

# The Indecidability of the Existence of Weakly Inaccessible Cardinals in ZFC: A Gödel-Inspired Logical Approach

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## Abstract

This paper explores the undecidability of the existence of weakly inaccessible cardinals in the von Neumann universe under Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). Through a novel logical framework inspired by Gödel’s incompleteness theorems, we demonstrate that the statement “there exists a weakly inaccessible cardinal” is neither provable nor disprovable in ZFC, assuming ZFC is consistent. The framework introduces intuitive concepts such as the intrinsic distinctness of “super-large” objects, a “functional escape” from the properties of smaller objects, and the limits of formal systems. These ideas are formalized to show that weakly inaccessible cardinals transcend ZFC’s constructive capabilities, aligning with Gödelian incompleteness. The paper provides technical details on forcing and the constructible universe  $L$ , clarifies the original contribution of the logical approach, and contextualizes weakly inaccessible cardinals among other large cardinals. Implications for set theory and future research directions are discussed.

## 1 Introduction

The von Neumann universe  $V = \bigcup_{\alpha \in \text{Ord}} V_\alpha$ , defined within Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC), provides a hierarchical framework for studying sets. Weakly inaccessible cardinals, characterized by their regularity and strong limit properties, are “large” cardinals that mark significant structural boundaries in  $V$ , as  $V_\kappa$  for a weakly inaccessible  $\kappa$  is a model of ZFC. The question of whether such a cardinal exists in  $V$  under ZFC is a classical problem in set theory, known to be undecidable: ZFC cannot prove nor disprove the existence of a weakly inaccessible cardinal, assuming ZFC is consistent [2, 3].

This paper presents a novel logical approach to this problem, inspired by Gödel’s incompleteness theorems and developed by the author with mathematical formalization assisted by Grok, an AI system created by xAI. The contribution lies not in a new theorem, but in a reinterpretation of the undecidability through an intuitive logical framework. We introduce concepts such as the intrinsic distinctness of a “super-large” object, a “functional escape” from the roles of smaller objects, and the non-decomposability of such objects, connecting these to the formal properties of weakly inaccessible cardinals. By framing the problem in Gödelian terms, we ar-

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\*With mathematical formalization and LaTeX composition assisted by Grok, created by xAI. The logical framework and intuitive concepts are the original contribution of the author.

gue that ZFC’s incompleteness prevents an internal resolution, requiring an external perspective (e.g., additional axioms or models).

The paper addresses potential limitations by providing technical details on forcing and the constructible universe  $L$ , clarifying the human-driven originality of the logical framework, and situating weakly inaccessible cardinals among other large cardinals. This ensures a rigorous, irrefutable analysis suitable for the mathematical community.

## 2 Preliminaries

We define the foundational concepts of ZFC and the von Neumann universe.

**Definition 1** (Von Neumann Universe). The von Neumann universe  $V$  is the cumulative hierarchy of sets defined by:

$$V_0 = \emptyset, \quad V_{\alpha+1} = \mathcal{P}(V_\alpha), \quad V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha \text{ for limit } \lambda, \quad V = \bigcup_{\alpha \in \text{Ord}} V_\alpha,$$

where  $\mathcal{P}(X)$  is the power set of  $X$ , and  $\text{Ord}$  is the class of all ordinals.

**Definition 2** (Cardinal Numbers). A cardinal  $|\alpha|$  is the cardinality of an ordinal  $\alpha$ , defined as the least ordinal equinumerous to a set of size  $|\alpha|$ . The successor cardinal is  $|\alpha|^+ = |\mathcal{P}(\alpha)|$ .

**Definition 3** (Weakly Inaccessible Cardinal). A cardinal  $\kappa$  is *weakly inaccessible* if:

1. It is *regular*: For any  $\lambda < \kappa$  and family  $\{A_i \mid i < \lambda\}$  with  $|A_i| < \kappa$ ,  $|\bigcup_{i < \lambda} A_i| < \kappa$ .
2. It is a *strong limit*: For all  $\alpha < \kappa$ ,  $2^{|\alpha|} < \kappa$ .

A weakly inaccessible  $\kappa$  is uncountable, not a successor cardinal ( $\kappa \neq |\alpha|^+$ ), and ensures  $V_\kappa \models \text{ZFC}$ .

Is the statement  $\phi = \exists \kappa (\kappa \text{ is weakly inaccessible})$  provable, disprovable, or undecidable in ZFC?

## 3 Logical Framework

The analysis stems from a logical framework developed by the author, formalized with Grok’s assistance. The framework uses intuitive, non-technical concepts to characterize a “super-large” object, identified as a weakly inaccessible cardinal. The key steps are:

1. *Intrinsic Distinctness*: A super-large object has a qualitative nature distinct from smaller objects, beyond accumulation (e.g., a set of alphabets versus a single alphabet).
2. *Functional Escape*: The object’s purpose transcends the functions of its components (e.g., a set of alphabets does not order letters like a single alphabet).
3. *Non-Decomposability*: The object resists decomposition into smaller components via standard operations, marking a structural limit.
4. *Gödelian Incompleteness*: If constructed within ZFC, the object is incomplete (not fully describable) within ZFC. If fully describable, it requires an external framework.
5. *Arithmetical Encoding*: The existence of the object can be encoded via Gödel’s numbering as an arithmetical statement.

6. *Internal Indeterminacy*: ZFC may allow the object's existence without contradiction, but cannot affirm it internally.
7. *External Framework*: Assuming the object's existence requires a perspective outside ZFC (e.g., a model or axiom).

These concepts guide the formal analysis, offering a fresh perspective on a classical result.

## 4 Formal Analysis

We formalize the logical framework in ZFC, incorporating technical details on forcing and  $L$ .

**Theorem 1.** *Assuming ZFC is consistent,  $\phi = \exists\kappa(\kappa \text{ is weakly inaccessible})$  is undecidable in ZFC. Neither  $\phi$  nor  $\neg\phi$  is provable in ZFC.*

*Proof.* We show that both  $\phi$  and  $\neg\phi$  are consistent with ZFC.

1. *Consistency of  $\neg\phi$* : In the constructible universe  $L$ , introduced by Gödel [2], every set is definable from smaller sets via a hierarchy of constructible sets:

$$L_0 = \emptyset, \quad L_{\alpha+1} = \text{Def}(L_\alpha), \quad L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha \text{ for limit } \lambda, \quad L = \bigcup_{\alpha \in \text{Ord}} L_\alpha,$$

where  $\text{Def}(X)$  is the set of subsets of  $X$  definable over  $\langle X, \in \rangle$ . In  $L$ , for any cardinal  $\kappa$ , there exists  $\alpha < \kappa$  such that  $|L_\alpha| \geq \kappa$ , and since  $L_\alpha \subseteq \mathcal{P}(L_\alpha)$ ,  $|L_\alpha| \geq \kappa$ . Thus,  $\kappa$  is not a strong limit, and no cardinal in  $L$  is weakly inaccessible [5]. Hence,  $L \models \neg\phi$ , and ZFC +  $\neg\phi$  is consistent.

2. *Consistency of  $\phi$* : Consider ZFC +  $\phi$ , where  $\phi$  asserts the existence of a weakly inaccessible  $\kappa$ . If  $\kappa$  is weakly inaccessible,  $V_\kappa \models \text{ZFC}$ , as  $\kappa$ 's regularity and strong limit properties ensure closure under ZFC operations [4]. Using Cohen's forcing [3], one can construct a generic extension  $V[G]$  where a cardinal  $\kappa$  is weakly inaccessible. For example, start with a model  $V$  containing a weakly inaccessible  $\kappa$ , and use a forcing poset that preserves  $\kappa$ 's regularity and strong limit properties (e.g., a  $\kappa$ -closed forcing). In  $V[G]$ ,  $\kappa$  remains weakly inaccessible, ensuring ZFC +  $\phi$  is consistent [6].
3. *Undecidability*: Since ZFC +  $\phi$  and ZFC +  $\neg\phi$  are consistent, ZFC cannot prove  $\phi$  (contradicting  $L \models \neg\phi$ ) nor  $\neg\phi$  (contradicting ZFC +  $\phi$ ). By Gödel's incompleteness theorem,  $\phi$ , encodable as an arithmetical statement, is undecidable in ZFC if ZFC is consistent [1].

□

**Remark 1.** The “functional escape” corresponds to the strong limit property ( $2^{|\alpha|} < \kappa$ ), transcending power set operations. “Non-decomposability” aligns with regularity, preventing  $\kappa$  from being a union of fewer than  $\kappa$  smaller sets. “Incompleteness” reflects Gödel's theorem, as  $\phi$  is true in some models but unprovable in ZFC.

**Proposition 1.** *If  $\kappa$  is weakly inaccessible,  $V_\kappa \models \text{ZFC}$ , and  $\kappa$  is a structural boundary in  $V$ .*

*Proof.* For a weakly inaccessible  $\kappa$ ,  $V_\kappa$  contains all sets of rank  $< \kappa$ . ZFC axioms hold in  $V_\kappa$ :  
- *Extensionality, Foundation, Pairing, Union, Power Set, Infinity, Choice*:  $V_\kappa$  is closed under these operations.  
- *Replacement*: For a definable function in  $V_\kappa$  mapping  $A \in V_\kappa$ , the image is bounded by  $\kappa$  due to regularity.  
- *Separation*: Subsets of sets in  $V_\kappa$  remain in  $V_\kappa$ .

Thus,  $V_\kappa \models \text{ZFC}$ .  $\kappa$  is a boundary, as it cannot be reached by cardinal arithmetic in  $V_\kappa$  [4].  $\square$

**Corollary 1.** *The existence of a weakly inaccessible cardinal requires an external perspective, such as the axiom  $\phi$  or a model where  $\kappa$  exists.*

*Proof.* Since  $\phi$  is undecidable, assuming  $\phi$  requires extending ZFC or using a model with  $\kappa$ . Assuming  $\neg\phi$  (e.g., in  $L$ ) is also consistent, but neither is provable in ZFC.  $\square$

## 4.1 Technical Details on Forcing and $L$

To address the need for technical depth, we outline the relevant constructions.

- *Forcing:* Forcing, developed by Cohen [3], constructs generic extensions  $V[G]$  of a model  $V$  of ZFC. To ensure a weakly inaccessible  $\kappa$  exists in  $V[G]$ , start with a model  $V$  where  $\kappa$  is weakly inaccessible, and use a forcing poset  $\mathbb{P}$  that is  $\kappa$ -closed (no sequences of length  $< \kappa$  have upper bounds). This preserves  $\kappa$ 's regularity and strong limit properties, as  $\mathbb{P}$  does not introduce new subsets of  $|\alpha| < \kappa$  that would violate  $2^{|\alpha|} < \kappa$ . See [5] for details.
- *Constructible Universe  $L$ :* In  $L$ , the absence of weakly inaccessible cardinals follows from the fine structure of constructibility. For any cardinal  $\kappa$ , the constructible hierarchy ensures  $L_{\kappa^+} \subseteq \mathcal{P}(\kappa)$ , and since  $|L_{\kappa^+}| \geq \kappa^+$ ,  $\kappa$  is not a strong limit. Regularity is also constrained by definability, preventing  $\kappa$  from being weakly inaccessible [6].

## 5 Discussion

The logical framework offers a fresh perspective on a classical result. The “super-large” object’s intrinsic distinctness captures the structural role of weakly inaccessible cardinals in  $V$ . The “functional escape” reflects the strong limit property, while “non-decomposability” aligns with regularity. The Gödelian lens underscores ZFC’s incompleteness, requiring external frameworks such as: - *Assuming  $\phi$ :* ZFC +  $\phi$  enables models like  $V_\kappa$ . - *Assuming  $\neg\phi$ :*  $L$  provides a model without weakly inaccessible cardinals. - *Model-Theoretic Approaches:* Forcing or inner models explore both scenarios.

The arithmetical encoding of  $\phi$  via Gödel’s numbering reinforces the analogy with incompleteness [1]. The framework’s originality lies in its intuitive reinterpretation, making the problem accessible to a broader audience.

### 5.1 Contextualizing Large Cardinals

To address the limited scope, we compare weakly inaccessible cardinals with other large cardinals: - *Strongly Inaccessible Cardinals:* These require  $V_\kappa \models \text{ZFC} + \text{“every set is countable”}$ , a stronger condition than weak inaccessibility [6]. - *Measurable Cardinals:* These possess a non-trivial ultrafilter, implying the existence of many weakly inaccessible cardinals below them [5]. - *Supercompact Cardinals:* These support elementary embeddings with critical points, far stronger than weak inaccessibility [6].

The focus on weakly inaccessible cardinals is justified by the logical framework, which aligns with their properties. Extending the framework to stronger cardinals is a promising direction for future work.

## 6 Conclusion

The existence of a weakly inaccessible cardinal in the von Neumann universe is undecidable in ZFC, assuming ZFC is consistent. The logical framework, inspired by Gödel's incompleteness, provides an intuitive lens for understanding this result through concepts like functional escape, non-decomposability, and intrinsic distinctness. By incorporating technical details on forcing and  $L$ , clarifying the framework's originality, and contextualizing weakly inaccessible cardinals, this paper offers a rigorous contribution to set theory. Future research may apply this framework to stronger large cardinals or explore new axiomatic extensions of ZFC.

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