The truth of hail conjecture

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Abstract— Assuming that the hail conjecture is wrong, there is a number set H, and none of which satisfies the conjecture. In H, there must be a smallest number. Because the minimum number cannot be found by the conjecture algorithm, thus proving that H must be empty.

Key words: H set; W set

1. Introduction

Hail conjecture is also called Collatz conjecture. The Collatz conjecture was proposed by German mathematician Lothar Collatz in 1937.

Let the set of natural numbers be N, $N = \{1, 2, 3 \cdots\}$

Collatz conjecture: $\forall n \in N$, $f(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod 2 \\ 3n + 1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$

$f(n)$ will lead the sequence starting from n to the cycle $\cdots 4 \to 2 \to 1 \to 4 \to$.

With the help of computers, it has been verified computationally that all numbers within 10^{30} conform to the conjecture, but to this day, no mathematical proof has been provided.

Current State of Research

Terence Tao's Probabilistic Proof (2019–2020):

Showed that almost all Collatz sequences eventually reach values below 200. Specifically, >99% of starting values below decay to 1810. This result—though not a full proof—is the strongest partial result to date, leveraging density and ergodic theory.

This paper uses proof by contradiction, assuming that the conjecture not holds true, there must be a set H $= \{h_1, h_2, h_3, h_4, \cdots h_i\}$ of numbers in which all the numbers do not satisfy the conjecture. Finding h_1 in this set is the main task of this paper, because in fact, h_1 cannot be found, so there is no set of numbers. Therefore, the conjecture holds true.

2. Definition

Assumes that the Collatz conjecture is incorrect, we have：

Definition: H set and W set.

1. *In N, the part that satisfies the "Collatz conjecture" is called the set* W.

W $= \{w_1, w_2, w_3, w_4, \cdots w_i\}$, $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, $w_4 = 4$ $\cdots w_i \to \infty$

2. *In N, the part that does not satisfy the "Collatz conjecture" is called the H set.*

H $= \{h_1, h_2, h_3, h_4, \cdots h_i\}$, h_1 *is the smallest element of H and satisfies and* $h_1 \equiv 1 \pmod 2$.

(if $h_1 \equiv 0 \pmod 2$, *then* $f(h_1) = \frac{h_1}{2} < h_1$, contradicting the minimality of h_1)

Clearly, (W $\cup$ H $=$ N, W $\cap$ H $= \emptyset$) *and* $\forall n$, *if* $n < h_1$, *then* $n \in$ W.

3. Proof of Lemmas

Lemma1: $h_1 = 2k + 1$, $k \equiv 1 \pmod 2$, $h_1 = 4k_1 + 3$.

Proof

$In\ h_1 = 2k + 1,$ *if* $k \equiv 0(\text{mod}2)$, *let* $k = 2k_1$, *then* $h_1 = 4k_1 + 1$.

$f(4k_1 + 1) = 3(4k_1 + 1) + 1 = 12k_1 + 4,$

$f(12k_1 + 4) = \frac{12k_1 + 4}{4} = 3k_1 + 1,$

$\because 3k_1 + 1 < 4k_1 + 1, \therefore 4k_1 + 1 \neq h_1$ [Definition] **#**

Lemma2: In $h_1 = 2k + 1 = 4k_1 + 3, k \neq 3k_i + 2.$

Proof

In $h_1 = 2k + 1,$ *if* $k = 3k_i + 2,$ *then* $h_1 = 2k + 1 = 2(3k_i + 2) + 1 = 6k_i + 5.$
$\because \exists q = 4k_1 + 3 < 6k_i + 5,$
$f(4k_1 + 3) = 3(4k_1 + 3) + 1 = 12k_1 + 10,$

$f(12k_1 + 10) = \frac{12k_1 + 10}{2} = 6k_i + 5,$ *and* $4k_1 + 3 < 6k_i + 5 = h_1.$

$\therefore 6k_i + 5 \neq h_1,$ [Definition] *that is* $h_1 = 2k + 1, k \neq 3k_i + 2.$ **#**

Lemma 3: $h_1 = 4k + 3$ can be expressed in the form of $2^n - (4k + 1)$

Proof

$2^n - (4k + 1) = 2^n - (4k + 1) - 3 + 3 = 2^n - 4(k + 1) + 3 = 4(2^{n-2} - (k + 1)) + 3$

$\because n \in N, k \in N, \therefore 2^n - (4k + 1) = (4k + 3).$ **#**

4. Proof of the Main Proposition

$h_1 = 4k + 3$ [lemma1]
Let $h_1 = 4k + 3 = 2^n - (4k + 1), k = 0,1,2 \dots, h_1 = 2^n - 4k - 1 = 2^2(2^{n-2} - k) - 1$ [lemma3]

Step1: when $k = 0, 2^2(2^{n-2} - k) - 1 = 2^n - 1$

$2^n - 1$ can be represented in one of the following three forms:
1. $2^n - 1 = 3m,$ 2. $2^n - 1 = 3m + 1,$ 3. $2^n - 1 = 3m + 2, m \in N$
$\because h_1 \neq 3m + 2,$ [lemma2]
$\therefore h_1 = 2^n - 1 = 3m$ or $h_1 = 2^n - 1 = 3m + 1$

4.1, $h_1 = 2^n - 1 = 3m$

$\because (2^n - 1) \equiv 1(mod2), \therefore m \equiv 1(mod2),$
let $m = 2m_1 + 1,$ then $3m = 6m_1 + 3$
have: $2^n - 1 = 6m_1 + 3, 2^n = 6m_1 + 4, 2^{n-1} = 3m_1 + 2, 2^{n-1} - 1 = 3m_1 + 1$
If $2^n - 1 = h_1,$ then $\exists 2^{n-1} - 1 < 2^n - 1,$ makes $2^n - 1 \neq h_1.$
That is $h_1 \neq 2^n - 1 = 3m$

4.2 . $2^n - 1 = 3m + 1$,

$\because (2^n - 1) \equiv 1(mod2), \therefore m \equiv 0(mod2)$

let $m = 2m_1$, then $3m + 1 = 6m_1 + 1$

Have: $2^n - 1 = 6m_1 + 1$, $2^n = 6m_1 + 2$, $2^{n-1} = 3m_1 + 1$. $2^{n-1} - 1 = 3m_1$

If $2^n - 1 = h_1$, then $\exists 2^{n-1} - 1 < 2^n - 1$, makes $2^n - 1 \neq h_1$.

That is $h_1 \neq 2^n - 1 = 3m+1$

Proved by 4.1 and 4.2 that $2^n - 1 \neq h_1$.

The significance of Step1 is that h_1 cannot be found in $2^n - 1$. However, if we assume h_1, $2^{n-1} - 1$ is generated through h_1, so that h_1 cannot satisfy the definition.

Step2: when k > 0, $h_1 = 2^2(2^{n-2} - k) - 1$

$\because h_1 = 2^2(2^{n-2} - k) - 1 \neq 3m + 2$, [lemma2]

$\therefore h_1 = 2^2(2^{n-2} - k) - 1 = 3m$ or $h_1 = 2^2(2^{n-2} - k) - 1 = 3m + 1$

4.3. $h_1 = 2^2(2^{n-2} - k) - 1 = 3m,$

$m \equiv 1 (mod 2)$, let $m = 2m_1 + 1$, then $3m = 6m_1 + 3$

$2^2(2^{n-2} - k) - 1 = 6m_1 + 3$, $2^2(2^{n-2} - k) - 1 = 6m_1 + 4$ $2^1(2^{n-2} - k) = 3m_1 + 2$,

$$2^1(2^{n-2} - k) - 1 = 3m_1 + 1$$

$\because 2^2(2^{n-2} - k) - 1 = h_1$, then $\exists 2^1(2^{n-2} - k) - 1 < 2^2(2^{n-2} - k) - 1$

makes $2^2(2^{n-2} - k) - 1 \neq h_1$.

$$\therefore h_1 \neq 2^2(2^{n-2} - k) - 1$$

4.4 . $2^2(2^{n-2} - k) - 1 = 3m + 1$

$m \equiv 0 (mod 2)$ let $m = 2m_1$, then $3m + 1 = 6m_1 + 1$

That is $2^2(2^{n-2} - k) - 1 = 6m_1 + 1$, $2^2(2^{n-2} - k) = 6m_1 + 2$, $2^{n-1} = 3m_1 + 1$

$$2^1(2^{n-2} - k) - 1 = 3m_1$$

$\because 2^2(2^{n-2} - k) - 1 = h_1$, then $\exists 2^1(2^{n-2} - k) - 1 < 2^2(2^{n-2} - k) - 1$

makes $2^2(2^{n-2} - k) - 1 \neq h_1$.

$\therefore h_1 \neq 2^2(2^{n-2} - k) - 1 = 3m$

Proved by 4.3 and 4.4 that $2^2(2^{n-2} - k) - 1 \neq h_1$.

The significance of the first step is that h_1 cannot be found in $2^2(2^{n-2} - k) - 1$. However, if we assume h_1, $2^1(2^{n-2} - k) - 1$ is generated through h_1, so that h_1 cannot satisfy the definition.

5. Conclusion

$\because \nexists h_1, \therefore \nexists$ H $= \{ h_1, h_2, h_3, h_4, \cdots h_i \}$

that is H=∅.

Collatz conjecture is correct. **End**

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