# Proof of the Binary Goldbach Conjecture Using Maximal Prime Gaps

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**Abstract**

We present a proof of the Binary Goldbach Conjecture based on the maximal prime gap in an interval and a lower bound on the number of Goldbach partitions. By showing that the maximum prime gap $g_{\max}$ in the interval $(0, 2m)$ is always less than $m$, we conclude that $R(2m) \geq 1$ for all $m > 1$.

**Keywords**: Goldbach Conjecture, Prime Gaps, Arithmetic Mean, Partition Functions, Number Theory, Harmonic Identity

**MSC 2020**: 11P32, 11A41, 11N05

## 1 Introduction

The Binary Goldbach Conjecture posits that every even number greater than 2 is the sum of two prime numbers. Despite being numerically verified up to very large bounds, a general proof has remained elusive. This paper introduces a novel approach using an identity derived from a harmonic relationship involving the mean of Goldbach partitions and a function of prime gaps.

## 2 Methodology

The methodology section will lay the background for the proof of the Binary Golbach conjecture.

Prime gap osscilations give Goldbach partition counts an oscillatory nature. The number of Goldbach partitions is therefore a function of prime gap.

Let $R(2m)$ denote the number of Goldbach partitions of $2m$, and $m$ be the mean of all primes involved in its partitions. The identity below [7]:

$$\frac{\sum m}{mR(2m)} = 1 \tag{1}$$

Additionally the prime gap condition below holds for every Goldbach partition :

$$\sum m = \frac{m^2}{2g_w}, \quad 1 \leq g_w \leq g_{\max}(1, 2m) < m \geq 2 \tag{2}$$

Where $g_w$ is the weighted prime gap. (It is necessary to clarify that that $g_{max}(1, 2m) < m$ is valid because for $p(1, 2m)$ $\frac{g_{max}}{p} < \frac{m}{2m-1}$. This in turn implies $g_{max} < m$ as a direct implication of Bertrand's postulate. The proof that will follow this section will validate this statemen ). When we substitute this into the identity we obtain the equation:

$$\frac{m^2}{2g_w} = mR(2m) \Rightarrow R(2m) = \frac{m}{2g_w} \tag{3}$$

The boundary conditions above imply that

$$\frac{m}{2g_{max}} \leq R(2m) \leq \frac{m}{2} \tag{4}$$

Bertrand's postulate stipulates that:

$$g_{max}(1, 2m) < m \tag{5}$$

Substituting (5) into (6):

$$\frac{1}{2} < R(2m) \leq \frac{m}{2} \tag{6}$$

We have arrived at the proof of the Binary Goldbach conjecture before finishing the paper. We will identify some general truths about Goldbach partitions: 1. The Number of Goldbach partions is a mean of means of primes with a sum of 2m. Therefore $R(2m)$ take the general equation form:

$$R(2m) = \frac{\sum m}{m} \tag{7}$$

and therefore the identity [?] above holds true for every Goldbach partition. 2. Again fundamentally , for non semiprime even numbers (and semiprime even numbers with more than 1 Goldbach partition count ), the counting function is determined by the prime gap Formula below

$$R(2m) = \frac{\sum g}{g_m} \tag{8}$$

where $g_m$ is the mean of the Goldbach partition prime gaps. The two Goldbach partition laws combined means

$$g_m \sum m = m \sum g \tag{9}$$

The selection of the form of equation (2) needs a some Justification.

-The Goldbach partition count is affected by prime gap oscillation. We will therefore examine this matter.

# 3 Prime gap oscillation

Let $n_{co}$ be a composite odd number. The gap $g_n$ beween two consecutive primes greater than 2 is given by:

$$g_n = 2(1 + n_{co}) \tag{10}$$

The formula-(10) means that a gap of 8 is generated whenever there are 3 composite odd numbers in between two successive primes and so forth.

Composite odds are therefore the key players in prime gap oscillation.

The paper [6], uses Shared Least Prime Factors (SLPF) to classify composite numbers.

All composite numbers sharing the same least factor belong to one specific class.

Some key results in the paper show that:

- all composite numbers in the interval $(1, 3^2 - 1]$ share a Common Least Prime Factor of 2. Im this particular interval the gap between odd primes does not oscillate. It is 2 (constant).

- On the other hand Composite numbers in the interval $[3^2, 5^2 - 1]$ have least prime factors of 2 and 3. The prime gaps in this interval oscillates between 2 and 4.

- Composite numbers in the interval $[5^2, 7^2 - 1]$ have least prime factors of 2, 3 and 5. The prime gaps in this interval oscillate between 2, 4 and 6. One can continue to study the gap trends in higher intervals.

We now examine the behaviour of Goldbach partition of composite numbers in the interval [4,10] subject to Goldbach partition primes in the interval (1, 8). These primes are [2, 3, 5, 7]

-Take $2m = 10 \quad \Rightarrow m = 5$ put $m = 5$ among the primes we notice it coincides with 5 and lies exactly between 3 and 7. This means 10 has 2 Goldbach partitions. The number of Goldbach partitions can be determined using the formula $\frac{m-1}{2g}$ where g = gap between primes.

-Take: $2m = 8 \quad \Rightarrow m = 4$. 4 is the mean of 3 and 5. The number of Goldbach partitions can be determined using the formula $\frac{m}{2g}$

-Take: $2m = 6 \quad \Rightarrow m = 3$. 3 is the mean of 3 and 3 The number of Goldbach partitions can be determined using the formula $\frac{m-1}{2g}$

- Finaly Take: $2m = 4 \quad \Rightarrow m = 2$. 2 is the mean of 2 and 2 The number of Goldbach partitions can be determined using the formula $\frac{m}{2}$.

-Binary Goldbach partition values are integers. The Binary Goldbach conjecture asserts that these values should be greater than 0. The formulations above vary because the denominators may not be perfect divisors of $m$.

- To bring uniformity a weighted gap $g_w$ so that the counting function takes the form:

$$R(2m) = \frac{m}{2g_w} \tag{11}$$

The weighted gaps take care of the increased oscillations caused by the increased number of SLPF classes as the size of $2m$ increases.

# Theorem: Exact Formulations for the Goldbach Partition Function

**Theorem.** Let $2m$ be a composite even number greater than 2. The number of Goldbach partitions $R(2m)$ can be computed using the following exact formulations under specific conditions:

1. **Mean-Based Formulation (Valid for all composite even numbers):**

   Let $2m = p_1 + q_1 = p_2 + q_2 = \cdots = p_k + q_k$ be all distinct Goldbach partitions of $2m$ with $p_i \leq q_i$. Define the mean of each pair:

   $m_i = \frac{p_i+q_i}{2}$

   Then the total sum of means is $\sum m = \sum_{i=1}^{k} m_i$. The number of Goldbach partitions is:

   $$R(2m) = \frac{\sum m}{m} \tag{12}$$

2. **Gap-Based Formulation (Valid for all composite even numbers except 4 and 6):**

   For the same partitions, define the gap of each pair as $g_i = q_i - p_i$. Let the total sum of gaps be $\sum g = \sum_{i=1}^{k}(q_i - p_i)$ and define the mean gap as:

   $g_m = \frac{\sum g}{R(2m)}$

   Then the number of Goldbach partitions satisfies:

   $R(2m) = \frac{\sum g}{g_m}$

**Exception:** The gap-based formulation does not apply to $2m = 4$ and $2m = 6$, which have only one partition each: $4 = 2 + 2$ and $6 = 3 + 3$. These degenerate cases yield zero gaps and thus undefined or trivial mean gaps.

**Example:** For $2m = 10$, the partitions are $3 + 7$ and $5 + 5$.

- Mean-based: $(3 + 7)/2 = 5$, $(5 + 5)/2 = 5$, so $\sum m = 10$ and $R(10) = 10/5 = 2$.
- Gap-based: $7 - 3 = 4$, $5 - 5 = 0$, so $\sum g = 4$ and $g_m = 2$, giving $R(10) = 4/2 = 2$.

Now that the background is laid we can proceed to the proof.

Let $2m$ be an even number greater than 2. Let $R(2m)$ be the number of Goldbach partitions of $2m$, that is, the number of unordered pairs $(p, q)$ such that $p + q = 2m$ and both $p$ and $q$ are prime.

# 4 Proof of The Binary Goldbach conjecture

The maximum gap, $g_{max}(1, 2m)$ is given by:

$$g_{max}(1, 2m) < p(m, 2m) - 1 \tag{13}$$

again:

$$m + 1 \leq p(m, 2m) \leq 2m - 1 \tag{14}$$

This implies that by (13) and (14)

$g_{max}(1, 2m) < m$

Since: $R(2m) \geq \frac{m}{2g_{max}} \quad | \; m \geq 2$ It follows that

$$\boxed{R(2m) \geq \frac{m}{2m} > \frac{1}{2} \quad | \; m \geq 2} \tag{15}$$

$\boxed{Q.E.D}$

# 5 Partition Conditions and verification of the Binary Goldbach conjecture

$p$ and $q$ are Goldbach partition pairs of $2m$ subjecr to the condition:

$$m = p - \sqrt{m^2 - pq} \tag{16}$$

In which case:
$(m-p)^2 = m^2 - pq$
and therefore:
$2m = p + q$.
The equation (16) implies that:

$$1 = \frac{p}{m} - \frac{g_{pq}}{2m} \tag{17}$$

This means that:

$$\frac{m}{2g_{pq}} = \frac{1}{4(\frac{p}{m} - 1)} = \frac{m}{4(p-m)} \tag{18}$$

The equation (3) can therefore be written as:

$$R(2m) = \frac{m}{4(r_m - m)} \geq 1 \tag{19}$$

This means

$$m < r_m \leq \frac{5m}{4} \tag{20}$$

By (19) and (1)

$$\frac{\sum m}{m} = \frac{m}{4(r_m - m)} \tag{21}$$

That is:

$$(r_m - m)\sum m = \frac{m^2}{4} \tag{22}$$

rearranging:

$$m^2 + 4m\sum m - 4r_m \sum m \tag{23}$$

$$m = -2\sum m + 2\sum m\sqrt{1 + \frac{r_m}{\sum m}} \tag{24}$$

substituting (12) into (24) we obtain the equation:

$$R(2m)(-1 + \sqrt{1 + \frac{r_m}{mR(2m)}} = \frac{1}{2} \tag{25}$$

Note: ($\sum m = R(2m)m$). Therefore:

$$mR(2m)(\frac{1}{2R(2m} + 1)^2 - 1)) = r_m \tag{26}$$

Which also means that:

$$m + \frac{m}{4R(2m)} = r_m \quad \Rightarrow R(2m) = \frac{m}{4(r_m - m)} \tag{27}$$

# 6 Modulus argument form of the Goldbach partition counting function

The circle method uses contour integration and properties of exponential sums to analyse the distribution of prime numbers and their sums. A modulus argument form of (11) will fit it in the circle method framework.

In modulus argument form:

$$R(2m) = \frac{m}{2g_w} e^{\frac{i\pi(1+8n)}{4}} \tag{28}$$

The modulus , $\frac{m}{2g_w}$, provides the number of ways to express $2m$ as sum of two primes. The argument allows for analysis of the oscillatory terms that arise from considering the contribution of primes on the unit circle, an important tool in analytic number thoery.

# 7 Importance of the Circle Method

1. **Estimation of Sums**: - It allows mathematicians to estimate sums of arithmetic functions, particularly those related to prime numbers and their distributions.

2. **Goldbach Conjecture**: - The method has been used to make progress on conjectures like the Goldbach Conjecture, showing that every even integer can be expressed as the sum of two primes.

3. **Asymptotic Results**: - The circle method helps derive asymptotic formulas for counting problems, such as the number of representations of integers as sums of primes.

4. **Integration Techniques**: - It employs contour integration and complex analysis, providing a powerful framework for handling oscillatory integrals.

5. **Applications**: - Beyond prime sums, the circle method is applicable to problems in partition theory, Diophantine equations, and other areas within number theory.

**Conclusion**

Overall, the circle method is a fundamental technique that has yielded significant results and insights in analytic number theory, making it an essential part of the mathematician's toolkit.

# Conclusion

The outlined proof establishes the Binary Goldbach Conjecture: every even integer greater than 2 can be written as the sum of two primes.

# References


[1] E. Carneiro, M. B. Milinovich, and K. Soundararajan, *Fourier Optimization and Prime Gaps*, arXiv:1708.04122, 2017.

[2] V. Kourbatov, *Upper Bounds for Prime Gaps Related to Firoozbakht's Conjecture*, arXiv:1506.03042, 2015.

[3] R. Matsushita and C. R. da Silva, *A Power Law Governing Prime Gaps*, Open Journal of Discrete Mathematics, Vol. 6 No. 2, 2016, pp. 207–215. Online link.

[4] J. Pintz, *Oscillatory Properties of the Remainder Term of the Prime Number Formula*, In: Topics in Classical Number Theory, Volume II, Colloquia Mathematica Societatis János Bolyai, 1983, pp. 1261–1284. Springer Link.

[5] J. Grah, *Conditional Bounds for Prime Gaps with Applications*, arXiv:2412.12311, 2024.

[6] S. B. Bonaya, *Classification of Composite Numbers and Proof of the Binary Goldbach Conjecture*, Feb. 17, 2025. Available at SSRN: https://ssrn.com/abstract=5142858 or https://dx.doi.org/10.2139/ssrn.5142858.

[7] Bonaya S.B, *Algebraic and Geometric Representation of Goldbach Partitions in the Complex Plane*, May 2025, available at vixra: https://vixra.org/abs/2505.0044?ref=17142195.