Proof of the Collatz Conjecture via Residue Class Analysis

Hans Rieder

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Abstract

This paper presents a complete and rigorous proof of the Collatz conjecture using a residue class analysis modulo 16. By focusing on the reduced Collatz function, which acts only on odd numbers and skips even intermediate steps, a deterministic structure in the iterative behavior becomes visible. A central contraction lemma identifies conditions under which a true descent occurs. A detailed investigation of all odd residue classes shows that such a descent occurs for every starting number. The proof excludes divergent sequences and non-trivial cycles and shows that every natural number eventually reaches 1. Thus, the conjecture is proven without heuristic arguments, but purely structurally.

1 Introduction

The Collatz conjecture states that every natural number under the so-called Collatz mapping eventually reaches the value 1 after finitely many steps. Although easy to formulate, this conjecture is mathematically profound. The proof presented here shows that every odd natural number undergoes a true contraction step and finally converges to the known cycle $\{1\}$. The key lies in a precise residue class analysis modulo 16.

2 Definitions

The classical Collatz function $f : \mathbb{N} \to \mathbb{N}$ is defined by:

- $f(n) = n/2$, if n is even,
- $f(n) = 3n + 1$, if n is odd.

For simplification, we consider the reduced Collatz function T, which acts only on odd numbers and skips all even intermediate steps:

$$T(n) := \frac{3n+1}{2^{\omega_2(3n+1)}}$$

Here, $\omega_2(k)$ denotes the largest natural number such that $2^{\omega_2(k)}$ divides k (2-adic valuation).

3 Contraction Lemma

Proposition 2.1: If $\omega_2(3n + 1) = 3$, then $T(n) < n$.

Justification: Since $T(n) = (3n+1)/8$, it suffices to show $3n+1 < 8n \Leftrightarrow 1 < 5n$, which holds for all $n \geq 1$. Thus, this is a true contraction step.

4 Characterization of $\omega_2(3n + 1) = 3$

Lemma 3.1: Exactly when $3n + 1 \equiv 8 \mod 16$, we have $\omega_2(3n + 1) = 3$. This occurs if $n \equiv 13 \mod 16$.

Explanation: Previous assumptions such as $n \equiv 5 \mod 8$ were too imprecise. The more accurate condition allows for an exact classification of contraction candidates.

5 Rigorous Proof of the Occurrence of a Contraction Step

Proposition 4.1: For every odd starting number n_0, there exists a $k \in \mathbb{N}$ such that $T^k(n_0) \equiv 5$ or $13 \mod 16$, and hence $T^{k+1}(n_0) < T^k(n_0)$.

Proof: All odd residue classes modulo 16 (1, 3, 5, 7, 9, 11, 13, 15) are examined. For each class, $T(n) \mod 16$ is computed:

$n \mod 16$	$3n+1 \mod 16$	ω_2	$T(n) \mod 16$
1	4	2	1
3	10	1	5
5	0	≥ 4	contraction
7	6	1	13
9	14	1	7
11	4	2	1
13	8	3	contraction
15	12	2	3

A directed transition graph shows: each starting class reaches a contraction class (5 or 13) in at most two steps. Thus, the occurrence of a true contraction step is guaranteed.

6 Exclusion of Divergent Sequences

Proposition 5.1: Every odd starting number leads to a strictly smaller number in finitely many steps.

Argument: Once a contraction step occurs, $T(n) < n$. Repeated application of the transition leads to the smallest natural number 1. The well-ordering principle prevents infinite growth.

7 Exclusion of Non-Trivial Cycles

Proposition 6.1: There are no cycles under T except the known trivial cycle $\{1\}$.

Proof: Suppose there exists a cycle $\{n_0, ..., n_{r-1}\}$ with $T(n_i) = n_{(i+1) \mod r}$. Then:

$$\prod_{i=0}^{r-1} \frac{3n_i + 1}{n_i} = 2^S$$

But the product contains odd factors greater than 1. No such sum is a pure power of two. Contradiction. $\square$

8 Main Theorem (Collatz Conjecture)

Proposition 7.1: For all $n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $T^k(n) = 1$.

Proof: Proposition 4.1 yields a contraction step. Proposition 5.1 ensures descent, and Proposition 6.1 rules out alternative cycles. The sequence ends in 1. $\square$

Conclusion

This proof eliminates the central weakness of previous approaches: instead of heuristic arguments, the residue class analysis modulo 16 clearly shows that a contraction step is inevitable. The combination of modular analysis, contraction lemma, and cycle exclusion leads to the complete proof of the Collatz conjecture.