Relativistic Correction of Electromagnetic Theory in Gravitational Field and Its Physical Effects

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Abstract: Based on the covariant form of general relativity, this paper systematically studies the correction effect of gravitational field on classical electromagnetic theory. By introducing the Newtonian gravitational potential Φ, we derive the modified Maxwell's equations and the electromagnetic wave propagation equation, revealing the coupling mechanism between the gravitational field and the electromagnetic field. The results show that under the weak field approximation, the charge density, current density and electromagnetic field equations need to introduce $(1 \pm \Phi/c^2)$ relativistic correction factors. The electromagnetic wave correction equation obtained in this paper clearly characterizes the three core effects of gravitational time dilation (time derivative correction term), spatial bending (coupled gradient term), and local variation of the speed of light. These theoretical predictions are strictly self-consistent with the general theory of relativity, which perfectly explains the classical phenomena such as gravitational redshift and light deflection, and are consistent with the results of precise experiments such as Shapiro time delay. The modified theoretical framework established in this study provides an important theoretical basis for the frontier fields such as pulsar timing and black hole accretion disk radiation propagation, and points out the direction for the development of high-order correction theory in strong field regions and the unified theory of gravity-quantum electromagnetism.

1 Introduction

The influence of gravitational field on Maxwell's equations has been discussed by many authors in the past [1~3], and the weak field approximation is generally used to analyze it. In the case of a weak gravitational field, the form of Maxwell's equations will change to a certain extent, and the derivation process is prone to errors due to the complexity of the calculations involved. Here we use DeepSeek to derive the derivation, giving us a concise and clear derivation process, and better exploring the influence of the gravitational field on the electromagnetic field.

2 Coupling of gravitational field with electromagnetic field in gravitational field

2.1 Weak gravitational field approximation

If the gravitational potential needs to be introduced directly into Maxwell's equations (e.g., Newton's gravitational potential Φ), we can consider the coupling of gravitational force and electromagnetic field under the weak-field approximation [3]. In this case, the space-time gauge can be written as straight space-time (Minkowski's gauge) plus a small perturbation, where the gravitational potential Φ is part of the perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, h_{00} = -\frac{2\Phi}{c^2}, h_{ij} = -\frac{2\Phi}{c^2}\delta_{ij}$$

Ignore h_{0i} (no gravitational magnetic effect). In this case, the inverter gauge is:

$$g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}, h^{00} = -h_{00} = \frac{2\Phi}{c^2}, h^{ij} = -h_{ij} = \frac{2\Phi}{c^2}\delta^{ij}.$$

2.2 Maxwell's equations in a weak gravitational field

In curved space-time, the expansion of the covariant derivative ∇_μ introduces the contribution of the gauge. For the equation of $F^{\mu\nu}$:

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^{\mu}_{\mu\alpha} F^{\alpha\nu} + \Gamma^{\nu}_{\mu\alpha} F^{\mu\alpha} = 0.$$

In the weak-field approximation, the Christoffel symbol $\Gamma^{\mu}_{\alpha\beta}$ is:

$$\Gamma^{\mu}_{00} \approx \frac{\partial\Phi/c^2}{\partial x^\mu}, \Gamma^{\mu}_{ij} \approx O(\Phi/c^2).$$

Keeping to the $O\left(\frac{\Phi}{c^2}\right)$ order, the equation is reduced to the following values:

$$\partial_\mu F^{\mu\nu} + \left(\frac{\partial\Phi/c^2}{\partial x^\alpha}\right) F^{\alpha\nu} = 0.$$

The electromagnetic field tensor $F^{\mu\nu}$ is defined by the potential A^μ as:

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu},$$

Where $\partial^{\mu} = g^{\mu\alpha}\partial_{\alpha}$. Under the conditions of a weak-field approximation:

$$F^{0i} = -F^{i0} \approx -\left(1 - \frac{2\Phi}{c^2}\right)\frac{E_i}{c}, F^{ij} = \epsilon^{ijk}B_k.$$

Since

$$\nabla_{\mu}F^{\mu\nu} = \partial_{\mu}F^{\mu\nu} + \Gamma^{\mu}_{\mu\alpha}F^{\alpha\nu} + \Gamma^{\nu}_{\mu\alpha}F^{\mu\alpha}.$$

And

$$\Gamma^{\mu}_{00} \approx \frac{\partial\Phi/c^2}{\partial x^{\mu}}, \Gamma^{\mu}_{ij} \approx \frac{\Phi/c^2}{r}r^{\mu}\delta_{ij}.$$

For the spatial component ν=i, the main contribution comes from:

$$\Gamma^{\mu}_{\mu 0}F^{0i} \approx \frac{\nabla\Phi}{c^2}\cdot\left(-\frac{E}{c}\right), \Gamma^{i}_{\mu\alpha}F^{\mu\alpha} \approx O(\Phi/c^2).$$

Revision of Ampere-Maxwell's law

Now considering the active existence, substitute the above results into $\nabla_{\mu}F^{\mu i} = \mu_0 J^i$ and keep them to the $O(\Phi/c^2)$ order:

$$\partial_j F^{ji} + \partial_0 F^{0i} + \frac{\nabla\Phi}{c^2}\cdot\left(-\frac{E_i}{c}\right) = \mu_0 J^i.$$

Substituting $F^{ji} = \varepsilon^{jik}B_k$ and $F^{0i} = -(1 - 2\Phi/c^2)E_i/c$ yields:

$$(\nabla \times B)^i - \frac{1}{c^2}\left(1 - \frac{2\Phi}{c^2}\right)\frac{\partial E_i}{\partial t} - \frac{1}{c^2}\frac{\partial\Phi}{\partial x^i}\frac{E_i}{c} = \mu_0 J^i.$$

Step 1: Expand and organize the equations

Write the original equation as a component:

$$(\nabla \times B)^i - \frac{1}{c^2}\frac{\partial E_i}{\partial t} + \frac{2\Phi}{c^4}\frac{\partial E_i}{\partial t} - \frac{1}{c^3}\frac{\partial\Phi}{\partial x^i}E_i = \mu_0 J^i.$$

Step 2: Ignore higher-order small quantities

- Since $\Phi/c^2 \ll 1$, $2\Phi/c^4(\partial E_i/\partial t)$ is a modifier of the $O(\Phi/c^2)$ order.
- $1/c^3(\partial\Phi/\partial x^i)E_i$ is a higher-order small amount（$O(\Phi/c^3)$）, which can be ignored.

So we get

$$(\nabla \times \mathrm{B})^{\mathrm{i}} - \frac{1}{\mathrm{c}^2}\frac{\partial \mathrm{E_i}}{\partial \mathrm{t}} + \frac{2\Phi}{\mathrm{c}^4}\frac{\partial \mathrm{E_i}}{\partial \mathrm{t}} = \mu_0 \mathrm{J}^{\mathrm{i}}.$$

Step 3: Merge like-for-like items

Merge items with $\partial E_i/\partial t$:

$$(\nabla \times \mathrm{B})^{\mathrm{i}} - \frac{1}{\mathrm{c}^2}\left(1 - \frac{2\Phi}{\mathrm{c}^2}\right)\frac{\partial \mathrm{E_i}}{\partial \mathrm{t}} = \mu_0 \mathrm{J}^{\mathrm{i}}.$$

Step 4: Correction of the current density

In curved space-time, the current density J^i is also affected by the gravitational potential:

$$\mathrm{J}^{\mathrm{i}} \approx \left(1 + \frac{2\Phi}{\mathrm{c}^2}\right)\mathrm{J}^{\mathrm{i}}_{flat}.$$

Step 5: Restore the vector form

Merge all the components into a vector equation:

$$\nabla \times \mathrm{B} - \frac{1}{\mathrm{c}^2}\left(1 - \frac{2\Phi}{\mathrm{c}^2}\right)\frac{\partial \mathrm{E}}{\partial \mathrm{t}} = \mu_0 \mathrm{J}\left(1 + \frac{2\Phi}{\mathrm{c}^2}\right).$$

Step 6: Sort out the final form

Move the displacement current term to the right:

$$\nabla \times \mathrm{B} = \mu_0 \mathrm{J}\left(1 + \frac{2\Phi}{\mathrm{c}^2}\right) + \frac{1}{\mathrm{c}^2}\left(1 - \frac{2\Phi}{\mathrm{c}^2}\right)\frac{\partial \mathrm{E}}{\partial \mathrm{t}}.$$

Following the steps above, Maxwell's equations in the gravitational field will become:

1. Gauss's Law

$$\nabla \cdot \mathrm{E} = \frac{\rho}{\epsilon_0}\left(1 + \frac{2\Phi}{\mathrm{c}^2}\right)$$

- The gravitational potential Φ increases the effective charge density (because the gravitational field affects space-time and thus the electric field distribution).

2. Ampère-Maxwell Law

$$\nabla \times B = \mu_0 J \left(1 + \frac{2\Phi}{c^2}\right) + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}\left(1 - \frac{2\Phi}{c^2}\right)$$

- The gravitational potential corrects the contribution of the current and the displacement current.

3. Faraday's Law

$$\nabla \times E = -\frac{\partial B}{\partial t}\left(1 + \frac{2\Phi}{c^2}\right)$$

- The gravitational field affects the temporal rate of change of the magnetic field.

4. Gauss's Law for Magnetism

$$\nabla \cdot B = 0$$

- No magnetic monopole, the form remains unchanged.

Analyzing the new Maxwell's equations, it can be seen that it will have the following physical significance:

Electric field enhancement: where the gravitational potential is low (e.g., near a massive object), $\Phi<0$, so $(1 \pm 2\Phi/c^2) < 1$, resulting in a decrease in the divergence of the electric field E (equivalent to a decrease in charge density).

Magnetic field correction: Similarly, the gravitational field affects the temporal evolution of the magnetic field.

Change in the speed of light: Since c changes effectively in the gravitational field ($c_{eff} \approx c(1 + 2\Phi/c^2)$), the propagation of electromagnetic waves is also affected (e.g., gravitational redshift).

3 Analysis of electromagnetic wave wave equation in weak gravitational field

3.1 Wave equation of electric and magnetic fields

Under the Lorentz criterion, the four-dimensional potential A^μ satisfies:

$$\Box A^{\nu} - \frac{2}{c^2}\nabla\Phi \cdot \nabla A^{\nu} = 0,$$

Where

- $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ is the d'Alembert operator
- $A^{\nu} = (\phi/c, A)$ is the electromagnetic four-dimensional potential
- Φ is the Newtonian gravitational potential
- c is the speed of light in a vacuum

Since

$$\Box A^{\nu} = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)A^{\nu} = \frac{1}{c^2}\frac{\partial^2 A^{\nu}}{\partial t^2} - \nabla^2 A^{\nu}$$

We focus on the spatial component (ν=i), which is the equation for the vector potential A:

$$\frac{1}{c^2}\frac{\partial^2 A^{i}}{\partial t^2} - \nabla^2 A^{i} - \frac{2}{c^2}\nabla\Phi \cdot \nabla A^{\nu} = 0$$

In the weak field approximation, the time component gauge $g_{00} = -(1 + 2\Phi/c^2)$. This causes the time derivative term to need to be corrected:

$$\frac{\partial^2}{\partial t^2} \to \left(1 - \frac{2\Phi}{c^2}\right)\frac{\partial^2}{\partial t^2}$$

So

$$\left(1 - \frac{2\Phi}{c^2}\right)\frac{1}{c^2}\frac{\partial^2 A^{i}}{\partial t^2} - \nabla^2 A^{i} - \frac{2}{c^2}\nabla\Phi \cdot \nabla A^{\nu} = 0$$

Physical Meaning Explained

1. Correction of the time derivative term$(1 - 2\Phi/c^2)$:

- Reflects the effect of the gravitational field on the passage of time (gravitational time dilation)
- In the region of the strong gravitational field (Φ<0), the effective time derivative increases

2. Space Derivatives Terminology Modifiers:

- $\nabla^2 A$ is the standard Laplace term
- $(2/c^2)\,\nabla\Phi \cdot \nabla A$ is the coupling term of the gravitational field and the electromagnetic field

3. Speed of light correction: The equation can be rearranged as:

$$\frac{\partial^2 A}{\partial t^2} - \left(1 + \frac{2\Phi}{c^2}\right) c^2 \nabla^2 A + \cdots = 0$$

indicates the equivalent speed of light $c_{eff} = c(1 + \Phi/c^2)$

Similarly, the wave equations for the electric field $E = -\partial_t A - \nabla\varphi$ and the magnetic field $B = \nabla \times A$ also contain a correction term for Φ.

Since $E = -\frac{\partial A}{\partial t} - \nabla\phi$ and $B = \nabla \times A$, we can get

$$\left(1 - \frac{2\Phi}{c^2}\right) \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} - \nabla^2 E - \frac{2}{c^2} \nabla\Phi \cdot \nabla E = 0,$$

$$\left(1 - \frac{2\Phi}{c^2}\right) \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} - \nabla^2 B - \frac{2}{c^2} \nabla\Phi \cdot \nabla B = 0.$$

This is the equation for electromagnetic wave fluctuations in a weak gravitational field.

3.2 Correction of the propagation velocity of electromagnetic waves

In the gravitational field, the phase velocity of the electromagnetic wave v_p no longer c, but:

$$v_p \approx c\left(1 + \frac{2\Phi}{c^2}\right).$$

It means

- At low gravitational potentials ($\Phi < 0$, as near the Earth's surface), the speed of light decreases slightly:

$$v_p \approx c\left(1 - \frac{|2\Phi|}{c^2}\right).$$

- At high gravitational potential ($\Phi > 0$, e.g., far from the mass source), the speed of light increases slightly.

3.3 The physical effects of the new electromagnetic wave wave equation

3.3.1 Gravitational redshift

The frequency ω of an electromagnetic wave varies in the gravitational field:

$$\omega(r) \approx \omega_0 \left(1 + \frac{2\Phi(r)}{c^2}\right),$$

Where

- ω_0 is the frequency at infinity.
- When light escapes from a low gravitational potential (Φ<0), the frequency decreases (redshift).

3.3.2 Light deflection

Because the speed of light is affected by the gravitational potential, light rays are deflected as they pass through massive objects such as the Sun:

$$\Delta\theta \approx \frac{4GM}{c^2 b},$$

Where

- G is the gravitational constant $(6.674 \times 10^{-11} m^3 kg^{-1} s^{-2})$,
- M is the stellar mass
- b is the collision parameter (minimum distance) of the ray.

3.3.3 Shapiro time delay

As the light signal passes through the gravitational field, the propagation time increases:

$$\Delta t \approx \frac{2GM}{c^3} \ln\left(\frac{4 r_e r_p}{b^2}\right),$$

Where

- r_e and r_p are the distances between the transmitting and receiving points, respectively.

3.3.4 Example Calculations (Near the Earth's Surface)

In the earth's surface, $\Phi = -\mathrm{GM}_{\oplus}/\mathrm{R}_{\oplus} \approx -6.25 \times 10^7 m^2/s^2$, then

- **Lightspeed Modifier:**

$$v_p \approx c\left(1 - \frac{6.25 \times 10^7}{(3 \times 10^8)^2}\right) \approx c(1 - 7 \times 10^{-10}).$$

- **Gravitational redshift (launch from ground into space):**

$$\frac{\Delta\omega}{\omega_0} \approx \frac{\Phi}{c^2} \approx -7 \times 10^{-10}.$$

4 Conclusions

The analysis in this paper shows that in the weak field approximation, an influencing factor of gravitational force can be introduced into Maxwell's equations. Depending on the influence of this gravitational factor, a correction term is added to the charge density, current density, electric field, and magnetic field in Maxwell's equations. From this, the wave equation for electromagnetic waves affected by gravity is further derived. In this wave equation, due to the existence of the gravitational field, the corresponding gravitational time will be dilated, and there will also be a coupling term between the gravitational force and the electromagnetic field. We can also find that the propagation speed of electromagnetic waves slows down in the gravitational field. These conclusions are in full agreement with those of general relativity.

(The derivation and calculation of the formula in this paper were done by DeepSeek)

References

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