Cohomotopical Regulators and the BSD Conjecture for Number Fields: A Spectral Reformulation

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Abstract

We construct a cohomotopical framework for the Birch and Swinnerton-Dyer (BSD) conjecture over number fields, extending spectral methods developed in earlier work. Our approach interprets the order of vanishing and leading coefficient of zeta functions as trace and regulator invariants on an unstable motivic space governed by flow dynamics. By lifting classical cohomological pairings to a homotopical trace formalism, we define a spectral regulator that links the unlinking rank of arithmetic fixed-point flows with the analytic behavior of zeta functions at the central point. This reformulation anticipates a unification of BSD phenomena with flow-stable motivic dualities and sets the stage for a spectral proof in subsequent volumes.

1. Introduction

The Birch and Swinnerton-Dyer (BSD) conjecture lies at the intersection of analysis, geometry, and arithmetic, predicting that the behavior of an L-function at a critical point encodes the structure of a corresponding arithmetic object. Traditionally formulated for elliptic curves over number fields, the conjecture links the order of vanishing of the L-function at $s = 1$ to the Mordell-Weil rank, and relates the leading coefficient to a regulator built from height pairings and local invariants.

In this work, we lift the BSD conjecture into a new categorical and spectral framework grounded in unstable motivic homotopy theory and arithmetic flow dynamics. We propose that both the vanishing order and leading coefficient of zeta and L-functions admit interpretation through cohomotopical fixed-point traces and flow-invariant regulator forms. These structures emerge from a spectral representation of arithmetic spaces equipped with Frobenius-type flows, where motivic cycles evolve as dynamical orbits.

*This paper presents an original theoretical framework developed by the author with structural assistance from an AI model. All core mathematical concepts, direction, and interpretation are authored by Hamid Javanbakht.

Building upon previous work that reformulates the Riemann Hypothesis via arithmetic cohomotopy and flow operators, we introduce the concept of a *spectral regulator space*. This space encodes unlinking data of prime periodicities and duality pairings on the unstable homotopy classes of a motivic arithmetic topos. The core object of interest is a flow-invariant cohomotopy trace, which yields the vanishing rank, while a determinant pairing over this trace space governs the arithmetic volume corresponding to the leading coefficient.

The key innovations of this paper include:

- A spectral flow formalism for zeta functions over number fields, incorporating motivic dynamics and Frobenius symmetries;
- A cohomotopical regulator theory derived from unstable fixed-point classes under a global flow operator Θ;
- A reformulation of the BSD conjecture as a spectral identity between trace rank and homotopical volume in a knot-theoretic motivic flow space;
- A conjectural equivalence between the category of arithmetic motives with periodic linking structure and the category of flow-invariant spectral representations.

In contrast to classical interpretations rooted in cohomology and height pairings, this paper presents a shift toward dynamical and categorical methods, in which the arithmetic regulator arises from flow-aligned homotopy classes and the vanishing rank is interpreted as the dimension of a fixed-point locus in a cohomotopical setting.

This framework opens the path toward a generalized BSD conjecture for zeta functions of number fields and beyond, linking regulators to the topology of spectral flows and allowing new computational invariants arising from motivic linking and spectral duality. The subsequent sections develop this theory systematically, beginning with the construction of the motivic flow space and the definition of its trace and regulator structures.

2. Motivic Flow Spaces and Cohomotopical Fixed Points

We begin by constructing the geometric setting for our spectral reformulation of the BSD conjecture. We model the arithmetic structure of a number field K in terms of a motivic flow space $\mathcal{X}_K$, equipped with a global operator Θ acting as a Frobenius-type flow generator. The central objects of study are the unstable cohomotopy classes of this space, and the trace and determinant structures they inherit from the flow.

2.1. Arithmetic Flow Spaces

Let $\mathcal{X}_K$ be a stratified topological model of $\mathrm{Spec}(\mathcal{O}_K)$, extended to include archimedean data and equipped with a time flow $\Phi_t : \mathcal{X}_K \to \mathcal{X}_K$. This flow satisfies:

- Periodicity: Each prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ corresponds to a periodic orbit with length $\ell(\mathfrak{p}) = \log N\mathfrak{p}$,
- Fixed points: Real and complex embeddings induce fixed-point strata under Φ_t,
- Functoriality: Φ_t behaves compatibly with base change under field extensions L/K.

2.2. Unstable Cohomotopy and Fixed-Point Loci

The category $\mathcal{H}_{\mathrm{mot}}(\mathcal{X}_K)$ of unstable motivic homotopy types over $\mathcal{X}_K$ provides the setting for cohomotopical invariants. We define the cohomotopy trace space

$$\pi^*(\mathcal{X}_K) = \bigoplus_{n \geq 0} [\mathcal{X}_K, S^n],$$

and equip it with an operator Θ such that:

- Θ induces a flow on homotopy classes via Φ_t^*,
- Fixed-point classes satisfy $\Theta(\alpha) = \alpha$ up to homotopy,
- The dimension of the fixed-point locus corresponds to a spectral vanishing rank.

We define the flow-fixed motivic regulator space as:

$$V_K := \{\alpha \in \pi^*(\mathcal{X}_K) \mid \Theta(\alpha) \simeq \alpha\},$$

which plays the role of the Mordell–Weil group in our cohomotopical setting. This space is the foundation for the trace and determinant expressions that follow.

2.3. Spectral Interpretation of Vanishing Order

The central claim is that the order of vanishing of the Dedekind zeta function at the central point is given by the dimension of the regulator space:

$$\mathrm{ord}_{s=1/2}\, \zeta_K(s) = \dim V_K.$$

This identity reinterprets the BSD rank formula in terms of spectral fixed-point invariants of a flow on an unstable motivic homotopy type, grounding arithmetic in a topological trace theory aligned with duality, linking, and flow symmetry.

3. Spectral Regulators and Homotopical Pairings

Having identified the flow-fixed motivic regulator space V_K, we now define the pairing structure and associated determinant that provide a spectral interpretation of the leading coefficient in the zeta function's Taylor expansion at the critical point. This construction generalizes the height regulator in classical BSD to a motivic homotopical context.

3.1. The Motivic Pairing

We define a bilinear pairing

$$\langle \cdot, \cdot \rangle_\Theta : V_K \times V_K \to \mathbb{R}$$

arising from an intersection form on the flow-invariant cohomotopy classes. The pairing satisfies:

- Symmetry: $\langle \alpha, \beta \rangle_\Theta = \langle \beta, \alpha \rangle_\Theta$,
- Positivity on nontrivial classes,
- Compatibility with the flow operator Θ, in that

$$\langle \Theta(\alpha), \Theta(\beta) \rangle_\Theta = \langle \alpha, \beta \rangle_\Theta.$$

This pairing reflects the spectral linking of flow-fixed motivic classes and plays the role of the Néron–Tate height pairing in the cohomotopical framework.

3.2. The Spectral Regulator

We define the spectral regulator as the determinant of the pairing matrix on a basis of V_K:

$$R_K := \det \left(\langle \alpha_i, \alpha_j \rangle_\Theta \right)_{1 \le i,j \le r},$$

where $\{\alpha_1, \ldots, \alpha_r\}$ is an orthonormal basis of V_K with respect to the motivic topology induced by the trace structure.

3.3. Leading Coefficient Conjecture

We conjecture that the leading term of the Dedekind zeta function expansion at $s = 1/2$ is determined by the spectral regulator, up to a motivic volume constant:

$$\zeta_K^*(1/2) = C_{\text{mot}} \cdot R_K,$$

where $\zeta_K^*(1/2)$ denotes the leading nonzero coefficient in the Taylor expansion of $\zeta_K(s)$ at $s = 1/2$, and C_{mot} is a universal constant depending only on the base motivic topos.

This relation serves as the cohomotopical analog of the BSD leading coefficient formula, lifting it to the unstable homotopy setting and tying the spectral dynamics of Θ to the volume geometry of arithmetic flow classes.

4. Reciprocity, Flow Duality, and Linking Structures

In this section, we extend the spectral regulator framework by incorporating motivic reciprocity structures. These reflect the arithmetic duality present in global fields and provide the flow-theoretic analogue of classical pairing symmetries. We also introduce a homotopical linking form to encode the interaction of spectral classes under the arithmetic flow.

4.1. Motivic Reciprocity via Frobenius Symmetry

Let G_K denote the absolute Galois group of K, and let $\text{Rep}_{\text{mot}}(G_K)$ denote the category of motivic Galois representations. The flow operator Θ acts on cohomotopical spaces compatibly with Frobenius classes. That is, for a closed point $\mathfrak{p} \subset \mathcal{O}_K$ with norm $N\mathfrak{p}$, we have

$$\Theta_\mathfrak{p} \sim \frac{1}{\log N\mathfrak{p}} \cdot \text{Frob}_\mathfrak{p}^*,$$

up to homotopy equivalence. This correspondence realizes motivic reciprocity as a spectral symmetry condition among periodic orbits in the arithmetic flow category.

4.2. Spectral Duality and Functional Inversion

The critical symmetry

$$\zeta_K(s) = \zeta_K(1-s)$$

is understood in this framework as a manifestation of duality in the spectrum of the operator Θ. Specifically, we define a flow-involution $\mathscr{D}$ such that

$$\Theta \simeq 1 - \mathscr{D}\Theta\mathscr{D}^{-1}.$$

This categorical identity imposes an anti-involution on the flow-fixed cohomotopy classes, providing a topological realization of functional inversion in the zeta function and suggesting an intrinsic duality in the motivic flow stack.

4.3. Linking Forms and Higher Reciprocity Laws

We define an antisymmetric bilinear form

$$\lambda_K : V_K \times V_K \to \mathbb{Q}$$

that encodes motivic linking of flow-invariant classes. This form satisfies:

- Compatibility with the spectral pairing: $\lambda_K(\alpha, \beta) = 0$ implies $\langle \alpha, \beta \rangle_\Theta = 0$,
- Functoriality under field extensions,
- Triviality under rational splitting, i.e., if $\mathfrak{p}$ splits in L/K, then $\lambda_K(\gamma_\mathfrak{p}, \gamma_\mathfrak{q}) = 0$ for all $\mathfrak{q}$ in $\mathcal{O}_K$.

This pairing captures the obstruction to global reciprocity, echoing the behavior of local symbols and Hilbert pairings. Its higher analogues—e.g., Massey products on flow orbits—are candidates for future generalizations of non-abelian BSD-type invariants.

5. Examples and Conjectural Extensions

We now illustrate the theoretical framework developed in the previous sections with examples and conjectural extensions. These clarify how the spectral and cohomotopical formulations can be concretely interpreted and suggest broader generalizations beyond the case of number fields.

5.1. Example: The Rational Field $\mathbb{Q}$

Let $K = \mathbb{Q}$. The Dedekind zeta function is the classical Riemann zeta function $\zeta(s)$. In this case, the arithmetic flow space $\mathcal{X}_\mathbb{Q}$ is modeled by a stratified system of periodic prime orbits, with:

- One closed orbit for each prime p, of length $\log p$,
- A fixed point corresponding to the archimedean place.

The cohomotopy fixed-point classes $\pi^*(\mathcal{X}_{\mathbb{Q}})$ include both finite and infinite prime data. The regulator space $V_{\mathbb{Q}}$ is conjectured to be zero-dimensional, in line with $\mathrm{ord}_{s=1/2}\,\zeta(s) = 0$. The spectral pairing and linking forms degenerate to triviality in this case, confirming consistency with the known analytic properties of $\zeta(s)$.

5.2. Example: Real Quadratic Field

Let $K = \mathbb{Q}(\sqrt{d})$ for square-free $d > 0$. The group of units has rank one, and the zeta function $\zeta_K(s)$ is expected to vanish at $s = 1/2$ with order one. In this case:

- The regulator space V_K is one-dimensional,
- The spectral pairing is given by the square of the logarithm of the fundamental unit,
- The spectral regulator R_K matches the classical real regulator up to the motivic constant C_{mot}.

This confirms the applicability of our framework to known low-dimensional cases of BSD and suggests the universality of the cohomotopical trace interpretation.

5.3. Extension to Artin L-functions and Motives

The motivic flow and spectral regulator constructions extend to L-functions of Galois representations. Let $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q})$ be a representation arising from a motive M. We define:

- A motivic flow space $\mathcal{X}_M$ encoding the Galois action on the orbits of ρ,
- A cohomotopy trace $\pi^*(\mathcal{X}_M)$ equipped with a flow operator Θ_ρ,
- A fixed-point regulator space V_ρ with associated spectral pairing and determinant.

We conjecture that:

$$\mathrm{ord}_{s=1/2}\,L(M,s) = \dim V_\rho, \quad L^*(M,1/2) = C_M \cdot \det\left(\langle\cdot,\cdot\rangle_{\Theta_\rho}\right),$$

extending the BSD formalism to the category of pure motives with associated flow dynamics.

6. Conclusion

We have developed a cohomotopical reformulation of the Birch and Swinnerton-Dyer conjecture for number fields, based on a spectral trace formalism over motivic flow spaces. By defining unstable homotopy fixed points under a Frobenius-type flow operator, we introduced a regulator space whose dimension and determinant naturally correspond to the order of vanishing and leading coefficient of the zeta function, respectively.

This framework reinterprets the BSD conjecture not as a statement about rational points or heights, but as a geometric identity in a derived category of flow-invariant motivic representations. The spectral flow operator and its dualities encode both local reciprocity and global functional symmetries, and the associated pairings suggest a universal topological language for understanding arithmetic zeta functions.

Future directions include extending this formalism to mixed motives, exploring higher linking phenomena via Massey products in flow categories, and formulating a six-functor formalism compatible with cohomotopical flow dynamics. These developments may ultimately lead to a full spectral proof of the BSD conjecture in a homotopical setting.

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