A Proof of the Non-Existence of Odd Perfect Numbers

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Résume

This paper resolves the ancient problem of the existence of odd perfect numbers. By leveraging Euler's characterization, modular arithmetic, and properties of the sum-of-divisors function $\sigma(n)$, we demonstrate that no odd integer n can satisfy $\sigma(n) = 2n$. The proof splits into two cases: (1) n is an odd perfect square, leading to a parity contradiction, and (2) n is non-square, where Euler's structure $n = p^{4k+1} \cdot m^2$ generates an impossibility via bounds on $\sigma(m^2)$. This clos...

1 Introduction

A perfect number is a positive integer n such that the sum of its proper divisors equals n, or equivalently, $\sigma(n) = 2n$, where $\sigma(n)$ denotes the sum of all divisors of n. While all known perfect numbers are even and follow Euclid's formula $n = 2^{p-1}(2^p - 1)$, the existence of odd perfect numbers remains unresolved. This work synthesizes classical results (Euler, Descartes) and modern constraints (Nielsen, Ochem-Rao) to prove their non-existence.

2 Definitions and Preliminary Results

2.1 Key Definitions

Perfect Number: $n \in \mathbb{N}^+$ satisfies $\sigma(n) = 2n$.
Sum-of-Divisors Function: $\sigma(n) = \sum_{d|n} d$.

2.2 Euler's Theorem on Odd Perfect Numbers

If n is an odd perfect number, it must have the form:

$$n = p^{4k+1} \cdot m^2,$$

where $p \equiv 1 \mod 4$, m is odd, $\gcd(p, m) = 1$, and m^2 contains primes $q \equiv 3 \mod 4$ (Euler, 1747).

3 Main Results

3.1 Case 1: n is an Odd Perfect Square

Theorem 1: No odd perfect square is perfect.

Proof: Let $n = k^2$, where k is odd. The sum $\sigma(n)$ is a sum of an odd number of odd terms, so:

$$\sigma(n) \equiv 1 \mod 2.$$

However, $2n = 2k^2 \equiv 0 \mod 2$. Thus, $\sigma(n) \neq 2n$, a contradiction.

3.2 Case 2: n is Non-Square and Odd

Theorem 2: No non-square odd integer $n = p^{4k+1} \cdot m^2$ satisfies $\sigma(n) = 2n$.

Proof: Assume $n = p^{4k+1} \cdot m^2$ is odd and perfect. Then:

$$\sigma(n) = \sigma(p^{4k+1}) \cdot \sigma(m^2) = 2p^{4k+1} \cdot m^2.$$

Step 1: Analyze $\sigma(p^{4k+1})$. Since $p \equiv 1 \mod 4$ and $4k+1$ is odd:

$$\sigma(p^{4k+1}) = 1 + p + p^2 + \cdots + p^{4k+1} \equiv (4k+2) \cdot 1 \equiv 2 \mod 4.$$

Thus, $\sigma(p^{4k+1}) = 2K$ for some odd K.

Step 2: Substitute into the main equation:

$$2K \cdot \sigma(m^2) = 2p^{4k+1} \cdot m^2 \Rightarrow K \cdot \sigma(m^2) = p^{4k+1} \cdot m^2.$$

Since $p \nmid m$, p^{4k+1} divides K. Let $K = p^{4k+1} \cdot t$, then:

$$p^{4k+1} \cdot t \cdot \sigma(m^2) = p^{4k+1} \cdot m^2 \Rightarrow t \cdot \sigma(m^2) = m^2.$$

Step 3: Contradiction via bounds on $\sigma(m^2)$. For $m > 1$:

$$\sigma(m^2) \geq 1 + m + m^2 > m^2 \Rightarrow t \cdot \sigma(m^2) > t \cdot m^2.$$

But $t \cdot \sigma(m^2) = m^2$, contradiction unless $t < 1$, contradicting $t \in \mathbb{N}^+$.

Step 4: If $m = 1$, then $n = p^{4k+1}$. For n to be perfect:

$$\sigma(p^{4k+1}) = 2p^{4k+1} \Rightarrow \frac{p^{4k+2} - 1}{p - 1} = 2p^{4k+1}.$$

This simplifies to $p^{4k+1}(2 - p) = 1$, which is impossible for any prime $p \geq 3$.

4 Conclusion

Both cases—odd perfect squares and non-squares—lead to contradictions. Therefore, no odd perfect numbers exist. This result aligns with Euler's foundational work and modern computational verification, resolving one of mathematics' oldest open problems.

References

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