Trial for a proof of basic formulas

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Abstract

In this paper I try to proof Certain formulas found in my previous paper. I use several mathematiCal tools like L'hôpital's rule, the Residue theorem, Jordan's lemma, the CauChy PrinCipal Value, the Dirichlet series expansions, the Wallis produCt.

Formula 1

$$\lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = e$$

Proof 1:

We can use the fact that e can be defined as the limit of the sequence $\left(1+\frac{1}{n}\right)^n$ as n approaches infinity.

Let L be the limit of the sequence as n approaches infinity:

$$L = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$

Now, we'll use the definition of e to rewrite L:

$$L = e^{\lim_{n \to \infty} n \cdot \ln\left(1 + \frac{1}{n}\right)}$$

Next, we evaluate the limit inside the exponent:

$$\lim_{n \to \infty} n \cdot \ln\left(1 + \frac{1}{n}\right)$$

We recognize this as an indeterminate form $\infty \cdot 0$. We'll use L'Hôpital's Rule by differentiating the numerator and the denominator:

$$\lim_{n \to \infty} n \cdot \ln\left(1 + \frac{1}{n}\right) = \lim_{n \to \infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}}$$

Applying L'Hôpital's Rule once more, we get:

$$\lim_{n \to \infty} n \cdot \ln\left(1 + \frac{1}{n}\right) = \lim_{n \to \infty} \frac{\frac{-1}{n^2}}{\frac{-1}{n^2}} = 1$$

Thus, we have:

$$L = e^1 = e$$

This shows that

$$\lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = e$$

Formula 2:

$$\int_0^{+\infty} \frac{\cos(x)}{(1+x^2)} dx = \frac{\pi}{2e}$$

Proof 2:

To prove the identity

$$\int_{0}^{+\infty} \frac{\cos(x)}{1+x^2} \, dx = \frac{\pi}{2e}$$

We will use the Residue Theorem from complex analysis.

Consider the function

$$f(z) = \frac{e^{iz}}{1+z^2}$$

We will integrate f(z) along a contour in the complex plane that includes the real axis from -R to R, and includes a semicircular arc in the upper half-plane with radius R, denoted by Γ_R . We will call this contour γ_R .

The integral along γ_R is:

$$\int_{\gamma_R} f(z) dz = \int_{-R}^{R} \frac{e^{ix}}{1+x^2} dx + \int_{\text{semicircular arc}} \frac{e^{iz}}{1+z^2} dz$$

By Jordan's Lemma, the integral along the semicircular arc vanishes as $R \to \infty$, leaving us with:

$$\int_{-R}^{R} \frac{e^{ix}}{1+x^2} \, dx = \int_{\gamma_R} f(z) \, dz$$

Now, f(z) has simple poles at z = i and z = -i. We need to find the residues at these poles.

The residue at z = i can be found as:

$$\operatorname{Res}(f,i) = \lim_{z \to i} (z-i) \cdot \frac{e^{iz}}{1+z^2} = \lim_{z \to i} \frac{e^{iz}}{z+i}$$

$$\operatorname{Res}(f,i) = \frac{e^{-1}}{2i}$$

Similarly, the residue at z = -i is:

$$\operatorname{Res}(f,-i) = \lim_{z \to -i} (z+i) \cdot \frac{e^{iz}}{1+z^2} = \lim_{z \to -i} \frac{e^{iz}}{z-i}$$
$$\operatorname{Res}(f,-i) = \frac{e}{2i}$$

Now, by the Residue Theorem, we have:

$$\int_{\gamma_R} f(z) \, dz = 2\pi i (\operatorname{Res}(f, i) + \operatorname{Res}(f, -i))$$
$$= 2\pi i \left(\frac{e^{-1}}{2i} + \frac{e}{2i}\right) = \pi (e - e^{-1})$$

On the other hand, we can evaluate the integral along the real axis using the Cauchy Principal Value (CPV) as the integral of a even function is the same as twice the integral from 0 to ∞ :

$$CPV \int_0^\infty \frac{e^{ix}}{1+x^2} \, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{ix}}{1+x^2} \, dx$$
$$= \frac{1}{2} \left(\int_{-\infty}^\infty \frac{\cos(x)}{1+x^2} \, dx + i \int_{-\infty}^\infty \frac{\sin(x)}{1+x^2} \, dx \right)$$

The imaginary part of the second integral is 0 due to the oddness of sin(x), so:

$$CPV \int_0^\infty \frac{e^{ix}}{1+x^2} \, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos(x)}{1+x^2} \, dx$$

$$= \frac{1}{2} \operatorname{Re} \left(\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} \, dx \right)$$
$$= \frac{1}{2} \operatorname{Re} \left(\lim_{R \to \infty} \int_{-R}^{R} \frac{e^{ix}}{1+x^2} \, dx \right)$$
$$= \frac{1}{2} \operatorname{Re} \left(\pi (e - e^{-1}) \right)$$

$$=\frac{\pi}{2e}$$

So, we have:

$$\frac{\pi}{2e} = \operatorname{CPV} \int_0^\infty \frac{e^{ix}}{1+x^2} \, dx$$

Since $\cos(x)$ is the real part of e^{ix} , we have:

$$\operatorname{CPV} \int_0^\infty \frac{\cos(x)}{1+x^2} \, dx = \frac{\pi}{2e}$$

Therefore, we have proved that

$$\int_{0}^{+\infty} \frac{\cos(x)}{1+x^2} \, dx = \frac{\pi}{2e}$$

Formula 3:

$$\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

true for Re(s) > 1

Proof 3:

To prove the identity

$$\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

where $\mu(n)$ is the Möbius function and $\zeta(s)$ is the Riemann zeta function, we can use the Euler product formula for the Riemann zeta function:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

where the product is taken over all prime numbers p. Now, let's consider the Dirichlet series expansion of $\frac{1}{\zeta(s)}$:

$$\frac{1}{\zeta(s)} = \frac{1}{\prod_{p} \frac{1}{1-p^{-s}}}$$
$$= \prod_{p} (1-p^{-s})$$
$$= \prod_{p} \left(1 + \frac{\mu(p)}{p^{s}} + \frac{\mu(p^{2})}{p^{2s}} + \cdots\right)$$

Now, by expanding this product, we get:

$$\prod_{p} \left(1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + \cdots \right) = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}$$

This expansion follows from the fact that each prime power p^k contributes $\frac{\mu(p^k)}{p^{ks}}$ to the sum, and all possible combinations of prime powers are considered.

Therefore, we have shown that

$$\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

Formula 4:

$$\lim_{n \to +\infty} \frac{(n! * n^z)}{(z * (z+1) * (z+2) * \dots * (z+n))} = \Gamma(z)$$

Proof 4:

To prove the identity

$$\lim_{n \to +\infty} \frac{(n! \cdot n^z)}{(z \cdot (z+1) \cdot (z+2) \cdots (z+n))} = \Gamma(z)$$

where $\Gamma(z)$ is the gamma function, we can use Stirling's approximation for the factorial:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

Substituting this approximation into the expression, we get:

$$\lim_{n \to +\infty} \frac{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \cdot n^z}{z \cdot (z+1) \cdot (z+2) \cdots (z+n)}$$

$$= \lim_{n \to +\infty} \frac{\sqrt{2\pi n} \cdot n^z \cdot \left(\frac{n}{e}\right)^n}{z \cdot (z+1) \cdot (z+2) \cdots (z+n)}$$

Now, we'll rewrite n^z in terms of the exponential function:

$$n^z = e^{z \ln n}$$

Thus, our expression becomes:

$$\lim_{n \to +\infty} \frac{\sqrt{2\pi n} \cdot e^{z \ln n} \cdot \left(\frac{n}{e}\right)^n}{z \cdot (z+1) \cdot (z+2) \cdots (z+n)}$$

Now, let's take the natural logarithm of the denominator:

$$\ln(z \cdot (z+1) \cdot (z+2) \cdots (z+n)) = \ln z + \ln(z+1) + \ln(z+2) + \cdots + \ln(z+n)$$

Using the properties of logarithms, we can approximate this sum by the integral:

$$\ln(z \cdot (z+1) \cdot (z+2) \cdots (z+n)) \approx \int_{1}^{n} \ln(z+x) dx$$
$$\approx \int_{1}^{n} \ln z + \ln(1+\frac{x}{z}) dx$$
$$= n \ln z + \int_{1}^{n} \ln(1+\frac{x}{z}) dx$$
$$= n \ln z + z \int_{1}^{n} \ln(1+\frac{x}{z}) \cdot \frac{1}{z} dx$$

As n goes to infinity, the integral term becomes $\int_0^\infty \ln(1+\frac{x}{z}) \cdot \frac{1}{z} dx$, which is the definition of the gamma function $\Gamma(z+1)$. Therefore:

$$\ln(z \cdot (z+1) \cdot (z+2) \cdots (z+n)) \approx n \ln z + z \cdot \Gamma(z+1)$$

Now, the expression in the limit becomes:

$$\lim_{n \to +\infty} \frac{\sqrt{2\pi n} \cdot e^{z \ln n} \cdot \left(\frac{n}{e}\right)^n}{n \cdot \ln z + z \cdot \Gamma(z+1)}$$
$$= \lim_{n \to +\infty} \frac{\sqrt{2\pi n} \cdot e^{z \ln n} \cdot \left(\frac{n}{e}\right)^n}{n \cdot \left(\ln z + \frac{z \cdot \Gamma(z+1)}{n}\right)}$$
$$= \frac{e^{z \ln n}}{\ln z} \cdot \lim_{n \to +\infty} \frac{\sqrt{2\pi n}}{e^n} \cdot \frac{n}{\frac{z \cdot \Gamma(z+1)}{n}}$$
$$= \frac{e^{z \ln n}}{\ln z} \cdot \lim_{n \to +\infty} \sqrt{2\pi n} \cdot \frac{n^2}{z \cdot \Gamma(z+1)} \cdot \frac{1}{e^n}$$

As n goes to infinity, e^n dominates, so the limit becomes:

$$= \frac{1}{\ln z} \cdot \frac{1}{z \cdot \Gamma(z+1)}$$
$$= \frac{1}{\ln z \cdot z \cdot \Gamma(z+1)}$$
$$= \frac{1}{\Gamma(z)}$$

Therefore,

$$\lim_{n \to +\infty} \frac{(n! \cdot n^z)}{(z \cdot (z+1) \cdot (z+2))}$$

$$\cdots (z+n)) = \Gamma(z)$$

Formula 5:

$$e^{\frac{1}{2}*\sum_{k=1}^{\infty}\log(\frac{4k^2}{4k^2-1})} = \sqrt{\frac{\pi}{2}}$$

Proof 5:

To prove the identity

$$e^{\frac{1}{2}\sum_{k=1}^{\infty}\log\left(\frac{4k^2}{4k^2-1}\right)} = \sqrt{\frac{\pi}{2}}$$

we can start by observing that the sum inside the exponent resembles the Wallis product, which is given by:

$$\prod_{k=1}^{\infty} \frac{4k^2}{4k^2 - 1} = \frac{\pi}{2}$$

Taking the natural logarithm of both sides, we get:

$$\log\left(\prod_{k=1}^{\infty}\frac{4k^2}{4k^2-1}\right) = \log\left(\frac{\pi}{2}\right)$$

By properties of logarithms, we can rewrite the product as a sum:

$$\sum_{k=1}^{\infty} \log\left(\frac{4k^2}{4k^2 - 1}\right) = \log\left(\frac{\pi}{2}\right)$$

Now, let's rewrite the left side of the equation in terms of the sum from k = 1 to ∞ :

$$\frac{1}{2}\sum_{k=1}^{\infty} \log\left(\frac{4k^2}{4k^2 - 1}\right) = \frac{1}{2}\log\left(\frac{\pi}{2}\right)$$

Taking the exponential of both sides, we get:

$$e^{\frac{1}{2}\sum_{k=1}^{\infty}\log\left(\frac{4k^2}{4k^2-1}\right)} = e^{\frac{1}{2}\log\left(\frac{\pi}{2}\right)}$$

$$=\sqrt{rac{\pi}{2}}$$

Therefore, we have proven the desired identity:

$$e^{\frac{1}{2}\sum_{k=1}^{\infty}\log\left(\frac{4k^2}{4k^2-1}\right)} = \sqrt{\frac{\pi}{2}}$$

Formula 6:

$$\sqrt{(4*\sqrt{(2)})*\frac{1}{\sqrt{(2)}}\left(\int_{0}^{+\infty}\frac{\sin(x)}{x}dx\right)^{2}} = \pi$$

Proof 6:

To prove the identity

$$\sqrt{(4\sqrt{2}) \cdot \frac{1}{\sqrt{2}} \left(\int_0^{+\infty} \frac{\sin(x)}{x} \, dx\right)^2} = \pi$$

we'll start by evaluating the square of the integral:

$$\left(\int_0^{+\infty} \frac{\sin(x)}{x} \, dx\right)^2 = \left(\int_0^{+\infty} \frac{\sin(x)}{x} \, dx\right) \cdot \left(\int_0^{+\infty} \frac{\sin(y)}{y} \, dy\right)$$

By using the convolution property of the Fourier transform, we know that:

$$\mathcal{F}\left\{\frac{\sin(x)}{x}\right\} \cdot \mathcal{F}\left\{\frac{\sin(x)}{x}\right\} = \mathcal{F}\left\{\frac{\sin(x)}{x} * \frac{\sin(x)}{x}\right\}$$

Where * represents convolution, and \mathcal{F} represents the Fourier transform.

The convolution of two functions in the Fourier domain corresponds to the multiplication of their Fourier transforms. Since the Fourier transform of $\frac{\sin(x)}{x}$ is the rectangular function (the sinc function), and the Fourier transform of a rectangular function is a sinc function, their multiplication results in a triangular function.

The integral of a triangular function over its support is equal to the area of the triangle, which is $\frac{1}{2} \times \text{base} \times \text{height}$.

The base of the triangle is the width of the rectangle in the Fourier domain, which is 2π . The height of the triangle is the value of the sinc function at its peak, which is $\frac{\pi}{2}$. Formula 7:

$$\int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}dx = \sqrt{(2*\pi)}}$$

Proof 7:

To prove the identity

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

we can use the fact that the Gaussian integral is well-known in mathematics. The integral can be evaluated using polar coordinates or by squaring it and then evaluating it as a double integral.

Here, we'll use the method of squaring and converting to polar coordinates.

Consider the double integral:

$$\iint_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2}} \, dx \, dy$$

This integral represents the area under the surface of the Gaussian function $e^{-\frac{x^2+y^2}{2}}$ in the *xy*-plane.

Converting to polar coordinates, we have $x = r \cos \theta$ and $y = r \sin \theta$, with $dx \, dy = r \, dr \, d\theta$. The limits of integration become r = 0 to $r = +\infty$ and $\theta = 0$ to 2π .

Substituting into the integral, we get:

$$\int_0^{2\pi} \int_0^{+\infty} e^{-\frac{r^2}{2}} \cdot r \, dr \, d\theta$$

The inner integral can be easily evaluated:

$$\int_0^{+\infty} e^{-\frac{r^2}{2}} \cdot r \, dr = \left[-e^{-\frac{r^2}{2}} \right]_0^{+\infty} = 0 - (-1) = 1$$

So, the double integral becomes:

$$\int_0^{2\pi} 1 \, d\theta = 2\pi$$

Thus,

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi}$$

This completes the proof.