

The Proof of Collatz conjecture

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Abstract

The Collatz conjecture remains one of the most well-known open problems in mathematics. In this paper, we propose a novel framework for analyzing the Collatz sequence using function composition and modular arithmetic. By defining the functions $F(x) = 3x + 1$ and $G(x) = x/2$, and encoding the iterative structure of the Collatz process as a composition of F and G operations, we construct a parametrized expression $J(j_1, j_2, \dots, j_l)(x)$. We show that for any odd integer x , there exists a finite sequence $j = (j_1, \dots, j_l) \in \mathbb{N}_{\geq 1}^l$ such that the composition yields $J(j)(x) = 2^n$ for some integer n . The proof is based on modular congruences and the fact that 2 is a multiplicative generator modulo powers of 3. This function-theoretic and number-theoretic approach provides an alternative structural interpretation of the Collatz process.

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Introduction

The Collatz conjecture, sometimes known as the $3x+1$ problem, is a deceptively simple yet famously unsolved question in number theory. The conjecture states that for any positive integer x , repeatedly applying the transformation

$$x \mapsto \begin{cases} \frac{x}{2} & \text{if } x \equiv 0 \pmod{2}, \\ 3x + 1 & \text{if } x \equiv 1 \pmod{2} \end{cases}$$

will eventually reach the value 1.

Despite its elementary formulation, the conjecture has resisted proof for decades. In this paper, we propose a new approach to analyzing the Collatz process through the lens of function composition and modular arithmetic. By representing sequences of the operations $3x+1$ and $x/2$ as structured compositions of two functions, we aim to encode the Collatz iterations in a precise algebraic form.

Our goal is to investigate whether, for every odd starting integer x , there exists a finite sequence of compositions that maps x to a power of two. We explore this through the definition of a parametric function $J(j_1, j_2, \dots, j_l)(x)$, and aim to characterize when this function outputs an exact power of two.

Rather than following the traditional recursive formulation of the problem, this work takes a structural and number-theoretic perspective, paving a pathway toward a more modular and algebraic understanding of the conjecture.

1 Proof

Notational Conventions

Before proceeding to the main argument, we establish the following notational conventions that will be used throughout the paper:

- Let $f(x) = 3x + 1$, and $g(x) = \frac{x}{2}$. These represent the two elementary steps in the Collatz iteration.
- A finite sequence of natural numbers $j = (j_1, j_2, \dots, j_l) \in \mathbb{N}^l$ encodes the number of g operations between successive f operations in a composition. This sequence determines the structure of a composed function starting and ending with g , such as:

$$f \circ g^{j_l} \circ f \circ \dots \circ f \circ g^{j_2} \circ f \circ g^{j_1} \circ f(x)$$

- The length of the sequence j is denoted by l , which is the number of g -groups between the f 's. Thus, the number of f 's in the full composition is $l + 1$.

- For each $k = 0, 1, \dots, l$, we define

$$S_k := \sum_{i=1}^k j_i, \quad \text{with the convention } S_0 := 0.$$

This represents the total number of G operations performed up to the k -th F operation.

- We define the function $J(j_1, j_2, \dots, j_l)(x)$ as:

$$J(j)(x) := \frac{3^{l+1}x + \sum_{k=0}^l 3^{l-k} \cdot 2^{S_k}}{2^{S_l}}$$

This expression corresponds to the result of applying the composition $f \circ g^{j_1} \circ f \circ \dots \circ g^{j_l} \circ f$ to the input x .

Additionally, let us agree to call the sequence determined by the algorithm for x described in this way the 'actual Collatz j -sequence.'

And, for readers who do not understand, I will provide the following example. Note that here, A starts with f and ends with f .

Example 1. Let us illustrate the concept with a concrete example. Consider $x = 3$. The standard Collatz trajectory is:

$$3 \xrightarrow{f} 10 \xrightarrow{g} 5 \xrightarrow{f} 16$$

Thus, the number of g 's between the f 's are: $j_1 = 1$. So the actual Collatz j -sequence is:

$$j = (1)$$

Then the composed function can be expressed as:

$$J(1)(3) = f \circ g^1 \circ f(3) = 1$$

This illustrates how the j -sequence directly encodes the structure of the Collatz path for a given input.

Example 2. Let us illustrate the concept with a concrete example. Consider $x = 3$. The standard Collatz trajectory is:

$$m \xrightarrow{f} (3m+1) \xrightarrow{g} \left(\frac{3m+1}{2}\right) \xrightarrow{f} \left(\frac{9m+5}{2}\right) \xrightarrow{g} \left(\frac{9m+5}{4}\right) \xrightarrow{g} \left(\frac{9m+5}{8}\right) \xrightarrow{f} \left(\frac{27m+23}{8}\right)$$

The j -sequence at this point is

$$j = (1, 2)$$

Then, at this point, $J(j_1, j_2, \dots, j_l)(x)$ can be generalized as follows.

$$J(j_1, j_2, \dots, j_l)(x) = \frac{3^{l+1}x + \sum_{k=0}^l 3^{l-k} 2^{S_k}}{2^{S_l}}$$

If we substitute $x = m, j = (1, 2)$ into this equation and calculate, we can see that it results in the same outcome as in Example 2.

Lemma 1 (Uniqueness of Integer-Valued Sequences).

Let $x \in 2\mathbb{N} + 1$ be an odd positive integer, and let $j = (j_1, j_2, \dots, j_l) \in \mathbb{N}_{\geq 1}^l$ be a sequence of natural numbers. Suppose that the composed function

$$J(j)(x) := \frac{3^{l+1}x + \sum_{k=0}^l 3^{l-k} \cdot 2^{S_k}}{2^{S_l}}, \quad \text{with } S_k = \sum_{i=1}^k j_i, S_0 = 0$$

is a positive even integer. Then j corresponds to a prefix of the actual Collatz j -sequence of x .

Lemma 2 (Integer output from actual Collatz sequence).

Let $x \in 2\mathbb{N} + 1$ be an odd positive integer, and let $j = (j_1, j_2, \dots, j_l) \in \mathbb{N}_{\geq 1}^l$ be the actual Collatz j -sequence of x . Then

$$J(j)(x) := \frac{3^{l+1}x + \sum_{k=0}^l 3^{l-k} \cdot 2^{S_k}}{2^{S_l}} \in 2\mathbb{N}, \quad \text{where } S_k := \sum_{i=1}^k j_i, S_0 := 0.$$

Proof. Since each $f(x) = 3x+1$ applied to an odd input yields an even output, and each g^{j_i} is only applied to an even number, the entire composition

$$f \circ g^{j_l} \circ f \circ \dots \circ f \circ g^{j_1} \circ f(x)$$

represents the actual Collatz trajectory and results in a valid integer. Moreover, since the final output comes from applying g^{j_l} to the even result of the last f , the final value is also an even integer.

$$\therefore J(j)(x) \in 2\mathbb{N}. \quad \blacksquare$$

Proof. Let us assume that $J(j)(x) \in 2\mathbb{N}$, i.e., the output is an even integer. We note the following:

1. The expression $J(j)(x)$ was constructed to reflect the outcome of applying the composition

$$f \circ g^{j_l} \circ f \circ \dots \circ f \circ g^{j_2} \circ f \circ g^{j_1} \circ f(x)$$

where $f(x) = 3x + 1$ and $g(x) = \frac{x}{2}$, with right-to-left composition.

2. Each $j_i \geq 1$, which ensures that no two f 's are composed consecutively (i.e., no invalid application of $3x + 1$ to an even number).
3. Since x is odd and the composition always begins and ends with f , and since each f produces an even result, each g^{j_i} is applied to even numbers, which is valid.
4. The only way the composition $J(j)(x)$ can result in an even integer is if every intermediate step is valid and corresponds to actual evaluations in the Collatz trajectory. This is because the division by 2^{S_i} in the denominator can only yield an integer if the numerator is divisible by that same power of two.
5. However, if j does not match the actual sequence of g -counts between f 's in the Collatz process starting from x , then at some point, an f would be applied to an even number, or an incorrect number of g 's would cause a mismatch in the value. This results in a non-integer value due to the mismatch between the structure of the formula and the actual path of the Collatz iteration.
6. Therefore, if $J(j)(x) \in \mathbb{N}$ and even, then each intermediate value must be consistent with the Collatz trajectory. Thus, j must match the actual sequence or a prefix of the actual sequence for x .

Conclusion: Any sequence $j \in \mathbb{N}_{\geq 1}^l$ such that $J(j)(x) \in 2\mathbb{N}$ must correspond to a valid initial segment of the Collatz sequence of x , since otherwise a mismatch would force the expression to be non-integer.

Example 3. For example, in the case of $21 \rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 25$ this is not an actual Collatz j -sequence, so the result of $J(j)(x)$ comes out as an odd number. And, when $g(x)$ is applied to an odd number, a decimal is produced, which contradicts the assumption that the result should be an even integer, and thus is naturally excluded.

Now, let's proceed with the main proof.

Main Argument: Existence of a Power of Two. Let $x \in 2\mathbb{N} + 1$ be a fixed odd integer, and let $j = (j_1, \dots, j_l) \in \mathbb{N}_{\geq 1}^l$ be the actual Collatz j -sequence for x . As defined earlier, we consider the function:

$$J(j)(x) = \frac{3^{l+1}x + \sum_{k=0}^l 3^{l-k} \cdot 2^{S_k}}{2^{S_l}}, \quad \text{with } S_k = \sum_{i=1}^k j_i, \quad S_0 = 0.$$

We wish to show that there exists a natural number $n \in \mathbb{N}$ such that $J(j)(x) = 2^n$, i.e.,

$$3^{l+1}x + D = 2^{S_l+n}, \quad \text{where } D := \sum_{k=0}^l 3^{l-k} \cdot 2^{S_k}.$$

Rewriting,

$$2^m = 3^{l+1}x + D, \quad \text{where } m = S_l + n.$$

Now consider the congruence modulo 3^{l+1} :

$$2^m \equiv D \pmod{3^{l+1}}.$$

Since 2 is a primitive root modulo 3^k for all $k \geq 1$, the multiplicative group $(\mathbb{Z}/3^{l+1}\mathbb{Z})^\times$ is cyclic and generated by 2. Thus, the congruence

$$2^m \equiv D \pmod{3^{l+1}}$$

has a solution in m if and only if D is coprime to 3 (which it is, since all terms in D are sums of powers of 2 multiplied by powers of 3).

Therefore, for every such D , there exists $m \in \mathbb{N}$ satisfying the above congruence. Hence, there exists some $n \in \mathbb{N}$ such that:

$$J(j)(x) = \frac{3^{l+1}x + D}{2^{S_l}} = 2^n.$$

Conclusion: By combining Lemma 1 (which guarantees that only actual Collatz j -sequences yield even integers through $J(j)(x)$) with the group-theoretic fact that powers of 2 modulo 3^{l+1} are complete, we conclude that for any odd x , there exists a sequence j and a natural number n such that $J(j)(x) = 2^n$. This completes the argument.

Termination from Powers of Two. Since we have shown that for any odd integer x , there exists a finite sequence j such that $J(j)(x) = 2^n$, and since the Collatz sequence for any 2^n deterministically terminates after n steps via division by 2, it follows that the full Collatz sequence for x terminates at 1. Thus, the Collatz conjecture holds for all positive integers.

Q.E.D.

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