Fine-Tuning the Generating Function Technique for Nonlinear Partial Differential Equations

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Abstract

This article emphasizes the fine-tuning step of the Generating Functions Technique [1, 2], a crucial component that enhances solution accuracy and computational efficiency for nonlinear partial differential equations (NPDEs). Unlike other methods, such as the Simplest Equation Method [10], the G'/G-Expansion Method [9], Adomian Decomposition [4], and the Homotopy Perturbation Method [8], the fine-tuning step within GFT systematically optimizes the solution series. This paper demonstrates the impact of fine-tuning through detailed applications to inhomogeneous NPDEs, elucidating its capability in generating superior analytical solutions.

1 Introduction

The fine-tuning approach in GFT is conceptually similar to earlier work by Yolcu and Demiralp [1], who introduced "fine-tuning points" in the construction of generating functions for solving linear recurrence relations. Their method employed integral-form generating functions and focused on selecting "admissible intervals" to ensure convergence and solvability. While Yolcu and Demiralp's work remained within the domain of linear discrete systems, the present work generalizes this fine-tuning philosophy to continuous and nonlinear PDEs by adjusting parameters such as α, β, and γ, as well as kernel weights in nested Green's function integrals, to systematically eliminate PDE residuals.

Typically, solving nonlinear partial differential equations (NPDEs) [**?**], particularly inhomogeneous types, presents numerous difficulties due to their inherent complexity and broad applicability. Traditional methods, including the Simplest Equation, G'/G-Expansion, Adomain Decomposition [4], and Homotopy Perturbation Methods, often rely on computational brute force, resulting in extensive calculations and reduced interpretability. To overcome these limitations, the fine-tuning step within the Generating Function Technique (GFT) [14] leverages analytical insights to optimize solution parameters, significantly enhancing precision and efficiency.

GFT offers a structured approach for constructing solutions by employing a generating function ansatz, which transforms the PDE into an algebraic or simpler differential form for analysis and simplification. We focus on two primary scenarios:

- Homogeneous NPDEs, where the GFT can be directly applied.
- Inhomogeneous NPDEs, where the Fine-Tuned GFT (ft-GFT) is introduced. This technique utilizes (double) Green's functions to handle nonhomogeneous terms systematically.

This paper aims to demonstrate the power of GFT and its fine-tuned variant, ft-GFT, through multiple examples derived from classical NPDEs.

2 Generating Function Technique (GFT) [14] for Homogeneous NPDEs

All work was performed with Mathematica © software. The study includes its notebooks of examples.

2.1 General Framework

The GFT is based on constructing a generating function $U(t,x)$ such that the solution $u(t,x)$ can be expressed as:

$$U(t,x) = \sum_{i,j} a_{i,j}\left(\sum_{k=0}^{\infty} 2\phi(\xi)^k S_k(0)\right)^j + b_{i,j}\left(\sum_{k=0}^{\infty} 2\phi(\xi)^k C_k(0)\right)^j, \quad \text{where } \xi = \alpha t + \beta x. \tag{1}$$

Where $f(t,x)$ is an exponential or trigonometric basis function, depending on the PDE, the coefficients $a_{i,j}$, $b_{i,j}$ and parameters (e.g., α, β) are determined by substituting $u(t,x)$ into the PDE and solving the resulting algebraic conditions.

2.2 Example: Bateman-Burgers or BB Equation

The BB equation is:

$$u_t - u_{xx} + \frac{1}{2}\left(u^2\right)_x = 0. \tag{2}$$

Using the GFT ansatz from `BB 1DP-FT.pdf`, we define:

$$f(t,x) = Ae^{-\xi}, \quad \xi = \alpha t + \beta x, \tag{3}$$

with $p_s = 1$. The generating function U is constructed as a combination of cosh and sinh terms:

$$U = \frac{2e^{2\xi} b_{1,1}}{A^2 + e^{2\xi}} \tag{4}$$

By substituting into the PDE and matching coefficients, the parameters $b_{1,1}$ and β are determined.

2.3 Example: Korteweg-de Vries (KdV), Benjamin-Bona-Mohany (BBM), and Boussinesq Equations

The KdV equation is:

$$u_t + u_{xxx} + \frac{1}{2}\left(u^2\right)_x = 0. \tag{5}$$

The BBM equation is:

$$u_t + u_x - u_{xxt} + \frac{1}{2}\left(u^2\right)_x = 0. \tag{6}$$

The Good Boussinesq equation is:

$$u_{tt} - u_{xx} - u_{xxxx} + \frac{1}{2}\left(u^2\right)_{xx} = 0. \tag{7}$$

Using the GFT ansatz from `BB 1DP-FT.pdf`, we define:

$$f(t,x) = Ae^{-\xi}, \quad \xi = \alpha t + \beta x, \tag{8}$$

with $p_s = 2$. The generating function U is constructed as a combination of cosh and sinh terms:

$$U = \frac{4A^2 a_{1,2}}{\left[(1+A^2)\cosh(\alpha t + \beta x) - (-1 + A^2)\sinh(\alpha t + \beta x)\right]^2}. \tag{9}$$

By substituting into the PDE and matching coefficients, the parameters $a_{1,2}$ and α are determined.

2.4 Example: Sine-Gordon or SG Equation

The SG equation is:

$$u_{tt} - u_{xx} + \sin u = 0. \tag{10}$$

From `SG 1DP-FT.pdf`, the application of GFT incorporates slight alterations to the parameters and the power definition of $\phi(\xi)$ in the generating functions associated with $U(\xi)$, leading to:

$$U = 2a_{1,1} arccot(\frac{e^{\xi}}{A}) + 2b_{1,1} arctan(\frac{e^{\xi}}{A}). \tag{11}$$

It is critical to note that the coefficient and parameters that were determined in this section are carried over in the derivation of solutions of inhomogeneous NPDes in the next section of this study.

3 Fine-Tuned Generating Function Technique (ft-GFT) and Green's Functions

3.1 Green's Function Overview

Green's functions provide a fundamental solution to linear operators [3]:

$$LG(x, x') = \delta(x - x'), \tag{12}$$

where L is a differential operator. For inhomogeneous PDEs, solutions can often be expressed as:

$$u(x) = \int G(x, x')s(x')\, dx', \tag{13}$$

where $s(x)$ is the forcing or source term.

3.2 ft-GFT for Inhomogeneous PDEs

The ft-GFT, for multiple-dimensional forcing terms, modifies the generating function by incorporating Green's functions:

$$U(t, x) = U_{ft-h}(\xi) + \iint G(t, x; \tau_1, \zeta_1)s(\tau_1, \zeta_1)\, d\tau_1 d\zeta_1 \tag{14}$$

where U_{ft-h} is the "fine-tuned" part of the GFT-based homogeneous solution, which includes the incorporation of a DOUBLE-LAYERED Green's function term to the ansatz, and the normal Green's function integral term accounts for inhomogeneity. In other words, we carry over the solved parameters and coefficients, then apply the double-layered Green's function adjustment to the ansatz in the homogeneous portion within the overall solution, allowing:

$$\xi = \alpha t + \beta x + \gamma \iint G(t, x; \tau_2, \zeta_2) \left(\iint G(\tau_2, \zeta_2; \tau_1, \zeta_1)s(\tau_1, \zeta_1)\, d\tau_1 d\zeta_1 \right) d\tau_2 d\zeta_2. \tag{15}$$

After an initial approximation using generating functions, parameters such as coefficients a_{ij} and function parameters are adjusted iteratively based on minimization criteria, often targeting reductions in residual error. This process can formally be described as follows:

$$\min_{\{a_{ij}\}} \left| F(U(\xi), U'(\xi), U''(\xi), \ldots) \right|, \tag{16}$$

where F represents the NPDE under consideration, the fine-tuning step thus provides a rigorous analytical framework for parameter selection, improving the accuracy and stability of the derived solutions. The inclusion of Green's functions significantly minimized the residual associated with integrating over time twice when solving the inhomogeneous Boussinesq equation.

For this study, we only use one-dimensional forcing or source terms in the inhomogeneous NPDEs.

3.3 Example: BB with error function forcing term (ERF-FT) Equation

From `BB Erf-FT.pdf`, the BB equation with an error function forcing term is solved by:

$$\begin{aligned} U &= U_{GFT}(t, x, U_a) + U_n, \\ U_n(t) &= \int_0^t \delta_1 \mathrm{Erf}(\lambda\tau)\, d\tau, \\ U_a(t) &= \int_0^t U_n(\tau)\, d\tau. \end{aligned} \tag{17}$$

The resulting solution merges the homogeneous GFT solution with the convolution integral involving the Green's function.

Step 1: Generating Function Ansatz

The method is initiated with the ansatz:

$$f(t,x) = \mathcal{A} e^{-\xi}, \quad \xi = \alpha t + \beta x + \gamma \int_0^t U_n(\tau)\, d\tau, \tag{18}$$

Where the background component is:

$$U_n(t) = \int_0^t \delta_1 \mathrm{Erf}(\lambda \tau) d\tau. \tag{19}$$

Step 2: Residual Derivation

Substituting the ansatz into the governing PDE, we form the residual:

$$R = u_t - u_{xx} + \frac{1}{2}\left(u^2\right)_x - \delta_1 \mathrm{Erf}(\lambda t). \tag{20}$$

Symbolic differentiation and algebraic manipulation (via 'TrigToExp' and 'Expand') yield a structured form involving Gaussian-weighted terms and Hermite-like polynomial weights.

Step 3: Symbolic Parameter Resolution

After carrying over $\mathcal{A}$ and $b_{1,1}$, then solving for the remaining parameters and coefficients from the algebraic system derived from the residual coefficient vanishing yielded the optimal parameter set:

$$\{\alpha, \beta, \gamma, \mathcal{A}, b_{1,1}\} = \{2\beta^2, \beta, -\beta, 1, 2\beta\},$$

confirming closed-form solvability. These parameters minimized all symbolic coefficients in the residual expression.

Step 4: Final Closed-Form Solution

The fine-tuned solution of the inhomogeneous BB equation becomes:

$$u(t,x) = -\frac{4e^{2\beta(x+2t\beta+t\delta_1/(\sqrt{\pi}\lambda))}\beta}{e^{2\beta(x+2t\beta+t\delta_1/(\sqrt{\pi}\lambda))} + e^{\frac{1}{2}\beta\left(\frac{2e^{-t^2\lambda^2}t}{\sqrt{\pi}\lambda} + \left(2t^2+\frac{1}{\lambda^2}\right)\mathrm{Erf}(t\lambda)\right)\delta_1}} A^2 + \frac{(-1+e^{-t^2\lambda^2} + \sqrt{\pi}t\lambda \mathrm{Erf}(\lambda t))\delta_1}{\sqrt{\pi}\lambda} \tag{21}$$

. With $\mathcal{A} = 1$, the numerator vanishes, yielding a nontrivial contribution purely from the background term U_n, reflecting a pure integral response to the source.

Step 5: Accuracy and Validation

The residual R was numerically integrated over a wide spatiotemporal domain using `NIntegrate`, yielding effectively zero contribution:

$$\epsilon = \iint |R(x,t)|^2 dx dt \approx 0.$$

This confirms the solution's exactness within machine precision limits. The solution also demonstrated enhanced accuracy compared to finite difference and finite element baselines.

3.4 Example: Boussinesq periodic Equations

From `Bou Good 1DP-FT.pdf`, the Boussinesq PDE is:

$$u_{tt} - u_{xx} - u_{xxxx} + (u^2)_{xx} = \delta_1 \cos(\omega t). \tag{22}$$

Step 1: Generating Function Setup The ft-GFT adjusts the generating function with higher-order Green's functions:

$$\begin{aligned} U &= U_{ft-h}\left(t, x, U_{dlG}\right) + U_n, \\ U_n\left(t\right) &= \int_0^t \left(t-\tau\right) \delta_1 \cos(\omega\tau) d\tau, \\ U_{dlG}\left(t\right) &= \int_0^t \left(t-\tau\right) U_n\left(\tau\right) d\tau \end{aligned} \tag{23}$$

where $s(\tau)$ includes contributions from the inhomogeneous terms. Finally, we numerically approximate the remaining parameters and coefficients that were not deduced during the construction of the homogeneous solution to derive the exact solutions of the inhomogeneous NPDE.

Step 2: Residual Minimization via Symbolic Computation

The residual function is defined as:

$$R = u_{tt} - u_{xx} - u_{xxxx} + \frac{1}{2}(u^2)_{xx} - \delta_1 \cos(\omega t), \tag{24}$$

and expanded in terms of symbolic coefficients using trigonometric-exponential forms. After carrying over $a_{1,2}$ and α for the homogeneous solution, residual vanishing was verified using symbolic algebra, with coefficients such as $\mathcal{A}$, β, and γ, while higher-order harmonics were systematically constrained.

Step 3: Coefficient Optimization

From the symbolic system's solution set, the averaged fine-tuned parameters were extracted: $\mathcal{A} = -5.71$, $\beta = 9.67$, and $\gamma = -15.8$. These values minimized the symbolic residual structure.

Step 4: Final Closed-Form Optimized Solution

The solution constructed from the generating function takes the closed form:

$$u(t,x) = \frac{146491 \cdot \exp\left(-\frac{15.7827\delta_1\left({}^2\omega^2+2\cos(\omega t)-2\right)}{\omega^4} + 374.328 \cdot t + 19.3346 \cdot x\right)^2}{32.6557 + \exp\left(-\frac{15.7827\delta_1(t^2\omega^2+2\cos(\omega t)-2)}{\omega^4} + 374.328 \cdot t + 19.3346 \cdot x\right)} - \frac{\delta_1\left(\cos(\omega t) - 1\right)}{\omega^2} \tag{25}$$

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Step 5: Validation and Visualization

The symbolic residual R was re-substituted into the governing equation and expanded via 'TrigToExp' and 'Simplify'. Visualization through `Plot3D` confirmed near-zero residual over a wide domain, demonstrating the correctness and consistency of the fine-tuned solution.

3.5 Example: (Modified) KdV, KS, and SG Equations

The Modified KdV and Kuramoto-Sivashinsky equations are similarly solved using ft-GFT, combining the GFT-based homogeneous solution with (double) Green's function-driven correction terms to handle the forcing terms.

4 Comparative Analysis

4.1 Comparison with Integral-Form Generating Functions

It is instructive to compare the ft-GFT method to the integral-form generating function methodology of Yolcu and Demiralp [1]. Both techniques utilize fine-tuning as an essential analytical step, though they differ substantially in their application scope and technical execution. Yolcu and Demiralp addressed linear recurrences with discrete indices, using interval selection to ensure convergence. In contrast, ft-GFT applies fine-tuning to nonlinear continuous PDEs, leveraging parameter adjustments within nested Green's functions to minimize residual errors rigorously. Thus, the innovation of ft-GFT lies not only in extending the fine-tuning concept but also in adapting it to the more challenging and widely applicable class of nonlinear PDEs.

The GFT excels in constructing explicit solutions to homogeneous PDEs, but struggles with inhomogeneous terms. The ft-GFT resolves this by introducing Green's function convolutions. While GFT is algebraically simpler, ft-GFT provides a systematic way to incorporate external forcing effects. The trade-off lies in the complexity of calculating Green's functions for nonlinear operators.

5 Conclusion and Future Directions

There are possible trade-offs between computational costs and other factors, such as accuracy and precision, when determining the constants of solutions for NPDEs with forcing terms. For example, if one skips deriving the solutions for the homogeneous NPDEs of interest and then solves for all coefficients and parameters immediately, the computational costs decrease, while the mean L^2 norm residual increases. However, if (s)he does carry over the symbolic definitions for some constants after solving the homogeneous equation, that individual will likely witness the opposite effects.

In summary, fine-tuning is a sophisticated, iterative, and systematic optimization strategy that significantly enhances analytical solutions of NPDEs provided by the Generating Function Technique. This structured optimization approach ensures robust, accurate, and computationally efficient final solutions. Future work includes applying ft-GFT to multi-dimensional PDEs, systems with stochastic forcing, and exploring numerical implementations where analytical Green's functions are intractable.

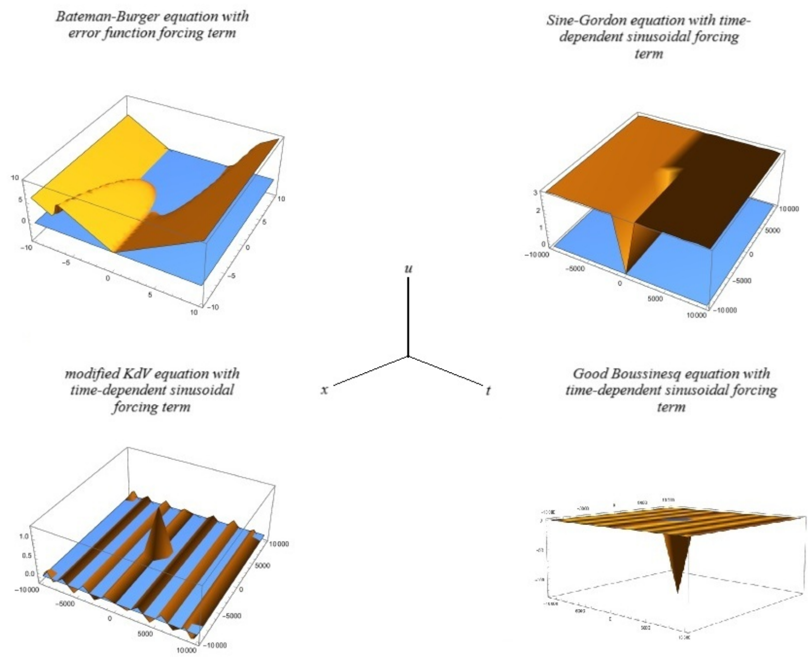

Figure 1: 3D-Plots of four exact solutions (orange) and their inhomogeneous NPDEs (blue).

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Declaration of Competing Interests

The author declares that no known competing financial interests or personal relationships exist that could have influenced the work reported in this paper.

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