# Proof of the Collatz Conjecture

## Abstract

*This paper presents a proof of the Collatz conjecture. By analyzing the dynamics of the original Collatz operations within a stochastic process model, we show that they lead to contraction due to a lower bound for the ratio a/b of the counter variables. Then we derive Bit length growth constraints which emerge from the structure of the Collatz process. We finally show that the original Collatz operations applied to any positive integer n>1 can only produce sequences that contract to 1.*

## 1. The Collatz conjecture

The Collatz conjecture asserts that the Collatz sequence defined by the rule

$$L(n) = \begin{Bmatrix} 3n+1 & in\ case\ n\ is\ odd \\ \frac{n}{2} & in\ case\ n\ is\ even \end{Bmatrix} \tag{1}$$

will eventually reach 1 for any positive integer n.

b: counter number for the amount of the operation (3n+1) in case n is odd

a: counter number for the amount of any operation (n/2) in case n is even

k: the total number of operations (steps), k=a+b

## 1.1 Derivation of an Evolution formula

We derive an evolution formula for the approximation of the resulting number $n_k$ after k=a+b Collatz operations. This formula approximates the numbers $n_k$ that evolve within the sequence by following the defined operations of the Collatz sequence.

We derive the evolution formula as follows: We recursively apply all steps, starting from $n_0$. The effect of applying b times the multiplication 3n in case n is odd, is:

$$n_0 \cdot 3^b$$

Then we apply all halving steps. But thus we need an additive correction term C, because each (3n+1) operation introduces an additional value 1.

$$n_k = n_0 \cdot \frac{3^b}{2^a} + C \tag{2}$$

The additive correction term C originates as follows:

$$n_k = n_0 \cdot \frac{3^b}{2^a} + \frac{1}{2^a} \sum_{j=0}^{b-1} 3^j \cdot 1 = n_0 \cdot \frac{3^b}{2^a} + \frac{1}{2^a} \cdot \frac{3^b - 1}{2} = n_0 \cdot \frac{3^b}{2^a} + \frac{3^b - 1}{2^{a+1}} \tag{3}$$

## 1.2 Derivation of the contraction requirement

It is possible to calculate the condition for which the application of an amount of “a” even operations and “b” odd operations defined in the Collatz operations (section 1), leads to contraction, i.e. $n_k<n_0$ after k=a+b total steps.

We start with formula (3), and keep in mind that for contraction we need $n_k<n_0$

$$n_k = n_0 \cdot \frac{3^b}{2^a} + \frac{3^b - 1}{2^{a+1}}$$

We define:

$$T \coloneqq n_0 \cdot \frac{3^b}{2^a}$$

$$C \coloneqq \frac{3^b - 1}{2^{a+1}}$$

Then (3) reads as follows:

$$n_k = T + C$$

No we are taking the logarithm on both sides of the equation

$$log_2(n_k) = log_2(T + C)$$

For large $n_0$ and T»C this means:

$$log_2(n_k) \approx log_2(T) + log_2\left(1 + \frac{C}{T}\right) = log_2(T) + \delta_1 \tag{4}$$

where

$$\delta_1 = log_2\left(1 + \frac{C}{T}\right) = log_2\left(1 + \frac{\frac{3^b - 1}{2^{a+1}}}{n_0 \cdot \frac{3^b}{2^a}}\right) = log_2\left(1 + \frac{1 - 3^{-b}}{n_0 \cdot 2}\right) \tag{5}$$

The correction term $\delta_1$ shrinks for growing b and $n_0$

Now we expand $log_2(T)$

$$log_2(T) = log_2(n_0) + b \cdot log_2(3) - a$$

Combining with (4) leads to

$$log_2(n_k) = log_2(T) + log_2\left(1 + \frac{C}{T}\right) = log_2(n_0) + b \cdot log_2(3) - a + \delta_1 \tag{6}$$

If we demand contraction $n_k<n_0$ then:

$$log_2(n_k) < log_2(n_0)$$

Thus

$$log_2(n_0) + b \cdot log_2(3) - a + \delta_1 < log_2(n_0)$$

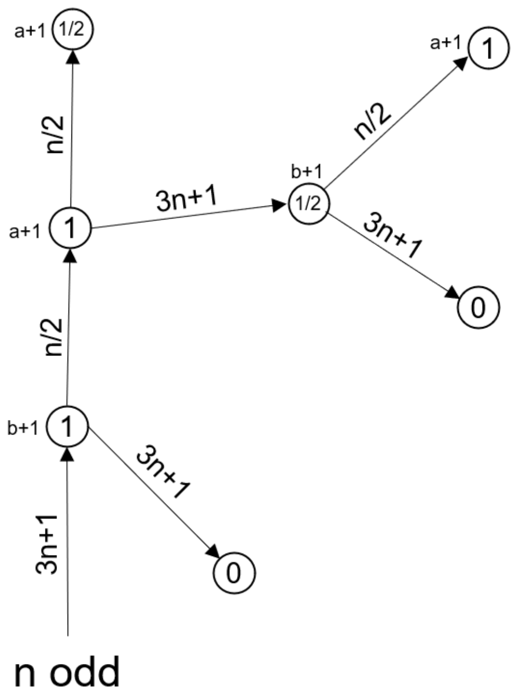
Subtract $log_2(n_0)$ from both sides and solve for the ratio a/b. Thus we get the condition for the ratio a/b that leads to contraction $n_k<n_0$:

$$\frac{a}{b} > log_2(3) + \frac{\delta_1}{b} = log_2(3) + \frac{log_2\left(1 + \frac{1 - 3^{-b}}{n_0 \cdot 2}\right)}{b} \tag{7}$$

# 2. Recursive stochastic process model

## 2.1 Probabilistic tree-structure of Collatz operations

We analyze a tree-like structure for the recursive succession of the original Collatz operations.
In the following exemplary tree like structure for an odd integer, the directed branches show the paths of Collatz operations, and the nodes are the probabilities that the operation assigned to the branch is executed. At each node with probability p>0, one of the counter variables “a”, “b” is incremented.




Picture 1: Tree-structure of Collatz operations

In this example, “a” was incremented by 3 in total, and “b” was incremented by 2.
This tree-structure of operations doesn’t take into account the recursive structure of the operations. Thus we need an improved recursive stochastic model.

## 2.2 Recursive stochastic process model

We appreciate the fact, there exists consecutive operations in case n is odd, i.e. the operation (3n+1) is directly followed by (n/2) with probability 1. We call this halving an "implicite halving". Afterwards there is probability ½, that the next operation is either (3n+1) or probability ½ that the next operation is (n/2), which we call an "explicite halving". Whereas after any (n/2) operation, the probability is ½, to be followed by (n/2) or ½ to be followed by (3n+1).
It is also proven, that the probability for the occurrence of a combined operation (3n+1)/2 and the probability of an explicite halving (n/2), is ½ for each. From this it must follow, that for large k=a+b, the total occurrence of (3n+1) is k/3, the total occurrence of explicite halvings is k/3 and the total occurrence of implicite halvings must be the same as the total occurrence of (3n+1), also k/3. All these statements are based on proven theorems.

In this decisive step we generate a model of the structure of the Collatz process with a Markov-chain approach, for which we need to introduce 5 distinct and recursive main operations $O_1$ ,…, $O_5$.

For each of the operations we assign its probability and its impact on the counter variables "a" and "b". We appreciate the fact that there is no deterministic succession or order of these operations. They are only applied recursively by the requirements of the original Collatz process described in section 1.
We also introduce the total occurrence of the operations for large k.

| Operation | Description | Probability P(Ox) $1 \leq x \leq 5$ | Total occurrence for large k | Δa | Δb | k |
|---|---|---|---|---|---|---|
| O1 | (3n+1) after implicite (n/2) | ½ | k/3 | 0 | 1 | 1 |
| O2 | (3n+1) after explicite (n/2) | ½ | | 0 | 1 | 2 |
| O3 | explicite (n/2) after explicite (n/2) | ½ | k/3 | 1 | 0 | 3 |
| O4 | explicite (n/2) after implicite (n/2) | ½ | | 1 | 0 | 4 |
| O5 | implicite (n/2) after (3n+1) | 1 | k/3 | 1 | 0 | 5 |

Table 1: The five main operations $O_1$ ,…, $O_5$

These 5 operations $O_1$ ,…, $O_5$. cover the whole Collatz process.

We can map these five operations in the order just defined by the Collatz conjecture onto any complete Collatz sequence. As we can see, this model correctly describes the behavior of the counter variables "a" and "b" within the original Collatz counting system. It reconciles that a+b=k.

The main advantage of this approach is, that we are able to calculate the expectation values for the counter variables "a" and "b", without the need for a special probability distribution.

By modeling Δa, and Δb as random variables, we are able to calculate Variances for "a" and "b".

## 2.2.1 Expectation values for a and b and the ratio EV(a)/EV(b)

We calculate the expectation values of “a” and “b” just by the counting as described in section 2.1 in case k is large, with the probabilities P(Ox) and its occurences.

$$EV(a) = \left(amount\ of\ explicite \frac{n}{2}\right) \cdot P(O3) + \left(amount\ of\ implicite \frac{n}{2}\right) \cdot P(O4) + (amount\ of\ 3n+1) \cdot P(O5) = \frac{k}{3} \cdot \frac{1}{2} + \frac{k}{3} \cdot \frac{1}{2} + \frac{k}{3} \cdot 1 = \frac{2}{3} k \tag{8}$$

$$EV(b) = \left(amount\ of\ implicite \frac{n}{2}\right) \cdot P(O1) + \left(amount\ of\ explicite \frac{n}{2}\right) \cdot P(O2) = \frac{k}{3} \cdot \frac{1}{2} + \frac{k}{3} \cdot \frac{1}{2} = \frac{1}{3} k \tag{9}$$

From (8) and (9) we are able to calculate the ratio of the expectation values EV(a)/EV(b) for large k

$$\frac{EV(a)}{EV(b)} = 2 \tag{10}$$

This must not be misinterpreted as the expectation value for the ratio a/b. But as a first result it shows that $\log_2(3)$ < EV(a)/EV(b) can be interpreted as a tendency which could favor contraction.

## 2.2.2 Variance analysis and expectation value EV(a)/EV(b)

From our recursive stochastic model, which is shown in table 1, we can directly follow that the probability for an increase in “a” is 3/5, and for “b” it is 2/5.
As we can map this stochastic process model onto any Collatz sequence, especially for large n, we can treat Δa and Δb as strongly dependent and correlated random variables.

$$\Delta a = \begin{cases} 0 & no\ increase\ of\ a \\ 1 & increase\ of\ a\ by + 1\ with\ probability\ P(\Delta a) = 3/5 \end{cases}$$

$$\Delta b = \begin{cases} 0 & no\ increase\ of\ b \\ 1 & increase\ of\ b\ by + 1\ with\ probability\ P(\Delta b) = 2/5 \end{cases}$$

Thus we know

$$P(\Delta a) = \frac{3}{5}$$

$$P(\Delta b) = \frac{2}{5}$$

The variances of these random variables are

$$Var(\Delta a) = EV(\Delta a^2) - EV(\Delta a)^2 = \frac{6}{25}$$

$$Var(\Delta b) = EV(\Delta b^2) - EV(\Delta b)^2 = \frac{6}{25}$$

We appreciate the fact that the total steps k=a+b and a=ΣΔa , b= ΣΔb.

To calculate the variances for Var(a) and Var(b) we must appreciate any dependency between the steps of the Collatz process. For large k we ascertain that the Δb are independent from each other, as well as the Δa are independent of each other.

From that it follows for the variances of a, and b

$$Var(a) = Var(\Delta a_1 + \Delta a_2 + \cdots + +\Delta a_k) = k \cdot \frac{6}{25} \tag{11}$$

$$Var(b) = Var(\Delta b_1 + \Delta b_2 + \cdots + +\Delta b_k) = k \cdot \frac{6}{25} \tag{12}$$

The approach of a joint distribution for both random variables is avoided in first place for the advance of the following analysis.

Now we are able to perform a sensitivity analysis of these variances onto the ratio EV(a)/EV(b):

$$\frac{a}{b} \approx \frac{EV(a) \pm \sqrt{Var(a)}}{EV(b) \pm \sqrt{Var(b)}}$$

We consider the 2 extreme cases of the upper limit for this ratio and the lower limit for it

The upper limit is

$$\left(\frac{EV(a)}{EV(b)}\right)_{max} = \frac{EV(a) + \sqrt{Var(a)}}{EV(b) - \sqrt{Var(b)}} = \frac{\frac{2}{3}k + \sqrt{\frac{6}{25}k}}{\frac{1}{3}k - \sqrt{\frac{6}{25}k}} \tag{13}$$

The lower limit

$$\left(\frac{EV(a)}{EV(b)}\right)_{min} = \frac{EV(a) - \sqrt{Var(a)}}{EV(b) + \sqrt{Var(b)}} = \frac{\frac{2}{3}k - \sqrt{\frac{6}{25}k}}{\frac{1}{3}k + \sqrt{\frac{6}{25}k}} \tag{14}$$

The following table shows the result for max and min ratios of EV(a)/EV(b) from k=3 to 243 exemplary, with some values skipped so the trend can be followed.

| k | max EV(a)/EV(b) | min EV(a)/EV(b) |
|---|---|---|
| 3 | 18,81 | 0,62 |
| 4 | 10,31 | 0,73 |
| 5 | 7,75 | 0,81 |
| 6 | 6,50 | 0,88 |
| 7 | 5,75 | 0,93 |
| 8 | 5,24 | 0,97 |
| 9 | 4,88 | 1,01 |
| 10 | 4,60 | 1,05 |
| 11 | 4,39 | 1,08 |
| 12 | 4,21 | 1,11 |
| 13 | 4,06 | 1,13 |
| 14 | 3,94 | 1,15 |
| 15 | 3,83 | 1,17 |
| 16 | 3,74 | 1,19 |
| 17 | 3,66 | 1,21 |
| 18 | 3,59 | 1,23 |
| 19 | 3,53 | 1,24 |
| 20 | 3,47 | 1,26 |
| ..... | ..... | ..... |
| 75 | 2,61 | 1,56 |
| 76 | 2,61 | 1,57 |
| 77 | 2,60 | 1,57 |
| 78 | 2,60 | 1,57 |
| 79 | 2,59 | 1,57 |
| 80 | 2,59 | 1,58 |
| 81 | 2,59 | 1,58 |
| 82 | 2,58 | 1,58 |
| 83 | 2,58 | 1,58 |
| 84 | 2,57 | 1,59 |
| 85 | 2,57 | 1,59 |
| ..... | ..... | ..... |
| 215 | 2,33 | 1,73 |
| 216 | 2,33 | 1,73 |
| 217 | 2,33 | 1,73 |
| 218 | 2,33 | 1,73 |
| 219 | 2,33 | 1,73 |
| 220 | 2,33 | 1,73 |
| 221 | 2,33 | 1,73 |
| 222 | 2,33 | 1,73 |
| 223 | 2,33 | 1,73 |
| 224 | 2,33 | 1,73 |
| 225 | 2,33 | 1,73 |
| 226 | 2,33 | 1,73 |
| 227 | 2,32 | 1,73 |
| 228 | 2,32 | 1,73 |
| 229 | 2,32 | 1,73 |
| 230 | 2,32 | 1,73 |
| 231 | 2,32 | 1,74 |
| 232 | 2,32 | 1,74 |
| 233 | 2,32 | 1,74 |
| 234 | 2,32 | 1,74 |
| 235 | 2,32 | 1,74 |
| 236 | 2,32 | 1,74 |
| 237 | 2,32 | 1,74 |
| 238 | 2,32 | 1,74 |
| 239 | 2,32 | 1,74 |
| 240 | 2,31 | 1,74 |
| 241 | 2,31 | 1,74 |
| 242 | 2,31 | 1,74 |
| 243 | 2,31 | 1,74 |

Table 2: Calculation of max and min EV(a)/EV(b) for $3 \leq k \leq 243$

The values from the table 2 give a clear indication that the Collatz sequence might produce only sequences for which the ratios for a/b are bigger than log2(3)+ $\delta_1$/b, which would mean contraction.

### 2.2.3 Derivation of a strong estimator for EV(a/b) and its variance Var(a/b)

Our main goal in this section is to derive a formula of the form

$$\frac{a}{b} = EV\left(\frac{a}{b}\right) \pm \sqrt{Var\left(\frac{a}{b}\right)} \tag{15}$$

Thus we need to calculate an estimator EV(a/b) and the variance of (a/b). Thus we are able to apply probabilistic inequalities, like Chebyshev's inequality to the result, as a/b can be treated like a random variable then.

In order to estimate EV(a/b) we use a second-order Taylor expansion of f(a,b)=a/b:

$$f(a,b) \approx f\big(EV(a),EV(b)\big) + f_a\big(EV(a),EV(b)\big)\big(a-EV(a)\big) + f_b\big(EV(a),EV(b)\big)\big(b-EV(b)\big) +$$
$$+\frac{1}{2}f_{aa}\big(EV(a),EV(b)\big)\big(a-EV(a)\big)^2 + f_{ab}\big(EV(a),EV(b)\big)\big(a-EV(a)\big)\big(b-EV(b)\big) + \frac{1}{2}f_{bb}\big(EV(a),EV(b)\big)\big(b-EV(b)\big)^2$$

Now we take expectations of this approximation. All the linear terms vanish (since EV(a-EV(a))=0) and we are left with

$$EV\left(\frac{a}{b}\right) \approx \frac{EV(a)}{EV(b)} + \frac{Cov(a,b)}{EV(b)^2} + \frac{EV(a)\cdot Var(b)}{EV(b)^3} \tag{16}$$

In order to continue we just need to derive a reasonable expression for the covariance Cov(a,b). For this we make use of the generalized variance propagation as follows

$$Var(f) = F_x \cdot C_y \cdot F_x^T$$

where $F_x$ is the scaled gradient row vector

$$F_x = \left[\sigma_{x_1}\frac{\partial f}{\partial x_1}, \dots, \sigma_{x_n}\frac{\partial f}{\partial x_n}\right]$$

$C_y$ is the diagonal correlation matrix, with elements $r_{xixj}$, which need not be symmetric.

Applied to our problem, we take into account that the two random variables Δa, Δb, thus the counter variables a and b are dependent and correlated.

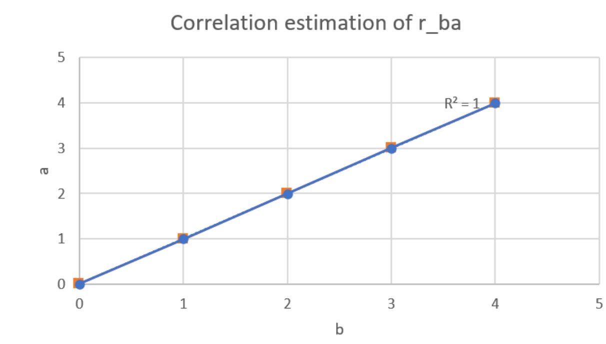
The generalized variance propagation turns out for f(a,b)=a/b, to be

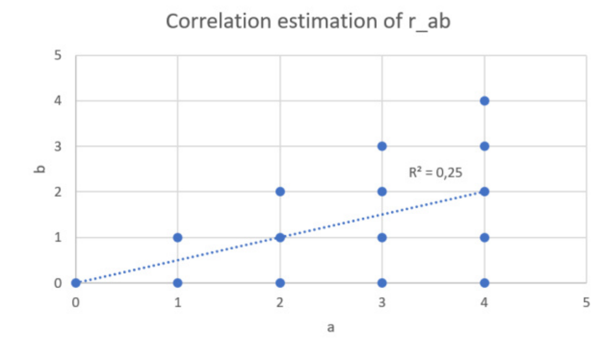
$$Var(f(a,b)) = \left(\frac{\partial f}{\partial a}\right)^2 Var(a) + \left(\frac{\partial f}{\partial b}\right)^2 Var(b) + \frac{\partial f}{\partial b}\frac{\partial f}{\partial a}\sqrt{Var(a)}\sqrt{Var(b)}(r_{ab} + r_{ba}) \tag{17}$$

where in the last expression after the "+" sign we find the Covariance as

$$Cov(a,b) = \sqrt{Var(a)}\sqrt{Var(b)}(r_{ab} + r_{ba}) \tag{18}$$

We need an estimation of $r_{ab}$ and $r_{ba}$ and find it by analyzing the behavior of a and b with respect to the Collatz operations:




| | |
|---|---|
| We find r_ba=1 | and r_ab=0.5 |

In order to calculate (17) and (18) symbolically we need following additional expressions

$$f(a,b) = \frac{a}{b}$$

$$\frac{\partial f}{\partial a} = \frac{1}{b} = \frac{1}{EV(b)} = \frac{3}{k}$$

$$\frac{\partial f}{\partial b} = -\frac{a}{b^2} = -\frac{EV(a)}{EV(b)^2} = -\frac{6}{k}$$

$$Var(a) = \frac{6}{25}k$$

$$Var(b) = \frac{6}{25}k$$

Our final result is

$$\frac{a}{b} = EV\left(\frac{a}{b}\right) \pm \sqrt{Var\left(\frac{a}{b}\right)} = \left(2 + \frac{231}{25k}\right) \pm \frac{\sqrt{108}}{5\sqrt{k}} \tag{19}$$

The following table shows the result for max and min expectation values for EV(a/b) from k=3 to 243 exemplary, with some values skipped so the trend can be followed.

| k | EV(a/b) | Sigma(a/b) | EV(a/b)+sigma(R) | EV(a/b)-sigma(R) |
|---|---|---|---|---|
| 3 | 5,08 | 1,20 | 6,28 | 3,88 |
| 4 | 4,31 | 1,04 | 5,35 | 3,27 |
| 5 | 3,85 | 0,93 | 4,78 | 2,92 |
| 6 | 3,54 | 0,85 | 4,39 | 2,69 |
| 7 | 3,32 | 0,79 | 4,11 | 2,53 |
| 8 | 3,16 | 0,73 | 3,89 | 2,42 |
| 9 | 3,03 | 0,69 | 3,72 | 2,33 |
| 10 | 2,92 | 0,66 | 3,58 | 2,27 |
| 11 | 2,84 | 0,63 | 3,47 | 2,21 |
| 12 | 2,77 | 0,60 | 3,37 | 2,17 |
| 13 | 2,71 | 0,58 | 3,29 | 2,13 |
| 14 | 2,66 | 0,56 | 3,22 | 2,10 |
| 15 | 2,62 | 0,54 | 3,15 | 2,08 |
| 16 | 2,58 | 0,52 | 3,10 | 2,06 |
| 17 | 2,54 | 0,50 | 3,05 | 2,04 |
| 18 | 2,51 | 0,49 | 3,00 | 2,02 |
| 19 | 2,49 | 0,48 | 2,96 | 2,01 |
| 20 | 2,46 | 0,46 | 2,93 | 2,00 |
| ... | ... | ... | ... | ... |
| 75 | 2,12 | 0,24 | 2,36 | 1,88 |
| 76 | 2,12 | 0,24 | 2,36 | 1,88 |
| 77 | 2,12 | 0,24 | 2,36 | 1,88 |
| 78 | 2,12 | 0,24 | 2,35 | 1,88 |
| 79 | 2,12 | 0,23 | 2,35 | 1,88 |
| 80 | 2,12 | 0,23 | 2,35 | 1,88 |
| 81 | 2,11 | 0,23 | 2,35 | 1,88 |
| 82 | 2,11 | 0,23 | 2,34 | 1,88 |
| 83 | 2,11 | 0,23 | 2,34 | 1,88 |
| 84 | 2,11 | 0,23 | 2,34 | 1,88 |
| 85 | 2,11 | 0,23 | 2,33 | 1,88 |
| ... | ... | ... | ... | ... |
| 215 | 2,04 | 0,14 | 2,18 | 1,90 |
| 216 | 2,04 | 0,14 | 2,18 | 1,90 |
| 217 | 2,04 | 0,14 | 2,18 | 1,90 |
| 218 | 2,04 | 0,14 | 2,18 | 1,90 |
| 219 | 2,04 | 0,14 | 2,18 | 1,90 |
| 220 | 2,04 | 0,14 | 2,18 | 1,90 |
| 221 | 2,04 | 0,14 | 2,18 | 1,90 |
| 222 | 2,04 | 0,14 | 2,18 | 1,90 |
| 223 | 2,04 | 0,14 | 2,18 | 1,90 |
| 224 | 2,04 | 0,14 | 2,18 | 1,90 |
| 225 | 2,04 | 0,14 | 2,18 | 1,90 |
| 226 | 2,04 | 0,14 | 2,18 | 1,90 |
| 227 | 2,04 | 0,14 | 2,18 | 1,90 |
| 228 | 2,04 | 0,14 | 2,18 | 1,90 |
| 229 | 2,04 | 0,14 | 2,18 | 1,90 |
| 230 | 2,04 | 0,14 | 2,18 | 1,90 |
| 231 | 2,04 | 0,14 | 2,18 | 1,90 |
| 232 | 2,04 | 0,14 | 2,18 | 1,90 |
| 233 | 2,04 | 0,14 | 2,18 | 1,90 |
| 234 | 2,04 | 0,14 | 2,18 | 1,90 |
| 235 | 2,04 | 0,14 | 2,17 | 1,90 |
| 236 | 2,04 | 0,14 | 2,17 | 1,90 |
| 237 | 2,04 | 0,14 | 2,17 | 1,90 |
| 238 | 2,04 | 0,13 | 2,17 | 1,90 |
| 239 | 2,04 | 0,13 | 2,17 | 1,90 |
| 240 | 2,04 | 0,13 | 2,17 | 1,90 |
| 241 | 2,04 | 0,13 | 2,17 | 1,90 |
| 242 | 2,04 | 0,13 | 2,17 | 1,90 |
| 243 | 2,04 | 0,13 | 2,17 | 1,90 |

Table 3: Calculation of EV(a/b) for $3 \leq k \leq 243$

## 2.2.4 Conclusive step: Contraction implications due to the existence of Var(a/b)

The result in section 2.2.3 means, that for any k=a+b, there must be at least one a/b which is within EV(a/b)±Sqrt(Var(a/b)), otherwise the existence of the variance would be contradicted at all.
In fact due to probability theory, for large k, there should be 68,3% of all possible ratios a/b for a given k=a+b, within that range.

From Cantelli's inequality it can be calculated that

$$P\left(\frac{a}{b} > EV\left(\frac{a}{b}\right) - \sqrt{Var\left(\frac{a}{b}\right)}\right) \geq \frac{1}{2}$$

This means that the probability that for a given k=a+b, the ratio a/b is bigger than EV(a/b)-Sqrt(Var(a/b)), is at least ½.

The amount of steps k=a+b is created by the Collatz process from a starting number $n_0$. There will be always be some $n_0$ for the same k, which do not fulfill to be within the range of EV(a/b)±Sqrt(Var(a/b)) yet, these are the sequences which haven't contracted yet, so for them $n_k \geq n_0$.

But in this case the Collatz process continues, a and b will be incremented according the Collatz process (see section 1), thus that larger k's emerge. As there are infinite numbers k=a+b possible, at least one of them must be mappable to at least one n0, such that for a sufficient large k=a+b, the ratio a/b must be within the range of EV(a/b)±Sqrt(Var(a/b)), which finally means contraction $n_k < n_0$ of the sequence for which the Collatz process was started at $n_0$.

# 3. Bit-length growth constraint

In the following step of the proof we derive a bit length-growth constraint formula from the Evolution formula for $n_k$ (3). Afterwards we compare it with the structural bound of bit-length growth that is maximally possible due to the defined Collatz operations in the original definition of the Collatz Conjecture (see section 1).

## 3.1 Bit-length evolution

Within the bit-length representation, the bit length Len(x) of any real number x can be calculated by

$$Len(x) = ceil(log_2(x)) + 1$$

where ceil() is the ceiling function.

From the evolution formula (3) we can calculate the bit length evolution after k=a+b operations:

$$Len(n_k) = ceil(log_2(n_0) + b \cdot log_2(3) - a + \delta_1) + 1 = ceil(log_2(n_0) + b \cdot log_2(3) + \delta_1) + 1 - a$$

where we applied the rule for the ceiling function, that ceil(x+a)=ceil(x)+a, in case x is a real number and a is a positive integer. Now we dissolve the ceiling function, and need to introduce a real number $R_1$, due to the neglected rounding effect of the ceiling function, where $0<R_1<1$.

$$Len(n_k) = log_2(n_0) + b \cdot log_2(3) + 1 + \delta_1 + R_1 - a \tag{20}$$

with the known correction term $\delta_1$ from (5).

$$\delta_1 = log_2\left(1 + \frac{1 - 3^{-b}}{n_0 \cdot 2}\right)$$

## 3.2 Bit-length growth by the structure of the Collatz process

Regarding the maximum possible bit length growth per Collatz operations, we consider the operation in case n is odd, which is the 3n+1 operation, that increments the counter b. Each such odd operation can increase the bit length by a maximum of 2bits. This operation will directly be followed by the operation (n/2), which may reduce the bit length by -1bit.

The maximum bit-length increase is therefore after k operations

$$Len(n_k) \leq Len(n_0) + 2b - a = ceil\big(log_2(n_0)\big) + 1 + 2b - a = log_2(n_0) + 1 + 2\mathrm{b} - \mathrm{a} + R_2 \tag{21}$$

where we dissolved the ceiling function, and needed to introduce another real number $R_2$, due to the neglected rounding effect of the ceiling function, where $0<R_2<1$

This is the maximum bit length increase and plays the role of a maximum upper bound to the bit-length evolution.

## 3.3 Conclusive step: Bit length growth constraint

Now we compare the right hand sides of the bit-length evolution formula (20) with the formula for the maximum possible bit-length increase (21) due to the structure of the Collatz operations defined in the Collatz Conjecture and get

$2b \geq b\,log_2(3) + \delta_1 + R_1 - R_2$

We divide this inequality by b

$$2 \geq log_2(3) + \frac{\delta_1}{b} + \frac{R_1}{b} - \frac{R_2}{b} \quad (22)$$

Without loss of generality, without circularity, without presuming contraction, we can make use of the fact that we already have shown for large enough and already known starting integer values n in the worst-case scenarios, see the result in (19), that for large k it must hold that

$$\frac{a}{b} \leq 2 \quad (23)$$

We substitute this into (22), and finally get an absolute minimum lower bound for the ratio a/b

$$\frac{a}{b} = log_2(3) + \frac{\delta_1}{b} + \frac{R_1}{b} - \frac{R_2}{b} \quad (24)$$

with $0<R_1<1$, $0<R_2<1$

That means that the structure of the Collatz process itself avoids the appearance of sequences which can harm this absolute lower bound of a/b for large k and thus large n.

Due to the fact that we derived this upper limit (30) for the bit length growth for large n, thus the smallest possible ratio a/b for any sequence, we must state that for any starting number n it must be valid that

$$\frac{a}{b} \geq log_2(3) + \frac{\delta_1}{b} + \frac{R_1}{b} - \frac{R_2}{b} \quad (25)$$

with $0<R_1<1$, $0<R_2<1$

# 4. The Proof of the Collatz conjecture

## 4.1 Conclusive Step: Forced Contraction

The contraction requirement $n_k<n_0$ resulted in the fulfillment of a needed inequality for the ratio a/b in (7) as follows:

$$\frac{a}{b} > log_2(3) + \frac{\delta_1}{b} = log_2(3) + \frac{log_2\left(1 + \frac{1-3^{-b}}{n_0 \cdot 2}\right)}{b} \tag{7}$$

So the correction term $\delta_1$ shrinks to zero for growing b, $n_0$

In section 3.3 we received (25) as an absolute lower bound for the ratio a/b

$$\frac{a}{b} \geq log_2(3) + \frac{log_2\left(1 + \frac{1-3^{-b}}{n_0 \cdot 2}\right)}{b} + \frac{R_1}{b} - \frac{R_2}{b} \tag{25}$$

with $0<R_1<1$, $0<R_2<1$

In case R1 > R2, the emerging sequence for that starting number is structurally forced to contract, as the ratio in (25) is then bigger than the requirement for contraction (7).

In section 2.2.3 we derived EV(a/b) and its variance Var(a/b) (19)

$$\frac{a}{b} = EV\left(\frac{a}{b}\right) \pm \sqrt{Var\left(\frac{a}{b}\right)} = \left(2 + \frac{231}{25k}\right) \pm \frac{\sqrt{108}}{5\sqrt{k}} \tag{19}$$

The result in section 2.2.3 means, that for any k=a+b, there must be at least one a/b which is within EV(a/b)±Sqrt(Var(a/b)), otherwise the existence of the variance Var(a/b) would be contradicted at all. The amount of steps k=a+b is created by the Collatz process from a starting number $n_0$. There are some $n_0$ for the same k, which do not fulfill to be within the range of EV(a/b)±Sqrt(Var(a/b)) yet, these are the sequences starting from $n_0$, which haven't contracted after k steps yet, so for them $n_k \geq n_0$.

But in this case the Collatz process continues, a and b will be incremented according the Collatz process (see section 1), thus that larger k's emerge. As there are infinite numbers k=a+b possible, at least one of them must be mappable to at least one $n_0$, such that for a sufficient large k=a+b, the ratio a/b must be within the range of EV(a/b)±Sqrt(Var(a/b)), which finally means contraction $n_k<n_0$ of the sequence for which the Collatz process was started at $n_0$.

## 4.2 Inductive Step: Full proof of the Collatz Conjecture

We will now prove that for any integer n>1, the Collatz sequence starting at n eventually reaches 1 by strong induction.

We define the Collatz Reachability Statement:

P(n): The Collatz sequence starting at n eventually reaches 1.

Base case:

We verify manually by applying the Collatz operations (see section 1)

P(n=4): 4→2→1

Inductive hypothesis:

If we have shown that P(j) holds for all 2≤j≤n. That is, for all j less than or equal to n, the Collatz sequence starting at j reaches 1.

Inductive step: Prove P(n+1)

From the conclusive steps in section 4.1 it is shown that the Collatz operations applied to any positive integer n+1 will finally lead to a positive number n<n+1 due to the fact that contraction is forced (see 4.1). Then by the inductive hypothesis, since all smaller numbers down to 2 are shown to reach 1, so must n+1.

We have finally and fully proven that the Collatz operations defined in Section 1 applied to any positive integer n>1 can only produce sequences that contract to 1.

# References

[1] Jeffrey C. Lagarias & Alan Weiss (1992)
"The $3x + 1$ problem: two stochastic models." Annals of Applied Probability, 2(1), 229–261.

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"The $3x + 1$ problem and its generalizations." American Mathematical Monthly, 92(1), 3–23.