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Miscellaneous formulas about number theory

Abstract:

This paper aims to present several formulas concerning number theory. In particular, I establish deep connections between various constants such as the Euler-Mascheroni constant, pi, square roots, logarithms, the exponential function, and the gamma function.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2\left(x+\exp(2\pi i k)\right)}=\frac{\pi^2}{6+6x}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n+\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}=e-\frac{\pi^2}{12}$$

$$3+\cfrac{1}{7+\cfrac{1}{16+\frac{1}{100}}}\approx 3.14159$$

$$\prod_{k=1}^{+\infty}\frac{4k^2}{4k^2-1}=\frac{\pi}{2}$$

$$\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{\left(\frac{(2n-3)!!}{(2n-2)!!}\right)^2*\left(\frac{\pi}{2}\right)}{n}=\frac{\Gamma(\frac{1}{4})^2}{2\sqrt{2\pi}}-\frac{2\sqrt{2}*\pi^{\frac{3}{2}}}{\Gamma(\frac{1}{4})^2}$$

$$\frac{4\,\Gamma\left(\frac{5}{4}\right)}{\left(2\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;-1\right)\Gamma\left(\frac{3}{4}\right)\right)^2}\Gamma\left(\frac{1}{4}\right)=\pi$$

$$\left(\int_0^1\int_0^1\frac{x}{1+x^2y^2}\,dx\,dy\right)\times 4+\log(4)=\pi$$

$$6\times\frac{\frac{\log^2(2)}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n\,(2n+1)}}{\log^2(2)}=\pi$$

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\left(\tfrac{1}{k}-ln(1+\tfrac{1}{k})+\tfrac{Cte}{n}\right)=\gamma+Cte$$

$$\int_0^{\infty}\frac{\cos(x)}{1+x^2}\,dx=\frac{\pi}{2e}$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n=e$$

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)}$$

$$\sum_{k=1}^{\infty}\log\left(\frac{4k^2}{4k^2-1}\right)=\log\left(\frac{\pi}{2}\right)\approx 0.4516$$

$$\frac{\sum_{n=1}^{\infty}\frac{1}{n!}\left(1+\frac{1}{n+1}\right)+3}{2}=e$$

Conjectures

Conjecture 1

Definitions:

Here I present a novel conjecture using basic mathematical tools like the sum of the divisors of an integer n called $\sigma(n)$, the sum of the squares of the positive divisors of n called $\sigma_2(n)$ I also use the prime-counting function which is the function counting the number of prime numbers less than or equal to some real number n. The prime-counting function is called $\pi(n)$

Conjecture:

We introduce the following expression called A:

$$A=\sigma_2(\pi(n)-\sigma(n+2))$$

We focus on numbers ends with 2. I calculate $A-1$ and so the new number ends with 1. Then I calculate the square root of this number ends with 1. When the number is an integer, it is always prime.

Example:

Let $n=100547$, we have $A=\sigma_2(9639-\sigma(100549))=8264809922$ We have $A-1=8264809921$ and we calculate the square root of 8264809921 and we have $\sqrt{A-1}=\sqrt{8264809921}=90911$ and 90911 is prime.

Conjecture 2

n is a natural number >1, $\varphi(n)$ denotes the Euler's totient function, P_n is the n^{th} prime number and $\sigma(n)$ is the sum of the divisors of n. Consider the expression:

$$F(n)=\varphi(|P_{n+2}-\sigma(n)|)+1$$

Conjecture: when $F(n)\equiv 3\pmod{20}$ then this number is a prime or not. When the number is not a prime it can be a power of prime by calculating $|P_{n+2}-\sigma(n)|=p^k$ (p prime, k a natural number >1).

Examples:

$n=10\,270\,001\,113$, we have:

$$F(10\,270\,001\,113)=\varphi(|P_{10\,270\,001\,115}-\sigma(10\,270\,001\,113)|)+1$$
$$=\varphi(259\,189\,944\,599-10\,468\,624\,896)+1=248\,721\,319\,703$$

which is prime because it ends by 03.

A counterexample is found with $n=680$:

$$F(680)=\varphi(|P_{682}-\sigma(680)|)+1=\varphi(5101-1620)+1=3423$$

which is not prime but we have $P_{n+2}-\sigma(n)=p^2$, more precisely it is the square of 59.

Interestingly for $n\leq 526\,388\,126$ (calculations with PARI/GP) all counterexamples are the power of prime.

Another example is found for $k=6$, this is $n=526\,388\,126$. In this case, we have:

$$F(n)=10\,549\,870\,323$$

which is not prime and $|P_{n+2}-\sigma(n)|=47^6$ (here $k=6$).

Conjecture 3

ϕ denotes the Euler's totient function, a denotes a natural number >1 and n denotes a natural number multiple of 4. If the remainder of the division of $\phi\,(a^n-2)+1$ by n is equal to $n-1$ then $\phi(a^n-2)+1$ is always a prime number.

Example with $a=119$ and $n=20$ The remainder of the division of $\phi\,(119^{20}-2)+1$ by 20 is equal to 19 then $\phi\,(119^{20}-2)+1$ is prime.

Miscellaneous formulas about number theory

$$\frac{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{1}{4}\right)^k}{2}=\sin\left(\frac{1}{2}\right)$$

$$\lim_{n\to\infty}\left(1+\frac{1}{2^{n+1}}\right)^{2^n}=\sqrt{e}$$

$$2\sqrt{\pi}=\frac{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\times\frac{2^k}{k!}\times\frac{\Gamma\left(k+\frac{1}{2}\right)}{(1+x)^k}}{\log\left(\frac{1}{2}\left(1+\sqrt{\frac{3+x}{1+x}}\right)\right)}\qquad\text{True for }|x+1|>2$$

$$\sqrt{2}=\frac{\sum_{k=1}^{\infty}\frac{\Gamma\left(k+\frac{1}{2}\right)}{k!\times 2^k}}{\sqrt{\pi}}+1$$

$$10\tan^{-1}\left(\frac{\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}}{1+\frac{1}{4}\left(1+\sqrt{5}\right)}\right)=\pi$$

$$\frac{\exp\left(\frac{\log(n)}{2}+\frac{1}{4}\log(\log(n))\right)}{\sqrt[4]{\log(n)}}=\sqrt{n}$$

$$\frac{10\sqrt{5+\sqrt{5}}\;\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right)}{100\times 2^{7/10}\,\Gamma\left(1+\frac{1}{10}\right)}=\sqrt{\pi}$$

$$\frac{2\,\Gamma\left(\frac{1}{2}+\frac{1}{10}\right)\sqrt{10}\;\Gamma\left(\frac{2}{5}\right)^2}{4\,\Gamma\left(\frac{1}{2}-\frac{1}{10}\right)\sqrt{5-\sqrt{5}}}=\pi$$

$$\left(\frac{256}{315}\times\frac{\Gamma\left(\frac{10+1}{2}\right)}{\Gamma\left(\frac{10}{2}\right)}\right)^2=\pi$$

$$2\cos^{-1}\left(\frac{\exp(-\zeta(3))}{2}\left(1+\frac{1}{\log(7+\sqrt{2})}\right)\right)+2\sin^{-1}\left(\frac{1+\log(7+\sqrt{2})}{2\exp(\zeta(3))\log(7+\sqrt{2})}\right)=\pi$$

$$\frac{\pi-\int_0^{\infty}(1-x^2)\times\frac{\log(1+x^2)}{x^2(1+x^2)}\,dx}{2\pi}=\log(2)$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n+\frac{1}{6}\sin\left(\frac{x}{n}\right)}\right)^n=\exp(x)$$

$$\pi=\left(\left(\int_0^1\frac{x^2}{\sqrt{1-x^4}}\,dx\right)\left(3\times\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}\right)\right)^2$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log\left(n+\frac{1}{2+\frac{1}{\log(n+1)}}\right)\right)=\gamma$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log\left(n+\frac{1}{2}+\frac{1}{12n}\right)\right)=\gamma$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log\left(n+\frac{1}{2}+\frac{1}{\sqrt{12n+\sqrt{n}}}\right)\right)=\gamma$$

$$\pi=-12\sum_{n=2}^{10}\frac{\mu(n)}{n}\sin^{-1}\left(\frac{1}{\phi(n)+1}\right)-\frac{12}{7}\sin^{-1}\left(\frac{1}{7}\right)-\frac{6}{5}\sin^{-1}\left(\frac{1}{5}\right)-2\sin^{-1}\left(\frac{1}{3}\right)$$

$$\frac{1}{2\sinh\left(\frac{3}{2}\left(\frac{8\operatorname{csch}^{-1}(2)}{\sqrt{5}}-\sum_{n=1}^{\infty}(-1)^{n+1}\times\frac{4-\frac{1}{2^n}}{n\binom{2n}{n}}\right)\right)}=\sqrt{2}$$

$$1+\frac{\log(n)}{2}\lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k}n^{j/(2k)}=\sqrt{n}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\times\frac{\sin\left(n\times\frac{\pi}{2}\right)}{n^2}=C$$

$$\int_0^1\tan^{-1}(x)\times\frac{1-x+\frac{x^2}{2}}{x}\,dx+\frac{1}{8}\left(2+\pi-\log(16)\right)=C$$

$$\ln(2)=1+\int_0^{\infty}\left(\frac{1}{e^t-1}-\frac{1}{te^t}\right)e^{-t}\,dt-\gamma$$

$$\lim_{n\to\infty}\frac{\left(-\pi+2\sqrt{3}\csc\left(\frac{\pi}{6n}\right)H_n-2\sqrt{3}\csc\left(\frac{\pi}{6n}\right)\log(n)\right)\sin\left(\frac{\pi}{6n}\right)}{2\sqrt{3}}=\gamma$$

$$e^x=\lim_{n\to\infty}\left(1+\frac{x}{n}+\frac{x^2}{2n^2}+\frac{x^3}{6n^3}\right)^n$$

$$\pi=\lim_{n\to\infty}\frac{2n\cdot\sin\left(\frac{\pi}{2n}\right)}{1+\frac{\ln(n)}{n}}$$

$$\sqrt{x}=\frac{1}{\sqrt{2\pi}}\cosh^{-1}\left(\sqrt{\pi}\cdot\sum_{n=0}^{\infty}\frac{(\pi x)^n}{2^n n!\,\Gamma(n+1/2)}\right)$$

$$e^2=\frac{1+\coth(1)}{\coth(1)-1}$$

$$e^{2n}=\frac{1+\coth(n)}{\coth(n)-1}$$

$$\frac{5}{\pi}=\frac{\left(\frac{2^{1/2}\cdot\left(2^{1/4}-1\right)}{\sin(1/2)}\right)\cdot\frac{5}{4}\cdot\left(2\sqrt{2}\right)}{\left(-1+2^{1/4}\right)\int_0^{\infty}\frac{t^{1/(2\pi)}}{t+t^2}\,dt}$$

$$\pi=6\left(\sqrt{6\log\left(\frac{3+\sqrt{5}}{2}\right)+2\arctan\left(\frac{1}{\sqrt{7+4\sqrt{3}}}\right)}-\sqrt{\cosh^{-1}(161)}\right)$$

$$e^x=\lim_{n\to\infty}\left(1+\frac{x}{n}+\frac{\sqrt{x^2+1}}{n^2}\right)^n$$

$$\ln(n)=\lim_{m\to\infty}\left[m\left(n^{1/m}-1\right)-\frac{m}{2}\left(n^{1/m}-1\right)^2\right]$$

$$\sqrt{2}=\frac{303\sqrt[5]{3}\sqrt[10]{11}\times 13^{9/10}\sqrt{17}\times 31^{3/10}\times 61^{3/5}\times 71^{7/10}\times 287^{2/5}}{80\,000\,000\sqrt[10]{\prod_{k=1}^{10}\left(1+\frac{1}{k\times 10}\right)^k}}$$

$$\frac{8\log(2)}{\log\left(\sqrt[\sqrt{2}]{256}+1\right)+\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\times 256^{-k/\sqrt{2}}}=\sqrt{2}$$