified.

A.8 Telicity, Computability, and Constructivity

Finally, this synthesis offers a philosophical view of constructive mathematics: telicity provides the layered architecture of formation; logoi provide the categorical scaffold of classification. Constructivity emerges not from denial of classical axioms per se, but from the decision to ground mathematical ontology in internally generable operations.

Theories, in this view, are not flat collections of axioms. They are telicly stratified logoi: categories of operations, groupoids of constructions, and internal contexts of transformation.

A.9 Outlook

This synthesis points toward a possible theory of *Telic Homotopy Type Theory*: a system in which homotopical, algebraic, and logical constructions are understood as stages in a stratified operational logos. Future directions may include:

- Developing an internal type theory for $\mathcal{S}_\pi$ or $\mathbf{Eff}_2$ directly.
- Expanding the D-ring formalism to cover synthetic spectra or sheaf models.
- Reconstructing classical homotopy invariants (e.g., characteristic classes, spectral sequences) as internal sections of stratified universes.

In this vision, the frontier between logic, topology, and computation dissolves—not into uniformity, but into a staged, coherent interplay of operations and spaces: a true classifying logos of mathematics.

Glossary

Algebraic Weak Factorization System (AWFS) A pair of classes of morphisms (cofibrations and fibrations) equipped with functorial factorizations and algebraic structure satisfying lifting and closure properties. Used to interpret identity types in type theory.

Classifying Logos A higher-categorical structure that serves as a universe classifying certain kinds of objects, such as types, operations, or groupoids. It generalizes the idea of a univalent universe and may internalize classifying maps for fibrations, type formers, or homotopy types.

Coaction An action of a coalgebra (e.g., a Hopf algebra like A^* or B^*) on a module, often used to describe how operations (e.g., Steenrod or Dyer–Lashof) interact with homology or bordism classes.

Cohesive/Coherent Groupoid An internal groupoid object in a topos or presheaf category satisfying compactness and diagonal coherence conditions. Used to model internal types with structured identity and substitution behavior.

D-Ring A commutative ring equipped with a formal group law of order two and a total squaring operation D_t satisfying Cartan-type and generalized Adem relations. Encodes Dyer–Lashof operations in bordism.

$\mathbf{E}_\infty$-space A topological or simplicial space equipped with a multiplication that is associative and commutative up to all higher homotopies. Serves as the domain for structured operations in homotopy theory.

Effective 2-Topos A proposed higher-dimensional analogue of the effective topos **Eff**, incorporating internal groupoids, coherence structures, and a realizability base. Supports homotopy type theory internally.

Formal Group Law A formal power series $F(x, y)$ satisfying associativity and identity conditions, used in algebraic topology to define characteristic classes and structured ring operations.

Identity Type In type theory, a type $x =_A y$ expressing the identity or path between terms of a type A. In homotopy type theory, these are interpreted as path spaces in a higher groupoid.

Logos (pl. Logoi) A finitely complete and cocomplete category (or higher category) with logical structure, often serving as a universe or context in which types and their relationships are defined and classified.

Milnor Hopf Algebra A^* The dual of the mod 2 Steenrod algebra. Used to coact on homology and define compatible operations in a homotopical setting.

Nishida Relations Relations expressing the compatibility between Dyer–Lashof operations and Steenrod (or Landweber–Novikov) operations. Formulated in terms of coaction diagrams and preserved under Thom reduction.

Partitioned Assembly A realizability-theoretic object consisting of a set with a partial recursive function labeling its elements. Forms the base category for constructing the effective topos.

π-Finite Space A homotopy type with finitely many connected components and finite homotopy groups in each component. Forms the objects of the category $\mathcal{S}_\pi$, an ∞-pretopos with good closure properties.

Q-Ring A D-ring whose formal group law is additive. Encodes Dyer–Lashof operations in homology, often modeled via the mod 2 homology of E_∞-spaces.

Realizability A semantics of constructive logic based on computational interpretation. The effective topos and its higher analogues model logical systems using categories of computable functions or indexed assemblies.

Stratification The layering of mathematical structures by dimension, truncation, or homotopical complexity. In this context, it refers to the internal structure of types as stratified by operations or logical complexity.

Telicity The quality of being directed toward an end or goal (from Greek *telos*). In this context, telicity refers to the staged generation of types, operations, or identities in a coherent, purpose-driven mathematical system.

Univalent Universe A universe object in a type-theoretic model satisfying Voevodsky's univalence axiom: equivalences between types correspond to identities. Enables models of HoTT where types behave like homotopy types.

Symbol Index

Symbol	Meaning
$N_*(X)$	Unoriented bordism ring of a space X
$H_*(X)$	Singular homology (mod 2) of X
D_t	Total squaring operation in a D-ring
Q_t	Total Dyer–Lashof operation in a Q-ring
$F(x, y)$	Formal group law (order two in D-ring theory)
A^*	Milnor Hopf algebra (dual of Steenrod algebra)
B^*	Faà di Bruno Hopf algebra (composition of series)
ϕ	Landweber–Novikov coaction map
α	Milnor coaction map on homology
ε	Thom reduction $N_*(X) \to H_*(X)$
U_π	Univalent universe classifying π-finite types
$\mathcal{S}_\pi$	∞-category of π-finite spaces
Eff	Effective topos (Hyland's realizability topos)
$\mathbf{Eff}_2$	Effective 2-topos (higher version with internal groupoids)
$\mathrm{Gpd}(\mathcal{E})$	Internal groupoids in category $\mathcal{E}$
$\mathrm{CohGpd}(\widehat{\mathcal{P}})$	Coherent groupoids over presheaves on partitioned assemblies
$[\mathcal{C}, \mathcal{D}]$	Internal hom category (functor category)
$\tau_{\leq n}(X)$	n-truncation of a space X
Path_A	Path object over type A (identity type)
AWFS	Algebraic weak factorization system
$x =_A y$	Identity type between x and y in type A
Δ	Diagonal map (e.g. $X \to X \times X$)
$\pi_n(X)$	n-th homotopy group of X
$B\mathrm{Aut}(F)$	Classifying space for automorphisms of F
$Q\langle M \rangle$	Free Q-ring generated by module M
$D\langle M \rangle$	Free D-ring generated by module M
Σ_n	Symmetric group on n letters
$p \circ q$	Composition of polynomial functors (or coverings)
$T(M)$	Thom reduction of M: $M \otimes_{N_*} \mathbb{Z}_2$

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