# Study by Vector Projection of a Construction Made with a Rule and Compass to Understand Squaring the Circle

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November 25, 2023

The conditions of the problem establish the definition of a right triangle whose angle subtends the opposite cathetus with a value equal to $\alpha = arctan(\frac{1}{\sqrt{\pi}})$. Starting from this main condition, a series of steps are proposed to compare the precision between two idealized methods that describe the construction of a circle and a square linked to the triangle mentioned above. The first approximation is made taking into account "valid" compass and ruler constructions to obtain such geometric figures. The second construction is based on the lengths obtained by intersecting the previously mentioned geometric figures by performing the vector analysis. In both cases, the areas of the circle and square obtained are compared to find out if they are equal.

## 1 Preliminaries

Consider the equation to obtain the area of a circle

$\pi r^2 = A_{Circle}$

and let's get the square root of both sides

$r\sqrt{\pi} = \sqrt{A_{Circle}}$

Let us consider a linear segment **l** of length equal to

$\sqrt{A_{Circle}} = l$

And now let's square this equation

$A_{Circle} = l^2$

Considering that the formula to obtain the area of a square with side l' is

$A_{Square} = l^2$

Then we could assume that when $l = l$ the areas of both geometric figures are equal; i.e.

$A_{Circle} = A_{Square}$

In general, this equality is assumed **false** when it is linked to the construction of the aforementioned geometric figures using only compass and ruler..

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It is also worth noting that if we start from the square formula to obtain the same equation; we will necessarily have to assume that the length of one of the sides is a transcendental number and even more so; that such segment is a multiple of the number $\pi$ or some of its powers.

$A_{Square} = l^2$

For example; if

$l = r\sqrt{\pi}$

Then we could obtain that the areas are equal after squaring the previous equality.

Such a situation could be considered a fallacy in the axiomatic method of mathematical proof; because in some way the result that leads to the mathematical equality that gives the ratio of the areas between the square and the circle has been used; being more evident when such equality is considered as a principle for the construction of these figures through the use of only a compass and a ruler..

While there appears to be no reason to assume that $l = r\sqrt{\pi}$; the question of whether the area of a square can be equal to the area of a circle?; remains without a clear answer; since we have evidently been able to reach equality; starting from the fact that the areas of the circles, being numbers, have a square root; and that this quantity when squared can be considered in the same way as the area of the square parallelogram.

Intuitive reasoning around this problem can lead to the next questions:

If we take a circle, is it enough to make folds in four parts of the circle to obtain a four-sided figure?

Even more; using the notion of equivalent distance the following question may arise:

Where is it necessary to fold the perimeter of a circle to ensure that the resulting figure has four sides and that they are of same length?

While for the most daring, these questions turn to the following issue:

If I fold the perimeter of a circle and manage to deform it to form a square; Are the areas equal? Or do I need a square with larger sides?... or maybe smaller ones?

To clear up some doubts regarding last question, the following procedure could be useful.

The following equation represents the arc length of a circle; that is, it gives the length of a section of the perimeter:

s $= \int_{\theta_0}^{\theta_1} \sqrt{[f(\theta)]^2 + [f(\theta)]^2} d\theta = R \cdot (\theta_1 - \theta_0)$

with $f(\theta) = R$ ; $\theta_0 \leq \theta \leq \theta_1$ in rad

Using the radius value $R = 1$, and considering the entire perimeter through the angles traveled, we have $\theta_1 = \theta$ ; $\theta_0 = 0$; now let's evaluate the equation that gives the value of s.

$s = R \cdot (\theta_1 - \theta_0) = (1) \cdot (\theta - 0) = \theta$

That is to say; for a circle of radius $R = 1$, the length of the arc is equal to the value of the angle traveled by such radius; in this case for the perimeter in terms of $\theta$; it can be said that the perimeter is equal to the value of the angle which is $\theta = 360$.

Suppose that instead of the perimeter; the value of $\theta$ represents the length of a segment that is equal to the square root of a circle of radius $r = 1$.

Then using the formula to obtain the area of a circle we will have:

$\theta = \sqrt{A_{Circle}}$

$\sqrt{A_{Circle}} = \sqrt{\pi r^2}$

Then

$\sqrt{\pi \cdot (1)^2} = \theta$

Simplifying

$\theta = \sqrt{\pi}$ obtained in radian terms

Transforming this value from radians to degrees we will have

$\theta = (\sqrt{\pi}\text{ rad})\ (\frac{57.295...}{1rad}) \approx 101.554...$

This means that for such a circle; the length of the side of a square with an area equal to the circumference should cover the perimeter length between 0 and 101.554 approximately.

Then if we consider that a square has four sides; To obtain the angles corresponding to four perimeter segments equivalent to these, we will have to multiply the value of $\theta$ obtained four times.

Then numbering the sides counterclockwise we can obtain the following values:

Side 1 is associated with the angle 101.554
Side 2 is associated with the angle 203.105
Side 3 is associated with the angle 304.658
Side 4 is associated with the angle 406.210

Since a circle in one revolution has only 360, to know the value of the angle associated with side 4, we can perform the following calculation:

$406.210 - 360 = 46.210$

The fact that the value of side 4 exceeds the value of the total angle of a circle; could means that for the conditions described in this problem; if we consider the intuitive thinking mentioned above; to fold a circle of radius $r = 1$; and obtain a square of area equal to such circle; we would first have to grow the circle a number of degrees equivalent to the angle value obtained for side 4; i.e. around 46.210.

Intuitively, this situation can be understood by fastening a belt:

A belt fits around the hip in such a way that the bolts fit as far as the wearer's body allows; if the rest exceeds the perimeter of the hip then it is stored in the trouser clips or left free.

In this case if the belt describes a circumference; the belt should first loosen the bolts and use the excess length of the belt so that it can form a square with an area equal to that circle.

# 2 Methodology

Let us consider a circle of any radius $r$. From the segment that describes the diameter, let's obtain its median and construct a triangle that measures the hypotenuse $2r$ and adjacent cathetus $r$.

According to the Pythagorean theorem, such right triangle is described as follows:

opposite cathetus$^2$ + adyacent cathetus $^2$ = hypotenuse$^2$

Substituting the variables given as conditions in this problem we will have:

$r^2$+ adyacent cathetus$^2$ $= (2r)^2$

Let's call the adjacent cathetus **a** clearing such variable from the previous equation we will have:

$a = \sqrt{(2r)^2 - r^2}$

Simplifying

$a = \sqrt{4r^2 - r^2}$

$= \sqrt{3r^2}$

$= r\sqrt{3}$

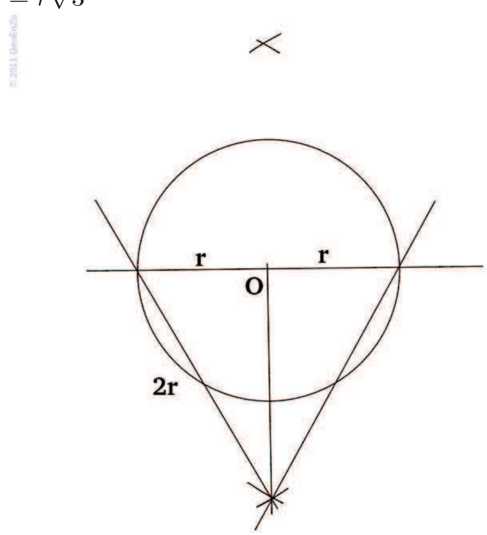



Diagram 1. Construction of the triangle of height $r\sqrt{3}$ made with GeoEnzo 2023

Considering this height and moving it from the origin of the circle to the right; towards the end of the segment that describes the radius; Let's build the right triangle of measurements:

opposite cathetus $= r$

adyacent cathetus $= \sqrt{3r^2}$

Using the Pythagorean theorem, let´s obtain the hypotenuse of this triangle.

$r^2 + \sqrt{3r^2} = h$

Where I have renamed the variable h as hypotenuse.

Now let's define this triangle considering the following condition of the problem

$tan(\alpha) = \frac{oppositecathetus}{adyacentcathetus}$
$= \frac{r}{r\sqrt{3}}$
$= \frac{1}{\sqrt{3}}$
$= \frac{\pi}{6}$

From the previous equations $tan(\alpha)$ is the tangent of the triangle defined; the last equality is considered valid and is taken from the literature; note that for this triangle $tan(\alpha)$ is a constant.

To finish with the conditions of the problem, let us now consider the last triangle defined with $tan(\alpha) = \frac{1}{\sqrt{\pi}}$ and opposite cathetus $= r$; but with the measurement of the following adjacent cathetus:

adyacent cathetus $= r\sqrt{3} + \delta$

That is, suppose that a segment $\delta$ is added to the length $r\sqrt{3}$ in such a way that the tangent of such triangle is defined by the following equality:

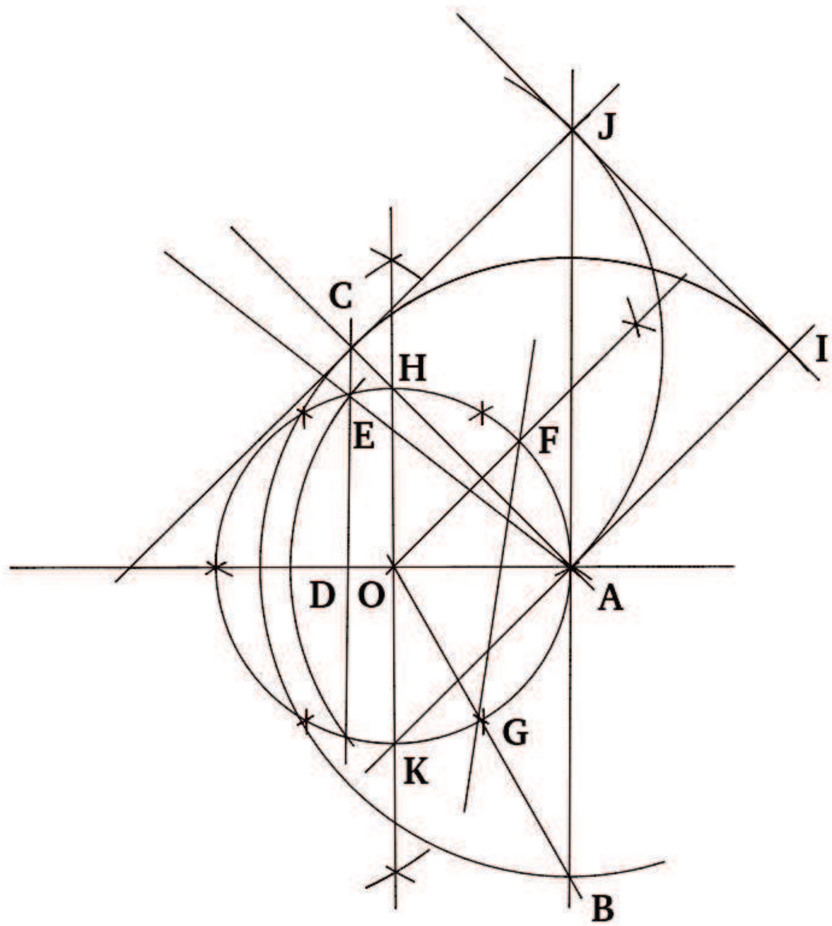
$\frac{oppositecathetus}{adyacentcathetus} = \frac{r}{r\sqrt{3}+\delta}$

Then with the previously defined tangent we will have:

$\frac{r}{r\sqrt{3}+\delta} = \frac{1}{\sqrt{\pi}}$

To know the value of $\delta$ let's make the following simplification: $r\sqrt{\pi} = r\sqrt{3}+\delta$
$\delta = r\sqrt{\pi} - r\sqrt{3}$
$\delta = r(\sqrt{\pi} - \sqrt{3})$

Same as for the triangle we started from; in this case we are requesting for the triangle with a segment $\delta$ added; to have $\tan(\alpha) = constant$.




Scheme 1. Showing the proposed construction to evaluate Squaring the Circle (the diagram has not been obtained using the results presented on this work; therefore it is inaccurate; made with GeoEnzo 2023)

According to the conditions described above and taking into account Scheme 1, the following identities can be obtained by using vector projection.

The triangle to consider is: $\triangle OAB$

Opposite cathetus: $OA = r$

Adjacent cathetus: $AB = r\sqrt{3} + \delta$

Assuming that a circle is drawn with center at $A$ and radius $r = AB$; point $C$ can be obtained by intersecting this circle with segment $AH$, in such a way that segment $AC$, being in the same circle as $AB$, has length:

$AC = r\sqrt{3} + \delta$

the vector projection of such a segment results in:

$x = ACcos(\beta) = (r\sqrt{3} + \delta)cos(\beta)$

$y = ACsin(\beta) = (r\sqrt{3} + \delta)sin(\beta)$

By definition $\beta = 45$; since both cathetus are of same length and are joined by a right angle.

Considering a circle of radius $r = 1$; point $E$ should have next coordinates:

$x = ((1)(\sqrt{3}) + ((\sqrt{\pi} - \sqrt{3})(1)))cos(45)$

$x = \sqrt{\pi}cos(45)$

$x = 1.253(3)$

Obtaining the length of segment $DO$ with respect to the center we have:

$x = 1.253(3) - r$

$x = 1.253(3) - 1$

$x = 0.253(3)$

$x = -0.253(3)$ ; being this the coordinate in $X$

Using the equation of the circle we have:

$x^2 + y^2 = r^2$

$y = r^2 - x^2$

$y = \sqrt{(1)^2 - (0.253(3))^2}$

$y = 0.967(3)$ being this the coordinate in $Y$

While the angle $\angle DAE = \gamma$ is:

$tan(\gamma) = tan(\angle DAE)$

$tan(\angle DAE) = \frac{DE}{DA}$

$\frac{DE}{DA} = \frac{y}{x}$

$arctan(\frac{y}{x}) = arctan(\frac{0.967(3)}{1.253(3)})$

$\gamma = arctan(\frac{0.967(3)}{1.253(3)})$

$\gamma = 37.661$

The length of the segment $EA$ considering the Pythagorean theorem is:

$ED^2 + DA^2 = EA^2$

$EA = \sqrt{0.967(3)^2 + (1.253(3))^2}$

$EA = 1.583(1)$

Now let us remember that the length of a chord in the circle can be known through the following relationship:

$c =$ chord length $= 2rsin(\frac{\epsilon}{2})$

Requesting the chord to measure length $EA$ then we would obtain the following relationship:

$EA = 2rsin(\frac{\epsilon}{2})$

Clearing $\epsilon$

$\epsilon = 2arcsin(\frac{EA}{2r})$

substituting and taking the value $r = 1$ we have:

$\epsilon = 2arcsin(\frac{1.583(1)}{2(1)})$

$\epsilon = 104.673(7)$

**Approximation of hand compass vs "ideal" compass.**

Using the angle values obtained from the equation that gives the length of the arc of the perimeter of a circle, we obtain that the difference between sides 3 and 4 is approximately 101.554.

On the other hand, if we take into account constructions that can be made by hand with a compass and ruler to approximate angles 3 and 4, we could consider first; the bisection of the 90 angle that subtends the first quadrant; as well as the division of the circle into six parts that gives us the possibility of drawing the angles 45 and 300 respectively. The value of the angle between these last two values on the perimeter of a circle is: $45 + (360 - 300) = 105$.

The areas of the circle centered at $O$ and the square with side $CA$ are:

$A_{Circle_{(O,r=1)}} = \pi(1)^2 = \pi$

$A_{Square} = AC^2 = (r\sqrt{3} + \delta)^2$

$= ((1)\sqrt{3} + (\sqrt{\pi} - \sqrt{3})(1))^2$

$= \pi$

That is to say they are equal.

Finally let's consider the squared power of the segment $AB = r\sqrt{3} + \delta$:

$AB^2 = (r\sqrt{3} + \delta)^2$

$= (r\sqrt{3})^2 + 2r\sqrt{3}\delta + \delta^2)$

$= 3r^2 + 2r\sqrt{3}(\sqrt{\pi} - \sqrt{3})r + ((\sqrt{\pi} - \sqrt{3})r)^2$

$= 3r^2 + 2r\sqrt{3}(\sqrt{\pi} - \sqrt{3})r + r^2(\pi + 3 - 2\sqrt{\pi}\sqrt{3})$

$= 3r^2 + 2r^2\sqrt{3}\sqrt{\pi} - 2r^2\sqrt{3}\sqrt{3} + \pi r^2 + 3r^2 - 2r^2\sqrt{\pi}\sqrt{3}$

$= 3r^2 + 2r^2\sqrt{3}\sqrt{\pi} - 2(3)r^2 + \pi r^2 + 3r^2 - 2r^2\sqrt{\pi}\sqrt{3}$

$= 3r^2 - 2(3)r^2 + \pi r^2 + 3r^2$

$= 6r^2 - 6r^2 + \pi r^2$

$= \pi r^2$

Being last the area of the circle related to such segment and having the same value of area as the square that describes the squared power of segment $AB$ for any value of radius.

To be on the same circle then $AB = CA$

That is, when we create the four-sided figure $CAIJ$ taking the segment $CA$ as the radius; we must draw the semicircle with center at $A$ that intersects the segment $KA$ at point $I$. By definition the angle $\angle KAH = 90$ therefore the angle $\angle CAI = 90$.

By creating the semicircle of radius $AC$ centered at point $C$, we can draw another semicircle centered at point $I$ that intersects the semicircle described above at point $J$.

Assuming that the intersections are made with an "ideal" compass; points $I$ and $J$ can be drawn with an ideal rule obtaining the tangents to the semicircles described above. Because $\angle CAI$ is a right angle and point I is tangent to the semicircle centered at A; the $\angle AIJ$ angle is also 90°.

Then, since segment $AB$ is parallel to segment $CO$; let us remember that we have previously defined the orthogonal projection of $CA$ segment on $OA$ segment; obtaining point $D$; furthermore as $CA = AI$; segment $AJ$ bisects angle $\angle CAI$.

Since $JI$ is perpendicular to segment $AI$ and the semicircle with center at $I$ has the same radius value as segment $CA$; the segment $JI = CA$.

Then as $CA = JI$ and also $\angle CAI = 90$ as well as $\angle AIJ = 90$ therefore $CJ = CA$; and also $CJ = AI$. It is also true that $CJ$ is parallel to $AI$.

That is, the four-sided figure with vertices at $C, A, I$ & $J$ forms an square.

Also for the algebraic reasons that describe the lengths of the triangle $\triangle OAB$; it can be asserted that the square $CAJI$ and the circle centered in $O$ of radius $OA$ are equal.

Finally, it must be concluded that the calculation results obtained; just as the algebraic relations and geometric ratios presented imply that it is possible to demonstrate overall that the ratio of the areas that describes their equality $A_{Circle} = A_{Square}$ is true.

On the other hand, it should be mentioned that this ratio is possible when it is considered that the radius of the circle and the side of the square of equal areas are part of a right-angled triangle that has a constant angle value $\alpha = arctan(\frac{1}{\sqrt{\pi}})$

The mathematical reasons described above are valid for any radius value from which you want to start; assuming a circle that you want to square; since it is possible to obtain the length of the side of the square associated with this figure with the equation for the value of $\delta = r(\sqrt{\pi} - \sqrt{3})$.

On the other hand, if you want to obtain the circle associated with a given square, you only have to establish the following algebraic ratio considering the length of the side L of the square from which you start.

$L = r\sqrt{3} + \delta$

$L - \delta = r\sqrt{3}$

$\frac{L-\delta}{\sqrt{3}} = r$

$\frac{L-(\sqrt{\pi}-\sqrt{3})r}{\sqrt{3}} = r$

$\frac{L}{\sqrt{3}} - \frac{(\sqrt{\pi}-\sqrt{3})r}{\sqrt{3}} = r$

$\frac{L}{\sqrt{3}} = r + \frac{(\sqrt{\pi}-\sqrt{3})r}{\sqrt{3}}$

$\frac{L}{\sqrt{3}} = (1 + \frac{(\sqrt{\pi}-\sqrt{3})}{\sqrt{3}})r$

$\frac{L}{\sqrt{3}} \div (1 + \frac{(\sqrt{\pi}-\sqrt{3})}{\sqrt{3}}) = r$

And thus obtain the value of the radius of the circle that has the same area value as the square from which it starts.

**An Exact Squaring the Circle Scheme made with Excel.**

Early assumptions establish that mathematical relation between areas of square and circle when they are equal; $A_{Circle} = A_{Square}$ is true.

A compass and straightedge proof made at hand must require to surpass next problems:

1) Establish some method to add $\delta = r(\sqrt{\pi}-\sqrt{3})$ to the cathetus adjacent of the triangle considered. Whatever this could made; will establish the hypotenuse between center of circle considered and the side that will give place to the related square.

2) Early condition establish next restriction for the angle formed with hypotenuse and radius of triangle

(from scheme 1)

$\angle BOA = \angle GOA$

$\angle BOA = 180 - 90 - arctan(\frac{1}{\sqrt{\pi}}) = 60.56876146$

Considering that we start since bisection of first quadrant then the angle obtained to form a chord that could be of use to draw an "exact scheme at hand" will be:

$\angle Chord = 45 + 60.56876146 = 105.56876146$

Compared to the value of angle for the "chord" needed to draw an exact squaring this imply an increment of

$\epsilon = 104.673(7)$

$\angle Chord - \epsilon = 105.56876146 - 104.673(7) = 0.895(0)$

Only to approximate early value obtained for $\angle Chord$ one degree less; i.e. to define with compass the value:

$\angle\psi = 105.568(7) - 1 = 104.568(7)$

Considering that triangle that relates square and circle establish a restriction on angles; then the only possibility to decrease in one degree the value of the chord must be start since the bisection of first quadrant angle.

To decrease by 1° the value of 45° could be necessary to establish next constructions with compass to reach such goal:

Construction to divide an angle into 5 parts; $\frac{45}{5} = 9$

Construction to divide an angle into 3 parts; $\frac{9}{3} = 3$

Construction to divide an angle into 3 parts; $\frac{3}{3} = 1$

Or any combination of constructions available to define a reduction of 1° to such angle obtained in bisection of 45°

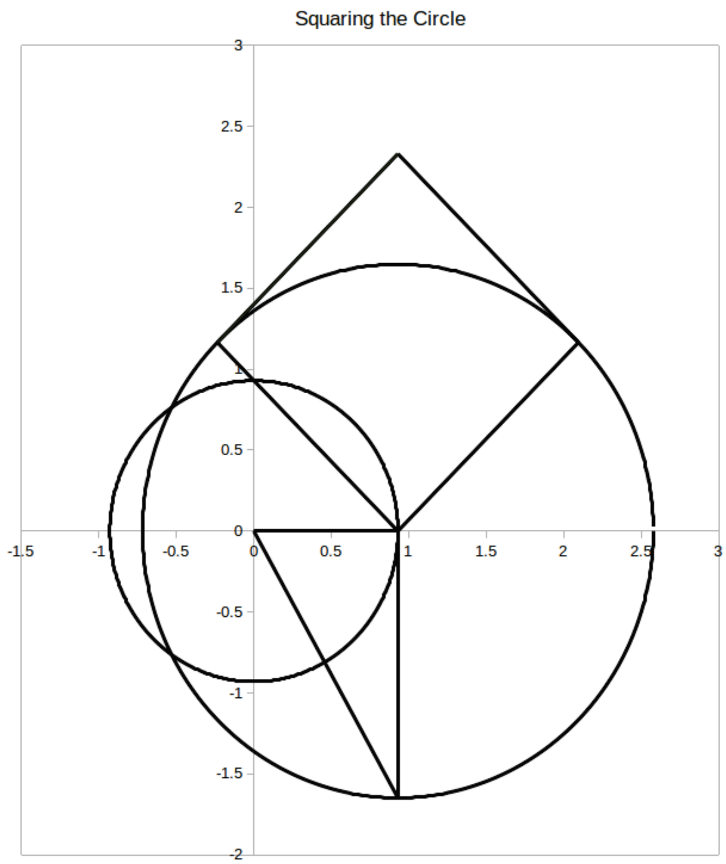
Even in that case the hand construction will not be exact.

On the other hand; the information obtained since scheme 1; using a compass and straightedge made know constructions; defines a "complex" graphic from which is possible to obtained the necessary information to establish the mathematics to understand "Squaring the Circle". It is possible to define intersections to assign "arbitrary values to each segment and angle" as well as "constitutes an scheme where the real angles and distances are convergent to the values of the exact Squaring".

The goal to obtain an exact scheme of construction could be reached using the mathematics exposed early; on a suitable software that could enable to establish the level of accuracy that shows the results of calculus.

Using irrational numbers to define the scheme proposed is convenient to establish a computational method to make things easier; unless a different use of compass could be discovered in the future.

In EXCEL is possible to define the equation of a circle and evaluate it numerically as well as to establish a value of $\delta$ to establish the necessary geometrical objects to state with vector projection a construction of squaring the circle that is faithful to the mathematics established to such purpose.




Graph 1. Squaring the Circle draw with EXCEL

Was used the value of radius r = 0.93 to obtain the small circle.

Equation of circle was used taking into account the next mathematical identities:

$x = rcos(\theta)$ $y = rsin(\theta)$

For values of $\theta$ in radians and taking them since the range of values for angle [0°, 360°]; using an increase of 1° per point.

Calculus of

$x^2 + y^2 = r^2$

was made to corroborate the values obtained early. Only for r = 1 this equality holds; while for any other value is not constant; however can not be cleared the reason of this error; maybe was caused by some error related with the use of radians instead of degrees. Like was established before this function on EXCEL can not be changed.

Big Circle was obtained using the equation:

$(x+1)^2+(y-0)^2=R^2$

Being

$R=r\sqrt{3}+\delta$

with

$\delta=r(\sqrt{\pi}-\sqrt{3})$

For same value of radius; as before; was used the equation:

$X=Rcos(\theta)+rcos(0)$

$Y=Rsin(\theta)+rsen(0)$

For $\theta$ between [0°,360°] with and increase of 1° per point and using the function Rad to convert degrees on radians. (Only for this part value of $rsin(0)$ was exclude to be zero).

Triangle hypotenuse; cathetus adjacent and opposite was obtained directly from graphic; i.e. using the values of:

Center like point $(0,0)$

Point $(0.93,0)$

And the point on big circle

$(X,Y)=(Rcos(270)+rcos(0),Rsin(270)+rsin(0))$

Square was made in a similar way considering besides next points:

$(X,Y)=(Rcos(135)+rcos(0),Rsin(135)+rsin(0))$

$(X,Y)=(Rcos(45)+rcos(0),Rsin(45)+rsin(0))$

$(X,Y)=(Rcos(45)+Rcos(135),Rcos(45)+Rsin(135))$

and again Point $(0.93,0)$

Requesting to add "Trend lines" between such points to obtain all lines for triangle and square.

Also was requested next data to corroborate the veracity of length, value for $\alpha$ angle, as well as areas of square and circle required; using next equations:

$A_{Circle}=\pi r^2$

$A_{Square}=sidesquare^2$

Length of such segments was measured like euclidean distance:

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$\alpha=arctan(\frac{radius}{sidesquare})$

Radius = 0.93 delta =0.037574830303074 Distance Side L = 1.64838208134213 Alpha Degrees = 29.4312385432435 Area Circle = 2.71716348608981 Area Square = 2.71716348608981

On this way could be possible to use EXCEL like a COMPASS; after establishing the already mentioned vector projection needed.

**Appendix 1. Steps to follow to draw the proposed construction.**

You can use any radius value

1) Draw a circle with center O and divide it into 6 parts with the help of the compass, placing it on the perimeter and opening equal to the radius of the circle.

2)Draw the diameter using the intersections of the previous point (extended segment $AO$) and bisect said segment (segment $KH$)

3) Select the first quadrant and bisect (segment $OF$)

4) Use the compass to get the opening $FG$

5) Draw a semicircle clockwise until it cuts the circle centered at $O$ to the left of the $Y$ axis (point $E$ and $L$)

6) Draw segment $EL$ (also mark point $D$)

7) Draw the segment $AH$ until it intersects the segment of the previous point (point $C$)

8) Use the compass to obtain the $AC$ opening and draw two semicircles; one with center in $A$ and the other with center $C$

9) Draw the segment $KA$ until it intersects the semicircle with center at $A$ (point $I$)

10) With the same opening and center at $I$, intersect the semicircle with center at $C$ (point $J$)

11) Draw segment $IJ$

12) Draw segment $JC$

13) Draw segment $AE$

**Appendix 2. Relationship of the triangle with $tan(\alpha) = \frac{1}{\sqrt{\pi}}$; vector product and Gauss integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.**

The vector product is described by the following equation:

$A \times B = ||A||||B||sin(\theta)$

Where $A$ and $B$ are a pair of vectors and $\theta$ the angle between them.

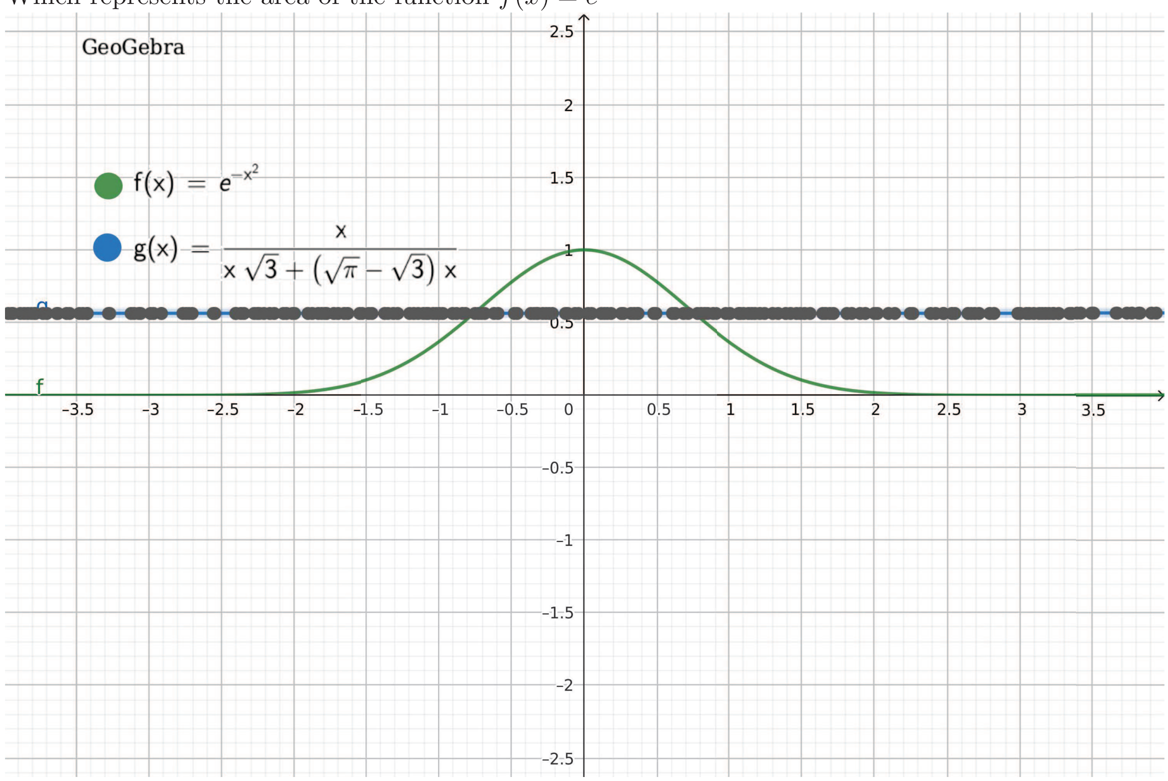
Let us consider the triangle defined for the proposed construction studied previously; and let's define the vectors $||A|| = r\sqrt{3}+\delta$ ; $||B|| = \sqrt{(r\sqrt{3}+\delta)^2 + r^2}$; with $\theta = arctan(\frac{1}{\sqrt{\pi}})$.

Let us also remember that the formula for the vector product describes the area of the parallelogram formed by $A$ and $B$; then from the vector product formula we will have:

$||A||||B||sin(\theta) =$ Parallelogram Area

Because the triangle has a right angle between the height and the base; the parallelogram formed follows the next relationship:

Parallelogram Area $=2$ Triangle Area
Substituting
$||A||||B||sin(\theta)=2$ Triangle Area
So
$\theta=arcsin(\frac{2TriangleArea}{||A||||B||})$
Like $\alpha=arctan(\frac{r}{r\sqrt{3}+\delta})$
and $\alpha=\theta$ then
$arcsin(\frac{2TriangleArea}{||A||||B||})=arctan(\frac{r}{r\sqrt{3}+\delta})$
Replacing the modules $||A||$ and $||B||$.
$arcsin(\frac{2TriangleArea}{(r\sqrt{3}+\delta)(\sqrt{(r\sqrt{3}+\delta)^2+r^2})})=arctan(\frac{r}{r\sqrt{3}+\delta})$
Solving the term that describes the area of the triangle we have:
2 Triangle Area $=(r\sqrt{3}+\delta)(\sqrt{(r\sqrt{3}+\delta)^2+r^2})sin(arctan(\frac{r}{r\sqrt{3}+\delta}))$
Evaluating for r = 1 and taking the $\delta$ defined above we will have:
2 Triangle Area $=(\sqrt{\pi})(\sqrt{\pi+1})sin(arctan(\frac{1}{\sqrt{\pi}}))$
2 Triangle Area $=1.772453851$
While $\sqrt{\pi}=1.772453851$
A function of known area equal to $\sqrt{\pi}$ is the Gaussian integral $\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi}$
Which represents the area of the function $f(x)=e^{-x^2}$



Graph 2. Intersection of $g(r) = \frac{r}{r\sqrt{3}+\delta} = \frac{r}{r\sqrt{3}+(\sqrt{\pi}-\sqrt{3})r}$ and $f(x) = e^{-x^2}$.

We see on Graph 2 that equation to define tangent is constant for any value of r and intersect with $f(x) = e^{-x^2}$ at around 0.5 (up from early value).

It means that tangent to define triangle that relates area of square and circle of same area is constant for any value of radius; and for r = 1 its value is $\frac{r}{r\sqrt{3}+\delta} = \frac{1}{\sqrt{\pi}}$.

Using the previous result, the following equality is proposed:

2 Triangle Area = Area of the Gaussian Function

$arcsin(\frac{AreaoftheGaussianFunction}{(r\sqrt{3}+\delta)(\sqrt{(r\sqrt{3}+\delta)^2+r^2})}) = arctan(\frac{r}{r\sqrt{3}+\delta})$

Substituting

$arcsin(\frac{\sqrt{\pi}}{(r\sqrt{3}+\delta)(\sqrt{(r\sqrt{3}+\delta)^2+r^2})}) = arctan(\frac{r}{r\sqrt{3}+\delta})$

Simplifying

$arcsin(\frac{\sqrt{\pi}}{(r\sqrt{\pi})(\sqrt{\pi r^2+r^2})}) = arctan(\frac{r}{r\sqrt{3}+\delta})$

$arcsin(\frac{1}{(r\sqrt{\pi r^2+r^2})}) = arctan(\frac{1}{\sqrt{\pi}})$

$arcsin(\frac{1}{(r^2\sqrt{\pi+1})}) = arctan(\frac{1}{\sqrt{\pi}})$

that represents the equality described above $\theta = \alpha$ and which is valid only for r=1:

$arcsin(\frac{1}{(\sqrt{\pi+1})}) = arctan(\frac{1}{\sqrt{\pi}})$

$\theta = 29.43123854 = 29.43123854 = \alpha$

Generalization of the previous equation and that would represent how the area of the triangle, that is related to the square and circle of equal areas varies; regarding the area of the Gaussian function; is the following:

$arcsin(\frac{Q(GaussianFunctionArea)}{(r\sqrt{3}+\delta)(\sqrt{(r\sqrt{3}+\delta)^2+r^2})}) = arctan(\frac{r}{r\sqrt{3}+\delta})$

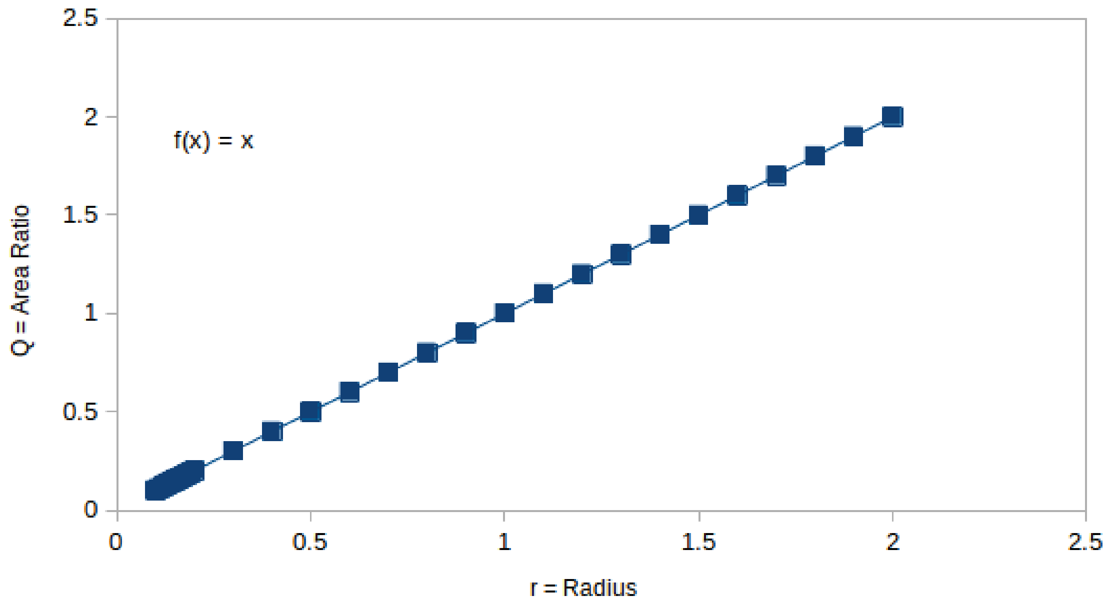
That is, the area of the triangle varying Q times with respect to the area of the Gaussian function is:

$arcsin(\frac{Q}{(r^2\sqrt{\pi+1})}) = arctan(\frac{1}{\sqrt{\pi}})$

Solving for Q we have:

$Q = (r^2\sqrt{\pi+1})sin(arctan(\frac{1}{\sqrt{\pi}}))$

The following graph shows the result to draw the $Q$ equation for an interval of r =[0 - 0.2] in a step of 0.01 added per trial; the rest of the points are obtained for the same equation with r = [0.2-2] with added step of 0.1 per trial; using EXCEL and taking into account that the sine function used for such task only take values in terms of radians as input data; since it is not possible to modify this feature (that is, enter values in terms of degrees and obtain the common result that can be obtained with any hand-held scientific calculator) into the software used.




Graph 3. Ratio of the radius that defines the triangle of $tan(\alpha) = \frac{1}{\sqrt{\pi}}$ vs proportion of the area of the Gauss integral ($\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$) with respect to the area of 2 area of the Triangle ("twice the area of the triangle of $tan(\alpha) = \frac{1}{\sqrt{\pi}}$).

It can be seen that the graph obtained describes a linear relationship of slope m = 1.

Showing a linear relation between the value of Area of Triangle of main condition and the Area of Gauss Integral.

**Appendix 3.**

Considering that Scheme 1 shown early to study the construction of Squaring the Circle; gives the possibility to define the intersections to establish the equality $A_{Circle} = A_{Square}$; then starting from this last fact; it is possible to state the following:

**Theorem**

Let be $A_Q :=$ Area of Square and $A_C :=$ Area of Circle

$A_Q = A_C$ if and only if $\exists$ a triangle rectangle such that $tan(\alpha) = \frac{r}{l} = \frac{1}{\sqrt{\pi}}$

where $\alpha$ is the angle adjacent to $l$ and opposite to $r$

$l :=$ side of square and $r :=$ radii of circle

**Proof**

1) If $\exists$ a triangle rectangle such that $tan(\alpha) = \frac{1}{\sqrt{\pi}}$ then $A_{Square} = A_{Circle}$

Lets take a triangle rectangle of hick opposite $r$ and hick adjacent $l$

Then

$tan(\alpha) = \frac{r}{l} = \frac{1}{\sqrt{\pi}}$

Hence reordering

$r\sqrt{\pi} = l$

squaring from both sides

$\pi r^2 = l^2$

Hence

$A_{Circle} = A_{Square}$

While from diagram we have that

if $r = OA$ and $l = AB$ then

$tan(\alpha) = \frac{OA}{AB} = \frac{1}{\sqrt{\pi}}$

Hence

$OA\sqrt{\pi} = AB$

Squaring from both sides

$\pi OA^2 = AB^2$

Remember that $AB = AC$ because they are on same circle (with center in $A$)

Hence
$\pi OA^2 = AC^2$

Like $C$ and $I$ are at same distance respect to $A$; and $A$ and $J$ are at same distance respect to $C$.
Hence

$A_{Circle}$ ; the area of the circle with center in $O$ is equal to $A_{Square}$ the square of sides $CA = AI = IJ = JC$

$\square$

2) $A_{Square} = A_{Circle}$ then $\exists$ a triangle rectangle such that $tan(\alpha) = \frac{1}{\sqrt{\pi}}$

Lets suppose $A_{Square}$ is the area of the square $ACJI$ while $A_{Circle}$ is the area of the circle with center in $O$ and radii $OA$

Now assume $A_{Square} = A_{Circle}$

Remember that $A_{Square} := l^2$ and $A_{Circle} := \pi\ r^2$

Using diagram we have for early equations of area:

Like $C$, $A$, $I$ and $J$ are on a regular square then:
$l = CA = AI = IJ = JC$

and

$r = OA$

Substituting $r$ and $l$ on its equations of area we have

$l^2 = \pi r^2$ ...(1)

$l^2 = CA^2 = AI^2 = IJ^2 = JC^2 = \pi r^2 = \pi OA^2$

Like $CA$ is on same circle that $AB$ then

$CA = AB$

Hence reordering (1) we have

$\frac{r^2}{l^2} = \frac{OA^2}{AB^2} = \frac{1}{\pi}$

Obtaining square root we have

$\frac{r}{l} = \frac{1}{\sqrt{\pi}} = \frac{OA}{AB}$

Like $\angle\ HAK$ is right angle and $HA = AK$
then $OA$ bisect $HK$ i.e. $\angle OAC$ is bisection of $\angle HAK$
Also $\angle OAC = \angle CAO$
Like $JA$ is the diagonal of square hence
$\angle JAC = \angle CAO$

Then

$\angle JAC + \angle CAO = \angle OAJ$

Hence $\angle OAJ$ is right angle

To be $B$ on same segment than $JA$ then

$\angle OAB$ is right (to be complementary to $\angle OAJ$)

Hence exist the right angle rectangle $\triangle OAB$

Hence

$\frac{OA}{AB} = \frac{1}{\sqrt{\pi}} = tan(\alpha)$

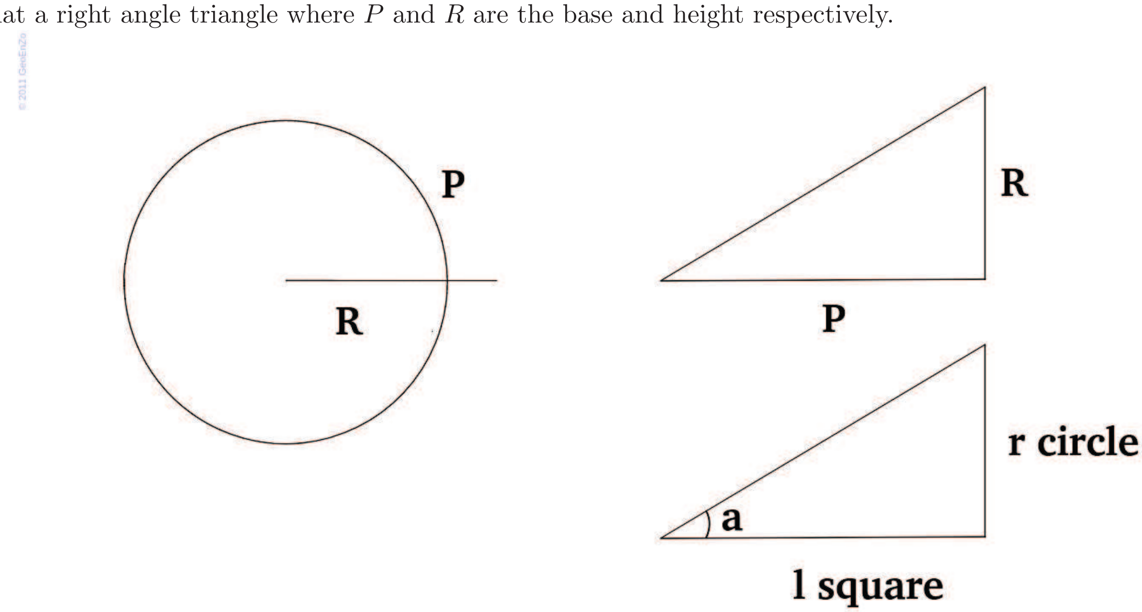
where $\alpha = \angle OBA$

$\square$

**Lemma Trivial.** Side $L$ and radius $R$ when $\pi R^2 = L^2$ (i.e Circle´s Area = Area´s Circle) not follow Archimedes's Pi approximation ; $P = 2\pi R$ with $P$ the length of the perimeter for a circle.

**Proof.**

Remember that for Archimedes´s Pi approximation circle has the same area that a right angle triangle where $P$ and $R$ are the base and height respectively.




Scheme 2. Circle and triangles considered in Archimedes Pi Approximation (made with GeoEnzo 2023)

Main condition to fulfill "Squaring the Circle" establish the existence of a right angle triangle of hick opposite to angle a equal to the radius ($r_{square}$) of the circle to square and hick adjacent equal to the side of such square ($l_{square}$).

Then

If $R = r_{circle}$ and $l_{square} = P$

taking $l_{square} = \sqrt{\pi} \cdot r_{circle}$

By Archimedes Pi condition we have

$l_{square} = P = 2\pi R$

then

$\sqrt{\pi} \cdot r_{circle} = 2\pi R = 2\pi r_{circle}$

Hence $\sqrt{\pi} = 2\pi$ (! contradiction)

□

**Lemma Golden.**

Squaring the Circle construction with compass and straightedge determines Golden ratio algebraic equation through length of side of the square ($l_{square}$).

**Construction with Compass and Straightedge.**

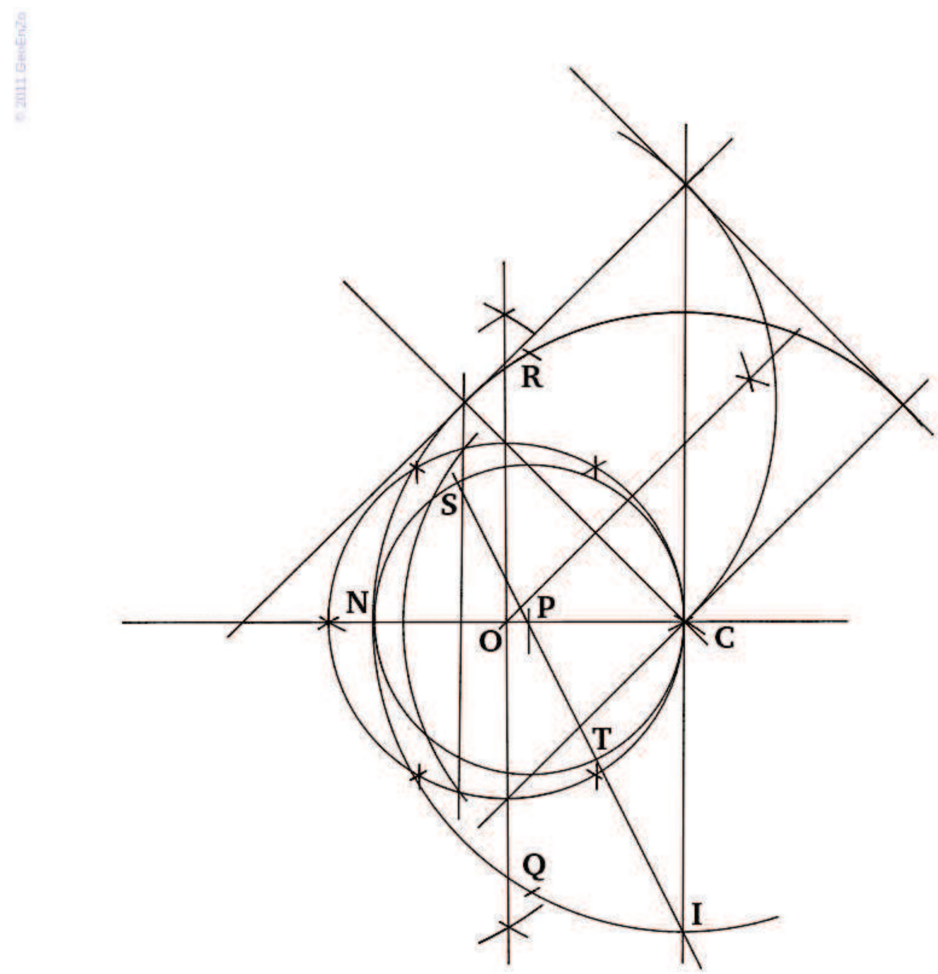
1) Take full diagram made for "Squaring the Circle"

2) Take distance $RC$ with compass and intersect diameter segment of circle with center in $O$ (point $N$)

3) Take distance $NC$ with compass and bisect (point $R$, $Q$ and $P$)

4) Draw a circle of radius $PC$ (or $NP$) with center in $P$

5) Draw segment $PI$ until intersect with the circle of center in $P$ (point $S$ and $T$)




Scheme 3. Construction of Golden Ratio for Squaring the Circle Construction with Compass and Rule (made with GeoEnzo 2023)

**Proof.**

Like points $I$ and $N$ are on the same circle (center in $C$) then
$NC = CI$ ... (1)
Quotient by NC of CI maintains next relation
$\frac{NC}{NC} = \frac{CI}{NC} = 1$
Like S and T are on the same segment and pass through point P then

$SP = TP$
Also
$SP + TP = 2SP = 2TP = ST$
Like $ST$ and $NC$ are diameters of same circle then
$ST = NC$
By (1)
$ST = CI$
Consider
$ST + TI > NC$
i.e.
$SI > NC$
Now consider
$\frac{SI+NC}{SI} > \frac{SI}{NC}$
Simplifying
$\frac{SI}{SI} + \frac{NC}{SI} > \frac{SI}{NC}$
$1 + \frac{NC}{SI} > \frac{SI}{NC}$ ...(2)
Name $\frac{SI}{NC} = \varphi$ and substitute on last equation (2)
$1 + \varphi^{-1} > \varphi$
Multiplying by $\varphi$
$\varphi + 1 > \varphi^2$
Reordering
$\varphi + 1 - \varphi^2 > 0$
Multiplying by -1
$\varphi^2 - \varphi - 1 < 0$ ...(3)

Solution to left side of algebraic inequity (3) is called Golden ratio and can be established using quadratic formula; i.e.

If
$\varphi^2 - \varphi - 1 = 0$
Then
$\varphi = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
Hence
$\varphi = \frac{-(-1) \pm \sqrt{(-1)^2-4(1)(-1)}}{2(1)}$
Simplifying
$\varphi = \frac{1 \pm \sqrt{1+4}}{2}$
$= \frac{1 \pm \sqrt{5}}{2}$
Taking positive solution we obtain the Golden ratio
$\varphi = \frac{1+\sqrt{5}}{2}$
Hence using solution to algebraic equation of Golden ratio we have
$0 < 0$ (! contradiction)
$\square$

**Appendix 4.**

Squaring the Circle construction made with compass and straightedge presented on this work was used also on "*Busto di uomo di profito con studio di proporzione*" made by Leonardo Da Vinci.

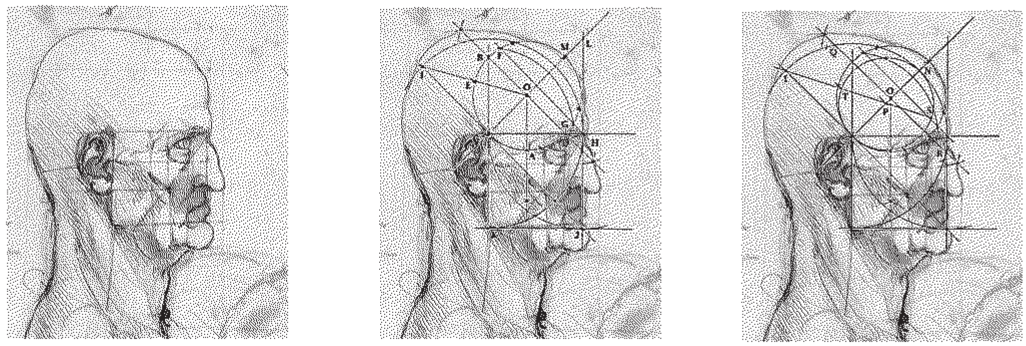


Image 1. Comparison between proportion study made by Da Vinci (left); proportion study with "Squaring the Circle" (middle) and proportion study with "Golden Ratio for Squaring the Circle (right). [ref. https://www.gallerieaccademia.it/en/bust-man-profile-study-proportions]

In the case of the "Construction of Golden Ratio with Squaring the Circle"; the diagram obtained was found similar to the geometry of Phase E (second or third contact) of an eclipse (Solar or Lunar) [ref. https://cdn.britannica.com/28/128-050-81620071/phases-total-eclipse-disk-west-Moon-east.jpg]

In conclusion it seems that the geometrical construction given here could have potential application on "Geometrical Analysis" for Astronomical, Physiology and Art studies".

# References

[1] On the Mechanization of Straightedge and Compass Constructions; Schreck; P.; J.Syst. Sci. Complex (2019); 32; 124-149

[2] Origin of Irrational Numbers and Their Approximations; Agarwal, R.P; Agarwal, H.; Computation 2021; 9(3), 29; https://doi.org/10.3390/computation9030029

[3] Recherches sur les moyens de reconnaítre si un Problém de Géometrie peut se résoudre avec la régle et le compas; Wantzel, M. L.; Journal de Mathématiques Pures et Appliquées; 1(2); 366-372.