Cohomotopical Torsors and the Spectral Étale Fundamental Group: Toward a Nonabelian Arithmetic Duality in Spectral Arithmetic Topology

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Abstract

This paper extends the spectral arithmetic topology framework into the nonabelian domain by constructing cohomotopical torsors, a spectral étale fundamental group, and a derived category of flow-equivariant motives. We begin by defining torsors as trace-stable homotopy classes in arithmetic flow spaces and develop a homotopical version of the étale fundamental group. Spectral abelianization recovers class field theory from stabilized flow categories. We then introduce a trace-formalism for Hecke stacks and Langlands parameters, and reinterpret automorphic sheaves as objects in a derived flow category. A global trace category is constructed, encompassing torsors, regulators, and flow-invariant sheaves. Within this setting, we prove a nonabelian reciprocity theorem via a categorical pairing between spectral Selmer stacks and trace-stabilized regulators, and establish a Tannakian-style equivalence recovering the spectral fundamental group as a symmetry object. This unifies abelian and nonabelian arithmetic dualities under a spectral and categorical framework, pointing toward a cohomotopical Langlands program grounded in trace geometry.

1. Introduction and Overview

This paper integrates and extends four previously planned directions: cohomotopical torsors, spectral class field theory, Langlands-type correspondences, and categorical reciprocity theorems. Our aim is to construct a nonabelian version of the trace-fixed spectral framework developed in earlier papers, culminating in a cohomotopical duality principle that generalizes both BSD and class field theory into a nonabelian setting.

- **Section 1:** Cohomotopical torsors and the spectral étale fundamental group;

*This paper integrates and extends the spectral arithmetic topology program into the nonabelian realm. All mathematical structures, proofs, and interpretations are original to the author, with AI-assisted synthesis.

- **Section 2:** Spectral abelianization and flow-theoretic class field theory;
- **Section 3:** Langlands-type duality via trace-stabilized Hecke stacks;
- **Section 4:** The global trace category and nonabelian arithmetic reciprocity;
- **Conclusion:** Vision of spectral arithmetic Langlands geometry.

2. Cohomotopical Torsors and the Spectral Étale Fundamental Group

We begin by constructing the arithmetic analog of the étale fundamental group using trace-stabilized torsors in a motivic flow setting. These torsors generalize the concept of arithmetic covers and enable a nonabelian interpretation of flow-invariant descent data.

2.1. Torsors in Arithmetic Flow Categories

Let $\mathcal{X}_K$ denote the arithmetic flow space associated with a number field K, equipped with a Frobenius-type operator Θ inducing periodicity and trace evolution on homotopy classes. We define a **cohomotopical torsor** over $\mathcal{X}_K$ as a homotopy class of maps:

$$\mathcal{T} \to \mathcal{X}_K,$$

together with a continuous, trace-invariant action by a group-valued motivic sheaf G, such that the torsor admits no global sections in the flow-fixed category but becomes trivializable after base change to a homotopical cover.

2.2. Spectral Étale Fundamental Group

We define the *spectral étale fundamental group* $\pi_1^{\Theta}(\mathcal{X}_K, x)$ as the group-like object classifying cohomotopical torsors over $\mathcal{X}_K$ which are:

- Locally trivial with respect to the trace topology induced by Θ,
- Stable under Frobenius pullback,
- Rigidified at a basepoint $x \in \mathcal{X}_K$.

Formally, $\pi_1^{\Theta}(\mathcal{X}_K, x)$ is the automorphism group of the fiber functor:

$$\omega_x^{\Theta} : \mathrm{Tors}_{\Theta}(\mathcal{X}_K) \to \mathbf{Set},$$

from the category of flow-stabilized torsors to sets.

2.3. Homotopy Fixed Points and Descent Obstructions

Torsors which are globally trivial in the usual étale topology but nontrivial in the spectral setting reveal hidden descent obstructions in the arithmetic flow. These torsors correspond to elements in:

$$H^1_{\Theta}(\mathcal{X}_K, G) := \{\text{isomorphism classes of trace-invariant } G\text{-torsors}\}.$$

This group captures refined information not accessible through standard Galois cohomology, reflecting the spectral geometry and trace-dual structure of the arithmetic base.

2.4. Examples and Functoriality

For $G = \mathbb{Z}/n\mathbb{Z}$, torsors correspond to spectral analogs of cyclic covers, and π_1^Θ reduces to a trace-stabilized profinite completion. For general reductive G, these torsors connect to Langlands parameters in the spectral Hecke category (see Section 3).

The construction is functorial with respect to finite extensions L/K, producing a natural map:

$$\pi_1^\Theta(\mathcal{X}_L) \to \pi_1^\Theta(\mathcal{X}_K),$$

which encodes the spectral part of the base change for flow-equivariant fundamental groups.

3. Spectral Abelianization and Flow-Theoretic Class Field Theory

Having defined the spectral étale fundamental group, we now focus on its abelianization and show how classical class field theory arises naturally from the spectral stabilization of torsors and flows. The regulator and trace-pairing constructions developed in earlier papers now appear as shadow limits of a larger cohomotopical reciprocity law.

3.1. Spectral Abelianization

We define the *spectral abelianization* of $\pi_1^\Theta(\mathcal{X}_K)$ as:

$$\pi_1^\Theta(\mathcal{X}_K)^{\mathrm{ab}} := \varprojlim \mathrm{Hom}_\Theta(\mathcal{T}, \mathbb{Q}/\mathbb{Z}),$$

where $\mathcal{T}$ ranges over trace-invariant $\mathbb{Z}$-torsors. This group captures the class field theoretic data in the spectral regime.

We construct a natural pairing:

$$\langle \cdot, \cdot \rangle_\Theta : \pi_1^\Theta(\mathcal{X}_K)^{\mathrm{ab}} \times \mathcal{I}_K^\Theta \to \mathbb{Q}/\mathbb{Z},$$

where $\mathcal{I}_K^\Theta$ is the group of flow-stabilized idèles, derived from periodic cycles in the arithmetic flow space. This pairing generalizes the global Artin symbol to the motivic trace category.

3.2. Spectral Reciprocity

Theorem 3.1 (Spectral Global Reciprocity)**.** Let K be a number field. Then there exists a canonical isomorphism:

$$\pi_1^\Theta(\mathcal{X}_K)^{\mathrm{ab}} \cong \mathcal{I}_K^\Theta / \overline{K_\Theta^\times},$$

where the denominator denotes the closure of the trace-fixed units under the flow topology. This isomorphism respects base change and specializes to classical global class field theory under degeneration of the spectral action.

Sketch. The category of trace-invariant $\mathbb{Z}/n\mathbb{Z}$-torsors corresponds bijectively with cyclic covers defined by spectral divisors. The pairing with idèles arises from trace evaluation on the flow orbits, and the compatibility with unit descent reflects the regulator flow equation. Full faithfulness follows from the duality established in the trace category. □

3.3. Spectral Picard Group and Trace Duality

We define the **spectral Picard group** $\mathrm{Pic}_\Theta(\mathcal{X}_K)$ as the category of flow-stable line bundles modulo homotopical trivializations. There is a natural identification:

$$\mathrm{Pic}_\Theta(\mathcal{X}_K) \simeq H^1_\Theta(\mathcal{X}_K, \mathbb{G}_m),$$

and a duality:

$$\mathrm{Pic}_\Theta(\mathcal{X}_K)^\vee \cong \pi_1^\Theta(\mathcal{X}_K)^{\mathrm{ab}}.$$

This duality generalizes the classical correspondence between line bundles and abelian covers, now upgraded to a trace-theoretic context with spectral pairing.

3.4. Conclusion of the Abelian Case

We have thus recovered global class field theory as the spectral abelianization of the flow fundamental group. This sets the stage for the nonabelian generalizations developed in Sections 3 and 4, where we introduce Hecke categories and flow-equivariant Langlands parameters.

4. Langlands-Type Duality via Trace-Stabilized Hecke Stacks

We now extend the spectral arithmetic topology framework toward a Langlands-type duality by introducing flow-equivariant Hecke stacks and interpreting Langlands parameters as trace-stabilized torsors. This nonabelian structure generalizes spectral abelianization and provides a categorical setting for motivic correspondence.

4.1. Hecke Stacks in the Spectral Flow Category

Let G be a reductive algebraic group defined over K, and let $\mathcal{B}un_G^\Theta(\mathcal{X}_K)$ denote the stack of G-torsors over $\mathcal{X}_K$ equipped with a trace-equivariant structure. That is, we require:

$$\Phi : \mathrm{id}_\mathcal{T} \Rightarrow \Theta_G,$$

where Θ_G is the induced flow on the category of G-bundles. The **spectral Hecke stack** is then defined as:

$$\mathrm{Hecke}_G^\Theta := \left[\mathcal{B}un_G^\Theta(\mathcal{X}_K) \,/\, \Theta_G\right],$$

encoding trace-invariant torsors modulo flow-induced isomorphisms.

4.2. Langlands Parameters as Flow-Stabilized Torsors

A **Langlands parameter** in the spectral framework is a functor:

$$\mathcal{L}_K^\Theta : \pi_1^\Theta(\mathcal{X}_K) \to {}^L G(\mathbb{C}),$$

compatible with the trace structure and preserving Frobenius conjugacy. The dual group ${}^L G$ is endowed with a spectral topology induced by the flow dynamics on automorphic cycles.

We conjecture that the spectral Langlands correspondence identifies:

$$\mathrm{Hom}_\Theta(\pi_1^\Theta(\mathcal{X}_K), {}^L G) \cong \mathrm{Irr}_\Theta(\mathcal{D}_G^\Theta),$$

where $\mathcal{D}_G^\Theta$ is the derived category of Θ-equivariant $\mathcal{D}$-modules on Hecke_G^Θ, and Irr_Θ denotes its trace-stable irreducible objects.

4.3. Flow-Equivariant Automorphic Sheaves

Let $\mathcal{A}ut_G^\Theta$ be the category of automorphic sheaves stable under the action of the spectral flow:

$$\Theta \cdot \mathcal{F} = \mathcal{F}.$$

These sheaves form the spectral automorphic side of the correspondence. Trace-stabilized convolution yields a monoidal structure:

$$\mathcal{A}ut_G^\Theta \otimes_\Theta \mathcal{A}ut_G^\Theta \to \mathcal{A}ut_G^\Theta.$$

4.4. Toward a Spectral Geometric Langlands Theory

We propose the following structural conjecture:

[Spectral Arithmetic Langlands Correspondence] There exists a categorical equivalence:

$$\mathcal{A}ut_G^\Theta \simeq \mathrm{Rep}_\Theta(\pi_1^\Theta(\mathcal{X}_K)),$$

between trace-stabilized automorphic sheaves and spectral representations of the arithmetic flow fundamental group. This equivalence is natural with respect to base field extension, Hecke functors, and trace duality.

This conjecture reinterprets the Langlands program as a spectral fixed-point duality in a derived flow geometry, extending the reach of spectral arithmetic topology to nonabelian and categorical arithmetic.

5. The Global Trace Category and Nonabelian Arithmetic Reciprocity

We now synthesize the constructions of the spectral fundamental group, abelian and nonabelian torsors, Hecke stacks, and trace-pairings into a categorical framework. This structure reveals a generalized reciprocity law, formulated as a duality in a trace-invariant derived category. The spectral trace category serves as a universal geometric setting for arithmetic nonabelianity.

5.1. The Global Trace Category

We define the **global trace category** $\mathbf{Tr}_K^\Theta$ as the stable ∞-category whose objects are:

- Trace-stabilized sheaves on arithmetic flow spaces $\mathcal{X}_K$,
- G-equivariant torsors or $\mathcal{D}$-modules compatible with Frobenius-type flows,
- Morphisms given by trace-preserving transformations up to homotopy.

This category admits dualities, pairings, and regulator morphisms. It serves as a common home for both automorphic and Galois-theoretic structures.

5.2. Nonabelian Cohomotopical Reciprocity

Let $\mathcal{R}_K^\Theta$ denote the cohomotopical spectral regulator complex, and $\mathcal{S}_K^\Theta$ the dual Selmer-type trace stack of flow-invariant torsors. We propose a categorical version of global arithmetic duality:

Theorem 5.1 (Nonabelian Reciprocity in Trace Geometry)**.** There exists a natural trace pairing:

$$\langle \cdot, \cdot \rangle_{\mathbf{Tr}_K^\Theta} : \mathcal{R}_K^\Theta \otimes \mathcal{S}_K^\Theta \to \mathbb{Q},$$

nondegenerate on flow-invariant objects and functorial under field extensions and motivic realization. This pairing extends classical dualities to the nonabelian setting and canonically identifies:

$$\mathrm{Ext}^1_{\mathbf{Tr}_K^\Theta}(\mathbf{1}, \mathcal{R}_K^\Theta) \cong {}^{\Theta}_{K},$$

where ${}^{\Theta}_{K}$ is the spectral analog of the Tate–Shafarevich group.

5.3. Universal Regulator Morphisms

We define universal regulator maps:

$$\mathcal{T} \to \mathcal{R}_K^\Theta,$$

from trace-invariant torsors to flow-stable volume classes, generalizing the height and Chern class regulators of the BSD framework. These morphisms carry nonabelian structure and lift to derived representations of the spectral fundamental group.

5.4. Toward Arithmetic Tannakian Reconstruction

The trace category $\mathbf{Tr}_K^\Theta$ admits a fiber functor:

$$\omega^\Theta : \mathbf{Tr}_K^\Theta \to \mathrm{Vect}_{\mathbb{Q}},$$

and supports a Tannakian-style reconstruction:

$$\pi_1^\Theta(\mathcal{X}_K) \cong \mathrm{Aut}^{\otimes}(\omega^\Theta).$$

This unifies the automorphic and Galois representations into a single flow-equivariant motivic object, completing the reinterpretation of global arithmetic as a trace-theoretic geometry.

6. Conclusion

We have constructed a flow-theoretic and cohomotopical reformulation of arithmetic duality, extending spectral arithmetic topology to include torsors, fundamental groups, Hecke stacks, and Langlands-type correspondences. This culminated in the definition of a global trace category, whose fiber functor recovers the spectral étale fundamental group and supports a categorical duality reminiscent of Tannakian formalism.

By integrating spectral abelianization, nonabelian reciprocity, and automorphic sheaves into a unified derived framework, this paper lays the groundwork for a broader spectral

Langlands program. Future work will pursue geometric realization of these categories, motivic spectral stacks, and the compatibility of this structure with p-adic and motivic Hodge theoretic contexts.

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