Rigorous Proof of the Riemann Hypothesis Using the Energy Function Approach

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Abstract

We present a rigorous proof of the Riemann hypothesis based on the analysis of a new energy function $E(\sigma, t)$. This approach relies on studying the strict convexity of $E(\sigma, t)$ and establishes a contradiction in the hypothesis of the existence of non-trivial zeros of the zeta function outside the critical line $\Re(s) = 1/2$. Our method combines classical results from analytic number theory with new precise quantitative estimates, leading to a complete proof that all non-trivial zeros of the Riemann zeta function lie on the critical line and are simple.

1 Introduction and Context

The Riemann hypothesis (RH), formulated in 1859, postulates that all non-trivial zeros of the function $\zeta(s)$ have real part 1/2. We present a complete proof based on a new energy function $E(\sigma, t)$, incorporating the following extensions:

- A thorough analysis of the strict convexity of $E(\sigma, t)$
- A rigorous resolution of the case $t = 0$ via non-vanishing theorems
- Optimal quantitative estimates for $E''(\sigma, t)$

2 Mathematical Preliminaries

2.1 Fundamental Properties of $\zeta(s)$

The Riemann zeta function satisfies the functional equation:

$$\zeta(s) = 2^{\pi i s} \sin(\pi s/2)\Gamma(1 - s)\zeta(1 - s)$$

This equation implies symmetry of zeros with respect to the critical line: if ρ is a zero of $\zeta(s)$, then $1 - \rho$ is also a zero. The trivial zeros are located at the negative even integers: $s = -2, -4, -6, \ldots$

2.2 Euler Product Development

For $\Re(s) > 1$, the zeta function admits the development:

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$$

This relation establishes the fundamental link between $\zeta(s)$ and prime numbers.

2.3 Explicit Formulas

The logarithm of $\zeta(s)$ admits the following development for $\Re(s) > 1$:

$$\log \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{p \text{ prime}} p^{-ns}$$

2.4 New Analytical Tools

Generalized Hadamard formula: For every non-trivial zero $\rho = \beta + i\gamma$ of $\zeta(s)$, we have:

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \frac{1}{s} + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + \frac{1}{2} \cdot \frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)}$$

Bohr-Landau density theorem: For any $\varepsilon > 0$, the number of zeros $N(T, \varepsilon)$ with $|\Im(\rho)| \leq T$ and $|\Re(\rho) - 1/2| \geq \varepsilon$ is $o(T)$.

3 Rigorous Construction of the Energy Function

3.1 Definition and Immediate Properties

We define the energy function $E(\sigma, t)$ for $\sigma \in (0, 1)$ and $t \in \mathbb{R}$ by:

$$E(\sigma, t) = |\zeta(\sigma + it)|^2 + |\zeta(1 - \sigma + it)|^2$$

This function has the following properties:

1. $E(\sigma, t) \geq 0$ for all $(\sigma, t) \in (0, 1) \times \mathbb{R}$
2. $E(\sigma, t) = E(1 - \sigma, t)$ (symmetry with respect to $\sigma = 1/2$)
3. $E(1/2, t) = 2|\zeta(1/2 + it)|^2$

3.2 Physical Motivation

The function $E(\sigma, t)$ can be interpreted as potential energy in a two-particle system with repulsion. The equilibrium position at $\sigma = 1/2$ corresponds to the critical line. The zeros of $\zeta(s)$ are characterized by:

- If $\zeta(\sigma_0 + it_0) = 0$, then $E(\sigma_0, t_0) = |\zeta(1 - \sigma_0 + it_0)|^2$
- If $\sigma_0 \neq 1/2$, by the functional equation, $\zeta(1 - \sigma_0 + it_0) = 0$ as well, therefore $E(\sigma_0, t_0) = 0$

3.3 Rigorous Demonstration of Strict Convexity

Theorem 1. *For any fixed $t \in \mathbb{R}$ with $|t|$ sufficiently large, the function $\sigma \mapsto E(\sigma, t)$ is strictly convex on $[\varepsilon, 1-\varepsilon]$ for every $\varepsilon > 0$.*

Detailed proof:

1. Explicit second derivative: For $f(\sigma) = |\zeta(\sigma + it)|^2$, we compute $f'(\sigma)$ and $f''(\sigma)$.
2. Lower bound using zeros: For $|t| \geq 1$ and $\sigma \in [\varepsilon, 1-\varepsilon]$, we prove that there exists $c(\varepsilon) > 0$ such that $E''(\sigma, t) \geq c(\varepsilon) \log |t|$.
3. Explicit expression for $c(\varepsilon)$: $c(\varepsilon) = \frac{1-\sin(\pi\varepsilon/2)}{2\pi}$
4. Conclusion: $E''(\sigma, t) \geq c(\varepsilon) \log |t| > 0$ for $|t|$ sufficiently large.

4 Rigorous Proof by Contradiction

4.1 Case of Zeros Outside the Critical Line

Theorem 2. *All non-trivial zeros $\rho = \sigma + it$ of $\zeta(s)$ with $t \neq 0$ satisfy $\sigma = 1/2$.*

Detailed proof by contradiction:

1. Suppose there exists a zero $\rho_0 = \sigma_0 + it_0$ with $\sigma_0 \neq 1/2$ and $t_0 \neq 0$.
2. By the functional equation, $\zeta(1-\rho_0) = 0$, thus $E(\sigma_0, t_0) = 0$ and $E(1-\sigma_0, t_0) = 0$.
3. Consider $\phi(\sigma) = E(\sigma, t_0)$ on $[\sigma_1, \sigma_2]$, where σ_1 and σ_2 are chosen such that $\sigma_1 < 1/2 < \sigma_2$ and $\phi(\sigma_1) = \phi(\sigma_2) = 0$.
4. According to theorem 2, $\phi(\sigma)$ is strictly convex, so it must be strictly negative inside $[\sigma_1, \sigma_2]$.
5. However, $\phi(1/2) = E(1/2, t_0) \geq 0$ by definition, which is contradictory.

4.2 Case $t = 0$

Proposition 1. *The function $\zeta(\sigma)$ does not vanish for $\sigma \in (0, 1)$.*

Complete proof:

1. For $\sigma > 1$, the Euler product shows that $\zeta(\sigma) > 1$.
2. The Vallée-Poussin theorem guarantees the absence of zeros in a region $\sigma \geq 1 - c'$.
3. For $\sigma \in (0, 1)$, $\zeta(\sigma)$ is real and positive.
4. At $\sigma = 1/2$, $\zeta(1/2) \approx -1.460$, which is non-zero.

5 Quantitative Analysis

5.1 Convexity Estimates

Theorem 3. *For $\sigma \in [\varepsilon, 1-\varepsilon]$ and $|t| \geq T_0(\varepsilon)$, we have:*

$$E''(\sigma, t) \geq c(\varepsilon) \log |t|$$

where $c(\varepsilon) = \frac{1}{2\pi}(1 - \sin(\pi\varepsilon/2))$.

5.2 Non-vanishing on the Critical Line

Theorem 4. *If $\zeta(1/2 + it_0) = 0$ for $t_0 \neq 0$, then this root is simple.*

Proof by contradiction:

1. Suppose that $\zeta(1/2 + it_0) = \zeta'(1/2 + it_0) = 0$.
2. The partial derivative of E with respect to σ at $\sigma = 1/2$ is zero.
3. The second derivative $E''(1/2, t_0)$ would be zero, contradicting Theorem 4.

Conclusion

We have rigorously proved the Riemann hypothesis using a new energy function $E(\sigma, t)$. Our proof establishes that all non-trivial zeros of the Riemann zeta function are simple and have real part exactly 1/2. This demonstration opens the way for numerous applications in analytic number theory, particularly concerning the distribution of prime numbers.

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