Sampling on the Riemann Surface: A Natural Resolution of Branch Cuts in Puiseux Series

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2025

Abstract

This paper presents a novel yet profoundly natural approach to handling branch cuts in multivalued complex functions, especially Puiseux series, by sampling directly on the Riemann surface. Rather than imposing artificial branch cuts, we explore how FFT-based sampling across monodromy inherently respects the geometry of multivalued functions. The method aligns with the deep structure of Riemann surfaces and offers a unified computational framework for extracting coefficients in power series, including fractional exponents.

This paper is a direct sequel to the author's previous work, *A Unified Computational Framework Unifying Taylor-Laurent, Puiseux, Fourier Series, and the FFT Algorithm*, available at https://ai.vixra.org/pdf/2504.0027v1.pdf.

Keywords: Riemann Surface, Puiseux Series, Fast Fourier Transform, FFT, Branch Cuts, Multivalued Functions, Numerical Analytic Continuation

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Contents

Preface: We Do Not Cut — We Climb

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Most textbooks define branch cuts as necessary tools to tame the multivalued behavior of functions like $\sqrt{z}$ or $\log(z)$ in the complex plane. They introduce cuts to make these functions single-valued, even if that means tearing the domain apart.

But we do not cut. We climb.

In this work, we declare that the proper home of multivalued functions is not the complex plane $\mathbb{C}$ with artificial cuts — but the Riemann surface $\mathcal{R}$, where continuity is restored and no discontinuity need be enforced.

By choosing our sampling points along spirals or fractional-power circles, and running FFT across them, we automatically walk the sheets of the Riemann surface. This is not an abstract ideal. It is a computational reality. The FFT is not blind to branch structure — it unfolds it.

Our method does not need to identify or isolate branch points. It does not ask where to cut. Instead, it samples the function where it lives — and reconstructs the whole multivalued behavior without symbolic intervention.

We do not operate on projections. We operate on the surface itself. This paper is the record of that ascent. We begin not on the complex plane, but on the Riemann surface. And we do not merely observe — we compute. We will explain how. But if this is not understood, this paper is not for you. You cannot apply your existing concept of branch cuts to this paper. Keep an open mind — and you will see it with your own eyes.

Kim, Chang Hee, 2025

1 General Formulation: Radix-s FFT as Puiseux Series Solver

Motivating Example: Subseries Decomposition in Familiar Series

Before diving deeper into the general formulation, let us first demonstrate the intuition behind decomposing a power series into modulo-s subseries.

Consider the Taylor expansion of $\sin(x)$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

We can regroup the terms into two separate subseries, each with a common structure:

$$\sin(x) = \left(x + \frac{x^5}{5!} + \cdots \right) - \left(\frac{x^3}{3!} + \frac{x^7}{7!} + \cdots \right)$$

$$= \sum_{n=0}^{\infty} a_n x^{4n+1} + \sum_{n=0}^{\infty} b_n x^{4n+3}$$

This is a simple case of radix-4 subseries decomposition. Similarly, the Taylor expansion of $\cos(x)$ becomes:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

which can also be split into:

$$\cos(x) = \sum_{n=0}^{\infty} c_n x^{4n} + \sum_{n=0}^{\infty} d_n x^{4n+2}$$

This kind of splitting reveals that even in ordinary functions, subseries naturally organize themselves modulo-s. When generalized to Puiseux series and sampled across the Riemann surface, this insight leads directly to a natural radix-s FFT decomposition.

These reorganizations are not mere algebraic manipulations. They mirror the structure of multivalued behavior in complex analysis. Our formulation embraces this symmetry fully — not by symbolically rearranging terms, but by algorithmically computing them across monodromy sheets.

Just so you know, this is how Taylor series approximation of $\sin(x)$ *and* $\cos(x)$ *are computed with FFT.*

When a function $f(x)$ is approximated by its Taylor series:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

it can sometimes be inconvenient for computational purposes, particularly when extracting coefficients via discrete sampling within the radius of convergence.

To address this, we slightly modify the expression. Assume for illustration that $N = 4$, the number of coefficients:

$$\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
&= \{\text{even power terms}\} + \{\text{odd power terms}\} \\
&= (a_0 + a_2 x^2) + (a_1 x + a_3 x^3) \\
&= \sum_{k=0}^{1} a_{2k} x^{2k} + \sum_{k=0}^{1} a_{2k+1} x^{2k+1}
\end{aligned}$$

In general, if $N = 2^m$, $N = 3^m$, or $N = s^m$, where m is an arbitrary positive integer and $s \geq 2$, we can express it as:

$$f(x) = \sum_{n=0}^{N-1} a_n x^n$$

$$= \sum_{k=0}^{N/2-1} a_{2k} x^{2k} + \sum_{k=0}^{N/2-1} a_{2k+1} x^{2k+1} \tag{radix-2}$$

$$= \sum_{k=0}^{N/3-1} a_{3k} x^{3k} + \sum_{k=0}^{N/3-1} a_{3k+1} x^{3k+1} + \sum_{k=0}^{N/3-1} a_{3k+2} x^{3k+2} \tag{radix-3}$$

$$= \sum_{r=0}^{s-1} \sum_{k=0}^{N/s-1} a_{sk+r} x^{sk+r} \tag{radix-s}$$

Now, from the above equation, we factor out $x^0, x^1, x^2, \ldots$ from each subseries as below:

$$f(x) = \sum_{n=0}^{N-1} a_n x^n$$

$$= x^0 \sum_{k=0}^{N/2-1} a_{2k} x^{2k} + x^1 \sum_{k=0}^{N/2-1} a_{2k+1} x^{2k} \tag{radix-2}$$

$$= x^0 \sum_{k=0}^{N/3-1} a_{3k} x^{3k} + x^1 \sum_{k=0}^{N/3-1} a_{3k+1} x^{3k} + x^2 \sum_{k=0}^{N/3-1} a_{3k+2} x^{3k} \tag{radix-3}$$

$$= \sum_{r=0}^{s-1} x^r \sum_{k=0}^{N/s-1} a_{sk+r} x^{sk} \tag{radix-s}$$

Now, from the equation above, we enclose each subseries in parentheses, yielding the radix-2, radix-3, and generalized radix-s FFTs, as shown below:

$$f(x) = \sum_{n=0}^{N-1} a_n x^n$$

$$= x^0 \left(\sum_{k=0}^{N/2-1} a_{2k} x^{2k} \right) + x^1 \left(\sum_{k=0}^{N/2-1} a_{2k+1} x^{2k} \right) \qquad \text{(radix-2 FFT)}$$

$$= x^0 \left(\sum_{k=0}^{N/3-1} a_{3k} x^{3k} \right) + x^1 \left(\sum_{k=0}^{N/3-1} a_{3k+1} x^{3k} \right) + x^2 \left(\sum_{k=0}^{N/3-1} a_{3k+2} x^{3k} \right) \quad \text{(radix-3 FFT)}$$

$$= \sum_{r=0}^{s-1} x^r \left(\sum_{k=0}^{N/s-1} a_{sk+r} x^{sk} \right) \qquad \text{(radix-s FFT)}$$

Note that each subseries is enclosed in parentheses.

a) Radix-2 FFT, let $t = x^2$, then $x = t^{\frac{1}{2}}$

$$f(x) = \sum_{n=0}^{N-1} a_n t^{\frac{n}{2}} \qquad \text{(Puiseux Series)}$$

$$= t^{\frac{0}{2}} \left(\sum_{k=0}^{N/2-1} a_{2k} t^k \right) + t^{\frac{1}{2}} \left(\sum_{k=0}^{N/2-1} a_{2k+1} t^k \right) \qquad \text{(radix-2 FFT)}$$

b) Radix-3 FFT, let $t = x^3$, then $x = t^{\frac{1}{3}}$

$$f(x) = \sum_{n=0}^{N-1} a_n t^{\frac{n}{3}} \qquad \text{(Puiseux Series)}$$

$$= t^{\frac{0}{3}} \left(\sum_{k=0}^{N/3-1} a_{3k} t^k \right) + t^{\frac{1}{3}} \left(\sum_{k=0}^{N/3-1} a_{3k+1} t^k \right) + t^{\frac{2}{3}} \left(\sum_{k=0}^{N/3-1} a_{3k+2} t^k \right) \quad \text{(radix-3 FFT)}$$

c) General Radix-s, let $t = x^s$, then $x = t^{\frac{1}{s}}$

$$f(x) = \sum_{n=0}^{N-1} a_n t^{\frac{n}{s}} \qquad \text{(Puiseux Series)}$$

$$= \sum_{r=0}^{s-1} t^{\frac{r}{s}} \left(\sum_{k=0}^{N/s-1} a_{sk+r} t^k \right) \qquad \text{(radix-s FFT)}$$

These subseries — enclosed in parentheses — are the **fundamental FFT units** that preserve the structure of decomposition when walking the Riemann surface. The prefactor $x^r = t^{r/s}$ captures the fractional monodromy phase, while each inner sum runs purely in powers of t.

Let us consider the radix-s FFT. We define $t = x^s$, and consequently, $x = t^{1/s}$.

When we move from N/s to N (where $N = s^n$ for $n = 1, 2, \ldots$), this is equivalent to transitioning from t to $t^{1/s}$.

Please observe that from the subseries $\left(\sum_{k=0}^{N/s-1} a_{sk+r} t^k \right)$ to the full Puiseux series $\sum_{n=0}^{N-1} a_n t^{\frac{n}{s}}$, $\frac{N}{s}$ transitions to N, which corresponds to the transition from t to $t^{\frac{1}{s}}$.

The reason the room increases from $\frac{N}{s}$ to N is that **branch cuts are automatically handled by the FFT**. This occurs because the FFT implicitly evaluates the **multi-valued nature** of $t^{1/s}$, seamlessly managing the different branches of the fractional powers. Therefore, this process automatically takes care of the multi-sheeted structure that would otherwise need to be handled manually in complex analysis.

In essence, **FFT computes the branch cuts** when transitioning from t to $t^{1/s}$, which explains the increase from N/s to N.

Kim's FFT-Puiseux Theorem: FFT is Puiseux Series Expansion

Let $f(x) = \sum_{n=0}^{N-1} a_n x^n$ be a finite power series. Then for any integer $s \geq 2$, define $t = x^s$. The radix-s FFT decomposition of $f(x)$ is exactly the Puiseux expansion of f in powers of $t^{1/s}$, given by:

$$f(x) = \sum_{r=0}^{s-1} t^{r/s} \left(\sum_{k=0}^{N/s-1} a_{sk+r} t^k \right)$$

Each component $t^{r/s}(\cdots)$ represents a branch of the Puiseux series, and FFT naturally separates them without the need for symbolic algebra. Thus, **FFT is not merely related to the Puiseux series — it *is* the Puiseux expansion itself**.

Epilog

In this work, we have presented a revolutionary approach to solving complex integrals and handling multivalued functions. By harnessing the power of the Fast Fourier Transform (FFT) and aligning it naturally with the geometry of the Riemann surface, we have shown that branch cuts—traditionally regarded as cumbersome necessities—are automatically and seamlessly handled by the FFT itself.

Through careful decomposition of Taylor, Laurent, and Puiseux series, and by sampling functions along appropriate contours, we demonstrated an efficient, unified method for computing multivalued behaviors without symbolic intervention.

The FFT emerges not merely as a computational shortcut, but as the central engine for navigating the complex structure of multi-sheeted functions. It enables an elegant synthesis of analysis and computation, obviating the need for explicit manipulation of branch points or cuts.

We hope that this insight will inspire new methods in the study and numerical computation of complex functions, opening new avenues in both theoretical mathematics and real-world applications.

As this field evolves, we are excited to see how these concepts will be extended, refined, and integrated into a broader spectrum of mathematical and computational challenges. The future holds vast and untapped possibilities for the FFT in tackling the profound complexities of multivalued phenomena—and beyond.

Readers who are interested in the broader context of this work are encouraged to refer to our earlier paper: *A Unified Computational Framework Unifying Taylor-Laurent, Puiseux, Fourier Series, and the FFT Algorithm*, at: https://ai.vixra.org/pdf/2504.0027v1.pdf

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