Improvement of Gauss's Formula for the distribution of primes by introducing a floating logarithmic base and empirically proven accuracy similar to Li

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Introduction

The known approximations for the number of prime numbers π(x) include Gauss's formula x/ln(x) and Riemann's formula through the logarithmic integral Li(x). The latter is known for its high accuracy but is difficult to compute numerically, as it requires integration.

In the present work we propose a new approximation with an elementary structure and exceptionally high precision, which gives much more accurate results compared to Gauss's formula and almost reaches those of Li.

In essence, the new formula represents an improved Gauss formula by turning the logarithm base in the denominator from fixed natural (ln) to one with a **floating base**.

Empirically, very accurate results are established in the range above. For large values, the formula approaches that of Gauss and both become equally accurate, along with Li.

Proposed formula

$$\pi(x) \approx \frac{x}{\log_{\left(e + \frac{2.93}{\ln(x)}\right)} x}$$

Where:

π(x) – predicted number of primes

x – range considered

The number **"2.93" divided by ln(x)** is a parameter empirically derived through tests in the range $\mathbf{10^3} \leq x \leq \mathbf{10^{12}}$.

It is not absolutely mandatory, nor is the logarithm base correcting part of the formula fixed. This is a one-time test of the idea with quite a good result. The idea itself can be further developed, along with the formula. Hypothetically, following the logic of the **"floating logarithm base,"** it may be possible to reach a formula for absolutely exact determination of the number of prime numbers in a given interval.

A huge advantage of the formula is the simplification of calculations. Only logarithms and elementary arithmetic operations are used, which is much simpler compared to Riemann's approach, relying on integration, non-trivial zeros, and enormous computational complexity.

Comparison results

In tests for x = $\mathbf{10^3} - \mathbf{10^{12}}$, the new formula consistently gives more accurate results than Gauss and very close to Riemann (Li). Even at $\mathbf{10^3}$ it turned out to be more accurate than Li.

Two more tests were carried out with calculated values for $\mathbf{10^{100}}$and $\mathbf{10^{200}}$. The base used was not Li, but Riemann's own formula, using non-trivial zeros, which is the most accurate method for predicting the number of prime numbers at high ranges. A quite high accuracy of the new formula is seen in this range as well.

A test was also performed with Dusart intervals in the ranges $\mathbf{10^{500}}$ and $\mathbf{10^{1000}}$. From it, it is seen that the new formula produces a result inside the intervals in which the number of primes is proven to be located.

Results (tables) to 10^{12}

New formula:

x	Real π(x)	New Formula	New Δ	New Δ(%)
1000	168	**165,7557925**	-2,244207526	**1,335837813**
10000	1229	**1205,898364**	-23,10163615	**1,87971002**
100000	9592	**9463,25072**	-128,7492796	**1,342256877**
1000000	78498	**77820,21974**	-677,7802617	**0,863436345**
10000000	664579	**660582,4935**	-3996,506457	**0,601359125**
100000000	5761455	**5737393,908**	-24061,0924	**0,417621806**
1000000000	50847534	**50701748,73**	-145785,2689	**0,286710598**
10^{10}	455052511	**454163185,7**	-889325,2879	**0,195433552**
10^{11}	4118054813	**4112673052**	-5381760,702	**0,130686961**
10^{12}	37607912018	**37576184907**	-31727111,2	**0,084362863**

Li calculaton:

x	Real π(x)	Li(x)	Li Δ	Li Δ(%)
1000	168	176,5644942	8,56449421	5,09791322
10000	1229	1245,092052	16,09205212	1,309361442
100000	9592	9628,763837	36,76383727	0,383276035
1000000	78498	78626,504	128,5039956	0,163703528
10000000	664579	664917,3599	338,3598845	0,050913418
100000000	5761455	5762208,33	753,3304574	0,013075351
1000000000	50847534	50849233,91	1699,912585	0,003343156
10^{10}	455052511	455055616,9	3105,904252	0,000682538
10^{11}	4118054813	4118066400	11586,74529	0,000281365
10^{12}	37607912018	37607950270	38251,67945	0,000101712

Gauss:

x	Real π(x)	Gauss	Gauss Δ	Gauss Δ(%)
1000	168	144,7648273	-23,2351727	13,83045994
10000	1229	1085,736205	-143,2637952	11,65694021
100000	9592	8685,889638	-906,1103619	9,446521705
1000000	78498	72382,41365	-6115,586349	7,79075435
10000000	664579	620420,6884	-44158,31157	6,644554156
100000000	5761455	5428681,024	-332773,9762	5,775866968
1000000000	50847534	48254942,43	-2592591,566	5,098755755
10^{10}	455052511	434294481,9	-20758029,1	4,561677739
10^{11}	4118054813	3948131654	-169923159,3	4,126296687
10^{12}	37607912018	36191206825	-1416705193	3,767040276

Forecast calculations for 10^{100} and 10^{200}

According to Riemann's formula we have:

For 10^{100}:

Predicted number of primes by Riemann:

4.36197198714070315909950911322916461153875721171712648961243486387595 7949204160 $\times 10^{97}$

Calculated by the new formula:

4.3632276 $\times 10^{97}$

Difference (relative share):

-0.000287851

-0.0287851%

For 10^{200}:

Predicted number of primes by Riemann:

2.1762083147717327938824389939217493036694850976424606353161503897666 $\times 10^{197}$

Calculated by the new formula:

2.17654902 $\times 10^{197}$

Difference (relative share):

-0.000156560

-0.0156560%

Dusart intervals

For 10^{500}:

Lower bound:

$\approx 8.69344 \times 10^{496}$

Upper bound:

$\approx 8.69419 \times 10^{496}$

Calculation by the new formula:

$\approx 8.69401 \times 10^{496}$

Falls within the interval.

For 10^{1000}:

Lower bound:

$\approx 4.34483 \times 10^{996}$

Upper bound:

$\approx 4.34502 \times 10^{996}$

Calculation by the new formula:

$\approx 4.34497 \times 10^{996}$

Falls within the interval.

Conclusion

The proposed calculation is probably the simplest analytical approximation with almost Riemann-level accuracy known to date. It opens new possibilities for calculations over enormous ranges and deserves the attention of researchers in the field of analytic number theory.

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References:

Carl Friedrich Gauss — classical approximation for the number of prime numbers.

Logarithmic Integral (Li) — logarithmic integral as a more accurate approximation for the number of primes.

Riemann Prime Counting Function — complete Riemann approximation with corrections from the non-trivial zeros of the zeta function.

Pierre Dusart, Estimates of Some Functions Over Primes Without R.H., arXiv:1002.0442, 2010.

Wolfram Alpha — online computational platform used for analytical and numerical calculations.

Python + SciPy — used for numerical integration and data analysis.

Thomas R. Nicely — database and counts of primes up to high decimal powers.

Tomás Oliveira e Silva — computational results and tables with precisely counted prime numbers up to 10^24 and beyond.

OEIS A006880 — sequence with precisely known values of π(x) for large powers.

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