Brace Height and Ordinal Length: A Duality in the Von Neumann Hierarchy

David L. Selke ChatGPT

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Abstract

We explore a structural duality within the von Neumann conception of ordinal numbers: between the flat, extensional enumeration of ordinals and the internal, intensional brace-nesting required to construct them. We argue that the transitivity of membership implies a direct tradeoff: the longer the ordinal list, the taller the brace structure required to individuate its elements. This perspective motivates a critique of large countable ordinals such as ω_1 and questions the ontological legitimacy of their construction under the iterative conception of set.

1 Introduction

A recent paper [1] has criticized the von Neumann construction of ω_2 based on apparent infinite increasing membership chains in many of its "elements". Here we will tighten this bound to ω_1 and below while elaborating the reasoning.

2 Extensional Parsing as First Principle

In ZFC set theory, the Axiom of Extensionality defines equality of sets in terms of their elements. But we stress a more primitive parsing assumption underlying this axiom: that the elements of a set are precisely those objects occurring at its top level of braces. For a set $A = \{x, y, z\}$, this is not mere syntax, but a semantic commitment that $x, y, z \in A$ and that no other elements are. We call this assumption **Extensional Parsing**.

Extensional Parsing is essential for coherent reasoning about sets, particularly ordinals. Without it, set membership becomes ambiguous, and comparison of sets breaks down. We therefore take it as foundational to any discussion of set structure.

3 Brace Height and Ordinal Identity

Each von Neumann ordinal α is defined as the set of all smaller ordinals: $\alpha = \{\beta \mid \beta < \alpha\}$. The transitivity of ordinals implies that their elements are subsets and that these subsets must be distinguishable by their internal brace structure.

We define the **brace-height dual** $B(\alpha)$ of an ordinal α to be a strictly ascending $\in$-chain of length α:

This dual represents the brace-nesting height required to construct or distinguish α. For finite α, the chain is finite. For the smallest transfinite α such as ω or ω^2 infinite membership chains do not exist so the dual may be defined variously. But for larger transfinite α, such as ω^ω or ϵ_0, the duals are already infinite, well before ω_1.

4 “You Want It Long, You Get It Tall”

The standard flat enumeration of ordinals—e.g., $\{0, 1, 2, 3, \dots, \omega, \omega + 1, \dots\}$—presents a surface-level list. But each element in this list requires internal height, realized via nested membership (or brace) structure, in order to be well-defined. Thus, the longer the ordinal list, the taller the underlying structures needed to support it.

> **Transitivity Constraint:** In a transitive ordinal structure, the flat inclusion of many elements demands matching brace-height structures to define those elements. *Long implies tall.*

Let δ be the least ordinal such that defining elements of greater brace-height requires infinite ascending $\in$-chains—i.e., beyond which Extensional Parsing breaks down. Then any purported set that includes ordinals up to height δ must implicitly support such infinite chains, violating the constraint that all sets be grounded by an outermost brace.

5 Implications for ω_1 and Beyond

Although ZFC proves the existence of ω_1 as the first uncountable ordinal, this rests on a hierarchy permitting unlimited nesting. From a foundational perspective rooted in brace-height, however, this becomes problematic: ω_1 must include ordinals of all countable heights, and hence structures of unbounded countable nesting.

But if brace-height is constrained below δ, then ω_1 is incoherent—it implies the existence of sets requiring infinite ascent in membership chains, without a top-level brace to ground them. Under this view, ω_1 violates the Transitivity Constraint and the principle of Extensional Parsing.

6 Conclusion

The flat set-theoretic enumeration of ordinals masks the nested complexity required to define them. By highlighting the duality between ordinal length and brace height (or membership height), we highlight the structural demands imposed by transitivity. If we take Extensional Parsing and the iterative conception of set seriously, we must carefully scrutinize the legitimacy of ordinals whose internal construction exceeds any finite limit.

7 References

References

[1] David Selke. A pathology of the von neumann ordinals. https://vixra.org/abs/2410.0123, 2024. Preprint, viXra:2410.0123.