The Inherent Mixed-Radix Structure of FFT: A General Framework for Puiseux Series and Branch Cut Computation

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Abstract

This paper presents a new foundational insight into the Fast Fourier Transform (FFT) algorithm: **mixed-radix decomposition is not an optional design, but an inherent structural property of FFT itself**.

We demonstrate that any power series $f(x)$, when decomposed into modulo-s subseries for any integer $s \geq 2$, naturally aligns with radix-s FFT computation — regardless of whether s is prime. This unification renders the need for specialized mixed-radix FFT frameworks obsolete.

Furthermore, we show that this structure enables seamless extraction of Puiseux series coefficients from a function, including those involving fractional powers and branch cuts. The FFT not only resolves monodromy behavior computationally, but also simplifies the treatment of multivalued functions across Riemann surfaces without symbolic intervention.

This paper is a direct sequel to the author's previous work, *Sampling on the Riemann Surface: A Natural Resolution of Branch Cuts in Puiseux Series*, available at https://ai.vixra.org/pdf/2504.0104v1.pdf, and the first foundational paper, *A Unified Computational Framework Unifying Taylor-Laurent, Puiseux, Fourier Series, and the FFT Algorithm*, available at https://ai.vixra.org/pdf/2504.0027v1.pdf.

Keywords: Riemann Surface, Puiseux Series, Fast Fourier Transform, mix-radix FFT, Branch Cuts, Multivalued Functions, Numerical Analytic Continuation

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Declaration of AI Assistance

Parts of this work benefited from computational assistance by Alice ChatGPT (AI Assistant). All creative insights, mathematical formulations, and final writing were solely directed, reviewed, and approved by the human author, Chang Hee Kim.

Preface: *We Do Not Mix — It Was Already Mixed*

In our first paper [2], we introduced a unified computational framework connecting Taylor, Laurent, Puiseux, and Fourier series using the FFT algorithm. We demonstrated how FFT could be used to extract power series coefficients efficiently and naturally, treating it as a solver of structured expansions rather than just a numerical tool.

In our second paper [1], we extended this framework to multivalued functions. By sampling on the Riemann surface rather than enforcing artificial branch cuts, we showed that FFT inherently respects monodromy and reconstructs the full structure of Puiseux series.

After submitting that paper, I proceeded to implement a radix-s FFT algorithm where s is a **prime number**, strictly following the way I decomposed the power series in the previous paper. The implementation worked as expected.

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1 Inherently Mixed-Radix Nature of FFT and Puiseux Decomposition

When a function $f(x)$ is approximated by its Taylor series:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

it can sometimes be inconvenient for computational purposes, particularly when extracting coefficients via discrete sampling within the radius of convergence.

To address this, we slightly modify the expression. Assume for illustration that $N = 4$, the number of coefficients:

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ &= \{\text{even power terms}\} + \{\text{odd power terms}\} \\ &= (a_0 + a_2 x^2) + (a_1 x + a_3 x^3) \\ &= \sum_{k=0}^{1} a_{2k} x^{2k} + \sum_{k=0}^{1} a_{2k+1} x^{2k+1} \end{aligned}$$

In general, if $N = 2^m$, $N = 3^m$, or $N = s^m$, where m is an arbitrary positive integer and $s \geq 2$, we can express it as:

$$\begin{aligned} f(x) &= \sum_{n=0}^{N-1} a_n x^n \\ &= \sum_{k=0}^{N/2-1} a_{2k} x^{2k} + \sum_{k=0}^{N/2-1} a_{2k+1} x^{2k+1} && \text{(radix-2)} \\ &= \sum_{k=0}^{N/3-1} a_{3k} x^{3k} + \sum_{k=0}^{N/3-1} a_{3k+1} x^{3k+1} + \sum_{k=0}^{N/3-1} a_{3k+2} x^{3k+2} && \text{(radix-3)} \\ &= \sum_{r=0}^{s-1} \sum_{k=0}^{N/s-1} a_{sk+r} x^{sk+r} && \text{(radix-s)} \end{aligned}$$

Now, from the above equation, we factor out $x^0, x^1, x^2, \ldots$ from each subseries as below:

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{N-1} a_n x^n \\
&= x^0 \sum_{k=0}^{N/2-1} a_{2k} x^{2k} + x^1 \sum_{k=0}^{N/2-1} a_{2k+1} x^{2k} && \text{(radix-2)} \\
&= x^0 \sum_{k=0}^{N/3-1} a_{3k} x^{3k} + x^1 \sum_{k=0}^{N/3-1} a_{3k+1} x^{3k} + x^2 \sum_{k=0}^{N/3-1} a_{3k+2} x^{3k} && \text{(radix-3)} \\
&= \sum_{r=0}^{s-1} x^r \sum_{k=0}^{N/s-1} a_{sk+r} x^{sk} && \text{(radix-s)}
\end{aligned}
$$

Now, from the equation above, we enclose each subseries in parentheses, yielding the radix-2, radix-3, and generalized radix-s FFTs, as shown below:

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{N-1} a_n x^n \\
&= x^0 \left(\sum_{k=0}^{N/2-1} a_{2k} x^{2k} \right) + x^1 \left(\sum_{k=0}^{N/2-1} a_{2k+1} x^{2k} \right) && \text{(radix-2 FFT)} \\
&= x^0 \left(\sum_{k=0}^{N/3-1} a_{3k} x^{3k} \right) + x^1 \left(\sum_{k=0}^{N/3-1} a_{3k+1} x^{3k} \right) + x^2 \left(\sum_{k=0}^{N/3-1} a_{3k+2} x^{3k} \right) && \text{(radix-3 FFT)} \\
&= \sum_{r=0}^{s-1} x^r \left(\sum_{k=0}^{N/s-1} a_{sk+r} x^{sk} \right) && \text{(radix-s FFT)}
\end{aligned}
$$

Note that each subseries is enclosed in parentheses. If we carefully examine the above power series decomposition (radix-s FFT), we can intuitively recognize that the original power series $f(x) = \sum_{n=0}^{N-1} a_n x^n$ is naturally partitioned into s subseries. This observation clearly demonstrates that s does not need to be a prime number for the decomposition to hold.

a) Radix-2 FFT, let $t = x^2$, then $x = t^{\frac{1}{2}}$

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{N-1} a_n t^{\frac{n}{2}} && \text{(Puiseux Series)} \\
&= t^{\frac{0}{2}} \left(\sum_{k=0}^{N/2-1} a_{2k} t^k \right) + t^{\frac{1}{2}} \left(\sum_{k=0}^{N/2-1} a_{2k+1} t^k \right) && \text{(radix-2 FFT)}
\end{aligned}
$$

b) Radix-3 FFT, let $t = x^3$, then $x = t^{\frac{1}{3}}$

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{N-1} a_n t^{\frac{n}{3}} && \text{(Puiseux Series)} \\
&= t^{\frac{0}{3}} \left(\sum_{k=0}^{N/3-1} a_{3k} t^k \right) + t^{\frac{1}{3}} \left(\sum_{k=0}^{N/3-1} a_{3k+1} t^k \right) + t^{\frac{2}{3}} \left(\sum_{k=0}^{N/3-1} a_{3k+2} t^k \right) && \text{(radix-3 FFT)}
\end{aligned}
$$

c) General Radix-s, let $t = x^s$, then $x = t^{\frac{1}{s}}$

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{N-1} a_n t^{\frac{n}{s}} && \text{(Puiseux Series)} \\
&= \sum_{r=0}^{s-1} t^{\frac{r}{s}} \left(\sum_{k=0}^{N/s-1} a_{sk+r} t^k \right) && \text{(radix-s FFT)}
\end{aligned}
$$

We observe once again that the Puiseux series $f(x) = \sum_{n=0}^{N-1} a_n t^{\frac{n}{s}}$, as presented in equation (Puiseux Series), is decomposed into s sub-power series in equation (radix-s FFT). This structure precisely mirrors the radix-s FFT algorithm and reinforces that s need not be a prime number.

These subseries — enclosed in parentheses — are the **fundamental FFT units** that preserve the structure of decomposition when walking the Riemann surface. The prefactor $x^r = t^{r/s}$ captures the fractional monodromy phase, while each inner sum runs purely in powers of t.

Let us consider the radix-s FFT. We define $t = x^s$, and consequently, $x = t^{1/s}$.

When we move from N/s to N (where $N = s^n$ for $n = 1, 2, \ldots$), this is equivalent to transitioning from t to $t^{1/s}$.

Please observe that from the subseries $\left(\sum_{k=0}^{N/s-1} a_{sk+r} t^k \right)$ to the full Puiseux series $\sum_{n=0}^{N-1} a_n t^{\frac{n}{s}}$, $\frac{N}{s}$ transitions to N, which corresponds to the transition from t to $t^{\frac{1}{s}}$.

The reason the room increases from $\frac{N}{s}$ to N is that **branch cuts are automatically handled by the FFT**. This occurs because the FFT implicitly evaluates the **multi-valued nature** of $t^{1/s}$, seamlessly managing the different branches of the fractional powers. Therefore, this process automatically takes care of the multi-sheeted structure that would otherwise need to be handled manually in complex analysis.

In essence, **FFT computes the branch cuts** when transitioning from t to $t^{1/s}$, which explains the increase from N/s to N.

Kim's FFT-Puiseux Theorem: FFT *is* the Puiseux Expansion

Let $f(x) = \sum_{n=0}^{N-1} a_n x^n$ be a finite power series. For any integer $s \geq 2$, define the substitution $t = x^s$. Then the radix-s FFT decomposition of $f(x)$ yields the Puiseux expansion of f in powers of $t^{1/s}$, explicitly:

$$f(x) = \sum_{r=0}^{s-1} t^{r/s} \left(\sum_{k=0}^{N/s-1} a_{sk+r} t^k \right)$$

Each term $t^{r/s}(\cdots)$ corresponds to a distinct branch of the Puiseux series. **FFT does not merely approximate the Puiseux structure — it *is* the Puiseux expansion, numerically resolved without symbolic algebra.**

The function $f(x)$ is naturally decomposed into s sub-power series $\left(\sum_{r=0}^{s-1} t^{r/s}(\cdots)\right)$, which constitutes the radix-s FFT algorithm, where s need not be a prime number but can be any integer $s \geq 2$.

Epilog

In this work, we have presented a foundational realization: the mixed-radix structure of the Fast Fourier Transform (FFT) is not a design choice but an inherent property of the algorithm itself. Any power series $f(x)$, when decomposed through radix-s FFT for any integer $s \geq 2$, aligns naturally with a Puiseux expansion in fractional powers of $t = x^s$. This insight renders traditional "mixed-radix FFT frameworks" unnecessary—they were already embedded within FFT from the beginning.

We demonstrated that this decomposition inherently segments $f(x)$ into s sub-power series branches $\left(\sum_{r=0}^{s-1} t^{r/s}(\cdots)\right)$, each capturing a distinct analytic continuation on the Riemann surface. The FFT, in its native numerical form, respects monodromy and fractional periodicity without any need for symbolic algebra or branch cut management.

Through careful unification of Taylor, Laurent, and Puiseux series, we have shown that FFT is not merely a computational engine—it is a complete solver for multivalued analytic structures. Its ability to traverse and reconcile multiple sheets of the Riemann surface is embedded in its radix structure and spectral resolution.

We hope this insight will inspire new paradigms in symbolic-free complex analysis, automated analytic continuation, and multibranch function decomposition. The future holds vast and untapped potential for FFT—not only as a numerical tool, but as a universal framework for exploring the layered topologies of mathematical functions.

Readers who are interested in the broader context of this work are encouraged to refer to our earlier paper: *A Unified Computational Framework Unifying Taylor-Laurent, Puiseux, Fourier Series, and the FFT Algorithm*, at: https://ai.vixra.org/pdf/2504.0027v1.pdf

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